Lecture No. Fifteen

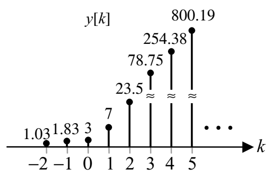


Fig. 13.5. Output response of the LTID system specified in Example 13.12.

Taking the inverse transform, we obtain

$$y[k] = \left[\frac{26}{15} \times 0.5^k - \frac{7}{3} \times 2^k + \frac{26}{15} \times 3^k\right] \text{ for } k > 0.$$

The output response is plotted in Fig. 13.5.

13.8. z-transfer function of LTID systems

In Chapters 10 and 11, we used the impulse response $h[k]$ and Fourier transfer function $H(\Omega)$ to represent an LTID system. An alternative representation for an LTID system is obtained by taking the z-transform of the impulse response:

$$h[k] \overset{z}{\longleftrightarrow} H(z).$$

The DTFT H(z) is referred to as the *z-transfer function* of the LTID system. In conjunction with the linear convolution property, Eq. (13.24), the z-transfer function H(z) may be used to determine the output response $y[k]$ of an LTID system when an input sequence $x[k]$ is applied at its input. In the time domain, the output response $y[k]$ is given by

$$y[k] = x[k] * h[k]. \tag{13.32}$$

Taking the z-transform of both sides of Eq. (13.32), we obtain

$$Y(z) = X(z)H(z), \tag{13.33}$$

where $Y(z)$ and $X(z)$ are, respectively, the z-transforms of the output response $y[k]$ and the input sequence $x[k]$. Equation (13.33) provides us with an alternative definition for the transfer function as the ratio of the z-transform of the output response and the z-transform of the input signal. Mathematically, the transfer function H(z) can be expressed as follows:

$$H(z) = \frac{Y(z)}{X(z)}. \tag{13.34}$$

The z-transfer function of an LTID system can be obtained from its difference equation representation, as described in the following.

Consider an LTID system whose input–output relationship is given by the following difference equation:

$$\begin{aligned} &y[k+n] + a_{n-1}y[k+n-1] + \cdots + a_0 y[k] \\ &\quad = b_m x[k+m] + b_{m-1}x[k+m-1] + \cdots + b_0 x[k]. \end{aligned} \tag{13.35}$$

By taking the z-transform of both sides of the above equation, we obtain

$$\left\{z^n + a_{n-1}z^{n-1} + \cdots + a_0 z\right\} Y(z) = \left\{b_m z^m + b_{m-1}z^{m-1} + \cdots + b_0\right\} X(z),$$

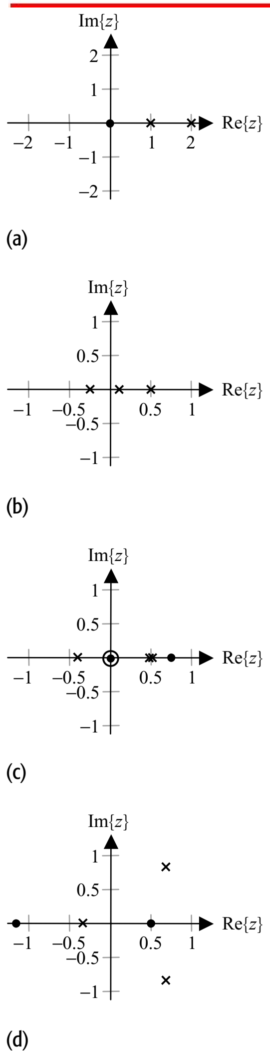


Fig. 13.6. Pole and zero plots for transfer functions in Example 13.13. Plot (a) corresponds to part (i) of Example 13.13; plot (b) corresponds to part (ii); plot (c) corresponds to part (iii); and plot (d) corresponds to part (iv). Also note that plot (c) includes double zeros at $z = 0$ and double poles at $z = 0.5$.

or alternatively as

$$H(z) = z^{m-n} \frac{(1 - z_1 z^{-1})(1 - z_2 z^{-1}) \cdots (1 - z_m z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1}) \cdots (1 - p_n z^{-1})}. \tag{13.39b}$$

