University of Anbar
College of Engineering
Dept. of Electrical Engineering

Signals and Systems III
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Lecture No. Five

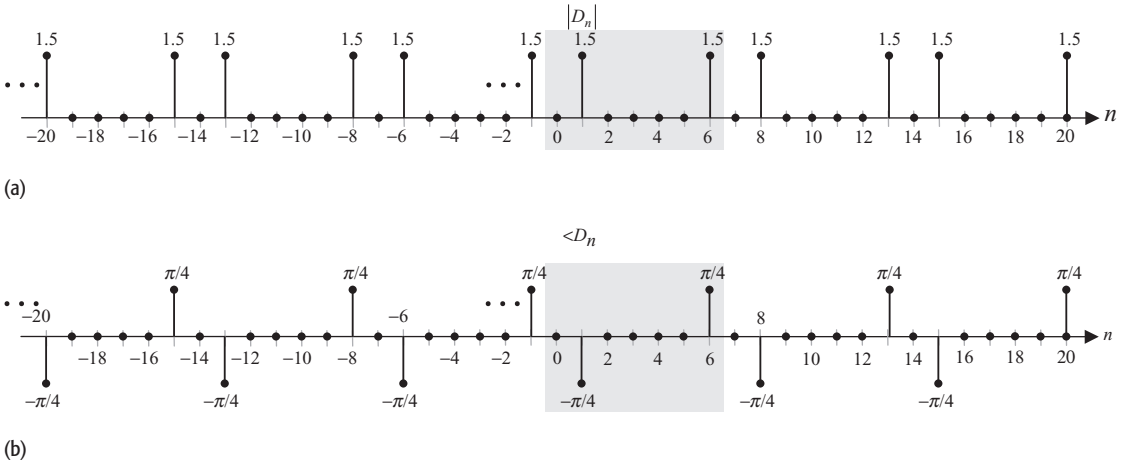


Fig. 11.5. (a) Magnitude spectrum and (b) phase spectrum of the DTFS coefficients in Example 11.5.

Corollary 11.2 The DTFS coefficients of the sinusoidal sequence $x[k] = B \sin(2\pi mk/K_0)$ are given by

$$D_n = \begin{cases} -\mathrm{j}\dfrac{B}{2} & \text{for } n = m, m \pm K_0, m \pm 2K_0, \ldots \\ \mathrm{j}\dfrac{B}{2} & \text{for } n = -m, -m \pm K_0, -m \pm 2K_0, \ldots \\ 0 & \text{elsewhere,} \end{cases} \tag{11.20}$$

provided that the greatest common divisor between integers m and K_0 is one.

11.4. Fourier transform for aperiodic functions

In Section 11.1, we used the exponential DTFS to derive the frequency representations for periodic sequences. In this section, we consider the frequency representations for aperiodic sequences. The resulting representation is called the DT Fourier transform (DTFT).

Figure 11.6(a) shows the waveform of an aperiodic sequence $x[k]$, which is zero outside the range $M_1 \le k \le M_2$. Such a sequence is referred to as a time-limited sequence having a length of $M_2 - M_1 + 1$ samples. As was the case for the CTFT, we consider periodic repetitions of $x[k]$ uniformly spaced with a duration of K_0 between each other; $K_0 \ge (M_2 - M_1 + 1)$ such that the adjacent replicas of $x[k]$ do not overlap with each other. The resulting sequence is referred to as the periodic extension of $x[k]$ and is denoted by $\tilde{x}_{K_0}[k]$. If we increase the value of K_0, in the limit, we obtain

$$\lim_{K_0 \to \infty} \tilde{x}_{K_0}[k] = x[k]. \tag{11.21}$$

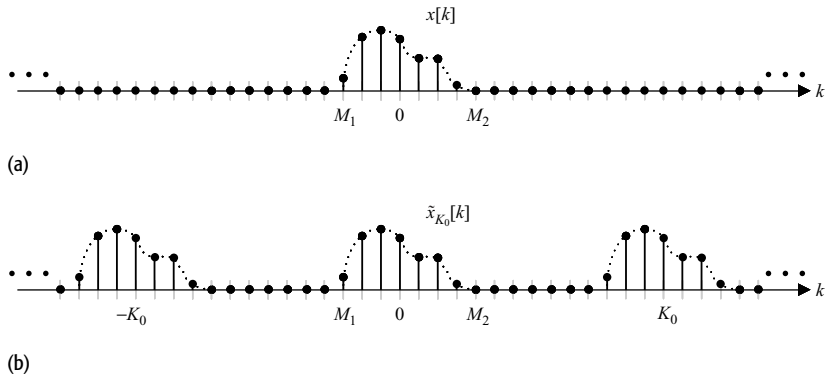


Fig. 11.6. (a) Time-limited sequence $x[k]$; (b) its periodic extension.

