Lecture No. Eleven

The Z-Transform

This chapter presents the following items as follow:

The Z-Transform

13.1. Introduction.

In Chapter 11, we introduced two frequency representations, namely the discrete-time Fourier series (DTFS) and the discrete-time Fourier transform (DTFT) for DT signals. These frequency representations are exploited to determine the output response of an LTID system. Unfortunately, the DTFT does not exist for all signals (e.g., periodic signals). In situations where the DTFT does not exist, an alternative transform, referred to as the *z-transform*, may be used for the analysis of LTID systems. Even for DT sequences for which the DTFT exists, the z-transforms are always real-valued, rational functions of the independent variable z provided that the DT sequences are real. In comparison, the DTFT is generally complex-valued. Therefore, using the z-transform simplifies the algebraic manipulations and leads to flow diagram representations of the DT systems, a pivotal step needed to fabricate the DT system in silicon. Finally, the DTFT can only be applied to a stable LTID system for which the impulse response is absolutely summable. Since the z-transform exists for both stable and unstable LTID systems, the z-transform can be used to analyze a broader range of LTID systems.

The difference between the DTFT and the z-transform lies in the choice of the independent variable used in the transformed domain. The DTFT $X(\Omega)$ of a DT sequence $x[k]$ uses the complex exponentials $e^{jk\Omega}$ as its basis function and maps $x[k]$ in terms of $e^{jk\Omega}$. The z-transform $X(z)$ expresses $x[k]$ in terms of z^k, where the independent variable z is given by $z = e^{(\sigma+j\Omega)k}$. The z-transform is, therefore, a generalization of the DTFT, just as the Laplace transform is a generalization of the CTFT. In this chapter, we introduce the z-transform and illustrate its applications in the analysis of LTID systems.

This chapter is organized as follows. Section 13.1 defines the bilateral, also referred to as the two-sided, z-transform and illustrates the steps involved in its computation through a series of examples. For causal signals, the bilateral z-transform reduces to the one-sided, or unilateral, z-transform, which is covered in Section 13.2. Section 13.3 presents inverse methods of calculating the time-domain representation of the z-transform. The properties of the z-transform are derived in Section 13.4. Sections 13.5–13.9 cover various applications of the z-transform. Section 13.5 applies the z-transform to calculate the output of an LTID system from the input sequence and the impulse response of the LTID system. The relationship between the Laplace transform and the z-transform is discussed in Section 13.6. Stability analysis of the LTID system in the z-domain is presented in Section 13.7, while graphical techniques to derive the frequency response from the z-transform are discussed in Section 13.8. Section 13.9 compares the DTFT and z-transform in calculating the steady state and transient responses of an LTID system. Section 13.10 introduces important MATLAB library functions useful in computing the z-transform and in the analysis of LTID systems. Finally, the chapter is concluded in Section 13.11 with a summary of important concepts.

13.2. Analytical development

Section 11.1 defines the synthesis and analysis equations of the DTFT pair $x[k] \xleftrightarrow{\text{DTFT}} X(\Omega)$ as follows:

DTFT synthesis equation $$x[k] = \frac{1}{2\pi} \int\limits_{\langle 2\pi \rangle} X(\Omega) e^{j\Omega k} d\Omega; \tag{13.1}$$

DTFT analysis equation $$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k}. \tag{13.2}$$

To derive the expression for the bilateral z-transform, we calculate the DTFT of the modified version $x[k]e^{-\sigma k}$ of the DT signal. Based on Eq. (13.2), the DTFT of the modified signal is given by

$$\Im\left\{x[k]e^{-\sigma k}\right\} = \sum_{k=-\infty}^{\infty} x[k] e^{-\sigma k} e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} x[k] e^{-(\sigma + j\Omega)k}. \tag{13.3}$$

Substituting $e^{\sigma + j\Omega} = z$ in Eq. (13.3) leads to the following definition for the bilateral z-transform:

z-analysis equation $$X(z) = \Im\left\{x[k]e^{-\sigma k}\right\} = \sum_{k=-\infty}^{\infty} x[k] z^{-k}. \tag{13.4}$$

It may be noted that the summation in Eq. (13.4) is absolutely summable only for selected values of z. For other values of z, the infinite sum in Eq. (13.4) may not converge to a finite value, and hence $X(z)$ becomes infinite. The region in the complex z-plane, where summation (13.4) is finite, is referred to as the region of convergence (ROC) of the z-transform $X(z)$.

