Lecture No. Two

9.3. Reconstruction of a band-limited signal from its samples

Figure 9.3(b) illustrates that the CTFT $X_s(\omega)$ of the sampled signal $x_s(t)$ is a periodic extension of the CTFT of the original signal $x(t)$. By eliminating the replicas in $X_s(\omega)$, we should be able to reconstruct $x(t)$. This is accomplished by applying the sampled signal $x_s(t)$ to the input of an ideal lowpass filter (LPF) with the following transfer function:

$$H(\omega) = \begin{cases} T_s & |\omega| \le \omega_s/2 \\ 0 & \text{elsewhere.} \end{cases} \tag{9.7}$$

The CTFT $Y(\omega)$ of the output $y(t)$ of the LPF is given by $Y(\omega) = X_s(\omega)H(\omega)$, and therefore all shifted replicas at frequencies $\omega > \omega_s/2$ are eliminated. All frequency components within the pass band $\omega \le \omega_s/2$ of the LPF are amplified by a factor of T_s to compensate for the attenuation of $1/T_s$ introduced during sampling. The process of reconstructing $x(t)$ from its samples in the frequency domain is illustrated in Fig. 9.4. We now proceed to analyze the reconstruction process in the time domain.

According to the convolution property, multiplication in the frequency domain transforms to convolution in the time domain. The output $y(t)$ of the lowpass filter is therefore the convolution of its impulse response $h(t)$ with the sampled signal $x_s(t)$. Based on entry (17) of Table 5.2, the impulse response of an ideal lowpass filter with the transfer function given in Eq. (9.7) is given by

$$h(t) = \text{sinc}\left(\frac{\omega_s t}{2\pi}\right). \tag{9.8}$$

Convolving the impulse response $h(t)$ with the sampled signal, $x_s(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t - kT_s)$ yields

$$y(t) = \text{sinc}\left(\frac{\omega_s t}{2\pi}\right) * \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t - kT_s), \tag{9.9}$$

which reduces to

$$y(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\left[\text{sinc}\left(\frac{\omega_s t}{2\pi}\right) * \delta(t - kT_s)\right] \tag{9.10}$$

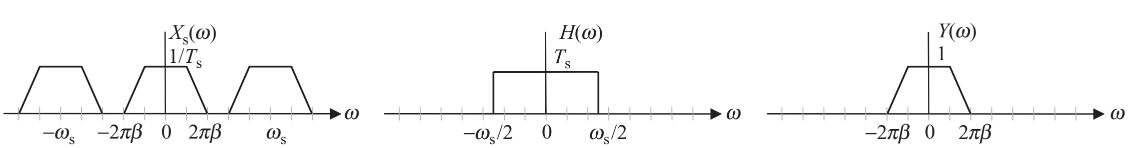


Fig. 9.4. Reconstruction of the original baseband signal $x(t)$ by ideal lowpass filtering. (a) Spectrum of the sampled signal $x_s(t)$; (b) transfer function $H(\omega)$ of the lowpass filter; (c) spectrum of the reconstructed signal $x(t)$.

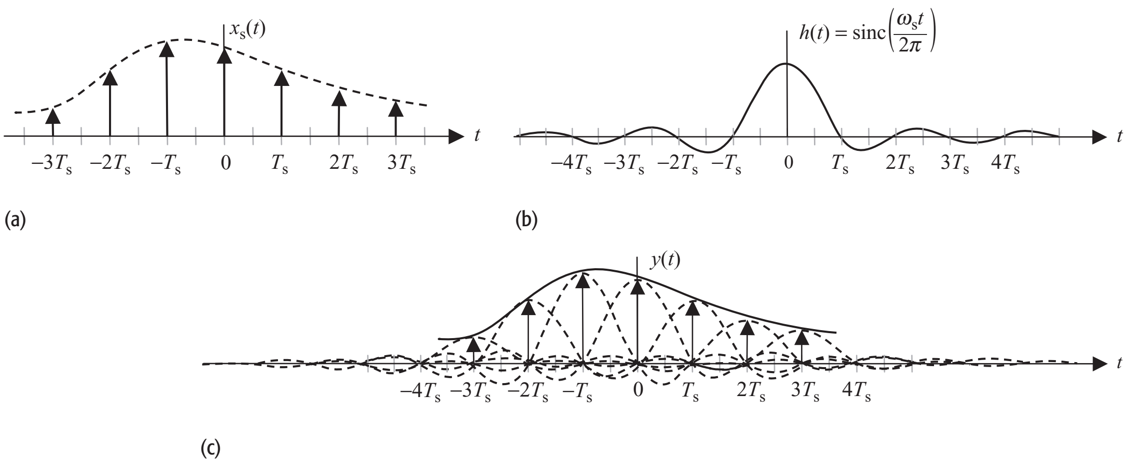


Fig. 9.5. Reconstruction of the band-limited signal in the time domain. (a) Sampled signal $x_s(t)$; (b) impulse response $h(t)$ of the lowpass filter; (c) reconstructed signal $x(t)$ obtained by convolving $x_s(t)$ with $h(t)$.