Since $\tilde{x}_{K_0}[k]$ is periodic with fundamental period K_0 (or fundamental frequency Ω_0), we can express it using the DTFS as follows:

$$\tilde{x}_{K_0}[k] = \sum_{n=\langle K_0 \rangle} D_n e^{jn\Omega_0 k}, \tag{11.22}$$

where the DTFS coefficients D_n are given by

$$D_n = \frac{1}{K_0} \sum_{k=\langle K_0 \rangle} \tilde{x}_{K_0}[k] e^{-jn\Omega_0 k},$$

for $1 \leq n \leq K_0$. Using Eq. (11.21), the above equation can be expressed as follows:

$$D_n = \lim_{K_0 \to \infty} \frac{1}{K_0} \sum_{k=-\infty}^{\infty} x[k] e^{-jn\Omega_0 k} \tag{11.23}$$

for $1 \leq n \leq K_0$. Let us now define a new function $X(\Omega)$, which is continuous with respect to the independent variable Ω:

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k}. \tag{11.24}$$

In Eq. (11.24), the independent variable Ω is continuous in the range $-\infty \leq \Omega \leq \infty$. In terms of $X(\Omega)$, Eq. (11.23) can be expressed as follows:

$$D_n = \lim_{K_0 \to \infty} \frac{1}{K_0} X(n\Omega_0). \tag{11.25}$$

The function $X(n\Omega_0)$ is obtained by sampling $X(\Omega)$ at discrete points $\Omega = n\Omega_0$.

Given the DTFS coefficients D_n of $\tilde{x}_{K_0}[k]$, the aperiodic sequence $x[k]$ can be obtained by substituting the values of D_n in Eq. (11.22) and solving for $M_1 \leq k \leq M_2$. The resulting expression is given by

$$x[k] = \lim_{K_0 \to \infty} \tilde{x}_{K_0}[k] = \lim_{K_0 \to \infty} \sum_{n=\langle K_0 \rangle} \frac{1}{K_0} X(n\Omega_0) e^{jn\Omega_0 k}. \tag{11.26a}$$

In the limit $K_0 \to \infty$, the angular frequency Ω_0 takes a very small value, say $\Delta\Omega$, with the fundamental period $K_0 = 2\pi/\Delta\Omega$. In the limit $K_0 \to \infty$, Eq. (11.26a) can, therefore, be expressed as follows:

$$x[k] = \lim_{\Delta\Omega \to 0} \sum_{n=\langle K_0 \rangle} \frac{1}{2\pi} X(n\Delta\Omega) e^{jnk\Delta\Omega} \Delta\Omega. \tag{11.26b}$$

Substituting $\Omega = n\Delta\Omega$ and applying the limit $\Delta\Omega \to 0$, Eq. (11.26b) reduces to the following integral:

$$x[k] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\Omega) e^{jk\Omega} d\Omega. \tag{11.27}$$

In Eq. (11.27), the limits of integration are derived by evaluating the duration $n = \langle K_0 \rangle$ in terms of Ω as follows:

$$\Omega = \langle n\Delta\Omega \rangle|_{n=\langle K_0 \rangle} = \left\langle n \left(\frac{2\pi}{K_0} \right) \right\rangle \Bigg|_{n=\langle K_0 \rangle} = \langle 2\pi \rangle,$$

implying that any frequency range of 2π may be used to solve the integral in Eq. (11.27). Collectively, Eq. (11.24), in conjunction with Eq. (11.27), is referred to as the DTFT pair.