By following a similar derivation for the DTFT synthesis equation, Eq. (13.1), the expression for the inverse z-transform is given by

z-synthesis equation $$x[k] = \frac{1}{2\pi j} \oint\limits_{C} X(z) z^{k-1} dz, \tag{13.5}$$

where C is a closed contour traversed in the counterclockwise direction within the ROC. Solving Eq. (13.5) involves the application of contour integration techniques and is, therefore, seldom used directly. In Section 13.3, we will consider alternative approaches based on the look-up table, partial fraction expansion, and power series expansion to evaluate the inverse z-transform.

Collectively, Eqs. (13.4) and (13.5) form the bilateral z-transform pair, which is denoted by

$$x[k] \overset{z}{\longleftrightarrow} X(z) \quad \text{or} \quad Z\{x[k]\} = X(z). \tag{13.6}$$

To illustrate the steps involved in computing the z-transform, we consider the following examples.

Example 13.1

Calculate the bilateral z-transform of the exponential sequence $x[k] = \alpha^k u[k]$.

Solution

Substituting $x[k] = \alpha^k u[k]$ in Eq. (13.4), we obtain

$$X(z) = \sum_{k=-\infty}^{\infty} \alpha^k u[k] z^{-k} = \sum_{k=0}^{\infty} (\alpha z^{-1})^k$$

$$= \begin{cases} \dfrac{1}{1-\alpha z^{-1}} & |\alpha z^{-1}| < 1 \\ \text{undefined} & \text{elsewhere.} \end{cases}$$

In the above expression, if $|\alpha z^{-1}| \geq 1$ the bilateral z-transform has an infinite value. In such cases, we say that the z-transform is not defined. The set of values of z over which the bilateral z-transform is defined is referred to as the region of convergence (ROC) associated with the z-transform. In this example, the ROC for the z-transform pair

$$\alpha^k u[k] \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}$$

is given by

$$\text{ROC: } \left|\alpha z^{-1}\right| < 1 \quad \text{or} \quad |z| > |\alpha|.$$

Figure 13.1 highlights the ROC by shading the appropriate region in the complex z-plane.

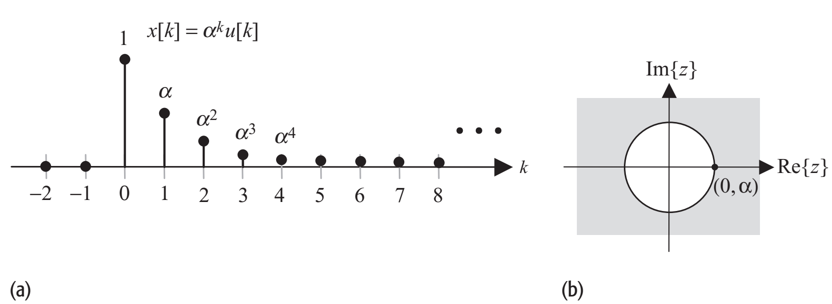

Fig. 13.1. (a) DT exponential sequence $x[k] = \alpha^k u[k]$; (b) the ROC, $|z| > |\alpha|$, associated with its bilateral z-transform. The ROC is shown as the shaded area and lies outside the circle of radius α.

Example 13.1 derives the bilateral z-transform of the exponential sequence $x[k] = \alpha^k u[k]$:

$$\alpha^k u[k] \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}, \quad \text{with} \quad \text{ROC } |z| > |\alpha|.$$

Since no restriction is imposed on the magnitude of α, the bilateral z-transform of the exponential sequence exists for all values of α within the specified ROC. Recall that the DTFT of an exponential sequence exists only for $\alpha < 1$. For $\alpha \geq 1$, the exponential sequence is not summable and its DTFT does not exist. This is an important distinction between the DTFT and the bilateral z-transform. While the DTFT exists for a limited number of absolutely summable sequences, no such restrictions exist for the z-transform. By associating an ROC with the bilateral z-transform, we can evaluate the z-transform for a much larger set of sequences.

Example 13.2

Calculate the bilateral z-transform of the left-hand-sided exponential sequence $x[k] = -\alpha^k u[-k-1]$.

