Chapter Three

Multiple Integrals

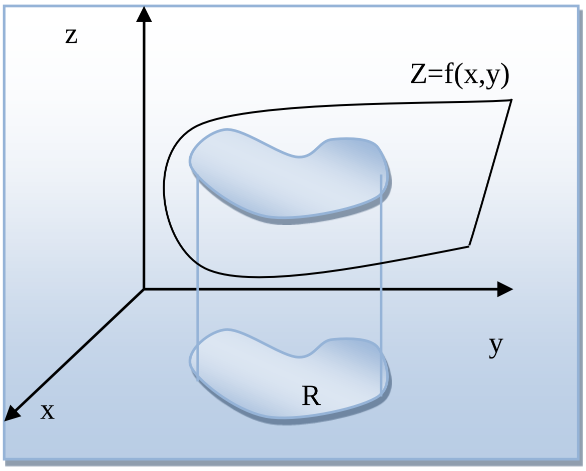
3.1 Introduction

Functions of one variable are usually integrated over intervals , function of two variables are usually integrated over regions in 2-space and functions of three variables over regions in 3- space .

In this chapter we will extend the concept of a definite (Single) integral to functions of two and three variables (Double and Triple Integrals).

3.2 Double Integrals

A function (f) of two variables is continuous and nonnegative on a region (R) in the xy plane. The volume (V) of the solid enclosed between the surface Z = f (x, y) and the region (R) is :-



$$v = \iint_R f(x,y)dA$$

3.2.1 Properties of Double Integrals

$= \iint_R cf(x,y)dA = c\iint_R f(x,y)dA$ C is constant

$= \iint_R [f(x,y) - g(x,y)]\,dA = \iint_R f(x,y)dA - \iint_R g(x,y)dA$

= If R= $R_1 + R_2$

$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$

3.2.2 Evaluation Double Integrals over Rectangular Region

- $\int_a^b f(x,y)dx$ **:** is partial integral with respect to (x).

It is evaluating by holding (y) fixed and integrating with respect to (x).

$\int_c^d f(x,y)dy$: is partial integral with respect to (y)

It is evaluating by holding (x) fixed and integrating with respect to (y)

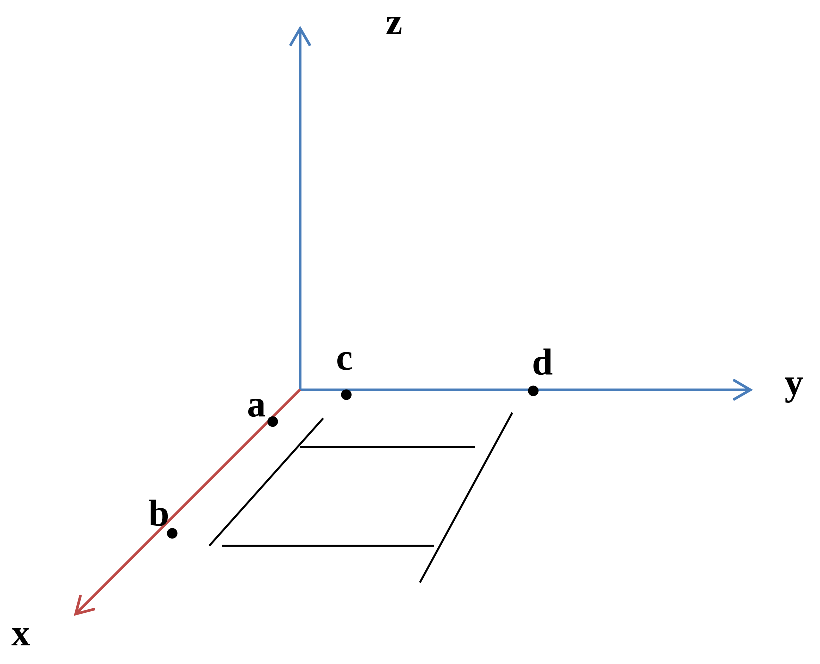
Example :

$\int_0^1 xy^2dx = y^2\int_0^1 xdx = y^2\frac{x^2}{2}]_{x=0}^{1} = \frac{y^2}{2}$ function of (y) can be integrated with respect to (y)

$=\int_0^1 xy^2dx = x\int_0^1 y^2dy = x\int_0^1 y^2dy = x\frac{y^3}{3}]_{y=0}^{1}$ function of (x) can be integrated with respect to (x)

$$=\int_c^d\int_a^b f(x,y)dxdy = \int_c^d\left[\int_a^b f(x,y)dx\right]dy$$

$$=\int_a^b\int_c^d f(x,y)dydx = \int_a^b\left[\int_c^d f(x,y)dy\right]dx$$



Example : Evaluate $\int_0^3 \int_1^2 (1+8xy)dydx$

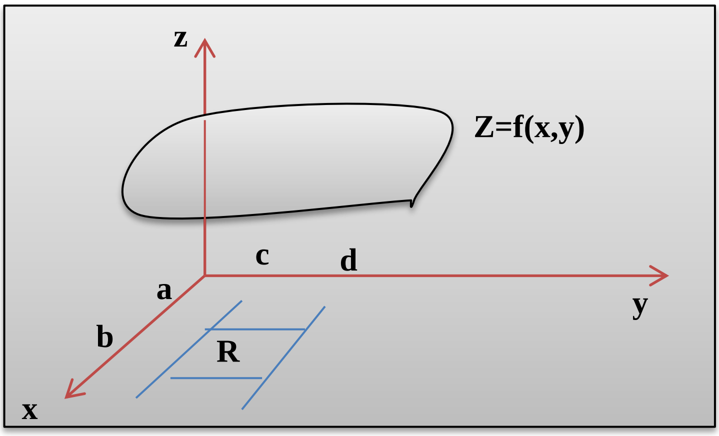
$=\int_0^3 \left[\int_1^2 (1+8xy)dydx\right] = \int_0^3 [y+8x\frac{y^2}{2}]_{y=1}^{2} dx$

