2.5 The Chain Role

If x = x(t) and y = y(t) are differentiable at t, and if z = f(x,y) is differentiable at the point (x,y) = { x(t) , y(t)} , then :

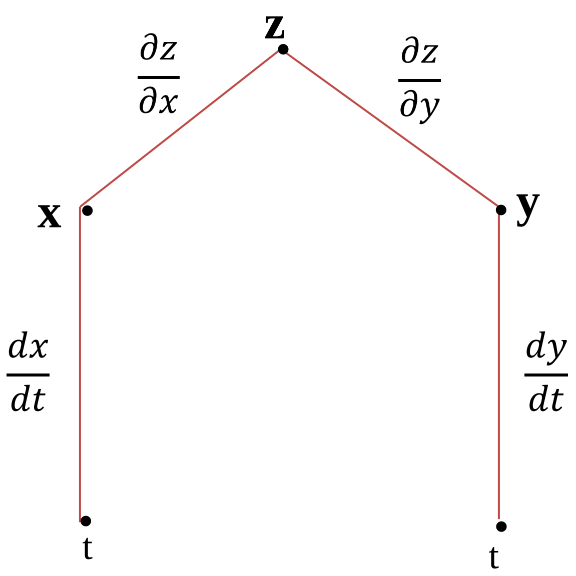
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$ Two – variable chain rule

Where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y)

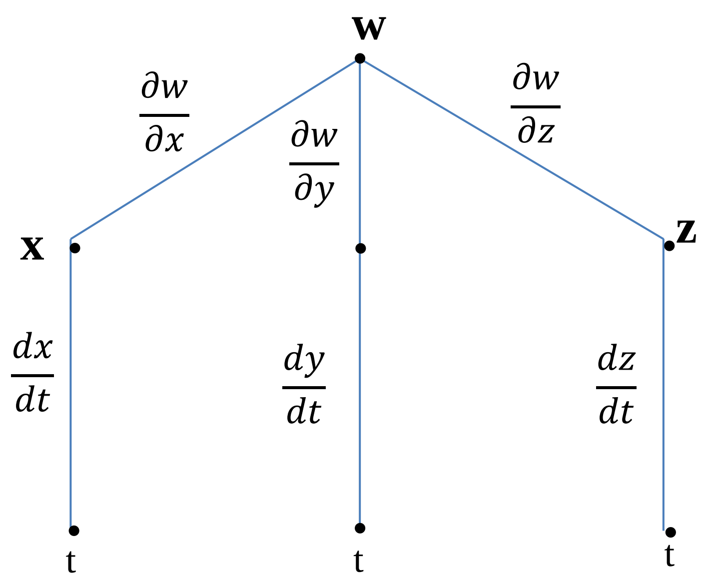
Also,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$ Three – variable chain rule

Where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y, z)



Tree diagram of two-variables chain rule



Tree diagram of Three – variables chain rule

Example : Suppose that $z= x^2y$, $x=t^2$, $y=t^3$, use the chain rule to find dz/dt ?

Solution :

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=(2xy)(2t)+(x^2)(3t^3)=7t^6$$

Alternatively , z = $x^2y=(t^2)^2(t^3)=t^7$ $then\ \frac{dz}{dt}=7t^6$

Example : Suppose that $z=\sqrt{xy+y}$, $x=\cos\theta$, $y=\sin\theta$ $find\ dz/d\theta$?

$$\frac{dz}{d\theta}=\frac{\partial z}{\partial x}\frac{dx}{d\theta}+\frac{\partial z}{\partial y}\frac{dy}{d\theta}=\frac{1}{2}(xy+y)^{-1/2}(y)(-\sin\theta)+\frac{1}{2}(xy+y)^{-\frac{1}{2}}(x+1)(\cos\theta)$$