or

$$y(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\left[\text{sinc}\left(\frac{\omega_s(t-kT_s)}{2\pi}\right)\right]. \qquad (9.11)$$

Equation (9.11) implies that the original signal $x(t)$ is reconstructed by adding a series of time-shifted sinc functions, whose amplitudes are scaled according to the values of the samples at the center location of the sinc functions. The sinc functions in Eq. (9.11) are called the interpolating functions and the overall process is referred to as the band-limited interpolation. The time-domain interpretation of the reconstruction of the original band-limited signal $x(t)$ is illustrated in Fig. 9.5. At $t = kT_s$, only the kth sinc function, with amplitude $x(kT_s)$, is non-zero. The remaining sinc functions are all zero. The value of the reconstructed signal at $t = kT_s$ is therefore given by $x(kT_s)$. In other words, the values of the reconstructed signal at the sampling instants are given by the respective samples. The values in between two samples are interpolated using a linear combination of the time-shifted sinc functions.

Example 9.1

Consider the following sinusoidal signal with the fundamental frequency f_0 of 4 kHz:

$$g(t) = 5\cos(2\pi f_0 t) = 5\cos(8000\pi t).$$

(i) The sinusoidal signal is sampled at a sampling rate f_s of 6000 samples/s and reconstructed with an ideal LPF with the following transfer function:

$$H_1(\omega) = \begin{cases} 1/6000 & |\omega| \le 6000\pi \\ 0 & \text{elsewhere.} \end{cases}$$

Determine the reconstructed signal.

(ii) Repeat (i) for a sampling rate f_s of 12 000 samples/s and an ideal LPF with the following transfer function:

$$H_2(\omega) = \begin{cases} 1/12\,000 & |\omega| \le 12\,000\pi \\ 0 & \text{elsewhere.} \end{cases}$$

Solution

(i) The CTFT $G(\omega)$ of the sinusoidal signal $g(t)$ is given by

$$G(\omega) = 5\pi[\delta(\omega - 8000\pi) + \delta(\omega + 8000\pi)].$$

Using Eq. (9.4), the CTFT $G_s(\omega)$ of the sampled signal with a sampling rate $\omega_s = 2\pi(6000)$ radians/s ($T_s = 1/6000$ s) is expressed as follows:

$$G_s(\omega) = 6000 \sum_{m=-\infty}^{\infty} G(\omega - 2\pi m(6000)) = 6000 \sum_{m=-\infty}^{\infty} G(\omega - 12\,000 m\pi).$$

Substituting the value of $G(\omega)$ in the above expression yields

$$\begin{aligned} G_s(\omega) &= 6000 \sum_{m=-\infty}^{\infty} 5\pi[\delta(\omega - 8000\pi - 12\,000\,m\pi) \\ &\qquad + \delta(\omega + 8000\pi - 12\,000\,m\pi)] \\ &= 6000(5\pi)\Bigg[\cdots + \underbrace{\delta(\omega + 16\,000\pi) + \delta(\omega + 32\,000\pi)}_{m=-2} \\ &\quad + \underbrace{\delta(\omega + 4000\pi) + \delta(\omega + 20\,000\pi)}_{m=-1} \\ &\quad + \underbrace{\delta(\omega - 8000\pi) + \delta(\omega + 8000\pi)}_{m=0} + \underbrace{\delta(\omega - 20\,000\pi) + \delta(\omega - 4000\pi)}_{m=1} \\ &\quad + \underbrace{\delta(\omega - 32\,000\pi) + \delta(\omega - 16\,000\pi)}_{m=2} + \cdots\Bigg]. \end{aligned}$$

When the sampled signal is passed through the ideal LPF with transfer function $H_1(\omega)$, all frequency components $|\omega| > 6000\pi$ radians/s) are eliminated from the output. The CTFT $Y(\omega)$ of the output $y(t)$ of the LPF is given by

$$Y(\omega) = H_1(\omega)G_s(\omega) = \frac{1}{6000} \cdot 6000(5\pi)[\delta(\omega + 4000\pi) + \delta(\omega - 4000\pi)].$$

Calculating the inverse CTFT, the reconstructed signal is given by $y(t) = 5\cos(4000\pi t)$.

