Simpson 3/8 Rule for Integration

Introduction

The main objective of this chapter is to develop appropriate formulas for approximating the integral of the form

$$I = \int_a^b f(x)dx \qquad (1)$$

Most (if not all) of the developed formulas for integration are based on a simple concept of approximating a given function $f(x)$ by a simpler function (usually a polynomial function) $f_i(x)$, where i represents the order of the polynomial function. In Chapter 07.03, Simpsons 1/3 rule for integration was derived by approximating the integrand $f(x)$ with a 2nd order (quadratic) polynomial function. $f_2(x)$

$$f_2(x) = a_0 + a_1x + a_2x^2 \qquad (2)$$

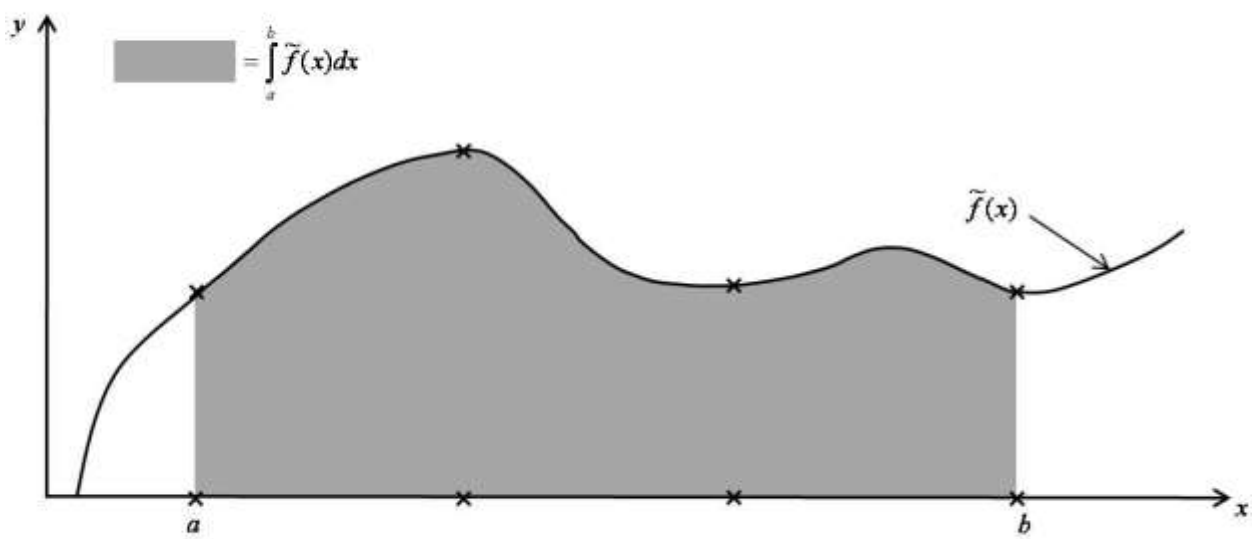

Figure 1 $\tilde{f}(x)$ Cubic function.

In a similar fashion, Simpson 3/8 rule for integration can be derived by approximating the given function $f(x)$ with the 3rd order (cubic) polynomial $f_3(x)$

$$
\left.\begin{aligned}
f_3(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\
&= \{1, x, x^2, x^3\} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}
\end{aligned}\right\} \tag{3}
$$

which can also be symbolically represented in Figure 1.

