Chapter Two

Partial Derivative

2.1 partial derivatives of function of two variables

Suppose that (x_O , y_O) is a point in the domain of a function f(x,y).

If we fix $y=y_O$ then $f(x ,y_O)$ is a function of a variable x alone .

The value of derivative is : $\frac{d}{dx}[f(x, y_O)]$

If we fix $x=x_O$, then $f(x_O,y)$ is a function of a variable y alone.

The value of derivative is : $\frac{d}{dy}[f(x_O, y)]$

If z= f(x,y) and (x_O, y_O) is a point in the domain of f , then the partial derivative of z with respect to x at (x_O, y_O) is :

$$fx(x_O, y_O) = \frac{d}{dx}[f(x, y_O)]_{x=x_O} = \lim_{x \to x_O} \frac{f(x , y_O) - f(x_O , y_O)}{x - x_O}$$

fx (x_O, y_O) = The slope of the surface z=f(x,y) in the x direction at (x_O, y_O).

=The rate of change of z with respect to x along the curve C_1.**Fig.(2.1)**

Similarly… $fy(x_O, y_O) = \frac{d}{dy}[f(x_O, y]_{y=y_O} = \lim_{y \to y_O} \frac{f(x_O,y) - f(x_O,y_O)}{y - y_O}$

$fy(x_O, y_O)$ = The slope of the surface z=f(x,y) in the y direction at (x_O ,y_O)

= The rate of change of z with respect to y along the curve C_2.**Fig(2.2)**

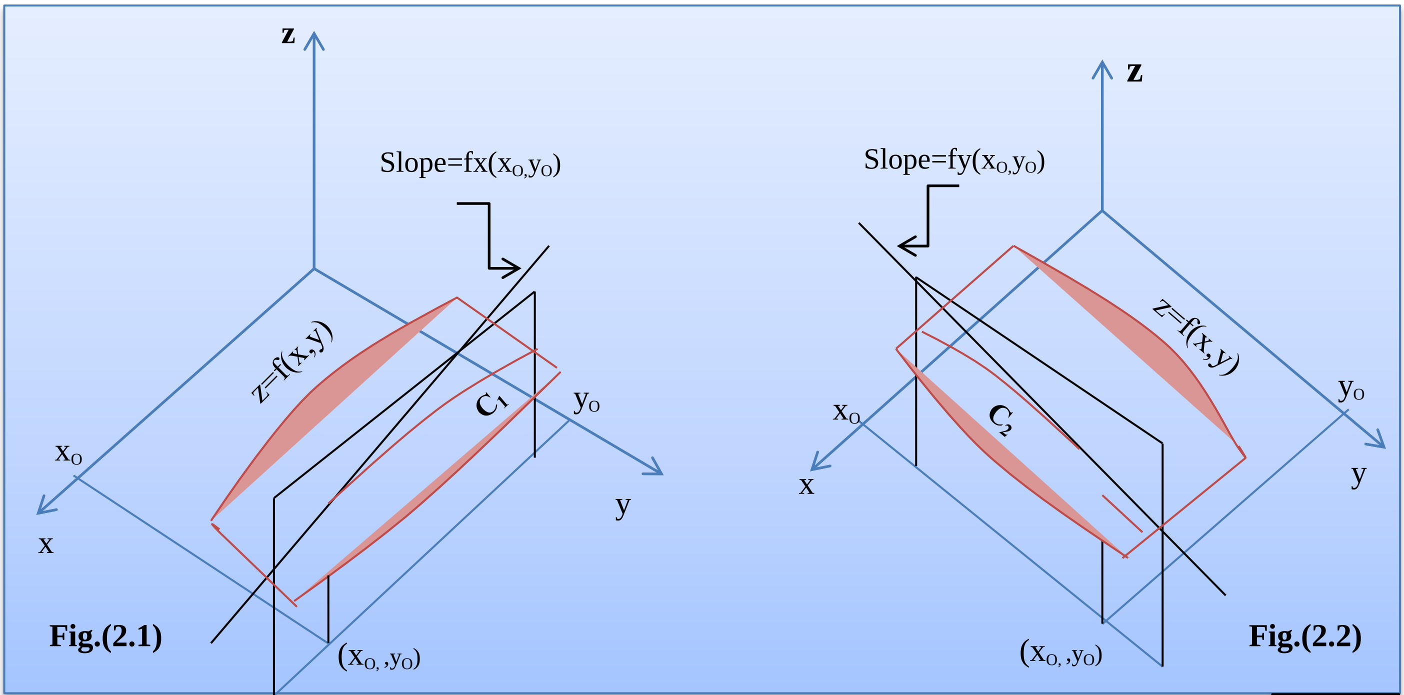
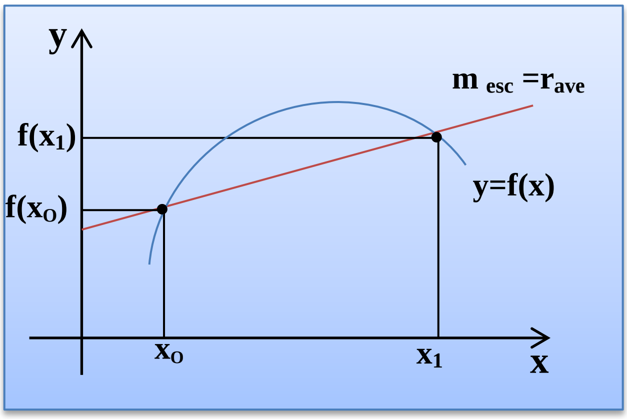