Example 13.13

Determine the poles and zeros of the following LTID systems:

$$\text{(i)}\quad H_1(z) = \frac{z}{z^2 - 3z + 2};$$

$$\text{(ii)}\quad H_2(z) = \frac{1}{(z - 0.1)(z - 0.5)(z + 0.2)};$$

$$\text{(iii)}\quad H_3(z) = \frac{z^2(2z - 1.5)}{(z + 0.4)(z - 0.5)^2};$$

$$\text{(iv)}\quad H_4(z) = \frac{z^2 + 0.7z + 1.6}{(z^2 - 1.2z + 1)(z + 0.3)}.$$

Solution

$$\text{(i)}\quad H_1(z) = \frac{z}{z^2 - 3z + 2} = \frac{z}{(z - 1)(z - 2)}.$$

There is one zero, at $z = 0$, and two poles, at $z = 1$ and 2.

$$\text{(ii)}\quad H_2(z) = \frac{1}{(z - 0.1)(z - 0.5)(z + 0.2)}.$$

There is no zero, but there are three poles, at $z = 0.1$, 0.5, and -0.2.

$$\text{(iii)}\quad H_3(z) = \frac{z^2(2z - 1.5)}{(z + 0.4)(z - 0.5)^2}.$$

There are three zeros, at $z = 0$, 0, and 0.75. There are three poles, at $z = -0.4$, 0.5, and 0.5.

$$\text{(iv)}\quad H_4(z) = \frac{(z - 0.5)(z + 1.2)}{((z - 0.6)^2 + 0.8^2)(z + 0.3)} = \frac{(z - 0.5)(z + 1.2)}{(z - 0.6 + \mathrm{j}0.8)(z - 0.6 - \mathrm{j}0.8)(z + 0.3)}.$$

There are two zeros, at $z = 0.5$ and -1.2. There are three poles, at $z = 0.6 - \mathrm{j}0.8$, $0.6 + \mathrm{j}0.8$, and -0.3.

The poles and zeros of the above four systems are shown in Fig. 13.6. In the plot, $\times$ marks the position of a pole and $\bullet$ marks the position of a zero.

13.8.1 Determination of impulse response

The impulse response $h[k]$ of an LTID system can be obtained by calculating the inverse z-transform of the transfer function $H(z)$. Example 13.14 explains the steps involved in determining the impulse response.

Example 13.14

The input–output relationship of an LTID system is given by the following difference equation:

$$y[k+2] - \frac{3}{4}y[k+1] + \frac{1}{8}y[k] = 2x[k+2]. \qquad (13.40)$$

Determine the transfer function and the impulse response of the system.

Solution

Substituting $m = k + 2$, Eq. (13.40) can be written as follows:

$$y[m] - \frac{3}{4}y[m-1] + \frac{1}{8}y[m] = 2x[m].$$

Calculating the z-transform on both sides of the equation yields

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = 2X(z),$$

which results in the following transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}.$$

To calculate the impulse response of the LTID system, consider the partial fraction expansion of $H(z)$ as

$$H(z) = \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)} \equiv \frac{4}{1 - \frac{1}{2}z^{-1}} - \frac{2}{1 - \frac{1}{4}z^{-1}}.$$

By calculating the inverse z-transform of both sides, the impulse response $h[k]$ is obtained:

$$h[k] = 4\left(\frac{1}{2}\right)^k u[k] - 2\left(\frac{1}{4}\right)^k u[k],$$

which is identical to the result obtained by Fourier technique in Example 11.18.

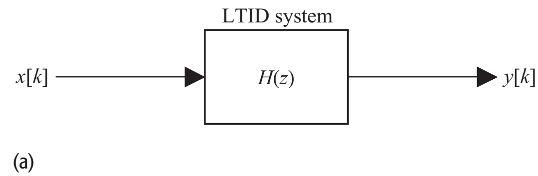
13.9. Relationship between Laplace and z-transforms

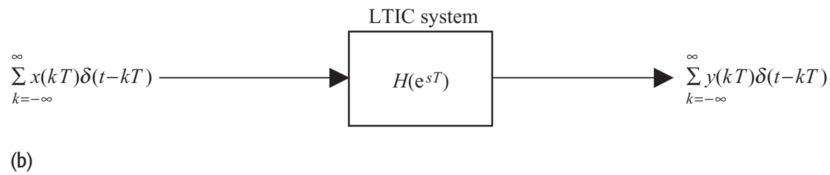
LTID signals and systems can be considered as special cases of LTIC signals and systems. Therefore, the Laplace transform can also be used to analyze such signals and systems. In this section, we derive the relationship between the Laplace and z-transforms.

