2.7.2 The Gradient

In the form of dot product

$$Duf(x_o, y_o) = (fx(x_o, y_o)i + fy(x_o, y_o)j).(u_1 i + u_2 j)$$

$$= (fx(x_o, y_o)i + fy(x_o, y_o)j)$$

Similarly

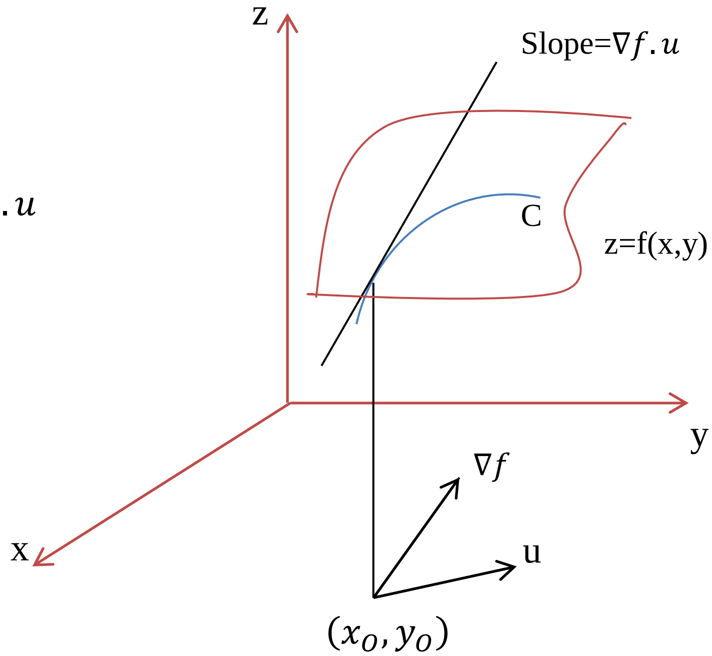
$$Duf\left((x_o, y_{o,} z_o\right) = (fx(x_o, y_o, z_o)i + fy(x_o, y_o, z_o)j) + fz(x_o, y_o, z_o)k\Big).u$$

- If f is a function of x and y , then the gradient of f is
  $$\nabla f(x, y) = fx(x, y)i + fy(x, y)j$$
- If f is a function of x, y and z , then the gradient of f is
  $\nabla f(x, y, z) = fx(x, y, z)i + fy(x, y, z)j + fz(x, y, z)k$

$\therefore$

$$Duf(x_o, y_o) = \nabla f(x_o, y_o).u$$

$$Duf(x_o, y_o, z_o) = \nabla f(x_o, y_o, z_o).u$$



Example : previous example

$$f(x, y, z) = x^2y - yz^3 + z\,, \qquad the\ point\ (1, -2, 0),\ \ u = \frac{2}{3}i + \frac{1}{3}j - \frac{2}{3}k$$

Solution :

$$fx(1, -2, 0) = -4\ , \qquad fy(1, -2, 0) = 1\,,\ \ fz(1, -2, 0) = 1$$

$$Duf(1, -2, 0) = \nabla f(1, -2, 0).u = (-4i + j + k).(\frac{2}{3}i + \frac{1}{2}j - \frac{2}{3}k)$$

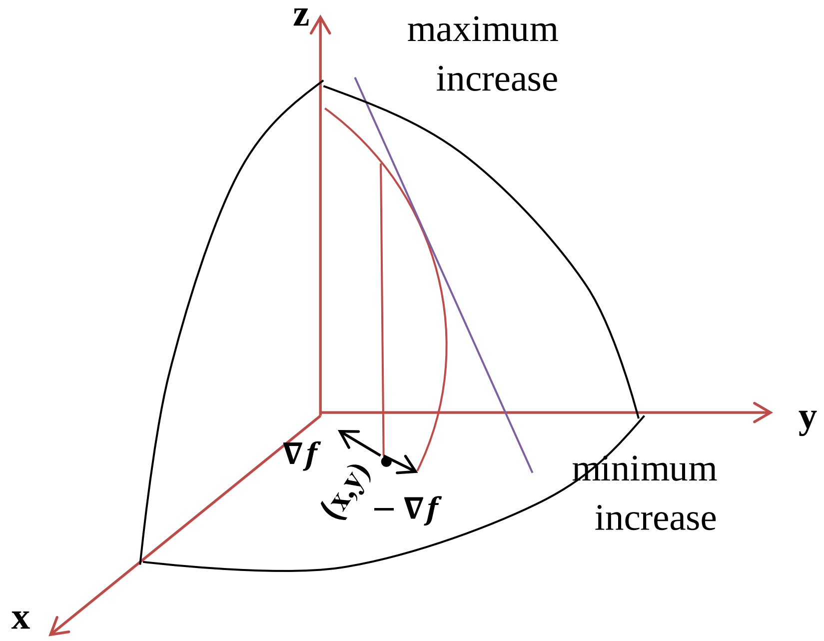
$$= (-4)\left(\frac{2}{3}\right) + \frac{1}{3} - \frac{2}{3} = -3$$