Solution

For the DT sequence $x[k] = -\alpha^k u[-k-1]$, Eq. (13.4) reduces to

$$X(z) = \sum_{k=-\infty}^{\infty} -\alpha^k u[-k-1] z^{-k} = -\sum_{k=-\infty}^{-1} (\alpha z^{-1})^k.$$

To make the limits of summation positive, we substitute $m = -k$ in the above equation to obtain

$$X(z) = -\sum_{m=1}^{\infty} (\alpha^{-1} z)^m = \begin{cases} -\dfrac{\alpha^{-1} z}{1-\alpha^{-1} z} & |\alpha^{-1} z| < 1 \\ \text{undefined} & \text{elsewhere,} \end{cases}$$

which simplifies to

$$X(z) = \begin{cases} \dfrac{1}{1-\alpha z^{-1}} & |z| < |\alpha| \\ \text{undefined} & \text{elsewhere.} \end{cases}$$

The DT sequence $x[k] = -\alpha^k u[-k-1]$ and the ROC associated with its z-transform are illustrated in Fig. 13.2.

In Examples 13.1 and 13.2, we have proved the following z-transform pairs:

$$\alpha^k u[k] \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}, \quad \text{with} \quad \text{ROC } |z| > |\alpha|,$$

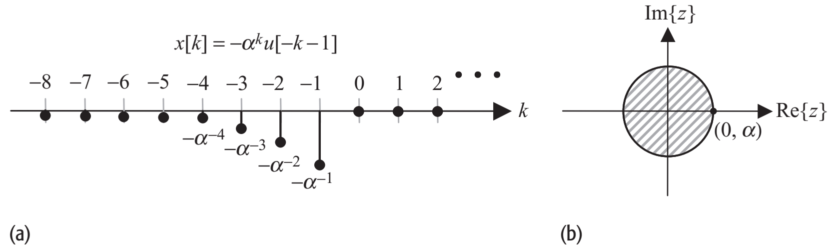

Fig. 13.2. (a) Non-causal function $x[k] = -\alpha^k u[-k-1]$; (b) its associated ROC, $|z| < |\alpha|$, shown as the shaded area excluding the circle, over which the bilateral z-transform exists.

and

$$-\alpha^k u[-k-1] \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}, \quad \text{with} \quad \text{ROC } |z| < |\alpha|.$$

Although the algebraic expressions for the bilateral z-transforms are the same for the two functions, the ROCs are different. This implies that a bilateral z-transform is completely specified only if both the algebraic expression and the associated ROC are included in its specification.

13.3. Unilateral z-transform

In Section 13.1, we introduced the bilateral z-transform, which may be used to analyze both causal and non-causal LTID systems. Since most physical systems in signal processing are causal, a simplified version of the bilateral z-transform exists in such cases. The simplified bilateral z-transform for causal signals and systems is referred to as the unilateral z-transform, and it is obtained by assuming $x[k] = 0$ for $k < 0$. The analysis equation, Eq. (13.4), simplifies as follows:

unilateral z-transform
$$X(z) = \sum_{k=0}^{\infty} x[k] z^{-k}. \tag{13.7}$$

Unless explicitly stated, we will, in subsequent discussion, assume the “unilateral” z-transform when referring to the z-transform. If the bilateral z-transform is being discussed, we will specifically state this. To clarify further the differences between the unilateral and bilateral z-transforms, we summarize the major points.

(1) The unilateral z-transform simplifies the analysis of causal LTID systems. Since most physical systems are naturally causal, we will mostly use unilateral z-transform in our computations. However, the unilateral z-transform cannot be used to analyze non-causal systems directly.
(2) For causal signals and systems, the unilateral and bilateral z-transforms are the same.

(3) The synthesis equation used for calculating the inverse of the unilateral z-transform is the same as Eq. (13.5) used for evaluating the inverse of the bilateral transform.

Example 13.3

Calculate the unilateral z-transform for the following sequences:

(i) unit impulse sequence, $x_1[k] = \delta[k]$;
(ii) unit step sequence, $x_2[k] = u[k]$;
(iii) exponential sequence, $x_3[k] = \alpha^k u[k]$;
(iv) first-order, time-rising, exponential sequence, $x_4[k] = k\alpha^k u[k]$;
(v) time-limited sequence, $x_5[k] = \begin{cases} 1 & k = 0, 1 \\ 2 & k = 2, 5 \\ 0 & \text{otherwise.} \end{cases}$