Definition 11.3 *The DTFT pair for an aperiodic sequence $x[k]$ is given by*

DTFT synthesis equation
$$x[k] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\Omega) e^{jk\Omega} d\Omega; \tag{11.28a}$$

DTFT analysis equation
$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k}. \tag{11.28b}$$

In the subsequent discussion, we will denote the DTFT pair as follows:

$$x[k] \xleftrightarrow{\text{DTFT}} X(\Omega). \tag{11.28c}$$

Example 11.6

Calculate the Fourier transform of the following functions:

(i) unit impulse sequence, $x_1[k] = \delta[k]$;

(ii) gate sequence, $x_2[k] = \text{rect}\left(\frac{k}{2N+1}\right) = \begin{cases} 1 & |k| \le N \\ 0 & \text{elsewhere;} \end{cases}$

(iii) decaying exponential sequence, $x_3[k] = p^k u[k]$ with $|p| < 1$.

Solution

(i) By definition,

$$X_1(\Omega) = \sum_{k=-\infty}^{\infty} \delta[k] e^{-j\Omega k} = e^{-j\Omega k}|_{k=0} = 1.$$

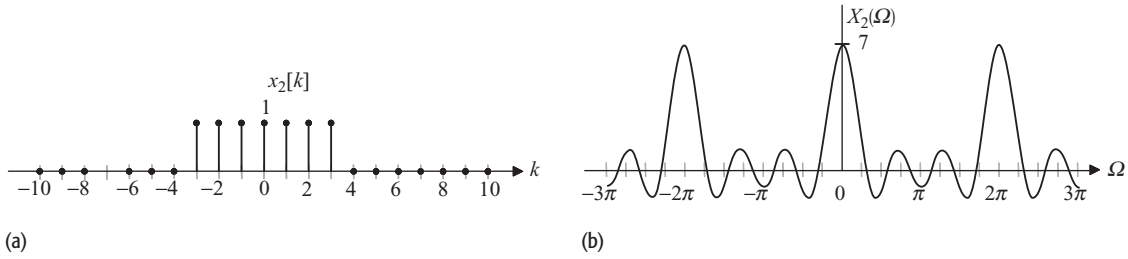


Fig. 11.7. (a) Rectangular sequence $x_2[k]$ with a width of seven samples. (b) DTFT of the rectangular sequence derived in Example 11.6(ii).

(ii) By definition,

$$X_2(\Omega) = \sum_{k=-\infty}^{\infty} x_2[k]\mathrm{e}^{-\mathrm{j}\Omega k} = \sum_{k=-N}^{N} 1 \cdot \mathrm{e}^{-\mathrm{j}\Omega k}.$$

The summation represents a GP series with $\exp(-\mathrm{j}\Omega)$ as the ratio between two consecutive terms. The GP series simplifies to

$$\begin{aligned} X_2(\Omega) &= (\mathrm{e}^{-\mathrm{j}\Omega})^{-N} \frac{1-(\mathrm{e}^{-\mathrm{j}\Omega})^{2N+1}}{1-\mathrm{e}^{-\mathrm{j}\Omega}} \\ &= (\mathrm{e}^{-\mathrm{j}\Omega})^{-N} \frac{(\mathrm{e}^{-\mathrm{j}\Omega})^{(2N+1)/2}}{(\mathrm{e}^{-\mathrm{j}\Omega})^{1/2}} \frac{(\mathrm{e}^{-\mathrm{j}\Omega})^{-(2N+1)/2} - (\mathrm{e}^{-\mathrm{j}\Omega})^{(2N+1)/2}}{(\mathrm{e}^{-\mathrm{j}\Omega})^{-1/2} - (\mathrm{e}^{-\mathrm{j}\Omega})^{1/2}} \\ &= \frac{\mathrm{e}^{\mathrm{j}(2N+1)\Omega/2} - \mathrm{e}^{-\mathrm{j}(2N-1)\Omega/2}}{\mathrm{e}^{\mathrm{j}\Omega/2} - \mathrm{e}^{-\mathrm{j}\Omega/2}} = \frac{\sin\left(\dfrac{2N+1}{2}\Omega\right)}{\sin\left(\dfrac{1}{2}\Omega\right)}. \end{aligned}$$

As a special case, we assume $N = 3$ and plot the rectangular sequence $x_2[k]$ and its DTFT $X_2(\Omega)$ in Fig. 11.7.

