كلية : التربية للعلوم الصرفة

القسم او الفرع :الرياضيات

المرحلة: الثالثة

أستاذ المادة : م.د. ناديه علي ناظم

اسم المادة بالغة العربية : التحليل الرياضي

اسم المادة باللغة الإنكليزية : **Mathematical Analysis**

اسم الحاضرة السابعة باللغة العربية: تقارب وتباعد المتتابعات في الفضاء المتري

اسم المحاضرة السابعة باللغة الإنكليزية: **converge and diverge the sequences in metric space**

...

Mathematical Analysis

Dr. Nadia Ali

Teaching at the University of Anbar
College of Education for Pure Sciences
Department of Mathematics

2022 – 2023

Example 3.2:

1. $< \frac{(-1)^{n+1}}{n} > = 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots$

   $|x_n| = \left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n} \leq 1 \implies < x_n >$ is bounded

   and $M = 1$

2. $< 5 + \frac{(-1)^{n+1}}{n} > = 6, \frac{9}{2}, \frac{16}{3}, \dots$

   $< x_n \geq 5 + \frac{1}{n} \leq 5 + 1 = 6 \implies < x_n >$ is bounded

   and $M = 6$

3. $< n + (-1)^n > = \begin{cases} < n-1 > , & if\ n\ is\ odd \\ < n+1 > , & if\ n\ is\ even \end{cases}$

4. $|x_n| = \begin{cases} |n-1| \geq 0 \\ |n+1| \geq 2 \end{cases}$

Theorem 3.2: In metric space. Every convergent sequence is bounded.

Proof:

Let $< x_n >$ be a convergent sequence in (X, d) and $x_n \to x$, to prove $< x_n >$ is bounded

Since $x_n \to x \implies \forall \epsilon > 0, \exists k \in N\ s.t\ d(x_n, x) < \epsilon, \forall n > k$

That $\epsilon = 1 \implies d(x_n, x) < 1, \forall n \in k.$

Let $r = \max\{1, d(x_1, x), d(x_2, x), \dots, d(x_n, x)\}$

$\implies d(x_n, x) < r$

$\therefore < x_n >$ is bounded and $M = 2r$

Remark 3.1: The convers of above theorem is not true.

Example 3.3: $<(-1)^n> = -1, 1, -1, 1, \ldots$

$|x_n| = |(-1)^n| = 1 \implies <x_n>$ is bounded and $M = 1$

$<(-1)^n>$ is divergent?

Remake 3.2: If $<x_n>$ unbounded, then $<x_n>$ is divergent.

Proof:

Suppose that $<x_n>$ converged and unbounded sequence.

Since $<x_n>$ Convergent $\to <x_n>$ bounded by theorem (In metric space, every conv. Seq. is bounded) $\to$ C! ,So $<x_n>$ unbounded is $<x_n>$ is divergent

Example 3.4:

- $<x_n> = <\sqrt{n-1}> = 0, \sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots$ unbounded $\implies <x_n>$ divergent

- $<x_n> = <n^2 - n> = 0, 2, 6, 11, \ldots$ unbounded $\implies <x_n>$ divergent

Definition 3.4: Let $<x_n>$ be a real sequence. Then it is called

- Non – decreasing. If $x_{n+1} \geq x_n, \forall n$
- Non – increasing. If $x_{n+1} \leq x_n, \forall n$.
- Not monotone. If it does not increasing and decreasing.

Example 3.5:

* $<x_n> = <\frac{1}{\sqrt{n}}>$

$x_n = \frac{1}{\sqrt{n}}$, $x_{n+1} = \frac{1}{\sqrt{n+1}}$

$\forall n, n+1 > n \implies \sqrt{n+1} > \sqrt{n} \to \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{2}} \to x_{n+1} \leq x_n$

$\therefore < x_n >$ is non – increasing

* $< x_n > = < \frac{n}{n+1} >$

  $x_n = \frac{n}{n+1}$, $x_{n+1} = \frac{n+1}{n+2}$

  $x_{n+1} - x_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)-n(n+2)}{(n+1)(n+2)} = \frac{n^2+2n+1-n^2-2n}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0$

  $\therefore x_{n+1} - x_n > 0 \rightarrow x_{n+1} > x_n , \forall n , \therefore < x_n >$ non – decreasing
* $< x_n > = < (-1)^n >$ not monotone
* $< x_n > = < \frac{(-1)^n}{\sin(n)} >$ not monotone.
* $< x_n > = < (-5)^n >$ not monotone.

Theorem 3.2: Every monotone bounded real seq. is convergent

Example 3.6: $< x_n > = \langle \frac{(-1)^n}{n} \rangle > 0$

$< x_n >$ Convergent seq. but not monotone.

Example 3.7: Show that $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ is convergent.

Theorem 3.3: Let (X, d) be a metric space and $S \subseteq X$:

i. If $< x_n >$ seq. in S and $x_n \rightarrow x$ then $x \in S$ or $x \in S'$
ii. If $x \in S$ or $x \in S'$, then there exists a sequence $< x_n >$ in S s.t $x_n \rightarrow x$

Definition 3.5: The sequence $< x_n >$ is a sub sequence of $< x_n >$, if $< m >$ is increasing sequence in N.