كلية : التربية للعلوم الصرفة

القسم او الفرع :الرياضيات

المرحلة: الثالثة

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اسم المادة بالغة العربية : التحليل الرياضي

اسم المادة باللغة الإنكليزية : Mathematical Analysis

اسم الحاضرة الخامسة باللغة العربية: خواص المجاميع المغلقه في الفضاء المتري والفضاء المرصوص

اسم المحاضرة الخامسة باللغة الإنكليزية : Properties of closed sets in metric space and compact space

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Mathematical Analysis

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Remark 2.1: the infinite union of closed sets is not necessary closed set

Example 2.6: let $S_n = \left\{ \left[\frac{-n}{n+1}, \frac{n}{n+1}\right] : n \in N \right\}$, S_n is closed interval, Is $\bigcup_{n=1}^{\infty} S_n$ is closed?

Solution:

If $n = 1 \rightarrow S_1 = \left[\frac{-1}{2}, , \frac{1}{2}\right]$

If $n = 2 \rightarrow S_2 = \left[\frac{-2}{3}, \frac{2}{3}\right]$

. .

. .

When $n \rightarrow \infty \Longrightarrow \lim_{n\rightarrow\infty} \frac{\pm n}{n+1} = \lim_{n\rightarrow\infty} \frac{\pm\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} = \pm 1$

$\therefore \bigcup_{n=1}^{\infty} s_n = (-1,1)$ open set

Theorem 2.5: The infinite intersection of closed set S is closed?

Def: let X be a metric space and $S \subseteq X, p \in X$, p is called an accumulation point of S if every open set contain p , contains another point q s.t $p \neq q , q \in S$.

i.e.: p is a cc. point of S if $\forall U$, U is open set $p \in U$, then $U - P \cap S \neq \phi$

Remark 2.2: Since every open set is Union balls. So, we can define acc. Point as following:

P is acc. Point of S, if $\forall r > 0 \; B(p,r) - \{p\} \cap S \neq \phi$

* S' is the closure of all acc. Point of S (Derived set)
* $\overline{S}$ is the closure of S and $\overline{S} = S \cup S'$
* P is not acc. Point, if $\exists U$, U is open and $p \in U$
  S.t $U - \{p\} \cap S = \phi$. (i.e. $\exists r > 0$, $B(r,p) - \{ p \} \cap S = \phi$

Example 2.7: let $s = \{1,5\}$, find S' *and* $\overline{S}$

Solution: TO find S' there are some cases

$x = 1$, $x = 5$, $x < 1$, $x > 5$, $1 < x < 5$

If $x = 1 \rightarrow$ x is not acc. Point since , $\exists r > 0$

$B(x,r) - \{x\} \cap S = \emptyset$, when $r = 1$

$B(1,1) - \{1\} \cap \{1,5\} = (0,2) - \{1\} \cap [1,5\} = \emptyset$

If $x = 5 \rightarrow$ x is not acc. Point, since $\exists r > 0$, $B(x,r) - \{x\} \cap S = \emptyset$, when $r = 1$

$\rightarrow B(5,1) - \{5\} \cap \{1,5\} = (4,6) - \{5\} \cap \{1,5\} = \emptyset$

If $x < 1 \rightarrow$ x are not acc. Point since $x \in (x - 1,1)$ and $(x - 1,1) \cap S = \emptyset$

If $x > 5 \rightarrow x$ are not acc. Point, since $x \in (5, x + 1)$ *and* $(5, x + 1) \cap S = \emptyset$

If $1 < x < 5$ are not acc. Point since, $x \in (1,5)$ *and* $(1,5) \cap S = \emptyset$

So, S has no a acc. Point then $S' = \emptyset$ and $\overline{S} = S \cup S' = S \cup \emptyset = S$.

Let $s = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots \right\} = \left\{\frac{1}{n}, n = 1,2,3, \dots . \right\}$ show that $S' = \{0\}$

If $S = (a,b)$, find S'

Solution:

If $x = a \rightarrow x$ is acc. Point since $\forall r > 0$,

$a \in B(0,r) = (a - r, a + r)$ and $B(a,r) - \{a\} \cap S \neq \emptyset$

If $x = b \rightarrow x$ is acc. Point, since $\forall r > 0$, $b \in B(b,r)$

$B(b,r) = (b - r, b + r)$ *and* $B(b,r) - \{b\} \cap (a,b) \neq \emptyset$

If $a < x < b \rightarrow x$ are acc. Point since $\forall\, r > 0$,
$x \in B(x,r) = (x-r, x+r)\ and\ \ B(x,r) - \{x\} \cap S \neq \emptyset$
That is $(x-r, x+r) - \{x\} \cap (a,b) \neq \emptyset$
If $x < a \rightarrow x$ are not acc. Point since $x \in (x-1, a)\ and\ (x-1, a) \cap S = \emptyset$
If $x > b \rightarrow x$ are not acc. Point, since $x \in (b, x+1)\ and\ (b, x+1) \cap (a,b) = \emptyset$
$\therefore S' = [a,b] \rightarrow \overline{S} = S \cup S' = [a,b]$