(iii) By definition,

$$X_3(\Omega) = \sum_{k=-\infty}^{\infty} p^k u[k]\mathrm{e}^{-\mathrm{j}\Omega k} = \sum_{k=0}^{\infty} (p\mathrm{e}^{-\mathrm{j}\Omega})^k.$$

The summation represents a GP series, which can be simplified to

$$X_3(\Omega) = \frac{1}{1-p\mathrm{e}^{-\mathrm{j}\Omega}} = \frac{1}{1-p\cos\Omega + \mathrm{j}p\sin\Omega}.$$

The DTFT $X_3(\Omega)$ is a complex-valued function of the angular frequency Ω. Its magnitude and phase spectra are determined below:

magnitude spectrum
$$\begin{aligned} |X_3(\Omega)| &= \frac{|1|}{|1-p\cos\Omega + \mathrm{j}p\sin\Omega|} \\ &= \frac{1}{\sqrt{1-2p\cos\Omega + p^2}}; \end{aligned}$$

phase spectrum
$$\begin{aligned} <X_3(\Omega) &= <1 - <(1-p\cos\Omega + \mathrm{j}p\sin\Omega) \\ &= -\tan^{-1}\left(\frac{p\sin\Omega}{1-p\cos\Omega}\right). \end{aligned}$$

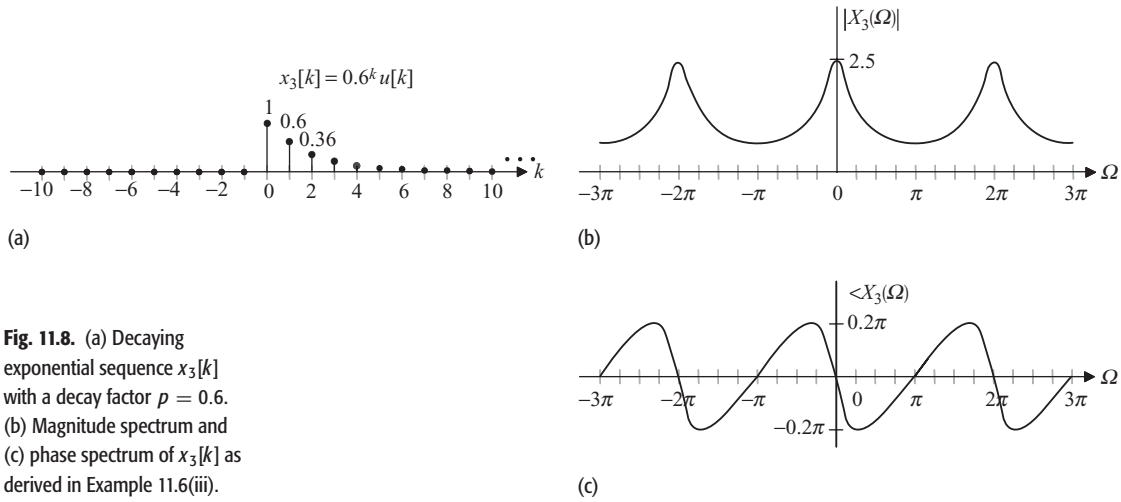


Fig. 11.8. (a) Decaying exponential sequence $x_3[k]$ with a decay factor $p = 0.6$. (b) Magnitude spectrum and (c) phase spectrum of $x_3[k]$ as derived in Example 11.6(iii).

As a special case, we plot the DT sequence $x_3[k]$ and its magnitude and phase spectra for $p = 0.6$ in Figs. 11.8(a)–(c).

In Example 11.6, we calculated the DTFTs for three different sequences and observed that all three DTFTs are periodic with period $\Omega_0 = 2\pi$. This property is referred to as the frequency–periodicity property and is satisfied by all DTFTs. In Section 11.4, we present a mathematical proof verifying the frequency–periodicity property.

Example 11.7

Calculate the DT sequences for the following DTFTs:

(i) $X_1(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2m\pi)$;

(ii) $X_2(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2m\pi)$.