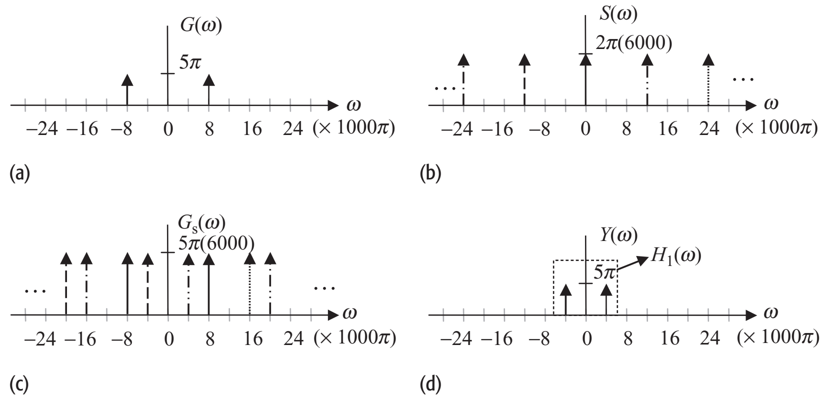


Fig. 9.6. Sampling and reconstruction of a sinusoidal signal $g(t) = 5\cos(8000\pi t)$ at a sampling rate of 6000 samples/s. CTFTs of: (a) the sinusoidal signal $g(t)$; (b) the impulse train $s(t)$; (c) the sampled signal $g_s(t)$; and (d) the signal reconstructed with an ideal LPF $H_1(\omega)$ with a cut-off frequency of 6000π radians/s.

The graphical representation of the sampling and reconstruction of the sinusoidal signal in the frequency domain is illustrated in Fig. 9.6. The CTFTs of the sinusoidal signal $g(t)$ and the impulse train $s(t)$ are plotted, respectively, in Fig. 9.6(a) and Fig. 9.6(b). Since the CTFT $S(\omega)$ of $s(t)$ consists of several impulses, the CTFT $G_s(\omega)$ of the sampled signal $g_s(t)$ is obtained by convolving the CTFT $G(\omega)$ of the sinusoidal signal $g(t)$ separately with each impulse in $G_s(\omega)$ and then applying the principle of superposition. To emphasize the results of individual convolutions, a different pattern is used in Fig. 9.6(b) for each impulse in $S(\omega)$. For example, the impulse $\delta(\omega)$ located at origin in $S(\omega)$ is shown in Fig. 9.6(b) by a solid line. Convolving $G(\omega)$ with $\delta(\omega)$ results in two impulses located at $\omega = \pm 8000\pi$, which are shown in Fig. 9.6(c) by solid lines. Similarly for the other impulses in $S(\omega)$.

The output $y(t)$ is obtained by applying $G_s(\omega)$ to the input of an ideal LPF with a cut-off frequency of 6000π radians/s. Clearly, only the two impulses at $\omega = \pm 4000\pi$, corresponding to the sinusoidal signal $\cos(4000\pi t)$, lie within the pass band of the lowpass filter. The remaining impulses are eliminated from the output. This results in an output, $y(t) = \cos(4000\pi t)$, which is different from the original signal.

(ii) The CTFT $G_s(\omega)$ of the sampled signal with $\omega_s = 2\pi(12\,000)$ radians/s ($T_s = 1/12\,000$ s) is given by

$$G_s(\omega) = 12\,000 \sum_{m=-\infty}^{\infty} G(\omega - 2\pi m(12\,000))$$

$$= 12\,000 \sum_{m=-\infty}^{\infty} G(\omega - 24\,000 m\pi).$$

Substituting the value of the CTFT $G(\omega) = 5\pi[\delta(\omega - 8000\pi) + \delta(\omega + 8000\pi)]$ in the above equation, we obtain

$$\begin{aligned}
G_s(\omega) &= 12\,000 \sum_{m=-\infty}^{\infty} 5\pi[\delta(\omega - 8000\pi - 24\,000m\pi) \\
&\quad + \delta(\omega + 8000\pi - 24\,000m\pi)] \\
&= 12\,000(5\pi)\Bigg[\cdots + \underbrace{\delta(\omega + 40\,000\pi) + \delta(\omega + 56\,000\pi)}_{m=-2} \\
&\quad + \underbrace{\delta(\omega + 16\,000\pi) + \delta(\omega + 32\,000\pi)}_{m=-1} \\
&\quad + \underbrace{\delta(\omega - 8000\pi) + \delta(\omega + 8000\pi)}_{m=0} + \underbrace{\delta(\omega - 32\,000\pi) + \delta(\omega - 16\,000\pi)}_{m=1} \\
&\quad + \underbrace{\delta(\omega - 56\,000\pi) + \delta(\omega - 40\,000\pi)}_{m=2} + \cdots\Bigg].
\end{aligned}$$

To reconstruct the original sinusoidal signal, the sampled signal is passed through an ideal LPF $H_2(\omega)$. The frequency components outside the pass-band range $|\omega| \le 12\,000\pi$ radians/s are eliminated from the ouput. The CTFT $Y(\omega)$ of the output $y(t)$ of the LPF is therefore given by

$$Y(\omega) = 5\pi[\delta(\omega + 8000\pi) + \delta(\omega - 8000\pi)],$$

which results in the reconstructed signal

$$y(t) = 5\cos(8000\pi t).$$

The graphical interpretation of the aforementioned sampling and reconstruction process is illustrated in Fig. 9.7.

As the signal g(t) is a sinusoidal signal with frequency 4 kHz, the Nyquist sampling rate is 8 kHz. In part (i), the sampling rate (6 kHz) is lower than the Nyquist rate, and consequently the reconstructed signal is different from the original signal due to the aliasing effect. In part (ii), the sampling rate is higher than the Nyquist rate, and as a result the original sinusoidal signal is accurately reconstructed.