2.7.2.1 Properties of the gradient

$Duf(x,y) = \nabla f(x,y).u = |\nabla f(x,y)||u| \cos\theta = |\nabla f(x,y)| \cos\theta$

Where θ is the angle between $\nabla f(x,y) and\ u$

- Where $\theta = 0$ the maximum value of Duf (x,y) is $|\nabla f(x,y)|$, this means that the surface z = f(x,y) has its maximum slop at point (x,y) in the direction of the gradient
- When $\theta = \pi$ the minimum value of Duf (x,y) is -$|\nabla f(x,y)|$, this means that the surface z= f(x,y) has its minimum slop of a point (x,y) in the direction that is opposite to the gradient



Example : let f(x,y) = x^2e^y . find the maximum value of a direction derivative at (-2,0) and find the unite vector in the direction in which the maximum value occurs.

Solution :

$\nabla f(x, yfx(x,y)i + fy(x,y)j = 2xe^y i + x^2e^y j$

$\nabla f(-2,0) = -4i + 4j \qquad maximum\ value\ \theta = 0$

$Duf(-2,0) = |\nabla f(-2,0)| = \sqrt{(-4)^2 + 4^2 =} \sqrt{3}2 = 4\sqrt{2}$

$$u = \frac{\nabla f(-2,0)}{|\nabla f(-2,0)|} = \frac{1}{4\sqrt{2}}(-4i, 4j) = -\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$$

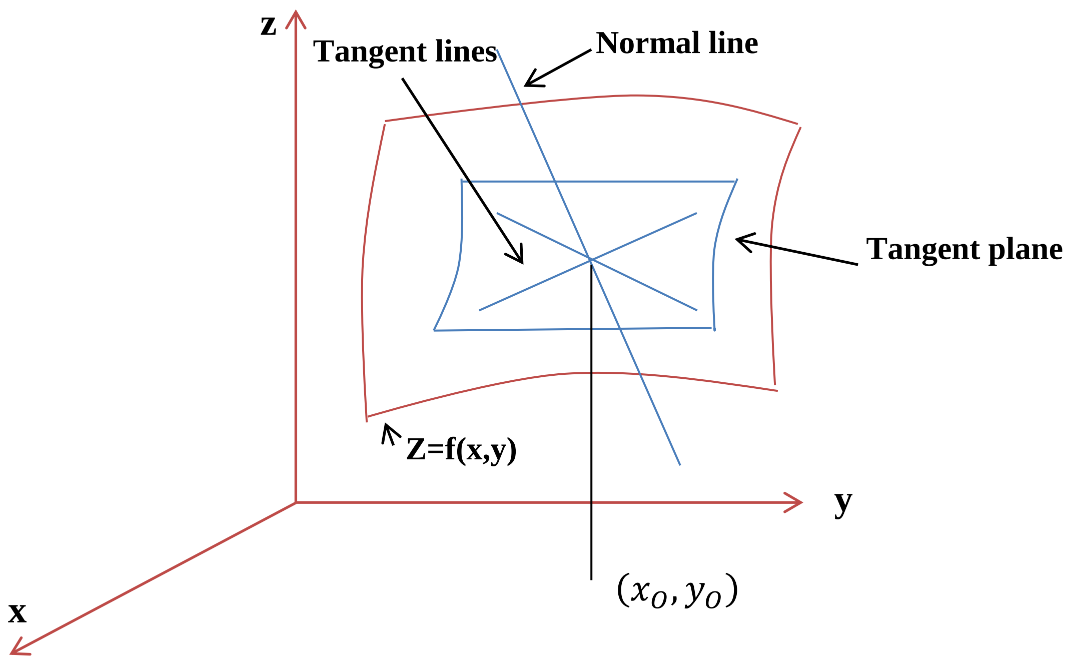
2.8 Tangent Planes and Normal Lines

If f (x,y) is differentiable at the point (x_O,y_O) , then the tangent plane to the surface z=f(x,y) at the point P_O(xO,yO), f(xO,yO) is the plane

$$z = f(x_O, y_O) + fx(x_O, y_O)(x - x_O) + fy(x_O, y_O)(y - y_O)$$

The line through the point P_O perpendicular to the tangent plane is normal to the surface z= f(x,y) at P_O this normal line can be expressed as :

$$x = x_O + f(x_O, y_O)t \ , \ y = y_O + fy(x_O, y_O)t \ , z = fx(x_O, y_O) - t$$