Solution

(i) Using the synthesis equation, Eq. (11.28a), the inverse DTFT of $X_1(\Omega)$ is given by

$$x_1[k] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X_1(\Omega) e^{jk\Omega} d\Omega = \frac{1}{2\pi} \int_{-2\pi}^{\pi} 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2m\pi) e^{jk\Omega} d\Omega$$

$$= \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} \delta(\Omega - 2m\pi) \underbrace{e^{jk2m\pi}}_{=1} d\Omega \quad [\because \delta(\Omega - \theta) f(\Omega) = \delta(\Omega - \theta) f(\theta)]$$

$$= \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} \delta(\Omega - 2m\pi) d\Omega.$$

The integral on the right-hand side of the above equation includes several impulse functions located at $\Omega = 0, \pm 2\pi, \pm 4\pi, \ldots$ Only the impulse function located at $\Omega = 0$ falls in the frequency range $\Omega = [-\pi, \pi]$. Therefore, $x_1[k]$ can be simplified as follows:

$$x_1[k] = \int_{-\pi}^{\pi} \delta(\Omega) \mathrm{d}\Omega = 1.$$

(ii) Using the synthesis equation, (11.28a), the inverse DTFT of $X_2(\Omega)$ is given by

$$\begin{aligned} x_1[k] &= \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X_2(\Omega) \mathrm{e}^{\mathrm{j}k\Omega} \mathrm{d}\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2m\pi) \mathrm{e}^{\mathrm{j}k\Omega} \mathrm{d}\Omega. \\ &= \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2m\pi) \mathrm{e}^{\mathrm{j}k(\Omega_0 - 2m\pi)} \, \mathrm{d}\Omega \\ &= \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2m\pi) \mathrm{e}^{\mathrm{j}k\Omega_0} \underbrace{\mathrm{e}^{\mathrm{j}k2m\pi}}_{=1} \, \mathrm{d}\Omega \\ &= \mathrm{e}^{\mathrm{j}k\Omega_0} \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2m\pi) \mathrm{d}\Omega. \end{aligned}$$

The integral on the right-hand side of the above equation includes several impulse functions located at $\Omega = \Omega_0 + 2m\pi$. Only one of these infinite number of impulse functions will be present in the frequency range $\Omega = [-\pi, \pi]$. Therefore, the integral will have a vaue of unity and the function $x_2[k]$ can be simplified as follows:

$$x_2[k] = \mathrm{e}^{\mathrm{j}k\Omega_0}.$$

Table 11.2 lists the DTFT and DTFS representations for several DT sequences. In situations where a DT sequence is aperiodic, the DTFS representation is not possible and therefore not included in the table. The DTFT of the periodic sequences is determined from its DTFS representation and is covered in Section 11.4.

Table 11.3 plots the DTFT for several DT sequences. In situations where a DT sequence or its DTFT is complex, we plot both the magnitude and phase components. The magnitude component is shown using a bold line, and the phase component is shown using a dashed line.

Example 11.7 illustrates the calculation of a DT function from its DTFT using Eq. (11.28a). In many cases, it may be easier to calculate a DT function from its DTFT using the partial fraction expansion and the DTFT pairs listed in Table 11.2. This procedure is explained in more detail in

Table 11.2. DTFTs and DTFSs for elementary DT sequences Note that the DTFS does not exist for aperiodic sequences