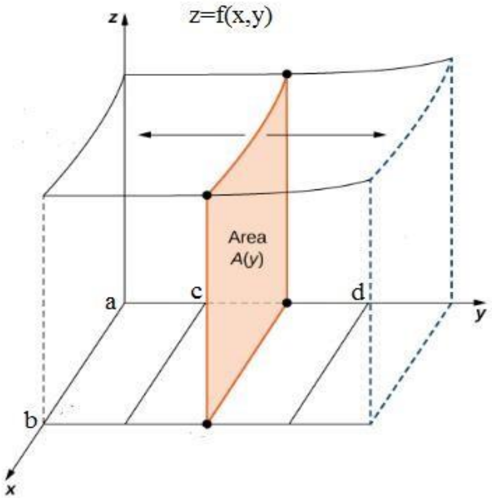
$=\int_0^3 [2+16x)-(1+4x)]\, dx = \int_0^3 (1+12x)dx$

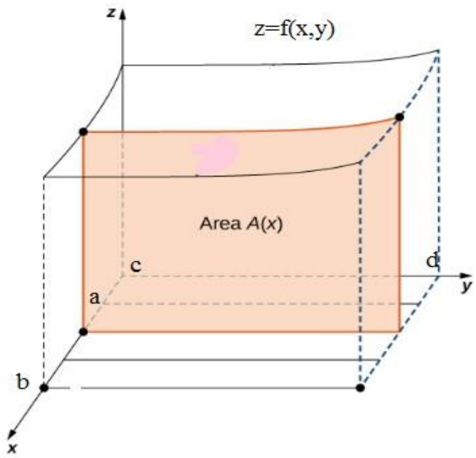
$=x+12\frac{x^2}{2}]_{x=0}^{3}$

$=3+12*\frac{9}{2}=57$ …….Let R be rectangular region , $a \le x \le b$, $c \le y \le d$ if f(x,y) is continuous function on R , then

$$\iint_R f(x,y)dA = \int_C^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$







A(y)= $\int_a^b f(x\ ,y)dx$

A(x)= $\int^b f(x\ ,y)$dy

V=$\int_c^d A(y)dy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy$

V= $\int_a^b A(x)dx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$

$\int_c^d \int_a^b f(x,y)dxdy$

$=\int_a^b \int_c^d f(x,y)dydx$

Example :Evaluate the double integral

$\iint_R \; y^2 x dA$, over the rectangular region

R=$\{(x, y): -3 \leq x \leq 2 \,, 0 \leq y \leq 1\}$

Solution:

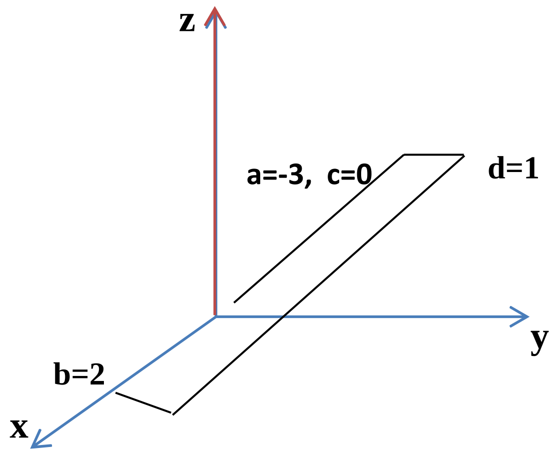
$$= \int_{a}^{b} \int_{c}^{d} \boldsymbol{f(x,y)} dydx \,, or \int_{c}^{d} \int_{a}^{b} f(x,y)dxdy$$

$$\int_0^1 \int_{-3}^2 y^2 xdxdy = \int_0^1 \left[\int_{-3}^2 y^2 xdx\right] dy$$

$$= \int_0^1 y^2 \frac{x^2}{2}\Big]_{x=-3}^2 dy$$

$$= \int_0^1 (2y^2 - 4.5y^{2)} dy = \int_0^1 -2.5y^2 dy$$

$$= -2.5 \frac{y^3}{3}\Big]_{y=0}^1 = \frac{-5}{6}$$



Example : use double integral to find the volume of the solid that is bounded above by the plane z 4 –x – y and below by the rectangular R =[0,1] ×[0,2]

Solution :

$$\int_c^d \int_a^b f(x,y)dxdy$$

$$= \int_0^2 \int_0^1 (4 - x - y)dxdy$$

$$= \int_0^2 4x - (x\verb|^|2/2) - xy]_0^1 \, dy$$

$$= \int_0^2 \left(4 - \frac{1}{2} - y\right) dy = 4y - \left(\frac{1}{2}y\right) - \frac{y^2}{2}\Big]_0^2$$

=8-1-2=5