Solution

(i) By definition,

$$X_1(z) = \sum_{k=0}^{\infty} \delta[k]z^{-k} = \delta[0]z^0 = 1, \qquad \text{ROC: entire z-plane.}$$

The z-transform pair for an impulse sequence is given by

$$\delta[k] \overset{z}{\longleftrightarrow} 1, \qquad \text{ROC: entire z-plane.}$$

(ii) By definition,

$$X_2(z) = \sum_{k=0}^{\infty} u[k]z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \begin{cases} \dfrac{1}{1 - z^{-1}} & \text{for } |z^{-1}| < 1 \\ \text{undefined} & \text{elsewhere.} \end{cases}$$

The z-transform pair for a unit step sequence is given by

$$u[k] \overset{z}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \qquad \text{ROC: } |z| > 1.$$

In the above transform pair, note that the ROC $|z^{-1}| < 1$ is equivalent to $|z| > 1$ and consists of the region outside a circle of unit radius in the complex z-plane. This circle of unit radius, with the origin of the z-plane as the center, is referred to as the unit circle and plays an important role in the determination of the stability of an LTID system. We will discuss stability issues in Section 13.7.

(iii) By definition,

$$X_3(z) = \sum_{k=0}^{\infty} \alpha^k u[k]z^{-k} = \sum_{k=0}^{\infty} (\alpha z^{-1})^k = \begin{cases} \dfrac{1}{1 - \alpha z^{-1}} & \text{for } \left|\alpha z^{-1}\right| < 1 \\ \text{undefined} & \text{elsewhere.} \end{cases}$$

The z-transform pair for an exponential sequence is therefore given by

$$\alpha^k u[k] \overset{z}{\longleftrightarrow} \frac{1}{1 - \alpha z^{-1}}, \qquad \text{ROC: } |z| > |\alpha|.$$

In the above transform pair, the ROC $|\alpha z^{-1}| < 1$ is equivalent to $|z| > \alpha$ and consists of the region outside the circle of radius $|z| = \alpha$ in the complex z-plane. Example 13.1 derives the bilateral z-transform for the function $x_3[k] = \alpha^k u[k]$. Since the function is causal, the bilateral and unilateral z-transforms are identical.

(iv) By definition,

$$X(z) = \sum_{k=0}^{\infty} k\alpha^k u[k] z^{-k} = \sum_{k=0}^{\infty} k(\alpha z^{-1})^k.$$

Using the following result:

$$\sum_{k=0}^{\infty} kr^k = \frac{r}{(1-r)^2}, \quad \text{provided } |\mathrm{r}| < 1,$$

the above summation reduces to

$$X(z) = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}, \quad \text{ROC: } |\alpha z^{-1}| < 1.$$

The z-transform pair for a time-rising, complex exponential is given by

$$k\alpha^k u[k] \overset{z}{\longleftrightarrow} \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} \text{ or } \frac{\alpha z}{(z-\alpha)^2}, \quad \text{ROC: } |z| > |\alpha|.$$

(v) Since the input sequence $x_5[k]$ is zero outside the range $0 \le k \le 5$, Eq. (13.4) reduces to

$$X(z) = \sum_{k=0}^{\infty} x[k] z^{-k} = x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + x[4]z^{-4} + x[5]z^{-5}.$$

Substituting the values of $x_5[k]$ for the range $0 \le k \le 5$, we obtain

$$X(z) = 1 + z^{-1} + 2z^{-2} + 2z^{-5} \quad \text{ROC: entire z-plane, except } z = 0.$$

For finite-duration sequences, the ROC is always the entire z-plane except for the possible exclusion of $z = 0$ and $z = \infty$.

13.4. Relationship between the DTFT and the z-transform

Comparing Eq. (13.2) with Eq. (13.4), the DTFT can be expressed in terms of the bilateral z-transform as follows:

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k] z^{-k} = X(z)|_{z=e^{j\Omega}}. \tag{13.8}$$

Since, for causal functions, the bilateral and unilateral z-transforms are the same, Eq. (13.8) is also valid for the unilateral z-transform for causal functions.