Sequence: $x[k]$	DTFS: $D_n = \frac{1}{K_0}\sum_{k=\langle K_0\rangle} x[k]e^{-jn\Omega_0 k}$	DTFT: $X(\Omega) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega k}$
(1) $x[k] = 1$	$D_n = 1$	$X(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2m\pi)$
(2) $x[k] = \delta[k]$	does not exist	$X(\Omega) = 1$
(3) $x[k] = \delta[k - k_0]$	does not exist	$X(\Omega) = e^{-j\Omega k_0}$
(4) $x[k] = \sum_{m=-\infty}^{\infty} \delta(k - mK_0)$	$D_n = \frac{1}{K_0}$ for all n	$X(\Omega) = \frac{2\pi}{K_0}\sum_{m=-\infty}^{\infty} \delta\left(\Omega - \frac{2m\pi}{K_0}\right)$
(5) $x[k] = u[k]$	does not exist	$X(\Omega) = \pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2m\pi) + \frac{1}{1 - e^{-j\Omega}}$
(6) $x[k] = p^k u[k]$ with $\lvert p\rvert < 1$	does not exist	$X(\Omega) = \frac{1}{1 - pe^{-j\Omega}}$
(7) First-order time-rising decaying exponential $x[k] = (k+1)p^k u[k]$, with $\lvert p\rvert < 1$.	does not exist	$X(\Omega) = \frac{1}{(1 - pe^{-j\Omega})^2}$
(8) Complex exponential (periodic) $x[k] = e^{jk\Omega_0}$ $K_0 = 2\pi p/\Omega_0$	$D_n = \begin{cases} 1 & n = p \pm rK_0 \\ 0 & \text{elsewhere} \end{cases}$ for $-\infty < r < \infty$	$X(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2m\pi)$
(9) Complex exponential (aperiodic) $x[k] = e^{jk\Omega_0}$, $2\pi/\Omega_0 \neq$ rational	does not exist	$X(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2m\pi)$
(10) Cosine (periodic) $x[k] = \cos(\Omega_0 k)$ $K_0 = 2\pi p/\Omega_0$	$D_n = \begin{cases} \frac{1}{2} & n = \pm p \pm rK_0 \\ 0 & \text{elsewhere} \end{cases}$ for $-\infty < r < \infty$	$X(\Omega) = \pi \sum_{m=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2m\pi) + \pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2m\pi)$
(11) Cosine (aperiodic) $x[k] = \cos(\Omega_0 k)$, $2\pi/\Omega_0 \neq$ rational	does not exist	$X(\Omega) = \pi \sum_{m=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2m\pi) + \pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2m\pi)$
(12) Sine (periodic) $x[k] = \sin(\Omega_0 k)$ $K_0 = 2\pi p/\Omega_0$	$D_n = \begin{cases} \frac{1}{2}j & n = \pm p \pm rK_0 \\ 0 & \text{elsewhere} \end{cases}$ for $-\infty < r < \infty$	$X(\Omega) = j\pi \sum_{m=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2m\pi) - j\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2m\pi)$
(13) Sine (aperiodic) $x[k] = \sin(\Omega_0 k)$, $2\pi/\Omega_0 \neq$ rational	does not exist	$X(\Omega) = j\pi \sum_{m=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2m\pi) - j\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2m\pi)$

(cont.)

Table 11.2. (*cont.*)

Sequence: $x[k]$	DTFS: $D_n = \frac{1}{K_0}\sum_{k=\langle K_0\rangle} x[k]e^{-jn\Omega_0 k}$	DTFT: $X(\Omega) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega k}$
(14) Rectangular (periodic) $x[k]=\begin{cases}1 & \lvert k\rvert \le N\\ 0 & N < \lvert k\rvert \le K_0/2\end{cases}$ $x[k]=x[k+K_0]$	$D_n = \begin{cases}(2N+1)/K_0 & k = rK_0\\ \frac{1}{K_0}\left[\frac{\sin\left(\frac{2N+1}{K_0}n\pi\right)}{\sin\left(\frac{1}{K_0}n\pi\right)}\right] & \text{elsewhere}\end{cases}$	$X(\Omega) = 2\pi\sum_{n=-\infty}^{\infty} D_n\delta\left(\Omega - \frac{2n\pi}{K_0}\right)$
(15) Rectangular (aperiodic) $x[k] = \begin{cases}1 & \lvert k\rvert \le N\\ 0 & \text{elsewhere}\end{cases}$	does not exist	$X(\Omega) = \frac{\sin\left(\frac{2N+1}{2}\Omega\right)}{\sin\left(\frac{1}{2}\Omega\right)}$
(16) Sinc $x[k] = \frac{W}{\pi}\,\mathrm{sinc}\left(\frac{Wk}{\pi}\right) = \frac{\sin(Wk)}{\pi k}$ for $0 < W < \pi$	does not exist	$X(\Omega) = \begin{cases}1 & \lvert\Omega\rvert \le W\\ 0 & W < \lvert\Omega\rvert \le \pi\end{cases}$ $X(\Omega) = X(\Omega + 2\pi)$
(17) Arbitrary periodic sequence with period K_0 $x[k] = \sum_{n=\langle K_0\rangle} D_n e^{jn\Omega_0 k}$	$D_n = \frac{1}{K_0}\sum_{k=\langle K_0\rangle} x[k]e^{-jn\Omega_0 k}$	$X(\Omega) = 2\pi\sum_{n=-\infty}^{\infty} D_n\delta\left(\Omega - \frac{2n\pi}{K_0}\right)$

Appendix D, and has been used later in this chapter in solving Examples 11.15 and 11.18.