Example : find an equation for the tangent plane and parametric equations for the normal line to the surface z= x^2y at the point (2,1,4)

Solution :

$$fx(x, y) = 2xy \ , \ fy(x, y) = x^2$$

$$fx(2,1) = 4 \ , \ fy(2,1) = 4$$

The tangent plane has equation

$$z = 4 + 4(x - 2) + 4(y - 1) = 4x + 4y - 8$$

The normal line has parametric equations

$$x = 2 + 4t \ , \qquad y = 1 + 4t \ , \qquad z = 4 - t$$

2.9 Maximum and Minimum of Function of two Variables

A function f of two variables is said to have a **relative maximum** at point (x_o,y_o) if $f(x_o,y_o) \geq f(x,y)$ for all points (x,y) that lie inside the point (x_o,y_o)

A function f of two variables is said to have a **relative minimum** at point (x_o,y_o) if $f(x_o,y_o) \leq f(x,y)$ for all points (x,y) that lie inside the points (x_o,y_o)

If f has a relative maximum or a relative minimum at (x_o,y_o) , then we say that f has a **relative extremum** at (x_o,y_o)

2.9.1 The Second Partials Test

- for any f(x,y) by putting $fx = fy = 0\left[\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0\right]$

We can be find the critical point (x_o,y_o)

- Let $D = f_{xx}(x_o,y_o)\, f_{yy}(x_o,y_o) - f_{xy}^2(x_o,y_o)$
  a) If $D > 0$ *and* $f_{xx}(x_o,y_o) > 0$, *then f has a relative minimum at* (x_o,y_o)
  b) If $D > 0$ *and* $f_{xx}(x_o,y_o) < 0$, *then f has a relative maximum at* (x_o,y_o)
  c) If $D < 0$, *then f has a saddle point at* (x_o,y_o)
  d) If $D = 0$, *then no conclusion can be drawn*

Example : locate all relative extrema and saddle points of
$f(x,y) = z = 3x^2 - 2xy + y^2 - 8y$

Solution :

$fx(x,y) = 6x - 2y \ \ and \ fy(x,y) = -2x + 2y - 8$

The critical points of f satisfy the equations

$$\left.\begin{array}{l} 6x-2y = 0 \\ -2x + 2y - 8 = 0 \end{array}\right\} \longrightarrow \text{ solving for } x \text{ and } y$$

x=2 , y=6 , so (2,6) is the only critical point $(x_{O,}y_O)$

$f_{xx}(x,y) = 6$, $f_{yy}(x,y) = 2$, $f_{xy}(x,y) = -2$

At the critical point $(x_{O,}y_O)$=(2,6)

$D = f_{xx}(2,6)f_{yy}(2,6) - f_{xy}^2(2,6) = (6)(2) - (-2)^2 = 8 > 0$

And

$f_{xx}(2,6) = 6 > 0$

f has a relative minimum at critical point (2,6)

Related Rates Problem

Example : at what rate the volume of a box changing if its length 8 ft and increasing at 3 ft/s , its width is 6ft and increasing at 2ft/s , and its heigh is 4ft and increasing at 1ft/s ?

Solution :

let x,y and z length , width and height of a box , respectively

let t time in seconds

$$\frac{dx}{dt} = 3 \ , \qquad \frac{dy}{dt} = 2 \ , \qquad and \ \frac{dz}{dt} = 1$$

X=8 , y = 6 , and z =4 , v = xyz

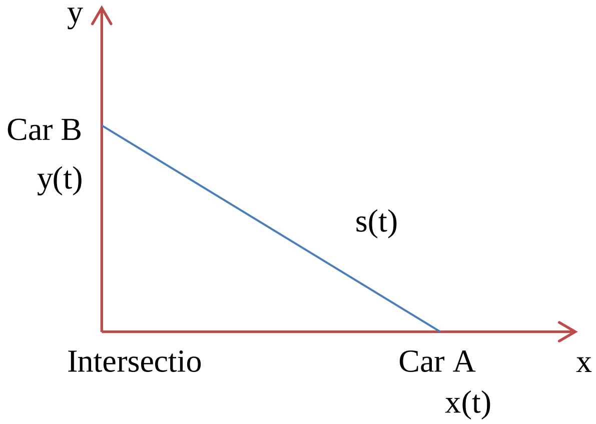
$$\frac{dv}{dt} = \frac{\partial v}{\partial x} . \frac{dx}{dt} + \frac{\partial v}{\partial y} . \frac{dy}{dt} + \frac{\partial v}{\partial z} . \frac{dz}{dt}$$

$$\frac{dv}{dt} = 6 * 4 * 3 + 8 * 4 * 2 + 8 * 6 * 1 = 184$$

The volume is increasing at a rate of 184 ft^3/s

Example : two straight roads intersect at right angles , car A , moving on one of the roads , approaches the intersection at 25 mi/hr and car B , moving on the other road, approaches the intersection at 30 mi/hr . At what rate is the distance between the car changing when A is 0.3 mile from the intersection and B is 0.4 mile from the intersection