When $\theta=\frac{\pi}{2}$ $,we\ have$ $,x=\cos\frac{\pi}{2}=0$, $y=\sin\frac{\pi}{2}=1$, $\theta=\frac{\pi}{2}$

$$\left.\frac{dz}{d\theta}\right|_{\theta=\pi/2}=\frac{1}{2}(1)(1)(-1)+\frac{1}{2}(1)(1)(0)=-\frac{1}{2}$$

2.5.1 Impact Differentiation

Consider the special case where z = f(x,y) is a function of x and y is a differentiable function of x , then

$$\frac{dz}{dx}=\frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{dy}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}$$

Suppose that the equation :

F(x,y) = c by differentiating both sides with respect to x

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0\ ,\qquad if\ \frac{\partial f}{\partial y}\neq 0\ ,we\ obtain$$

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{fx}{fy}$$

Example : given that $x^3+y^2-3=0\ ,\ find\frac{dy}{dx}$?

Solution :

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}\qquad f(x,y)=x^3+y^2-3=0$$

$$\partial f/\partial x=3x^2+y^2\qquad \partial f/\partial y=2yx$$

$$\frac{dy}{dx}=-\frac{3x^2+y^2}{2yx}$$

$$\boldsymbol{f(x,y,z)=0\ \ if\ \ \partial f/\partial z\neq 0\ then}$$

$$\boldsymbol{\frac{\partial z}{\partial x}=-\frac{\partial f/\partial x}{\partial f/\partial z}}$$

$$\boldsymbol{\frac{\partial z}{\partial y}=-\frac{\partial f/\partial y}{\partial f/\partial z}}$$

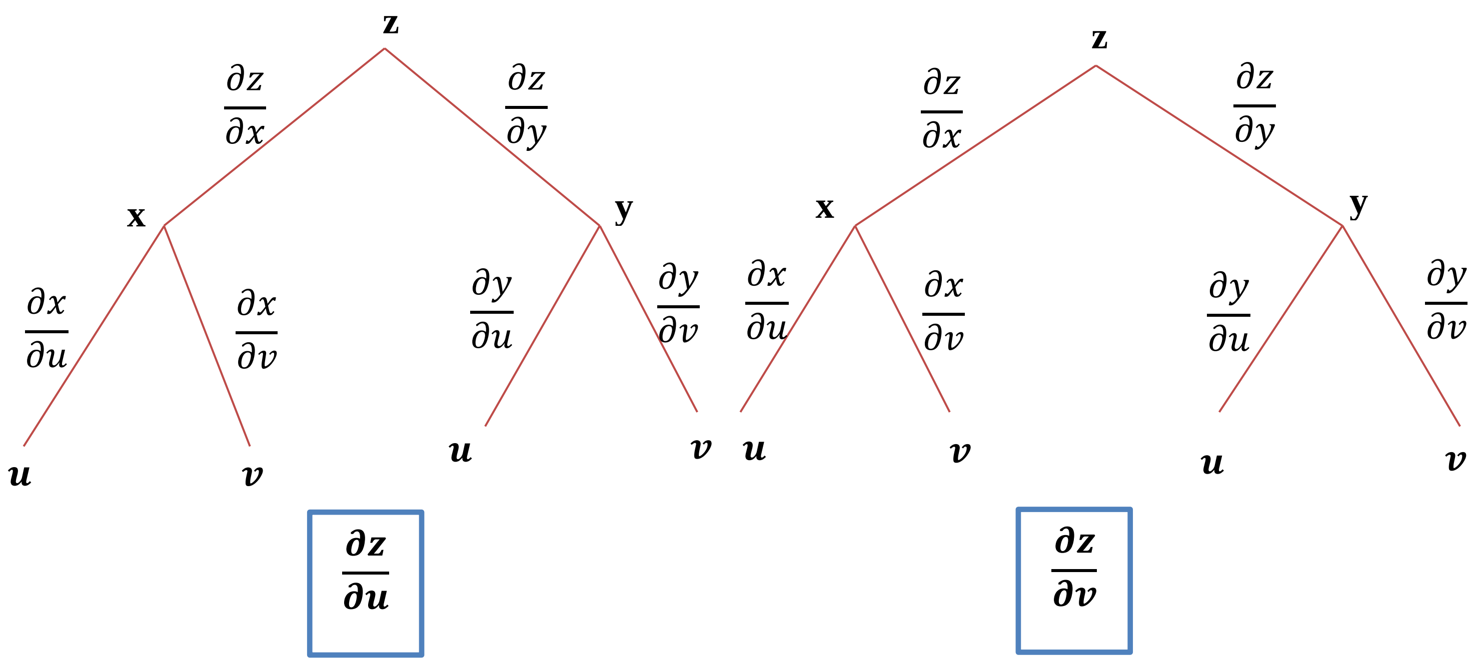
2.6 The chain rule for partial derivatives