9.4. Zero-order hold

A second practical implementation of sampling is achieved by the sample-and-hold circuit, which samples the band-limited input signal $x(t)$ at discrete time ($t = kT_s$) and maintains the sampled value for the next T_s seconds. To prevent aliasing, the sampling interval T_s must satisfy the sampling theorem. This

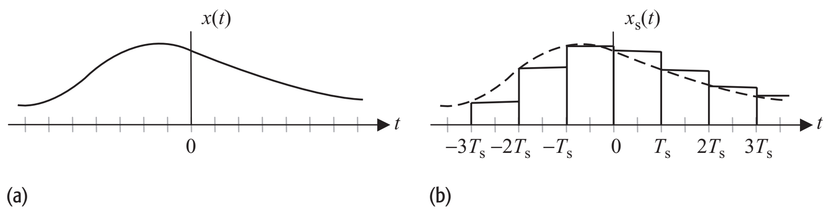


Fig. 9.10. Time-domain illustration of the zero-order hold operation for a CT signal. (a) Original signal $x(t)$; (b) zero-order hold output $x_s(t)$.

zero-order hold operation is illustrated in Fig. 9.10. Unlike the pulse-train sampling, the amplitude of the sampled signal is maintained constant for T_s seconds until the next sample is taken.

For mathematical analysis, the zero-order hold operation can be modeled by the following expression:

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT_s)p_2(t - kT_s) \tag{9.22a}$$

or

$$x_s(t) = p_2(t) * \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t - kT_s) = p_2(t) * \left[x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s)\right], \tag{9.22b}$$

where $p_2(t)$ represents a rectangular pulse given by

$$p_2(t) = \text{rect}\left(\frac{t - 0.5\,T_s}{T_s}\right). \tag{9.23}$$

Equation (9.22b) models the zero-hold operation and is different from Eq. (9.18) in two ways. First, the duration of the pulse $p_2(t)$ in Eq. (9.22b) is the same as the sampling interval T_s, whereas the duration of the pulse $p_1(t)$ is much smaller than T_s in pulse-train sampling. Secondly, the order of operation in the sampled signal $x_s(t)$ is different from that used in the corresponding sampled signal in pulse-train sampling. In Eq. (9.22b), the sampled signal $x_s(t)$ is obtained by convolving $p_2(t)$ with a periodic impulse train, which is scaled by the values of the reference signal at the location of the impulse functions. In Eq. (9.18), on the other hand, $x_s(t)$ is obtained by multiplying the original signal directly by the periodic pulse train $r(t)$.

The CTFT of Eq. (9.22b) is given by

$$X_s(\omega) = P_2(\omega) \cdot \frac{1}{2\pi}\left[X(\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2k\pi}{T_s}\right)\right], \tag{9.24}$$

where $P_2(\omega)$ denotes the CTFT of the rectangular pulse $p_2(t)$. Based on entry (16) of Table 5.2, the CTFT of $p_2(t)$ is given by the following transform pair:

$$\text{rect}\left(\frac{t - 0.5\,T_s}{T_s}\right) \xleftrightarrow{\text{CTFT}} T_s\,\text{sinc}\left(\frac{\omega T_s}{2\pi}\right) e^{-j0.5\,\omega T_s}.$$

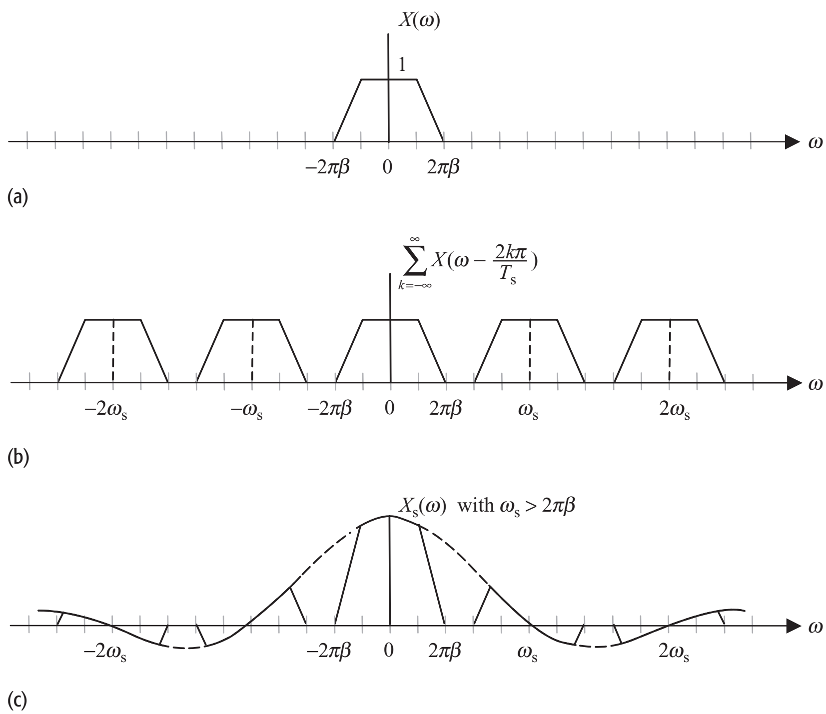


Fig. 9.11. Frequency-domain illustration of the zero-order hold operation for a CT signal. CTFTs of the: (a) original signal $x(t)$; (b) periodic replicas; and (c) the sampled signal $x_s(t)$.