Fig.(2.1) **Fig.(2.2)**

Slopes and Rates of Change

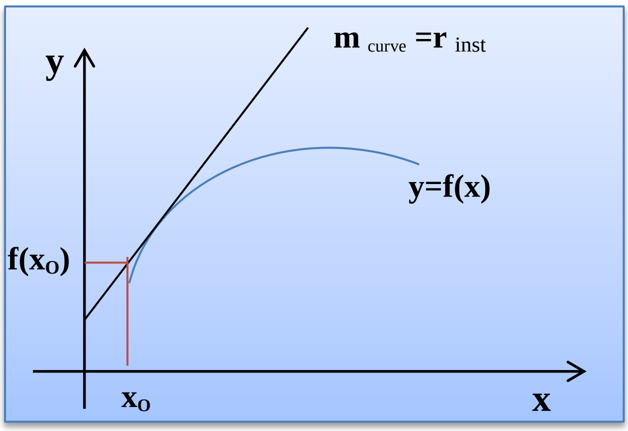
If y=f(x) , then the average rate of change of y with respect to x over the interval $[x_{o\,,}x_1]$ is



$$r_{ave} = \frac{f(x_1)-f(x_o)}{x_1-x_o}$$

If y=f(x) , then the instantaneous rate of change of y with respect to x when $x=x_o$ is

$$r_{inst} = \lim_{x_1\to x_\circ} \frac{f(x_1)-f(x_\circ)}{x_1-x_\circ}$$

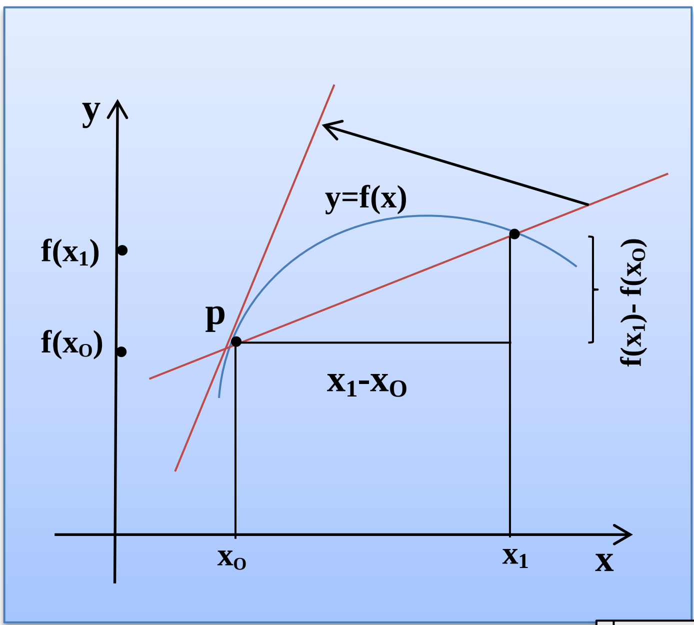


$\lim_{x_1\to x_\circ} \frac{f(x_1)-f(x_\circ)}{x_1-x_\circ}$ exists , then

The value of this limit is called the ***derivative of f at*** $x=x_o$ and if denoted by $\bar{f}(x_o)$

$$\bar{f}(x_o) = \lim_{x_1\to x_\circ} \frac{f(x_1)-f(x_o)}{x_1-x_o}$$

$\bar{f}(x_o)$ is the slope of the graph of f at point $p(x_o, f(x_o))$



Example :let $f(x, y) = x^2y + 5y^3$

1) Find the slope of the surface z =f(x,y) in the x – direction at the point (1,-2)
2) Find the slope of the surface z = f(x,y) in the y –direction at the point (1,-2)

Solution :

1) Differentiating f with respect to x with y held fixed yields

$$fx(x, y) = 2xy$$

The slope in the x- direction is fx(1,-2)= -4 , that is z is decreasing at the rate of 4 unit per unit increase in x .