<u>Method 1</u>

The unknown coefficients a_0, a_1, a_2 and a_3 in Equation (3) can be obtained by substituting 4 known coordinate data points $\{x_0, f(x_0)\}, \{x_1, f(x_1)\}, \{x_2, f(x_2)\}$ and $\{x_3, f(x_3)\}$ into Equation (3) as follows.

$$
\left.\begin{aligned}
f(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^2 \\
f(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^2 \\
f(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^2 \\
f(x_3) &= a_0 + a_1 x_3 + a_2 x_3^2 + a_3 x_3^2
\end{aligned}\right\} \tag{4}
$$

Equation (4) can be expressed in matrix notation as

$$
\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} \tag{5}
$$

The above Equation (5) can symbolically be represented as

$$
[A]_{4\times 4} \vec{a}_{4\times 1} = \vec{f}_{4\times 1} \tag{6}
$$

Thus,

$$
\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = [A]^{-1} \times \vec{f} \tag{7}
$$

Substituting Equation (7) into Equation (3), one gets

$$
f_3(x) = \{1, x, x^2, x^3\} \times [A]^{-1} \times \vec{f} \tag{8}
$$

As indicated in Figure 1, one has

$$\left.\begin{aligned}
x_0 &= a \\
x_1 &= a + h \\
&= a + \frac{b-a}{3} \\
&= \frac{2a+b}{3} \\
x_2 &= a + 2h \\
&= a + \frac{2b-2a}{3} \\
&= \frac{a+2b}{3} \\
x_3 &= a + 3h \\
&= a + \frac{3b-3a}{3} \\
&= b
\end{aligned}\right\} \tag{9}$$

With the help from MATLAB [Ref. 2], the unknown vector $\vec{a}$ (shown in Equation 7) can be solved for symbolically.

<u>Method 2</u>

Using Lagrange interpolation, the cubic polynomial function $f_3(x)$ that passes through 4 data points (see Figure 1) can be explicitly given as

$$\begin{aligned}
f_3(x) = {} & \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1) \\
& + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_3) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3)
\end{aligned} \tag{10}$$

Simpsons 3/8 Rule for Integration

Substituting the form of $f_3(x)$ from Method (1) or Method (2),

$$\begin{aligned}
I &= \int_a^b f(x)dx \\
&\approx \int_a^b f_3(x)dx \\
&= (b-a) \times \frac{\{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\}}{8}
\end{aligned} \tag{11}$$

Since

$$h = \frac{b-a}{3}$$

$$b-a=3h$$

and Equation (11) becomes

$$I \approx \frac{3h}{8} \times \{ f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \} \tag{12}$$

Note the 3/8 in the formula, and hence the name of method as the Simpson's 3/8 rule.
The true error in Simpson 3/8 rule can be derived as [Ref. 1]

$$E_t = -\frac{(b-a)^5}{6480} \times f''''(\zeta) \text{ , where } a \le \zeta \le b \tag{13}$$

Example 1

The vertical distance in meters covered by a rocket from $t=8$ to $t=30$ seconds is given by

$$s = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Use Simpson 3/8 rule to find the approximate value of the integral.

Solution

$$h = \frac{b-a}{n}$$

$$= \frac{b-a}{3}$$

$$= \frac{30-8}{3}$$

$$= 7.3333$$

$$f(t) = 2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t$$

$$I \approx \frac{3h}{8} \times \{ f(t_0) + 3f(t_1) + 3f(t_2) + f(t_3) \}$$

$$t_0 = 8$$

$$f(t_0) = 2000 \ln \left(\frac{140000}{140000 - 2100 \times 8} \right) - 9.8 \times 8$$

$$= 177.2667$$

$$\left\{ \begin{aligned} t_1 &= t_0 + h \\ &= 8 + 7.3333 \\ &= 15.3333 \\ f(t_1) &= 2000 \ln \left(\frac{140000}{140000 - 2100 \times 15.3333} \right) - 9.8 \times 15.3333 \\ &= 372.4629 \end{aligned} \right.$$

$$\begin{cases} t_2 = t_0 + 2h \\ \quad = 8 + 2(7.3333) \\ \quad = 22.6666 \\ f(t_2) = 2000\ln\left(\dfrac{140000}{140000 - 2100 \times 22.6666}\right) - 9.8 \times 22.6666 \\ \qquad = 608.8976 \end{cases}$$

$$\begin{cases} t_3 = t_0 + 3h \\ \quad = 8 + 3(7.3333) \\ \quad = 30 \\ f(t_3) = 2000\ln\left(\dfrac{140000}{140000 - 2100 \times 30}\right) - 9.8 \times 30 \\ \qquad = 901.6740 \end{cases}$$