Substituting the value of $P_2(\omega)$, Eq. (9.23) can be expressed as follows:

$$X_s(\omega) = e^{-j\,0.5\,\omega T_s}\,\mathrm{sinc}\left(\frac{\omega T_s}{2\pi}\right) \cdot \sum_{k=-\infty}^{\infty} X\left(\omega - \frac{2k\pi}{T_s}\right). \tag{9.25}$$

Based on Eq. (9.25), Fig. 9.11 illustrates the frequency-domain interpretation of the zero-hold operation. The spectrum $X_s(\omega)$ of the sampled signal is shown in Fig. 9.11(c), which contains scaled replicas of the CTFT of the original baseband signal. Unlike the pulse-train sampling, some distortion in the amplitude is introduced in the central replica located at $\omega = 0$. This distortion can be minimized by increasing the width of the main lobe of the sinc function in Eq. (9.25). Since the width of the main lobe is given by $2\pi/T_s$, it is equivalent to reducing the sampling interval T_s.

To recover the original CT signal, the sampled signal is filtered with an LPF having a cut-off frequency $\omega_c = \omega_s/2$. Due to the amplitude distortion introduced in the central replica, ideal lowpass filtering recovers an approximate version of the original CT signal. For perfect reconstruction, the filter with the transfer function given by

$$H(\omega) = \begin{cases} \dfrac{1}{\mathrm{sinc}(\omega T_s/2\pi)} & |\omega| \le \omega_s/2 \\ 0 & \text{elsewhere} \end{cases} \tag{9.26}$$

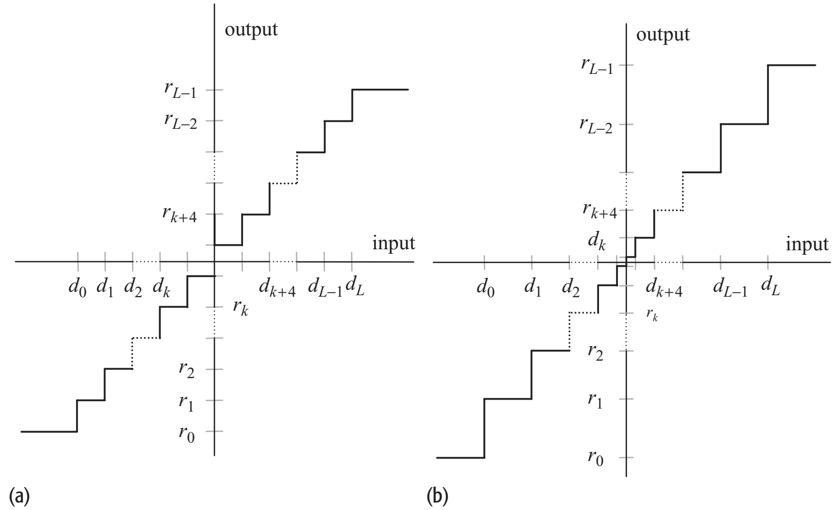


Fig. 9.12. Input–output relationship of an L-level quantizer used to discretize the sample values $x[kT_s]$ of a DT sequence $x[k]$. (a) Uniform quantizer; (b) non-uniform quantizer.

is used. The above filter is referred to as the compensation, or anti-imaging, filter. Filtering $X_s(\omega)$ with the anti-imaging filter introduces a linear phase $-\omega T_s$ corresponding to the exponential term $\exp(-j\omega T_s)$. Inclusion of a linear phase in the frequency domain is equivalent to a delay in the time domain and is therefore harmless and not considered as a distortion.

9.5. Quantization (Uniform and non-uniform)

The process of sampling, discussed in Sections 9.1 and 9.2, converts a CT signal $x(t)$ into a DT sequence $x[k]$, with each sample representing the amplitude of the CT signal $x(t)$ at a particular instant $t = kT_s$. The amplitude $x[kT_s]$ of a sample in $x[k]$ can still have an infinite number of possible values. To produce a true digital sequence, each sample in $x[k]$ is approximated to a finite set of values. The last step is referred to as *quantization* and is the focus of our discussion in this section.