Equation (13.8) shows that the DTFT is a special case of the z-transform with $z = e^{j\Omega}$. The equality $z = e^{j\Omega}$ corresponds to the circle of unit radius ($|z| = 1$) in the complex z-plane. Equation (13.8) therefore implies that the

Table 13.1. Unilateral z-transform pairs for several causal DT sequences

DT sequence $x[k] = \frac{1}{2\pi j}\oint_C X(z)z^{k-1}dz$	z-transform with ROC $X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$
(1) Unit impulse $x[k] = \delta[k]$	1, ROC: entire z-plane
(2) Delayed unit impulse $x[k] = \delta[k-k_0]$	z^{-k_0}, ROC: entire z-plane, except $z = 0$
(3) Unit step $x[k] = u[k]$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$, ROC: $\lvert z\rvert > 1$
(4) Exponential $x[k] = \alpha^k u[k]$	$\frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$, ROC: $\lvert z\rvert > \lvert\alpha\rvert$
(5) Delayed exponential $x[k] = \alpha^{k-1}u[k-1]$	$\frac{z^{-1}}{1-\alpha z^{-1}} = \frac{1}{z-\alpha}$, ROC: $\lvert z\rvert > \lvert\alpha\rvert$
(6) Ramp $x[k] = ku[k]$	$\frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$, ROC: $\lvert z\rvert > 1$
(7) Time-rising exponential $x[k] = k\alpha^k u[k]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} = \frac{\alpha z}{(z-\alpha)^2}$, ROC: $\lvert z\rvert > \lvert\alpha\rvert$
(8) Causal cosine $x[k] = \cos(\Omega_0 k)u[k]$	$\frac{1-z^{-1}\cos\Omega_0}{1-2z^{-1}\cos\Omega_0+z^{-2}} = \frac{z[z-\cos\Omega_0]}{z^2-2z\cos\Omega_0+1}$, ROC: $\lvert z\rvert > 1$
(9) Causal sine $x[k] = \sin(\Omega_0 k)u[k]$	$\frac{z^{-1}\sin\Omega_0}{1-2z^{-1}\cos\Omega_0+z^{-2}} = \frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$, ROC: $\lvert z\rvert > 1$
(10) Exponentially modulated cosine $x[k] = \alpha^k\cos(\Omega_0 k)u[k]$	$\frac{1-\alpha z^{-1}\cos\Omega_0}{1-2\alpha z^{-1}\cos\Omega_0+\alpha^2 z^{-2}} = \frac{z[z-\alpha\cos\Omega_0]}{z^2-2\alpha z\cos\Omega_0+\alpha^2}$, ROC: $\lvert z\rvert > \lvert\alpha\rvert$
(11) Exponentially modulated sine I $x[k] = \alpha^k\sin(\Omega_0 k)u[k]$	$\frac{\alpha z^{-1}\sin\Omega_0}{1-2\alpha z^{-1}\cos\Omega_0+\alpha^2 z^{-2}} = \frac{\alpha z\sin\Omega_0}{z^2-2\alpha z\cos\Omega_0+\alpha^2}$, ROC: $\lvert z\rvert > \alpha$
(12) Exponentially modulated sine II $x[k] = r\alpha^k\sin(\Omega_0 k+\theta)u[k]$, with $\alpha \in R$.	$\frac{A+Bz^{-1}}{1+2\gamma z^{-1}+\alpha^2 z^{-2}} = \frac{z(Az+B)}{z^2+2\gamma z+\gamma^2}$, ROC: $\lvert z\rvert \le \lvert\alpha\rvert^{(a)}$

[a] Where $r = \sqrt{\frac{A^2\alpha^2+B^2-2AB\gamma}{\alpha^2-\gamma^2}}$, $\Omega_0 = \cos^{-1}\left(\frac{-\gamma}{\alpha}\right)$, and $\theta = \tan^{-1}\left(\frac{A\sqrt{\alpha^2-\gamma^2}}{B-A\gamma}\right)$.

DTFT is obtained by computing the z-transform along the unit circle in the complex z-plane.

Table 13.1 lists the z-transforms for several commonly used sequences. Comparing Table 13.1 with Table 11.2, we observe that when the sequence is causal and its DTFT exists, the DTFT can be obtained from the z-transform by substituting $z = e^{j\Omega}$. Since the substitution $z = e^{j\Omega}$ can only be made if the ROC contains the unit circle, an alternative condition for the existence of the DTFT is the inclusion of the unit circle within the ROC of the z-transform. If the ROC of a z-transform does not include the unit circle, we cannot substitute $z = e^{j\Omega}$ and we say that its DTFT cannot be obtained from Eq. (13.8). For example, the ROC of the unit step function is given by $|z| > 1$, which does not contain the