Applying Equation (12), one has

$$I = \frac{3}{8} \times 7.3333 \times \{177.2667 + 3 \times 372.4629 + 3 \times 608.8976 + 901.6740\}$$
$$= 11063.3104m$$

The exact answer can be computed as

$$I_{exact} = 11061.34m$$

Multiple Segments for Simpson 3/8 Rule

Using n = number of equal segments, the width h can be defined as

$$h = \frac{b-a}{n} \tag{14}$$

The number of segments need to be an integer multiple of 3 as a single application of Simpson 3/8 rule requires 3 segments.

The integral shown in Equation (1) can be expressed as

$$\begin{aligned} I &= \int_a^b f(x)dx \\ &\approx \int_a^b f_3(x)dx \\ &\approx \int_{x_0=a}^{x_3} f_3(x)dx + \int_{x_3}^{x_6} f_3(x)dx + \text{........} + \int_{x_{n-3}}^{x_n=b} f_3(x)dx \end{aligned} \tag{15}$$

Using Simpson 3/8 rule (See Equation 12) into Equation (15), one gets

$$I=\frac{3h}{8}\left\{\begin{aligned}&f(x_0)+3f(x_1)+3f(x_2)+f(x_3)+f(x_3)+3f(x_4)+3f(x_5)+f(x_6)\\&+.....+f(x_{n-3})+3f(x_{n-2})+3f(x_{n-1})+f(x_n)\end{aligned}\right\} \tag{16}$$

$$=\frac{3h}{8}\left\{f(x_0)+3\sum_{i=1,4,7,..}^{n-2}f(x_i)+3\sum_{i=2,5,8,..}^{n-1}f(x_i)+2\sum_{i=3,6,9,..}^{n-3}f(x_i)+f(x_n)\right\} \tag{17}$$

Example 2

The vertical distance in meters covered by a rocket from $t=8$ to $t=30$ seconds is given by

$$s=\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

Use Simpson 3/8 multiple segments rule with six segments to estimate the vertical distance.

Solution

In this example, one has (see Equation 14):

$$f(t)=2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t$$

$$h=\frac{30-8}{6}=3.6666$$

$$\{t_0,f(t_0)\}=\{8,177.2667\}$$

$$\{t_1,f(t_1)\}=\{11.6666,270.4104\}\text{ where } t_1=t_0+h=8+3.6666=11.6666$$

$$\{t_2,f(t_2)\}=\{15.3333,372.4629\}\text{ where } t_2=t_0+2h=15.3333$$

$$\{t_3,f(t_3)\}=\{19,484.7455\}\text{ where } t_3=t_0+3h=19$$

$$\{t_4,f(t_4)\}=\{22.6666,608.8976\}\text{ where } t_4=t_0+4h=22.6666$$

$$\{t_5,f(t_5)\}=\{26.3333,746.9870\}\text{ where } t_5=t_0+5h=26.3333$$

$$\{t_6,f(t_6)\}=\{30,901.6740\}\text{ where } t_6=t_0+6h=30$$

Applying Equation (17), one obtains:

$$I=\frac{3}{8}(3.6666)\left\{177.2667+3\sum_{i=1,4,..}^{n-2=4}f(t_i)+3\sum_{i=2,5,..}^{n-1=5}f(t_i)+2\sum_{i=3,6,..}^{n-3=3}f(t_i)+901.6740\right\}$$

$$=(1.3750)\left\{\begin{aligned}&177.2667+3(270.4104+608.8976)\\&+3(372.4629+746.9870)+2(484.7455)+901.6740\end{aligned}\right\}$$

$$=11{,}601.4696m$$

Example 3

Compute

$$I=\int_8^{30}\left\{2000\ln\left(\frac{140000}{140000-2100t}\right)-9.8t\right\}dt,$$

using Simpson 1/3 rule (with $n_1=4$), and Simpson 3/8 rule (with $n_2=3$).