Figure 9.12(a) illustrates the input–output relationship for an L-level uniform quantizer. The peak-to-peak range of the input sequence $x[k]$ is divided uniformly into $(L + 1)$ quantization levels $\{d_0, d_1, \ldots, d_L\}$ such that the separation $\Delta = (d_{m+1} - d_m)$ is the same between any two consecutive levels. The separation Δ between two quantization levels is referred to as the *quantile interval* or quantization step size. For a given input, the output of the quantizer is calculated from the following relationship:

$$y[k] = r_m = \frac{1}{2}[d_m + d_{m+1}] \quad \text{for} \quad d_m \le x[k] < d_{m+1} \quad \text{and} \quad 0 \le m < L. \qquad (9.27)$$

In other words, the quantized value of the input lying within the levels d_m and d_{m+1} is given by r_m, which equals $0.5(d_m + d_{m+1})$. The quantization levels $\{d_0, d_1, \ldots, d_L\}$ are referred to as the *decision levels*, while the output levels $\{r_0, r_1, \ldots, r_{L-1}\}$ are referred to as the *reconstruction levels*.

Equation (9.27) approximates the analog sample values by using a finite number of quantization levels. The approximation introduces a distortion, which is referred to as the quantization error. The peak value of the quantization error is one-half of the quantile interval in the positive or negative direction.

The quantizer illustrated in Fig. 9.12(a) is called a uniform quantizer because the quantization levels are uniformly distributed between the minimum and maximum ranges of the input sequence. In most practical applications, the distribution of the amplitude of the input sequence is skewed towards low values. In speech communication, for example, low speech volumes dominate the sequence most of the time. Large-amplitude values are extremely rare and typically occupy only 15% to 25% of the communication time. A uniform quantizer will be wasteful, with most of the quantization levels rarely used. In such applications, we use non-uniform quantization, which provides fine quantization at frequently occurring lower volumes and coarse quantization at higher volumes. The input–output relationship of a non-uniform quantizer is shown in Fig. 9.12(b). The quantile interval is small at low values of the input sequence and large at high values of the sequence.

Example 9.3

Consider an audio recording system where the microphone generates a CT voltage signal within the range $[-1, 1]$ volts. Calculate the decision and reconstruction levels for an eight-level uniform quantizer.

Solution

For an $L = 8$ level quantizer with peak-to-peak range of $[-1, 1]$ volts, the quantile interval Δ is given by

$$\Delta = \frac{1 - (-1)}{8} = 0.25\ \text{V}.$$

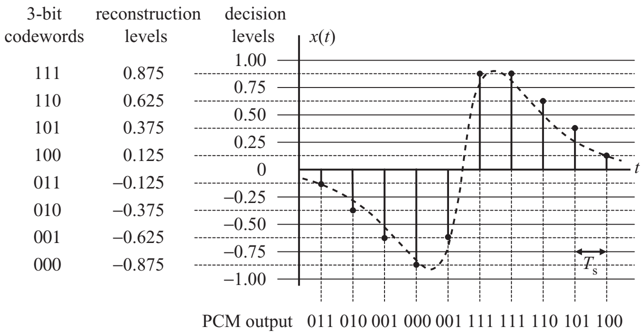
Starting with the minimum voltage of -1 V, the decision levels d_m are uniformly distributed between -1 V and 1 V. In other words,

$$d_m = -1 + m\Delta \quad \text{for} \quad 0 \le m \le L.$$

Substituting different values of m, we obtain

$$d_m = -1\ \text{V}, -0.75\ \text{V}, -0.5\ \text{V}, -0.25\ \text{V}, 0\ \text{V}, 0.25\ \text{V}, \\ 0.50\ \text{V}, 0.75\ \text{V}, 1\ \text{V}.$$

Fig. 9.13. Derivation of a PCM sequence from a CT signal $x(t)$. The original CT signal $x(t)$ is shown by the dotted line, while the PCM sequence is shown as a stem plot.



Using Eq. (9.27), the reconstruction levels r_m are given by

$$r_m = -0.875\,\text{V}, -0.625\,\text{V}, -0.375\,\text{V}, -0.125\,\text{V}, 0.125\,\text{V}, \\ 0.375\,\text{V}, 0.625\,\text{V}, 0.875\,\text{V}.$$

The maximum quantization error is one-half of the quantile interval Δ and is given by 0.125 V.

Time-domain analysis of discrete-time systems

This chapter presents the following items as follow:

10.1. Introduction.

10.2 Finite-difference equation representation of LTID systems

10.3 Representation of sequences using Dirac delta functions

10.4 Impulse response of a system

10.5 Convolution sum

10.6 Graphical method for evaluating the convolution sum

10.7 Properties of the convolution sum

Time-domain analysis of discrete-time systems

10.1. Introduction.

An important subset of discrete-time (DT) systems satisfies the linearity and time-invariance properties, discussed in Chapter 2. Such DT systems are referred to as linear, time-invariant, discrete-time (LTID) systems. In this chapter, we will develop techniques for analyzing LTID systems. As was the case for the LTIC systems discussed in Part II, we are primarily interested in calculating the output response $y[k]$ of an LTID system to a DT sequence $x[k]$ applied at the input of the system.