If a DT sequence $x[k]$ is obtained by sampling a CT signal $x(t)$ with a sampling interval T, the CT sampled signal $x_s(t)$ may be expressed as follows:

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT),$$

Fig. 13.7. Using Laplace transform techniques to analyze LTID systems. (a) Reference LTID system; (b) equivalent LTIC system with CT input and output signals.





where $x(kT)$ are the sampled values of $x(t)$ which equals the DT sequence $x[k]$. Calculating the Laplace transform of $x_s(t)$, we obtain

$$X(s) = L\{x_s(t)\} = \sum_{k=-\infty}^{\infty} x(kT)L\{\delta(t-kT)\} = \sum_{k=-\infty}^{\infty} x(kT)e^{-kTs}.$$

Comparing $X(s)$ with the z-analysis equation,

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k},$$

it is clear that

$$X(s) = X(z)|_{z=e^{sT}} \tag{13.41a}$$

since $x[k] = x(kT)$. Equation (13.41a) illustrates the relationship between the Laplace transform $X(s)$ of a sampled function and the z-transform $X(z)$ of the DT sequence obtained from the samples. As illustrated in Fig. 13.7, an LTID system can be analyzed using an equivalent LTIC system. Figure 13.7(a) shows an LTID system with the z-transfer function $H(z)$ and sequence $x[k]$ applied at its input. The analysis of the LTID system can be completed in the s-domain with the LTIC system shown in Fig. 13.7(b). The transfer function of the LTIC system is given by

$$H(s) = H(z)|_{z=e^{sT}} \tag{13.41b}$$

and the DT input is transformed to an equivalent CT input of the form

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT).$$

The output in Fig. 13.7(b) can be calculated using CT analysis techniques. The resulting output $y(t)$ can then be transformed back into the DT domain using the relationship $y[k] = x(t)$ at $t = kT$.

Example 13.15

A DT system is represented by the following impulse response function:

$$h[k] = 0.5^5 u[k]. \tag{13.42}$$

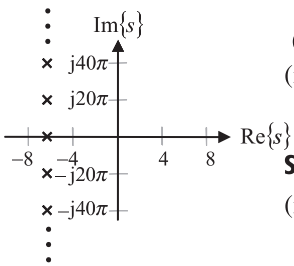


Fig. 13.8. Location of poles in the s-plane for the system in Example 13.15 with $T = 0.1$.

(i) Determine the z-transfer function of the system.
(ii) Determine the equivalent Laplace transfer function of the system.
(iii) Using the Laplace domain approach, determine if the system is stable.

Solution

(i) $H(z) = Z\{0.5^k u[k]\} = \dfrac{1}{1 - 0.5z^{-1}}$, or $\dfrac{z}{z - 0.5}$, ROC: $|z| > 0.5$.

(ii) Using Eq. (13.41b), the Laplace transfer function is given by

$$H(s) = H(z)|_{z=e^{sT}} = \frac{e^{sT}}{e^{sT} - 0.5}, \tag{13.43}$$

where T is the sampling interval.

(iii) A causal LTIC system is stable if all the poles corresponding to the Laplace transfer function lie in the left-hand half of the s-plane. Therefore, we will first calculate the pole locations in the s-plane, and then determine if the system is stable. The poles of the transfer function, Eq (13.43), are calculated from the characteristic equation as follows:

$$e^{sT} - 0.5 = 0 \Rightarrow e^{sT} = 0.5 \Rightarrow e^{(sT \pm j2\pi m)} = 0.5,$$

where $m = 0, 1, 2, \ldots$ Solving for the roots of this equation yields

$$s = \frac{1}{T}[\ln 0.5 \pm j2\pi m] \approx \frac{1}{T}[-0.693 \pm j2\pi m].$$

It is observed that an LTID system has an infinite number of poles in the s-domain. The locations of these poles for $T = 0.1$ are shown in Fig. 13.8. It is clear that these poles would lie in the left-half of the s-plane, irrespective of the value of the sampling interval T. The LTID system is, therefore, causal and stable.

Alternatively, the stability of the LTID system can be determined from its impulse response by noting that

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} 0.5^k = 2 < \infty,$$

which satisfies the BIBO stability requirement derived in Chapter 10.