Solution

The segment width is

$$
\begin{aligned}
h &= \frac{b-a}{n} \\
&= \frac{b-a}{n_1+n_2} \\
&= \frac{30-8}{(4+3)} \\
&= 3.1429
\end{aligned}
$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t$$

$$
\left.
\begin{aligned}
t_0 &= a = 8 \\
t_1 &= x_0 + 1h = 8 + 3.1429 = 11.1429 \\
t_2 &= t_0 + 2h = 8 + 2(3.1429) = 14.2857 \\
t_3 &= t_0 + 3h = 8 + 3(3.1429) = 17.4286 \\
t_4 &= t_0 + 4h = 8 + 4(3.1429) = 20.5714
\end{aligned}
\right\} \text{Simpson's } 1/3 \text{ rule}
$$

$$
\begin{aligned}
t_5 &= t_0 + 5h = 8 + 5(3.1429) = 23.7143 \\
t_6 &= t_0 + 6h = 8 + 6(3.1429) = 26.8571 \\
t_7 &= t_0 + 7h = 8 + 7(3.1429) = 30
\end{aligned}
$$

Now

$$
\begin{aligned}
f(t_0 = 8) &= 2000\ln\left(\frac{140{,}000}{140{,}000-2100\times 8}\right)-9.8\times 8 \\
&= 177.2667
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
f(t_1) &= 256.5863 \\
f(t_2) &= 342.3241 \\
f(t_3) &= 435.2749 \\
f(t_4) &= 536.3909 \\
f(t_5) &= 646.8260 \\
f(t_6) &= 767.9978 \\
f(t_7) &= 901.6740
\end{aligned}
$$

For multiple segments $(n_1 = \text{first } 4 \text{ segments})$, using Simpson 1/3 rule, one obtains (See Equation 19):

$$I_1 = \left(\frac{h}{3}\right)\left\{ f(t_0) + 4\sum_{i=1,3,\ldots}^{n_1-1=3} f(t_i) + 2\sum_{i=2,\ldots}^{n_1-2=2} f(t_i) + f(t_{n_1}) \right\}$$

$$= \left(\frac{h}{3}\right)\{ f(t_0) + 4(f(t_1) + f(t_3)) + 2f(t_2) + f(t_4) \}$$

$$= \left(\frac{3.1429}{3}\right)\{177.2667 + 4(256.5863 + 435.2749) + 2(342.3241) + 536.3909\}$$

$$= 4364.1197$$

For multiple segments $(n_2 = \text{last 3 segments})$, using Simpson 3/8 rule, one obtains (See Equation 17):

$$I_2 = \left(\frac{3h}{8}\right)\left\{ f(t_0) + 3\sum_{i=1,3,\ldots}^{n_2-2=1} f(t_i) + 3\sum_{i=2,\ldots}^{n_2-1=2} f(t_i) + 2\sum_{i=3,6,..}^{n_2-3=0} f(t_i) + f(t_{n_1}) \right\}$$

$$= \left(\frac{3h}{8}\right)\{ f(t_0) + 3f(t_1) + 3f(t_2) + 2(\text{no contribution}) + f(t_3) \}$$

$$= \left(\frac{3h}{8}\right)\{ f(t_4) + 3f(t_5) + 3f(t_6) + f(t_7) \}$$

$$= \left(\frac{3}{8} \times 3.1429\right)\{536.3909 + 3(646.8260) + 3(767.9978) + 901.6740\}$$

$$= 6697.3663$$

The mixed (combined) Simpson 1/3 and 3/8 rules give

$$I = I_1 + I_2$$

$$= 4364.1197 + 6697.3663$$

$$= 11061m$$

Comparing the truncated error of Simpson 1/3 rule

$$E_t = -\frac{(b-a)^5}{2880} \times f''''(\zeta) \tag{18}$$

With Simpson 3/8 rule (See Equation 12), it seems to offer slightly more accurate answer than the former. However, the cost associated with Simpson 3/8 rule (using 3rd order polynomial function) is significantly higher than the one associated with Simpson 1/3 rule (using 2nd order polynomial function).