2.6.1 consider the case where x and y are each functions of two variables let :

$z = f(x, y)$ *and* $x = x(u, v)$, $y = y(u, v)$: *then*

$Zz = f\big(x\ (u, v)\ ,\ \ y(u, v)\big)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \qquad and \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$



Example: given that , $z = e^{xy}$, $x = 2u + v$, $y = u/v$. Find $\partial z/\partial u$ *and* $\partial z/\partial v$ using the chain rule.

Solution:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (ye^{xy})(2) + (xe^{xy})\left(\frac{1}{v}\right) = \left[2y + \frac{x}{v}\right]e^{xy}$$

$$= \left[\frac{2u}{v} + \frac{2u+v}{v}\right]e^{(2u+v)\left(\frac{u}{v}\right)} = \left[\frac{4u}{v} + 1\right]e^{(2u+v)\left(\frac{u}{v}\right)}$$

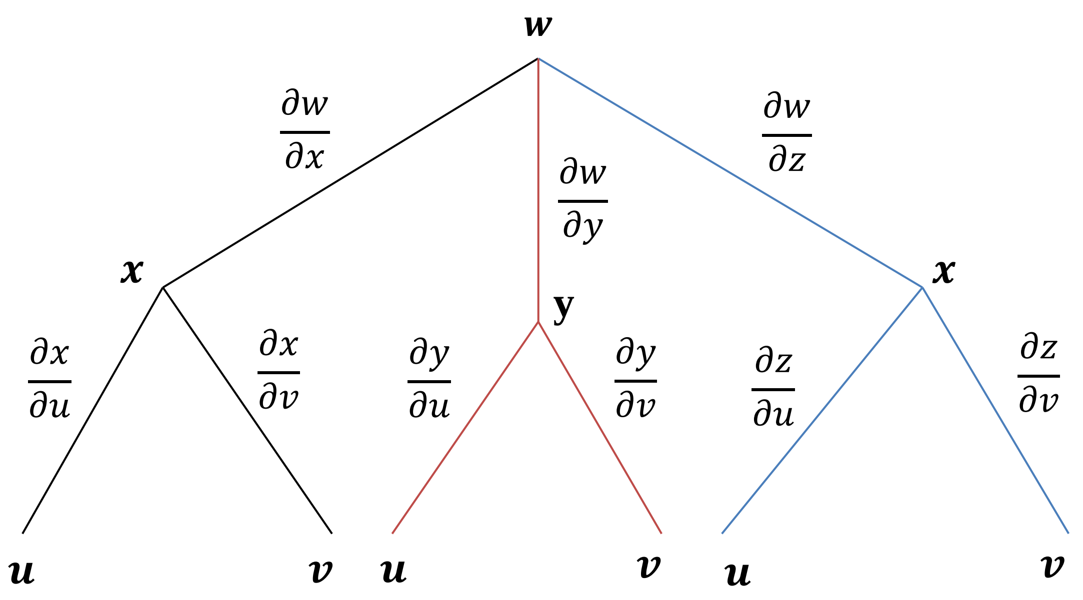
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (ye^{xy})(1) + (xe^{xy})\left(-\frac{u}{v^2}\right)$$

$$= \left[y - x\left(\frac{u}{v^2}\right)\right]e^{xy} = \left[\frac{u}{v} - (2u+v)\left(\frac{u}{v^2}\right)\right]e^{(2u+v)\left(\frac{u}{v}\right)} = -\frac{2u^2}{v^2}e^{(2u+v)\left(\frac{u}{v}\right)}$$

