Ordinary Differential Equations

A differential equation is an equation involving derivatives of an unknown function. If the unknown can be assumed to be a function of only one variable (so the derivatives are the "ordinary"), then we say the differential equation is an ***ordinary differential equation (ODE)***. Otherwise, the equation is a ***partial differential equation (PDE)***. Our interest will just be in ODEs. In these notes, the variable will usually be denoted by x and the unknown function by y or $y(x)$. So, for any given ordinary differential equation.

1. The ***order*** of a differential equation is the order of the highest order derivative present in the equation. Differential equations are often classified with respect to order.
2. The ***degree*** of a differential equation is the power of the highest order derivative in the equation.
3. The ***solution*** of a differential equation is any function that satisfies the equation. A general solution is a formula that describes all solutions to the equation.

Example 1. $\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = sinx$, is of order 2 degree 3.

Example 2. $y = e^{2x}$ is a solution of $y' - 2y = 0$.

Example 3. $y = sinx + c$ is a general solution of $y' = cosx$, $y = sinx - 2$ is a particular solution of $y' = cosx$ with the condition $y(0) = -2$.

<u>What is the purpose of differential equations?</u>

Many physical laws and relations appear mathematically in the form of such equations. For example, electronic circuit, falling stone, vibration, etc. Any physical situation involved motion or measure rates of change can be described by a mathematical model, the model is just a differential equation. The transition from the physical problem to a corresponding mathematical model is called modeling In this course, we shall pay our attention to solve differential equations and don't care of modeling.

$$\boxed{\text{Physical Situation}} \xrightarrow{\text{Modeling}} \boxed{\text{DE}} \xrightarrow{\text{Solving}} \boxed{\text{Solution}}$$

Some Methods for Solving First-Order ODEs

1. Separable Equations :

A ***first-order ODE*** is ***separable*** if it can be written as :

$$\frac{dy}{dx} = g(x)h(y).$$

Such a differential equation can be solved by the following procedure :

1. Get it into the above form (i.e., the derivative equaling the product of a function of x the $g(x)$ above, with a function of y the above $h(y)$.
2. Divide through by $h(y)$ (but also consider the possibility that $h(y) = 0$.
3. Integrate both sides with respect to x.
4. Solve the last equation for $y(x)$.

Example 1.1. Consider finding the general solution to $\frac{dy}{dx} = 2x(y^2+1)$.

Sol: going through the above steps:

$$\frac{1}{y^2+1}\frac{dy}{dx} = 2x$$

$$\int \frac{1}{y^2+1}\frac{dy}{dx}\, dx = \int 2x\, dx$$

$$tan^{-1}y = x^2 + C \quad \rightarrow \quad y = \tan(x^2 + C).$$

Example 1.2. Solve $(1+x)e^{3y}\frac{dy}{dx} = 1$

Sol: $\int e^{3y}dy = \int \frac{dx}{1+x} \rightarrow \frac{1}{3}e^{3y} = ln|1+x| + C_1 \rightarrow e^{3y} = 3ln|1+x| + C$

$\rightarrow 3y = ln(3ln|1+x| + C) \rightarrow y = \frac{1}{3}ln(3ln|1+x| + C).$

Example 1.3. Solve $\frac{dy}{dx} = 3x^2y^2$, given $y(0) = \frac{1}{2}$.

Sol: $\int \frac{dy}{y^2} = \int 3x^2\, dx \rightarrow \frac{-1}{y} = x^3 + C \rightarrow y = \frac{-1}{x^3+C}$. Since $y(0) = \frac{1}{2} \rightarrow \frac{1}{2} = \frac{-1}{C}$

$C = -2$. Hence, $y = \frac{-1}{x^3-2}$.

Reduction to separable forms:

Certain first-order differential equation are not separable but can be made separable by a simple change of variables .The equation of the form $y' = g\left(\frac{y}{x}\right)$ can be made separable. Such a differential equation can be solved by the following procedure :

1. Set $u = \frac{y}{x}$, then $y = ux$ (change of variable).
2. Differential $y' = u + xu'$ (product differentiation formula).
3. The original DE $y' = g\left(\frac{y}{x}\right) \to u + xu' = g(u) \to xu' = g(u) - u \to \frac{du}{dx} = \frac{g(u)-u}{x} \to \frac{du}{g(u)-u} = \frac{dx}{x}$.
4. Integrate both sides of the equation.
5. Replace u by y/x.

Example 1.4. Solve $2xyy' = y^2 - x^2$.

Sol: Dividing by x^2 we have, $2\frac{y}{x}\,y' - \left(\frac{y}{x}\right)^2 + 1 = 0$

By setting $u = \frac{y}{x} \to y = ux \to y' = u + xu'$. Then $2u\,(u + xu') - u^2 + 1 = 0$.