In the time domain, an LTID system is modeled either with a linear, constant-coefficient difference equation or with its impulse response $h[k]$. Section 10.1 covers linear, constant-coefficient difference equations and develops numerical techniques for solving such equations. Section 10.2 defines the impulse response $h[k]$ as the output of an LTID system to an unit impulse function $\delta[k]$ applied at the input of the system and shows how the impulse response can be derived from a linear, constant-coefficient difference equation. Section 10.3 proves that any arbitrary DT sequence can be represented as a linear combination of time-shifted DT impulse functions. This development leads to a second approach for calculating the output $y[k]$ based on convolving the applied input sequence $x[k]$ with the impulse response $h[k]$ in the DT domain. The resulting operation is referred to as the convolution sum and is defined in Section 10.4. Section 10.5 introduces two graphical methods for calculating the convolution sum, and Section 10.6 lists several important properties of the convolution sum. A special case of convolution sum, referred to as the periodic or circular convolution, occurs when the two operands are periodic sequences. Section 10.7 develops techniques for computing the periodic convolution and shows how it may be used to compute the linear convolution. In Section 10.8, we revisit the causality, stability, and invertibility properties of LTID systems and express these properties in terms of the impulse response $h[k]$. MATLAB instructions for computing the convolution sum are listed in Section 10.9. The chapter is concluded in Section 10.10 with a summary of the important concepts covered in the chapter.

10.2 Finite-difference equation representation of LTID systems

As discussed in Section 3.1, an LTIC system can be modeled using a linear, constant-coefficient differential equation. Likewise, the input–output relationship of a linear DT system can be described using a difference equation, which takes the following form:

$$\begin{aligned} y[k+n] + a_{n-1}y[k+n-1] + \cdots + a_0 y[k] \\ = b_m x[k+m] + b_{m-1}x[k+m-1] + \cdots + b_0 x[k], \end{aligned} \tag{10.1}$$

where $x[k]$ denotes the input sequence and $y[k]$ denotes the resulting output sequence, and coefficients a_r (for $0 \le r \le n-1$), and b_r (for $0 \le r \le m$) are parameters that characterize the DT system. The coefficients a_r and b_r are constants if the DT system is also time-invariant. For causal signals and systems analysis, the following n initial (or ancillary) conditions must be specified in order to obtain the solution of the nth-order difference equation in Eq. (10.1):

$$y[-1], y[-2], \ldots, y[-n].$$

We now consider an iterative procedure for solving linear, constant-coefficient difference equations.

Example 10.1

The DT sequence $x[k] = 2ku[k]$ is applied at the input of a DT system described by the following difference equation:

$$y[k+1] - 0.4y[k] = x[k].$$

By iterating the difference equation from the ancillary condition $y[-1] = 4$, compute the output response $y[k]$ of the DT system for $0 \le k \le 5$.

Solution

Express $y[k+1] - 0.4y[k] = x[k]$ as follows:

$$\begin{aligned} y[k] &= 0.4y[k-1] + x[k-1] \\ &= 0.4y[k-1] + 2(k-1)\,u(k-1) \quad \{\because\ x[k] = 2k\,u[k]\}, \end{aligned}$$

which can alternatively be expressed as

$$y[k] = \begin{cases} 0.4y[k-1] & k = 0 \\ 0.4y[k-1] + 2(k-1) & k \ge 1. \end{cases}$$

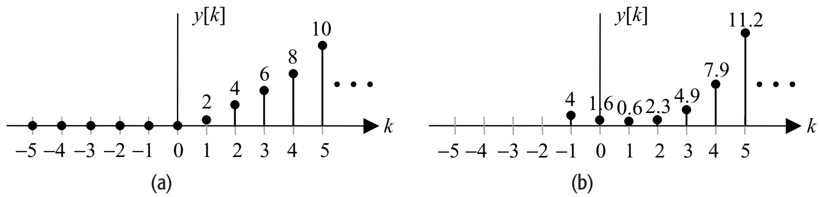


Fig. 10.1. Input and output sequences for Example 10.1.(a) Input sequence $x[k]$; (b) output sequence $y[k]$.

By iterating from $k = 0$, the output response is computed as follows:

$$y[0] = 0.4y[-1] = 1.6,$$
$$y[1] = 0.4y[0] + 2 \times 0 = 0.64,$$
$$y[2] = 0.4y[1] + 2 \times 1 = 2.256,$$
$$y[3] = 0.4y[2] + 2 \times 2 = 4.902,$$
$$y[4] = 0.4y[3] + 2 \times 3 = 7.961,$$
$$y[5] = 0.4y[4] + 2 \times 4 = 11.184.$$

Additional values of the output sequence for $k > 5$ can be similarly evaluated from further iterations with respect to k. The input and output sequences are plotted in Fig. 10.1 for $0 \leq k \leq 5$.

In Chapter 3, we showed that the output response of a CT system, represented by the differential equation in Eq. (3.1), can be decomposed into two components: the zero-state response and the zero-input response. This is also valid for the DT systems represented by the difference equation in Eq. (10.1). The output response $y[k]$ can be expressed as

$$y[k] = \underbrace{y_{zi}[k]}_{\text{zero-input response}} + \underbrace{y_{zs}[k],}_{\text{zero-state response}} \tag{10.2}$$

where $y_{zi}[k]$ denotes the *zero-input response* (or the *natural response*) of the system and $y_{zs}[k]$ denotes the *zero-state response* (or the *forced response*) of the DT system.