2.6.2

if the functions $x = x(u,v)$, $y = y(u,v)$ and $z = z(u,v)$ have first order partial derivatives at the point (u,v) , and if the function w= f(x,y,z) is differentiable at the point $(x(u,v),\ y(u,v),\ z(u,v))$, then :

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \quad and \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$



Tree diagram for three –variable chain rule

Example : suppose that w=e^{xyz}, $x = 3u + v$, $z = u^2 v$ $find$ $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$

Solution :

$$\frac{\partial w}{\partial u} = yze^{xyz}(3) + xze^{xyz}(3) + xye^{xyz}(2uv)$$

$$= e^{xyz}(3yz + 3xz + 2xyuv)$$

$$\frac{\partial w}{\partial v} = yze^{xyz}(1) + xyze^{xyz}(-1) + xye^{xyz}(u^2)$$

$$= e^{xyz}(yz - xz + xyu^2)$$

2.7 Direction derivatives and the gradient

2.7.1 Direction Derivatives

Directional derivatives allow as to compute the rates of change of a function with respect to distance in any direction

Duf: direction derivatives of f in the direction of u

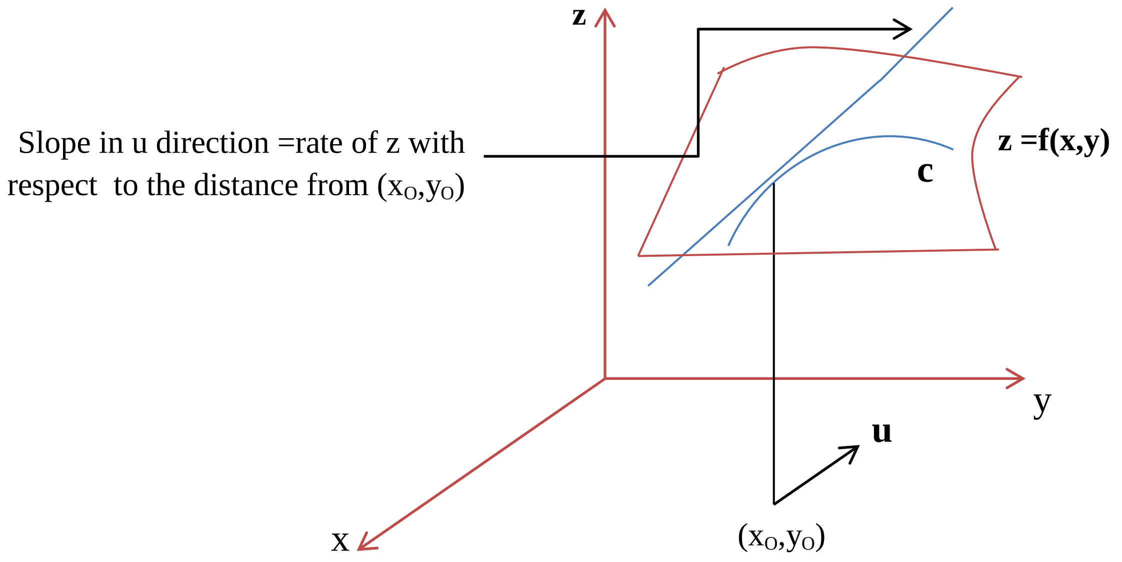
Geometrically :

Duf (x_o,y_o) is the slope of the surface z=f(x,y) in the direction of u at the point

$(x_o,y_o , f(x_o,y_o))$

Analytically :