$\to 2u^2 + 2uxu' - u^2 + 1 = 0 \to u^2 + 2uxu' + 1 = 0 \to -2uxu' = u^2 + 1 \to$

$\frac{2u\,du}{u^2+1} = \frac{-dx}{x} \to \ln(u^2+1) = -\ln|x| + C_1 \to u^2 + 1 = \frac{C}{x} \to \left(\frac{y}{x}\right)^2 + 1 = \frac{C}{x} \to$

$x^2 + y^2 = Cx$.

Example 1.5. Solve initial value problem

$$y' = \frac{y}{x} + \frac{2x^3\cos(x^2)}{y}, \quad y(\sqrt{\pi}) = 0.$$

Sol: Change of variable $u = \frac{y}{x} \to y = ux \to y' = u + xu'$. Then $u + xu' = u + \frac{2x^2\cos(x^2)}{u} \to uu' = 2x\cos(x^2) \to \frac{u^2}{2} = \sin(x^2) + C_1 \to y = x\sqrt{2x\cos(x^2) + C}$

Since $y(\sqrt{\pi}) = 0 \to C = 0 \to y = x\sqrt{2x\cos(x^2)}$.

2. Linear Equations:

A ***first-order ODE*** is said to be ***linear*** if it can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x).$$

where $p(x)$ and $q(x)$ are known functions of x. Such a differential equation can be solved by the following procedure:

1. Get it into the above form.
2. Compute the integrating factor $\mu(x) = e^{\int p(x)dx}$.
3. Find $\int \mu(x)q(x)\,dx$.
4. Then $y = \frac{1}{\mu(x)}\left[\int \mu(x)q(x)\,dx + C\right]$.

Example 2.1. Consider finding the general solution to $x\frac{dy}{dx} + 4y = 21x^3$.

Sol: going through the above steps:

Dividing through by x gives $\frac{dy}{dx} + \frac{4}{x}y = 21x^2$.

So the integrating factor is $\mu(x) = e^{\int p(x)dx} = e^{\int (4/x)dx} = e^{4\ln x} = x^4$.

$\int \mu(x)q(x)\, dx = \int 21x^6\, dx = 3x^7$. Then $y = \frac{1}{x^4}\,[3x^7 + C] = 3x^3 + Cx^{-4}$.

Example 2.2. Solve $\frac{dy}{dx} + y = e^x$.

Sol: $\mu(x) = e^{\int p(x)dx} = e^{\int dx} = e^x$.

$\int \mu(x)q(x)\, dx = \int e^{2x}\, dx = \frac{1}{2}e^{2x}$. Then $y = \frac{1}{e^x}\left[\frac{1}{2}e^{2x} + C\right] = \frac{1}{2}e^x + Ce^{-x}$.

Example 2.3. Solve $\frac{dy}{dx} - y = \sin x$.

Sol: $\mu(x) = e^{\int p(x)dx} = e^{\int -dx} = e^{-x}$.

$\int \mu(x)q(x)\, dx = \int e^{-x} \sin x\, dx = \frac{-1}{2}e^{-x}(\sin x + \cos x)$ (integration by parts).

Then $y = e^x\left[\frac{-1}{2}e^{-x}(\sin x + \cos x) + C\right] = \frac{-1}{2}(\sin x + \cos x) + Ce^x$.

3.Exact differential equations :

Now we want to consider a DE as

$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$. That is $M(x,y)\, dx + N(x,y)dy = 0$. The solving principle can be .

Method (1) : transform this equation to be separable or reduction to separable forms;

Method (2) : to find a function $u(x,y)$ such that the total differential du is equal to $Mdx + Ndy$.

In the latter strategy, if u exists, then equation $Mdx + Ndy$ is called ***exact***, and $u(x,y)$ is called a ***potential*** function for this differential equation. We know that $du = 0$ $\rightarrow u(x,y) = c$, it is just the general solution of the differential equation.

<u>Now how to find such u ?</u>

Since u as a function of two independent variables x and y.

Then, $du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y} dy = M\, dx + N\, dy \;\rightarrow\; \frac{\partial u}{\partial x} = M \;\&\; \frac{\partial u}{\partial y} = N$.

step1. To integrate M w.r.t. x or integrate N w.r.t. y to obtain u . Assume u is obtained by integrating M, then $u(x,y) = \int M\, dx + K(y)$.

step2. Partial differentiate u w.r.t. y (i.e. $\frac{\partial u}{\partial y}$) and to compare with N to find K function.

How to test $Mdx + Ndy = 0$ is exact or not ?

Proposition 3.1. (Test for exactness)
If $M, N, \frac{\partial N}{\partial x}$, and $\frac{\partial M}{\partial y}$ are continuous over a rectangular region R, then $Mdx + Ndy = 0$ is exact for (x, y) in R if and only if , $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ in R.

Example 3.1. Solve $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0.$

Sol: 1st step: Testing for exactness

$M = x^3 + 3xy^2$, $N = 3x^2y + y^3 \rightarrow \frac{\partial M}{\partial y} = 6xy = \frac{\partial N}{\partial x}$. The equation is exact.