The zero-input component $y_{zi}[k]$ for a DT system is the response produced by the system because of the initial conditions, and is not due to any external input. To calculate the zero-input component $y_{zi}[k]$, we assume that the applied input sequence $x[k] = 0$. On the other hand, the zero-state response $y_{zs}[k]$ arises due to the input sequence and does not depend on the initial conditions of the system. To calculate the zero-state response $y_{zs}[k]$, the initial conditions are assumed to be zero. Based on Eq. (10.2), a DT system represented by Eq. (10.1) can be considered as an incrementally linear system (see Section 2.2.1) where the additive offset is caused by the initial conditions (see Fig. 2.10). If the initial conditions are zero, the DT system becomes on LTID system. We now solve Example 10.1 in terms of the zero-input and zero-state components of the output.

Example 10.2

Repeat Example 10.1 to calculate (i) the zero-input response $y_{zi}[k]$, (ii) the zero-state response $y_{zs}[k]$, and (iii) the overall output response $y[k]$ for $0 \leq k \leq 5$.

Solution

(i) The zero-input response of the system is obtained by solving the following difference equation:

$$y[k+1] - 0.4y[k] = x[k],$$

with input $x[k] = 0$ and ancillary condition $y[-1] = 4$. The difference equation reduces to

$$y_{\mathrm{zi}}[k] = 0.4y_{\mathrm{zi}}[k-1],$$

with ancillary condition $y_{\mathrm{zi}}[-1] = 4$. Iterating for $k = 0, 1, 2, 3, 4,$ and 5 yields

$$\begin{aligned}
y_{\mathrm{zi}}[0] &= 0.4y_{\mathrm{zi}}[-1] = 1.6,\\
y_{\mathrm{zi}}[1] &= 0.4y_{\mathrm{zi}}[0] = 0.64,\\
y_{\mathrm{zi}}[2] &= 0.4y_{\mathrm{zi}}[1] = 0.256,\\
y_{\mathrm{zi}}[3] &= 0.4y_{\mathrm{zi}}[2] = 0.1024,\\
y_{\mathrm{zi}}[4] &= 0.4y_{\mathrm{zi}}[3] = 0.0410,\\
y_{\mathrm{zi}}[5] &= 0.4y_{\mathrm{zi}}[4] = 0.0164.
\end{aligned}$$

(ii) The zero-state response of the system is calculated by solving the following difference equation:

$$y_{\mathrm{zs}}[k] = 0.4y_{\mathrm{zs}}[k-1] + 2(k-1)u[k-1],$$

with ancillary condition $y_{\mathrm{zs}}[-1] = 0$. Iterating the difference equation for $k = 0, 1, 2, 3, 4,$ and 5 yields

$$\begin{aligned}
y_{\mathrm{zs}}[0] &= 0.4y_{\mathrm{zs}}[-1] + 2 \times (-1) \times 0 = 0,\\
y_{\mathrm{zs}}[1] &= 0.4y_{\mathrm{zs}}[0] + 2 \times 0 \times 1 = 0,\\
y_{\mathrm{zs}}[2] &= 0.4y_{\mathrm{zs}}[1] + 2 \times 1 \times 1 = 2,\\
y_{\mathrm{zs}}[3] &= 0.4y_{\mathrm{zs}}[2] + 2 \times 2 \times 1 = 4.8,\\
y_{\mathrm{zs}}[4] &= 0.4y_{\mathrm{zs}}[3] + 2 \times 3 \times 1 = 7.92,\\
y_{\mathrm{zs}}[5] &= 0.4y_{\mathrm{zs}}[4] + 2 \times 4 \times 1 = 11.168.
\end{aligned}$$

(iii) Adding the zero-input and zero-state components obtained in parts (i) and (ii), yields

$$\begin{aligned}
y[0] &= y_{\mathrm{zi}}[0] + y_{\mathrm{zs}}[0] = 1.6,\\
y[1] &= y_{\mathrm{zi}}[1] + y_{\mathrm{zs}}[1] = 0.64,\\
y[2] &= y_{\mathrm{zi}}[2] + y_{\mathrm{zs}}[2] = 2.256,\\
y[3] &= y_{\mathrm{zi}}[3] + y_{\mathrm{zs}}[3] = 4.902,\\
y[4] &= y_{\mathrm{zi}}[4] + y_{\mathrm{zs}}[4] = 7.961,\\
y[5] &= y_{\mathrm{zi}}[5] + y_{\mathrm{zs}}[5] = 11.184.
\end{aligned}$$

Note that the overall output response $y[k]$ is identical to the output response obtained in Example 10.1. By iterating with respect to k, additional values for the output response $y[k]$ for $k > 5$ can be computed.