كلية : التربية للعلوم الصرفة

القسم او الفرع :الرياضيات

المرحلة: الثالثة

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اسم المادة بالغة العربية : التحليل الرياضي

اسم المادة باللغة الإنكليزية : **Mathematical Analysis**

اسم الحاضرة الثالثة باللغة العربية: الفضاء المتري

اسم المحاضرة الثالثة باللغة الإنكليزية: **metric space**

Mathematical Analysis

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اسم الحاضرة الثالثة باللغة العربية:الفضاء المتري

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Metric space

Definition 2.1: let X be anon-empty set and $d: X \times X \to R^+$ be a mapping. We say that order (X, d) is metric space if it is satisfying the following:

1. $d(x,y) \geq 0 , \forall x, y \in X$
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq d(x,y) + d(y,z)$
4. $d(x,y) = 0 \leftrightarrow x = y$

Not: d is called metric mapping

$d(x,y)$ is a distance between x and y

Remark: A mapping $d: X \times X \to R^+$ is called a pseudo metric for X iff d satisfies (1,2,3) in the above definition and $d(x,x) = 0 , \forall x \in X$

Cauchy - Shwarz inequality

Let $a = (a_1, a_2 , \dots , a_n)$ and $b = (b_1 , b_2 , \dots b_n)$ be two tripe of complex number , then:

$$\sum_{i=1}^{n} |a_i . b_i| \leq \left(\sum_{i=1}^{n} |a_i|^2\right)^{\frac{1}{2}} . \left(\sum_{i=1}^{n} |b_i|^2\right)^{\frac{1}{2}}$$

Minkowskis inequality

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}} \quad , p \geq 1$$

Example 2.1: if $X = R$ and $d(x,y) = |x - y|$, show that (X,d) is a metric space.

Solution:

1. $d(x,y) = |x - y| \geq 0$ by def. of Absolute value
2. $d(x,y) = |x - y| = |-(y - x)| = |y - x| = d(y,x)$
3. $d(x,z) = |x - z| = |x - y + y - z|$

   $\leq |x - y| + |y - z|$

   $= d(x,y) + d(y,z)$
4. $d(x,y) = 0$ iff $x = y$

   $d(x,y) = 0$ iff $|x - y| = 0$

   iff $x - y = 0$

   iff $x = y$

   $\therefore (X,d)$ is a metric space

Discrete metric space

Let $X \neq \emptyset$ and $d: X \times X \to R$ s.t

$$d(x,y) = \begin{cases} 0 & if\ x = y \\ 1 & if\ x \neq y \end{cases}$$

$\forall x, y$, show that (X,d) is metric space

Solution:

1. $d(x,y) \geq 0$, $\forall x, y \in X$ (by def. d)
2. $d(x,y) = d(y,x)$?

   if $x = y \to d(x,y) = 0 = d(x,y)$

   if $x \neq y \to d(x,y) = 1 = d(y,x)$
3. Let $x, y, z \in X$ T.P $d(x,y) \leq d(x,y) + d(y,z)$?

   if $x = z$ then $d(x,z) = 0$

   since $d(x,y) \geq 0$ and $d(y,z) \geq 0$ then

   $d(x,z) \leq d(x,y) + d(y,z)$

   if $x \neq z$ then $d(x,z) = 0$

   since $d(x,z) = 1$ and either $x \neq y$ or $x \neq z$, $y = z$

   either: $d(x,z) = d(x,y) = d(y,z) = 1$

or: $d(x,z) = d(x,y) = 1$ and $d(y,z) = 0$

then: $d(x,z) \leq d(x,y) + d(y,z)$

$1 \quad \leq \quad 1 \quad + \quad 1$

$1 \quad \leq \quad 1 \quad + \quad 0$

Example 2.2: show that (X,d) is pseudo metric space but not metric where $d: X \times X \to R \, , d(x,y) = |x^2 - y^2| \; , forall \; x,y \in R$.

Solution:

Let $x,y,z, \in R$

1- $d(x,y) = |x^2 - y^2| \geq 0$, by def Abs. Value

2- $d(x,y) = |x^2 - y^2| = |-(y^2 - x^2)| = |y^2 - x^2| = d(y,x)$

3- $d(x,y) = |x^2 - y^2| = |x^2 - z^2 + z^2 - y^2| \leq |x^2 - z^2| + |z^2 - y^2|$

$\leq d(x,z) + d(z,y)$

4- $d(x,x) = |x^2 - x^2| = 0 \, , \forall x \in R$

$\therefore$ (X,d) pseudo metric space but not metric space,

since, if $d(x,y) = 0 \to |x^2 - y^2| = 0 \to \; x^2 - y^2 = 0 \to x^2 = y^2$

$\nrightarrow x = y$

ex: let $x = 1 \, , y = -1$

then $d(x,y) = d(1,-1) = |\, 1^2 - (-1)^2| = 0$, but $1 \neq -1$

Definition 2.2: let (X,d) be a metric space $S,T \subseteq X \, , p \in S$ then

1- The distance between p and S is

$d(p,S) = \inf\{d(p,x) : x \in S\}$

2- The distance between S and T is

$d(S,T) = \inf\{d(x,y) : x \in S \, , y \in T\}$

3- Diameter of S is $d(S) = \sup\{d(x,y) : x,y \in S\}$

4- S is called bounded, if $\exists \; M \in R^{++}$, s.t $d(x,y) \leq M \, , \forall x,y \in S$.

Definition 2.3: let (X, d) be a metric space and $S \subseteq X$, S is called open set, if $\forall x \in S$, $\exists r > 0$ s.t $B(x, r) \subset S$

Example 2.3: if $S = \emptyset$, then S is open set

If $x \in S \rightarrow \exists r > 0 \ s.t \ B(x,r) \subset S$

$F \rightarrow F \ or \ T : T$

If S = X , then S is open set

Solution:

Since all balls is contains in X

Any open interval is open set. But the convers is not true

Solution:

Let $x \in s \rightarrow x \in (a,b) \subseteq (a,b) = S.$

So. S is open set

Example 2.4: Let $S = (-1,1) \cup (2,3)$

Let $x \in s$, then $x \in (-1,1) \ or \ x \in (2,3)$

Then $x \in (-1,1) \subset S \ or \ x \in (2,3) \subset S$

∴ S is open set. But is not open interval

Any ball is open set.

Proof:

$\forall y \in B(x,r), \exists w > 0, s.t \ B(y,w) \subset B(x,r)$?

Let $w = r - d(x,y) > 0$

Let $Z \in B(y,w) \longrightarrow d(z,y) < w$

$d(Z,y) \leq d(x,y) + d(y,z)$

$\leq d(x,y + w$

$= d(x,y) + r - d(x,y)$

$= r$

Then $Z \in B(x,r) \longrightarrow B(y,w) \subset B(x,r)$
This is true for all y in B(x,r)
So B(x,r) is open set

$\boldsymbol{S = \{x\}, x \in R}$ is not open set

Since there is not open interval in S Containing x and Contained in S
i.e (($\forall r > 0, \exists B(x,r) = (x-r, x+r) \subset S$))