The number of multiple segments that can be used in the conjunction with Simpson 1/3 rule is 2, 4, 6, 8, … (any even numbers) for

$$I = \int_a^b f(x)dx$$

$$\approx \left(\frac{h}{3}\right)\{f(x_0)+4f(x_1)+f(x_2)+f(x_2)+4f(x_3)+f(x_4)+.....+f(x_{n-2})+4f(x_{n-1})+f(x_n)\}$$

$$= \left(\frac{h}{3}\right)\left\{f(x_0)+4\sum_{i=1,3,...}^{n-1} f(x_i)+2\sum_{i=2,4,6...}^{n-2} f(x_i)+f(x_n)\right\} \tag{19}$$

However, Simpson 3/8 rule can be used with the number of segments equal to 3,6,9,12,.. (can be certain integers that are multiples of 3).

If the user wishes to use, say 7 segments, then the mixed Simpson 1/3 rule (for the first 4 segments), and Simpson 3/8 rule (for the last 3 segments) would be appropriate.

Computer Algorithm for Mixed Simpson 1/3 and 3/8 Rule for Integration

Based on the earlier discussion on (single and multiple segments) Simpson 1/3 and 3/8 rules, the following "pseudo" step-by-step mixed Simpson rules for estimating

$$I = \int_a^b f(x)dx$$

can be given as

Step 1

User inputs information, such as

$f(x)$ = integrand

n_1 = number of segments in conjunction with Simpson 1/3 rule (a multiple of 2 (any even numbers)

n_2 = number of segments in conjunction with Simpson 3/8 rule (a multiple of 3)

Step 2

Compute

$$n = n_1 + n_2$$

$$h = \frac{b-a}{n}$$

$$x_0 = a$$

$$x_1 = a + 1h$$

$$x_2 = a + 2h$$

.

.

$$x_i = a + ih$$

.

.

$$x_n = a + nh = b$$

Step 3

Compute result from multiple-segment Simpson 1/3 rule (See Equation 19)

$$I_1 = \left(\frac{h}{3}\right)\left\{ f\left(x_0\right) + 4\sum_{i=1,3,\ldots}^{n_1-1} f\left(x_i\right) + 2\sum_{i=2,4,6\ldots}^{n_1-2} f\left(x_i\right) + f\left(x_{n_1}\right)\right\} \quad \text{(19, repeated)}$$

<u>Step 4</u>

Compute result from multiple segment Simpson 3/8 rule (See Equation 17)

$$I_2 = \left(\frac{3h}{8}\right)\left\{ f\left(x_0\right) + 3\sum_{i=1,4,7\ldots}^{n_2-2} f\left(x_i\right) + 3\sum_{i=2,5,8\ldots}^{n_2-1} f\left(x_i\right) + 2\sum_{i=3,6,9,\ldots}^{n_2-3} f\left(x_i\right) + f\left(x_{n_2}\right)\right\} \quad \text{(17, repeated)}$$

<u>Step 5</u>

$$I \approx I_1 + I_2 \quad (20)$$

and print out the final approximated answer for I.

Reference

SIMPSON'S 3/8 RULE FOR INTEGRATION	
Topic	Simpson 3/8 Rule for Integration
Summary	Textbook Chapter of Simpson's 3/8 Rule for Integration
Major	General Engineering
Authors	Duc Nguyen
Date	March 28, 2022