2nd step: $u = \int Mdx + K(y) = \int (x^3 + 3xy^2)dx + K(y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + K(y).$

3rd step: Since $\frac{\partial u}{\partial y} = N \rightarrow 3x^2y + K'(y) = 3x^2y + y^3 \rightarrow K'(y) = y^3 \rightarrow$

$K(y) = \frac{1}{4}y^4 + C_1.$

Thus $u = \frac{1}{4}(x^4 + 6x^2y^2 + y^4) + C_1$. Then the solution is $\frac{1}{4}(x^4 + 6x^2y^2 + y^4) = C.$
This is an implicit solution of the original DE.

4th step: checking solution for $Mdx + Ndy = 0.$

$\frac{d}{dx}\left[\frac{1}{4}(x^4 + 6x^2y^2 + y^4)\right] = \frac{dC}{dx} \rightarrow \frac{1}{4}(4x^3 + 12xy^2 + 12x^2yy' + 4y^3y') = 0.$

$\rightarrow (x^3 + 3xy^2) + (3x^2y + y^3)y' = 0 \rightarrow (x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0.$

Example 3.2. Solve $(sinx\ coshy)dx - (cosx\ sinhy)dy = 0, y(0) = 3.$

Sol: $M = sinx\ coshy,\ N = -cosx\ sinhy \rightarrow \frac{\partial M}{\partial y} = sinx\ sinhy = \frac{\partial N}{\partial x}.$

The DE is exact.

If $u = \int sinx\ coshy\ dx + k(y) = -cosx\ coshy + k(y)$

$\frac{\partial u}{\partial y} = -cosx\ coshy \rightarrow k =$ constant $\rightarrow$ Solution is $cosx\ coshy = c.$

Since $y(0) = 3, cos0\ cosh 3 = c \rightarrow cosx\ coshy = cosh 3.$

Integrating factors:

If a DE $M(x,y)dx + N(x,y)dy = 0$ is not exact, then we can sometimes find a nonzero function $F(x,y)$ such that $F(x,y)M(x,y)dx + F(x,y)N(x,y)dy = 0$ is exact. We call $F(x,y)$ an integrating factor for $Mdx + Ndy = 0$.

Note:

1. Integrating factor is not unique.
2. The integrating factor is independent of the solution.

Example 3.3. Solve $ydx - xdy = 0$.

Sol: $M = y, N = -x \rightarrow \frac{\partial M}{\partial y} = 1$ & $\frac{\partial N}{\partial x} = -1$. Then a DE is non- exact.

If you solve the equation by the same method.

$u = \int Mdx + k(y) = xy + k(y) \rightarrow \frac{\partial u}{\partial y} = x + k'(y) = N = -x \rightarrow k'(y) = -2x.$

Since $k(y)$ depends only on y; we cannot find the solution. Try $u = \int Ndy + k(x)$ also gets the same contradiction. Truly, the DE is separable.

How to find integrating factors?

We can find integrating factors by using one of the following two cases:

Case 1. If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ be a function of x only. Then $F(x) = e^{\int\left[\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\right]dx}$.

Case 2. If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ be a function of y only. Then $F(y) = e^{\int\left[\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\right]dy}$.

Example 3.4. Solve $2xydx + (4y + 3x^2)dy = 0$

Sol: $M = 2xy, N = 4y + 3x^2 \rightarrow \frac{\partial M}{\partial y} = 2x \neq \frac{\partial N}{\partial x} = 6x \rightarrow$ non- exact.

Testing whether $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ depends only on x or not.

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-4x}{4y+3x^2}$ depends on both x and y.

Testing whether $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ depends only on y or not.

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{2}{y}$ depends only on y.

Thus, $F(y) = e^{\int\frac{2}{y}dy} = y^2$. The original DE becomes

$2xy^3dx + (4y^3 + 3x^2y^2)dy = 0$ exact

$u = \int Mdx + K(y) = \int 2xy^3dx + K(y) = x^2y^3 + K(y)$

$$\frac{\partial u}{\partial y} = 3x^2y^2 + k'(y) = 4y^3 + 3x^2y^2 \rightarrow k'(y) = 4y^3 \rightarrow k(y) = y^4 + C_1$$

$u = x^2y^3 + y^4 + C_1$ → The solution is $x^2y^3 + y^4 = C$.

Exercises: Solve the following ODEs:

1. $y\frac{dy}{dx} - (1 + y^2)x^2 = 0.$
2. $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}.$
3. $\frac{dy}{dx} = \frac{x+2y}{3y-2x}.$
4. $\frac{dy}{dx} = -\frac{x+y}{x}.$
5. $\frac{dy}{dx} + \frac{1}{3}y = 1.$
6. $x\frac{dy}{dx} + y = x.$
7. $(y - x^3)dx + (x + y^3)dy = 0.$
8. $(cosy + ycosx)dx + (sinx - xsiny)dy = 0.$
9. $xy^3dx + (x^2y^2 + 1)dy = 0.$