2) Differentiating f with respect to y with x held fixed yields

$$fy(x, y) = x^2 + 15y^2$$

The slope in the y –direction is fy (1,-2)=61 , that is , z is increasing at the rate of 61 units per unit increase in y .

Example : determined fx(1,3) and fy(1,3)for the function

$$f(x, y) = 2x^3y^2 + 2y + 4x$$

Solution :

$$fx(x, y) = \frac{d}{dx}[2x^3y^2 + 2y + 4x] = 6x^2y^2 + 4$$

$$fx(1,3) = 6(1)^2(3)^2 + 4 = 58$$

$$fy(x, y) = \frac{d}{dy}[2x^3y^2 + 2y + 4x] = 4x^3y + 2$$

$$fy(1,3) = 4(1)^3(3) + 2 = 14$$

2.2 Partial Derivative notation

If z = f(x,y) , then the partial derivatives fx and fy are also denoted by the symbols

$$\frac{\partial f}{\partial x},\frac{\partial z}{\partial x}\ and\ \frac{\partial f}{\partial y},\frac{\partial z}{\partial y}$$

Some typical notations for the partial derivatives of z = f(x,y) at a point (x_O, y_O) are

$$\left.\frac{\partial f}{\partial x}\right|x=x_O,y=y_O,\qquad \frac{\partial z}{\partial x}|(x_O,y_0),\quad \left.\frac{\partial y}{\partial x}\right|(x_O,y_O),\qquad \frac{\partial f}{\partial x}(x_O,y_O),\qquad \frac{\partial z}{\partial x}(x_O,y_O)$$

Example : find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y} if\ z=x^4\sin(xy^3)$

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}[x^4\sin(xy^3)]=x^4\frac{\partial}{\partial x}[\sin(xy^3)]+\sin(xy^3)\frac{\partial}{\partial x}[x^4]$$

$$=x^4\cos(xy^3)*y^3+\sin(xy^3)*4x^3=x^4y^3\cos(xy^3)+4x^3\sin(xy^3)$$

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}[x^4\sin(xy^3]=x^4\frac{\partial}{\partial y}[\sin(xy^3)]+\sin(xy^3)\frac{\partial}{\partial y}[x^4]$$

$$=x^4\cos(xy^3)*3xy^2+\sin(xy^3)*0=3x^5y^2\cos(xy^3)$$

Note :

The partial derivatives of a function give the instantaneous rates of change of that function in directions parallel to the coordinate .

2.3 Partial Derivatives of functions with more than two variables

For a function f (x,y,z) of three variables , there are three partial derivatives

$$fx(x,y,z)\ , \qquad fy(x,y,z)\ , \qquad fz(x,y,z)$$

The partial derivative fx is calculated by holding y and z constant and different rating with respect to x. For fy the variables x and y are held constant . if w =(x,y,z)

The three partial derivatives of f can be denoted by

$$\frac{\partial w}{\partial x}, \ \frac{\partial w}{\partial y}\ , and\ \frac{\partial w}{\partial z}$$

Example :if $f = (x,y,z) = x^3y^2z^4 + 2xy + z,$ then

$$fx(x,y,z) = 3x^2y^2z^4 + zy$$

$$fy(x,y,z) = 2x^3yz^4 + zx$$

$$fz(x,y,z) = 4x^3y^2z^3 + 1$$

2.4 Higher – Order Partial Derivatives

The partial derivatives $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$, themselves have partial derivatives. This gives rise to four possible second – order partial derivatives of f , which are defined by :

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$ differential twice with respect to x

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$ differential twice with respect to y

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$ differential first with respect to x and then with respect to y

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$ differential first with respect to y and then with respect to x

The last two cases are called the **mixed second –order partial derivatives** .

Also , the derivatives $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ are called the **first – order partial derivatives**

Some possibilities of third –order , fourth – order are :

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = f_{xxx} \quad , \quad \frac{\partial^4 f}{\partial y^4} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y^3}\right) = f_{yyy}$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = f_{xyy} , \quad \frac{\partial^4 f}{\partial y^2 \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y \partial x^2}\right) = f_{xxyy}$$

Example :find the second –order partial derivative of $f(x,y) = x^2y^2 + x^{4y}$

Solution :

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y \quad and \quad \frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy^3 + 4x^3y) = 2y^3 + 12x^2y = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2y^2 + x^4) = 6x^2y = f_{yy}$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2 + x^4) = 6xy^2 + 4x^3 = f_{yx}$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy^3 + 4x^3y) = 6xy^2 + 4x^3 = f_{xy}$$

Example :let $f(x,y) = y^2e^x + y \quad find\ f_{xyy}$

Solution :

$$f_{xyy} = \frac{\partial^3 f}{\partial y^2\partial x} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y^2}(y^2e^x) = \frac{\partial}{\partial y}(2ye^x) = 2e^x$$