11.5. Existence of the DTFT

Definition 11.4 *The DTFT $X(\Omega)$ of a DT sequence $x[k]$ is said to exist if*

$$|X(\Omega)| < \infty, \quad \text{for } -\infty < \Omega < \infty. \tag{11.29}$$

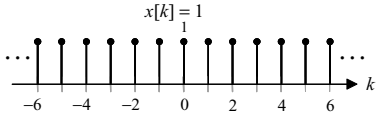
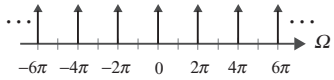
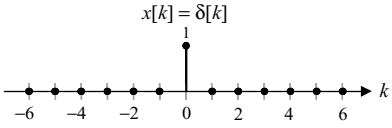
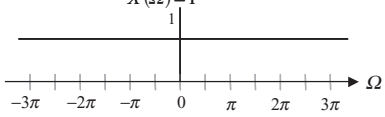
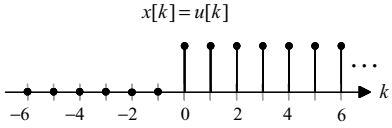
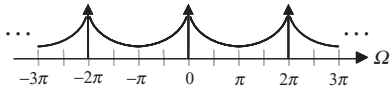
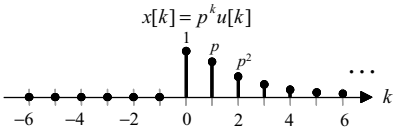
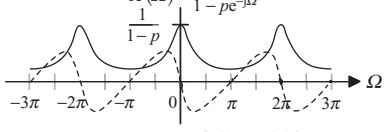
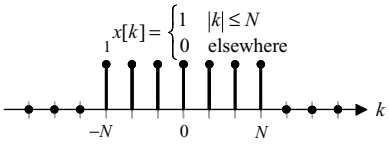
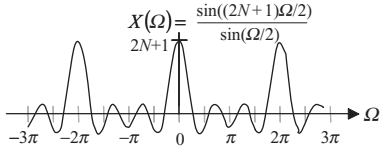
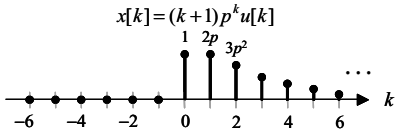
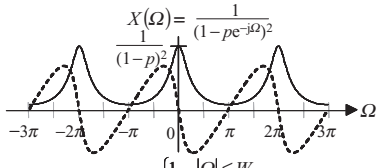
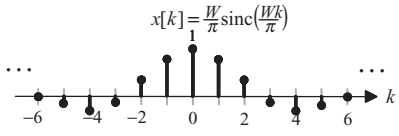
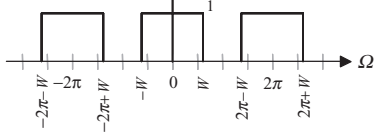
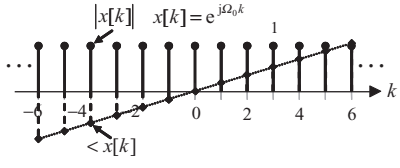
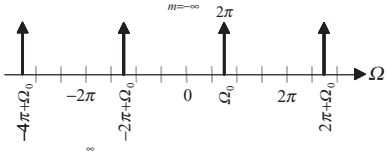
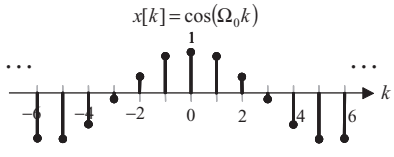
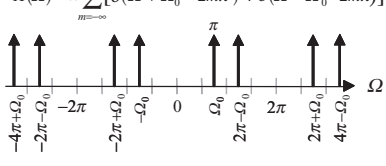
The above definition for the existence of the DTFT satisfies our intuition that a valid function should be finite for all values of the independent variable. Substituting the value of the DTFT $X(\Omega)$ from Eq. (11.28b), Eq. (11.29) can be expressed as follows:

$$\left|\sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega k}\right| < \infty,$$