Duf (x_o,y_o) represents the instantaneous rate of change of f(x,y) with respect to distance in the direction of u at the point (x_o,y_o)



Theorem :

a) If f(x,y) is differentiable at (x_o,y_o), and if u= u_1i+u_2j is a unit vector then :

$$Duf(x_o,y_o) = fx(x_o,y_o)u_1 + fy(x_o,y_o)u_2$$

b) If f(x,y,z) is differentiable at (x_o,y_o,z_o), and if u=$u_1i+u_2j+u_3k$ is a unite vector , then

$$Duf(x_o,y_o,z_o) = fx(x_o,y_o,z_o)u_1 + fy(x_o,y_o,z_o)u_2 + fz(x_o,y_o,z_o)u_3$$

Example : let f (x,y) =xy and find Duf (1,2), where u= $\frac{\sqrt{3}}{2}i+\frac{1}{2}j$

Solution :

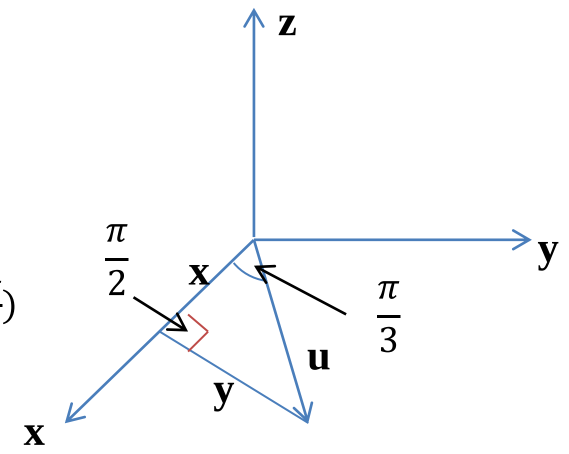
$$fx = y = 2 \ , \qquad fy = x = 1\,, \qquad u = \frac{\sqrt{3}}{2}i+\frac{1}{2}j$$

$$Duf(1,2) = 2\left(\frac{\sqrt{3}}{2}\right)+1\left(\frac{1}{2}\right) = \sqrt{3}+\frac{1}{2} \approx 2.23$$

We conclude that if we move a small distance from the point (1,2) in the direction of u , the function f (x,y) =xy will increase by about 2.23 times the distance moved .

Example :find the directional derivative of f(x,y) = $e^{xy} at\ (-2,0)$ in the direction of the unit vector that makes an angle of $\pi/3$ with the positive x –axis

Solution : the partial derivatives of f are



$fx(x,y) = ye^{xy}, \qquad fy(x,y) = xe^{xy}$

$fx(-2,0) = 0 \quad , \quad fy(-2,0) = -2$

$u = \cos\left(\frac{\pi}{3}\right)i + \sin\left(\frac{\pi}{3}\right)j = \frac{1}{2}i+\frac{\sqrt{3}}{2}j$

$$Duf(-2,0) = fx(-2,0)\cos\left(\frac{\pi}{3}\right) + fy(-2,0)\sin\left(\frac{\pi}{3}\right)$$

$$= 0\left(\frac{1}{2}\right)+(-2)\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$$

Example : find Duf for f(x,y,z) =$x^2y - yz^3 + z\ at$ the point (1,-2,0) in the direction of the vector a=2i+j-2k

Solution :

$fx(x,y,z) = zxy, \qquad fy(x,y,z) = x^2 - z^3, \quad fz(x,y,z) = -3yz^2+1$

$fx(1,-2,0) = -4\,, \quad fy(1,-2,0) = 1\,, \quad fz(1,-2,0) = 1$

$$u = \frac{a}{|a|} = \frac{1}{\sqrt{9}}(2i+j-2k) = \frac{2}{3}i+\frac{1}{3}j-\frac{2}{3}k$$

$$Duf(1,-2,0) = (-4)\left(\frac{2}{3}\right)+\frac{1}{3}-\frac{2}{3} = -3$$