13.10. DTFT and the z-transform

In Chapter 11 and in this chapter, we presented two different frequency-domain approaches to analyze DT signals and systems. The DTFT-based approach, introduced in Chapter 11, uses the real frequency Ω, whereas the z-transform-based approach uses the complex frequency $\sigma + j\Omega$. The output response of

an LTID system can be computed using the convolution property of either the DTFT or the z-transform. In addition, the frequency-domain approach offers insight about the system characteristics, which is not readily available from the time-domain approach. However, an important issue is to determine which of the two transforms should be used to analyze the LTID system. Both approaches have their own advantages. Depending upon the application under consideration, the appropriate transform should be selected.

Example 13.19

Consider an LTID system represented by the unit impulse response $h[k] = 0.8^k u[k]$. Calculate the overall output and steady state output of the LTID system for the input sequence $x[k] = \cos(\pi k/3)u[k]$.

Solution

z-transform method Using Table 13.1, the z-transforms of the impulse response $h[k]$ and the input $x[k]$ are given by

$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$

and

$$X(z) = \frac{1 - z^{-1}\cos(\pi/3)}{1 - 2z^{-1}\cos(\pi/3) + z^{-2}} = \frac{1 - 0.5z^{-1}}{1 - z^{-1} + z^{-2}}.$$

Using the convolution property, the z-transform of the output response is given by

$$Y(z) = H(z)X(z) = \frac{1 - 0.5z^{-1}}{(1 - 0.8z^{-1})(1 - z^{-1} + z^{-2})}.$$

By partial fraction expansion, the above expression becomes

$$\begin{aligned} Y(z) &= \frac{2}{7} \times \frac{1}{1 - 0.8z^{-1}} + \frac{5}{7} \times \frac{1 + 0.5z^{-1}}{1 - z^{-1} + z^{-2}} \\ &= \frac{2}{7} \times \frac{1}{1 - 0.8z^{-1}} + \frac{5}{7} \times \frac{1 - 0.5z^{-1}}{1 - z^{-1} + z^{-2}} + \frac{5}{7} \times \frac{z^{-1}}{1 - z^{-1} + z^{-2}}. \end{aligned}$$

Taking the inverse z-transform, the output response is given by

$$\begin{aligned} y[k] &= \frac{2}{7} \times 0.8^k u[k] + \frac{5}{7} \times \cos\left(\frac{\pi k}{3}\right) u[k] + \frac{10}{7\sqrt{3}} \times \sin\left(\frac{\pi k}{3}\right) u[k] \\ &= \left[0.287(0.8)^k + 1.091 \cos\left(\frac{\pi k}{3} - 0.857^{\mathrm{r}}\right)\right] u[k] \end{aligned}$$

where the superscript r indicates that the angle is expressed in radians.

The steady state output $y_{ss}[k]$ is computed by neglecting the transient term $(0.8)^k$, which decays to zero with time. The steady state output response is, therefore, given by

$$y_{ss}[k] = 1.091 \cos\left(\frac{\pi k}{3} - 0.857^{\mathrm{r}}\right) u[k].$$

DTFT method As in the CT case, the calculation of the actual output is difficult using the DTFT. However, the steady state value of the output can be easily calculated using DTFT. We have

$$H(\Omega) = \frac{1}{1 - 0.8\mathrm{e}^{-\mathrm{j}\Omega}}.$$

The value of the DTFT transfer function at $\Omega = \pi/3$, the fundamental frequency of the sinusoidal input, is given by

$$H(\Omega)|_{\Omega=\pi/3} = \frac{1}{1 - 0.8\mathrm{e}^{-\mathrm{j}(\pi/3)}} = 0.714 - \mathrm{j}0.285 = 1.091\mathrm{e}^{-\mathrm{j}0.857},$$

implying that $|H(\Omega)| = 1.091$ and $<H(\Omega) = -0.857$ radians. Therefore, the steady state output response is given by

$$\begin{aligned} y_{ss}[k] &= |H(\Omega)| \times \cos\left(\frac{\pi k}{3} + \ <H(\Omega)\right) u[k] \\ &= 1.091 \cos\left(\frac{\pi k}{3} - 0.857^{\mathrm{r}}\right) u[k]. \end{aligned}$$

Example 13.19 shows that the z-transform is a more convenient tool for transient analysis. For the steady state analysis, the z-transform does not offer much advantage over the DTFT. In signal processing applications, such as audio, image and video processing, the transients are generally ignored. In such applications, the DTFT is sufficient to analyze the steady state response. On the other hand, the transient analysis is important for applications such as control systems and process control. This is precisely the reason for the widespread use of the z-transform in digital control and system design, whereas the DTFT is preferred in signal processing applications.