which is satisfied if

$$\sum_{k=-\infty}^{\infty} |x[k]| \cdot |e^{-j\Omega k}| < \infty.$$

Table 11.3. DTFT spectra for elementary DT sequences

Sequence	Time-domain waveform	Magnitude and phase spectra
(1) Constant $x[k] = 1$		$X(\Omega) = 2\pi\sum_{m=-\infty}^{\infty}\delta(\Omega-2m\pi)$ 
(2) Unit impulse $x[k] = \delta[k]$		$X(\Omega)=1$ 
(3) Unit step $x[k] = u[k]$		$X(\Omega)=\pi\sum_{m=-\infty}^{\infty}\delta(\Omega-2m\pi)+\frac{1}{1-e^{-j\Omega}}$ 
(4) Decaying exponential $x[k] = p^k u[k]$ with $\lvert p\rvert < 1$		$X(\Omega)=\frac{1}{1-pe^{-j\Omega}}$ 
(5) Rectangular $x[k] = \begin{cases}1 & \lvert k\rvert \le N\\ 0 & \text{elsewhere}\end{cases}$		$X(\Omega)=\frac{\sin((2N+1)\Omega/2)}{\sin(\Omega/2)}$ 
(6) First-order time-rising decaying exponential $x[k] = (k+1)p^k u[k]$ with $\lvert p\rvert < 1$		$X(\Omega)=\frac{1}{(1-pe^{-j\Omega})^2}$ 
(7) Sinc $x[k] = \frac{W}{\pi}\text{sinc}\left(\frac{Wk}{\pi}\right)$		$X(\Omega)=\begin{cases}1 & \lvert\Omega\rvert\le W\\ 0 & W<\lvert\Omega\rvert\le\pi\end{cases}$ 
(8) Complex exponential $x[k] = e^{jk\Omega_0}$		$X(\Omega)=2\pi\sum_{m=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi m)$ 
(9) Cosine $x[k] = \cos(\Omega_0 k)$		$X(\Omega)=\pi\sum_{m=-\infty}^{\infty}[\delta(\Omega+\Omega_0-2m\pi)+\delta(\Omega-\Omega_0-2m\pi)]$ 

From the Euler's formula, we know that $|\exp(-j\Omega k)| = 1$. Therefore, an alternative expression to verify the existence of the DTFT is given by

$$\sum_{k=-\infty}^{\infty} |x[k]| < \infty.$$

Condition for the existence of DTFT The DTFT $X(\Omega)$ of a DT sequence $x[k]$ exists if

$$\sum_{k=-\infty}^{\infty} |x[k]| < \infty. \tag{11.2830}$$

Equation (11.2830) is a sufficient condition to verify the existence of the DTFT.

Example 11.8

Determine if the DTFTs exist for the following functions:

(i) causal exponential function, $x_1[k] = p^k u[k]$.
(ii) cosine waveform, $x_2[k] = \cos(\Omega_0 k)$.

Solution

(i) Equation (11.2830) yields

$$\sum_{k=-\infty}^{\infty} |x_1[k]| = \sum_{k=-\infty}^{\infty} |p^k u[k]| = \sum_{k=0}^{\infty} |p^k| = \begin{cases} \dfrac{1}{1-p} & 0 < |p| > 1 \\ \infty & |p| \geq 1. \end{cases}$$

Therefore, the DTFT of the exponential sequence $x_1[k] = p^k u[k]$ exists if $0 < |p| < 1$. Under such a condition, $x_1[k]$ is a decaying exponential sequence with the summation in Eq. (11.2830) having a finite value.

(ii) Equation (11.2830) yields

$$\sum_{k=-\infty}^{\infty} |x_2[k]| = \sum_{k=-\infty}^{\infty} |\cos(\Omega_0 k)| \to \infty.$$

Therefore, the DTFT does not exist for the cosine waveform. However, this appears to be in violation of Table 11.2, which lists the following DTFT pair for the cosine sequence:

$$\cos(\Omega_0 k) \xleftrightarrow{\text{DTFT}} \pi \sum_{m=-\infty}^{\infty} [\delta(\Omega + \Omega_0 - 2\pi m) + \delta(\Omega - \Omega_0 - 2\pi m)].$$

Looking closely at the above DTFT pair, we note that the DTFT $X(\Omega)$ of the cosine function consists of continuous impulse functions at discrete frequencies $\Omega = (\pm\Omega_0 - 2\pi m)$, for $-\infty < m < \infty$. Since the magnitude of a continuous impulse function is infinite, $|X(\Omega)|$ is infinite at the location of the impulses. The infinite magnitude of the impulses in the DTFT $X(\Omega)$ leads to the violation of the existence condition stated in Eq. (11.2830).

Example 11.8 has introduced a confusing situation for the cosine sequence. We proved that the condition of the existence of the DTFT is violated by the