Definition 2.7: A sub set A of a metric space X is said to be dense if $\overline{A} = X$
Ex: prove that $\bar{Q} = R$ (i.e., Q dense set in R)
Solution:

If $x \in R$, then x is acc. Point in Q.
Since any open interval Contain x Contains infinitely rational and irrationals
Then $Q' = R$
So $\overline{Q} = Q \cup Q' = Q \cup R = R$

Definition 2.8: a metric space is called separable if it has a countable dense subset.
Ex: R separable since Q countable and $Q \subseteq R$, with Q dense in R

Theorem 2.6: let X be a metric space, $S \subseteq X$ then

1- S is closed iff $S' \subset X$
2- $\overline{S}$ is closed set
3- $\overline{S} = S$ iff S closed set
4- $\overline{S}$ is smallest closed set contains S.

Compact Space

Definition 2.9: let (X,d) be a metric space, $\emptyset \neq S \subseteq X$, if the set $\{U_\lambda : U_\lambda \text{ } open \text{ } set, \lambda \in \wedge\}$ is a family of open subsets of X such that $S \subseteq \bigcup_{\lambda\in\wedge} U_\lambda$, then the family $\{U_\lambda\}$ is called open cover for S in X.

- If the family $\{U_\lambda\}$ is finite and $S \subseteq \bigcup_{\lambda\in\wedge} U_\lambda$ then $\{U_\lambda\}$ is called finite cover.
- Let $\{U_\lambda\}$ and $\{U_\alpha\}$ be to open cover for S and $U_\lambda \in \{U_\alpha\}\ \forall\lambda$, then $\{U_\lambda\}$ is called subcover for $\{U_\alpha\}$

Def: let A be a subset of a metric space (X,d), A is called compact set if every open cover for A in X has a finite subcover.

Example 2.8: Any finite subset B of matric space (X, d) is **compact set**

Example 2.9: R is not compact

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Example 2.10: Any open interval A=(a,b) is not compact

Example 2.11: Any closed interval A=[a,b] is Compact.

Proof :

Since we can restrict any open cover for A to finite subcover such as :

Let $\epsilon > 0, B = \{(a-\epsilon, a+\epsilon, (a,b), (b-\epsilon, b+\epsilon)\}$

Theorem: ((Bolzano weir strass theorem))

In compact space X, every infinite subset S of X has at least one accumulation point.

Theorem 2.7: In compact metric space, every closed subset is compact.

Proof : X be a compact metric space, and A be a closed subset of X, then

A^c is open. T.P A is compact.

Let $B = \{U_\lambda : U_\lambda \; is \; open \; set \; in \; X, \forall \lambda \in ^\wedge \}$ be open cover for A.

Then $A \subseteq \bigcup_{\lambda \in ^\wedge} U_\lambda$

Sine $X = A \cup A^c \subseteq (\bigcup_{\lambda \in ^\wedge} U_\lambda) \cup A^c$,

But A^c is open set then $\bigcup_{\lambda \in ^\wedge} U_\lambda \cup A^c$ is open cover for X, since X is compact set , then there exists a finite member $\lambda_1, \lambda_2, \dots \dots, \lambda_n$ such that

$$X = A^c \cup \left(\bigcup_{i=1}^{n} U_{\lambda i} \right)$$

Since that $X = A^c \cup (\bigcup_{i=1}^{n} U_{\lambda i})$ .Since $\cap A^c = \emptyset$, then $A \subseteq \bigcup_{i=1}^{n} U_{\lambda i}$

$\Rightarrow$ B has a finite subcover $\{U_{\lambda 1}, U_{\lambda 2}, \dots \dots, U_{\lambda n}\}$. For A, $\Rightarrow$ A is compact.

Theorem 2.8: Let (X, d) be a metric space, $A \subseteq X$, If A is compact, Then A is closed

Theorem 3.3: Let (X, d) be a metric space, $A \subseteq X$, If A is compact, Then A is bounded

Remark 2.3: In metric space

Compact → Closed + bounded

$\nleftarrow$

Theorem 2.9: Let $\{I_n : n = 1,2,3, \ldots\}$ be a family of closed interval if $I_{n+1} \subset I_n, \forall n, then \ \bigcap_{n=1}^{\infty} I_n = \emptyset$

Theorem 3.5: (**Hien-Bord Theorem**)
Every closed and bounded subset of R^n, $n \geq 1$, is compact.