Solution :



x(t) = position of car A

y(t) = position of B

s(t) =distance between cars

x=0.3 mile , y = 0.4 mile , $\frac{dx}{dt} = 25\, {}^{mi}/_{hr}$, $\frac{dy}{dt} = 30\, {}^{mi}/_{hr}$

$$\frac{ds}{dt} = ?$$

$$s = \sqrt{x^2 + y^2}$$

$$\frac{ds}{dt} = \frac{\partial s}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial s}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{2}(x^2 + y^2)^{\frac{-1}{2}} . 2x(25) + \frac{1}{2}(x^2 + y^2)^{\frac{-1}{2}} . 2y(30)$$

$$= 39\, {}^{mi}/_{hr}$$

$$= -39\, {}^{mi}/_{hr}$$

Exercises

1. Let $f(x,y) = 3x^3y^2$ find :
   a) fx(x,1) , b) fy (1,y) , c) fx(1,2) , d) fy(1,2)

Ans. a) $9x^2$ b) 6y c)36 d) 12

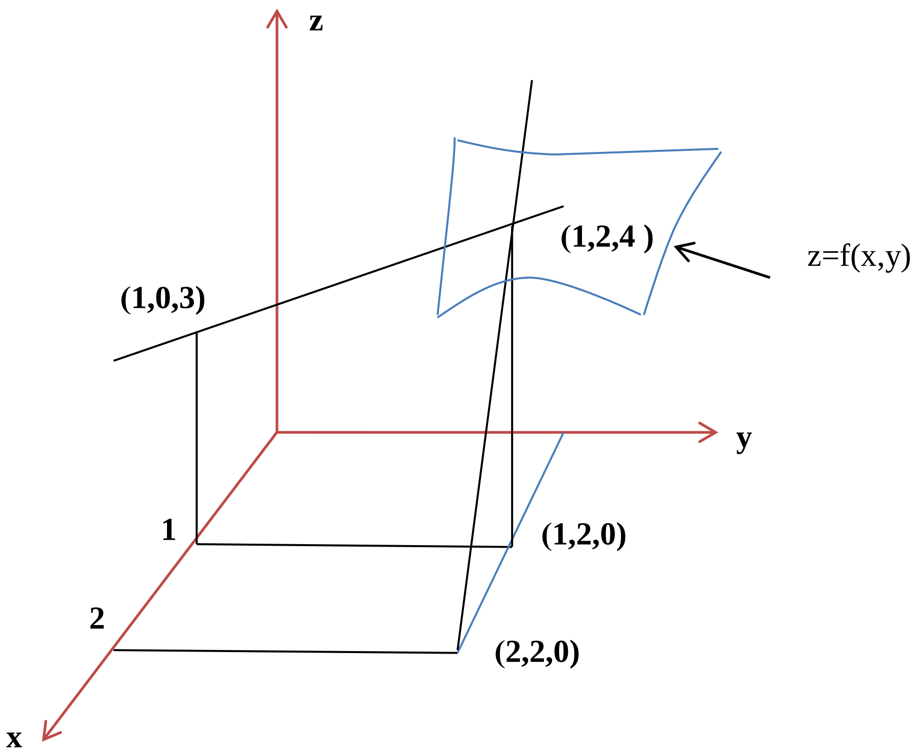
2. let $f(x,y) =(3x+2y)^{1/2}$ find :
   a) the slope of the surface z = f(x,y) in the x-direction at point (4,2)
   b) the slope of the surface z=f(x,y) in the y-direction at point (4,2)

Ans. a)3/8, b)1/4

3. let $z= \sin (y^2-4x)$ find :
   a) the rate of change of z with respect to x at the point (2,1) with y held fixed
   b) the rate of change of z with respect to y at the point (2,1) with xnheld fixed

Ans. a)-4cos7 , b)2cos7

4. use the information in the figure below to find the values of the first –order partial derivatives of f at the point (1,2)



Ans.

$$\frac{\partial z}{\partial x} = fx = -4$$

$$\frac{\partial z}{\partial y} = fy = \frac{1}{2}$$