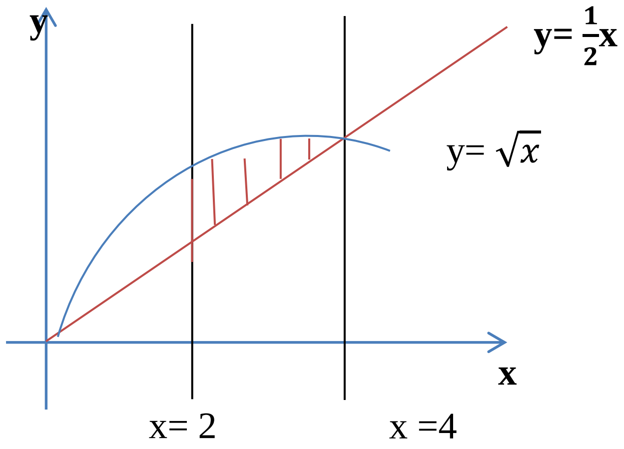
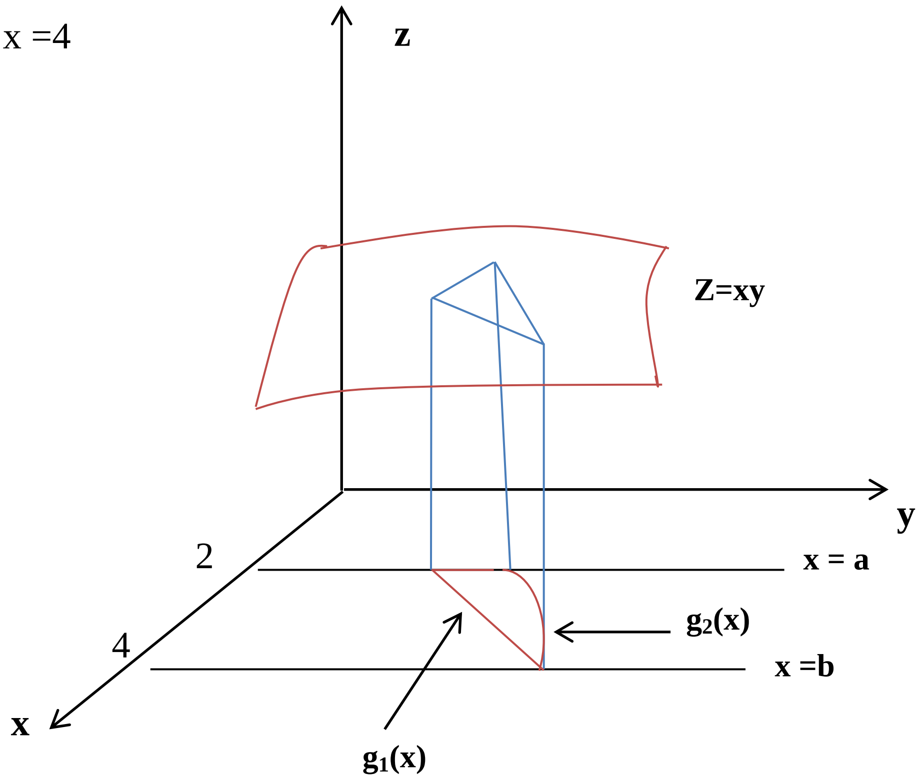
Example : evaluate $\iint_R xydA$, over the region R enclosed between y= $\frac{1}{2}$x , y= $\sqrt{x}$, x= 2, and x =4 .

Solution :





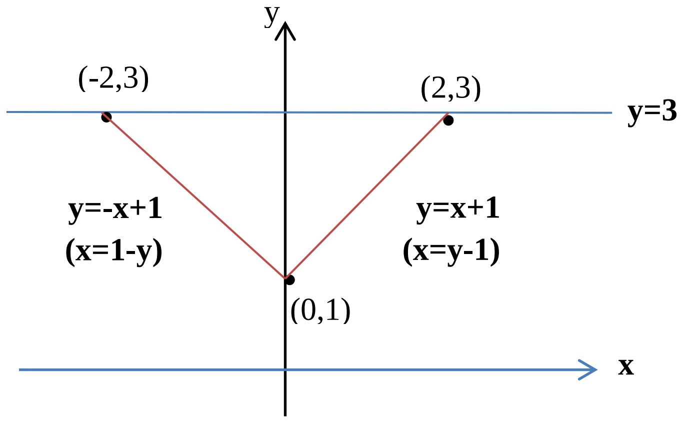
$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx \qquad \text{Type I}$$

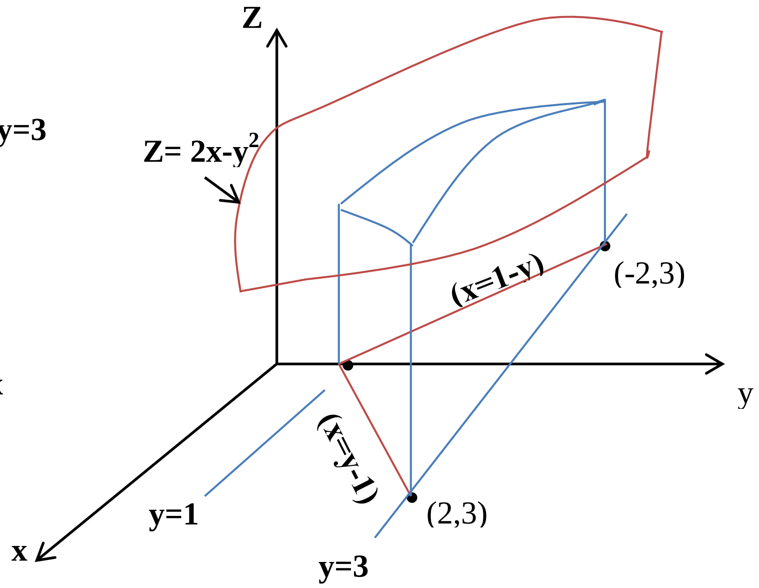
$$V = \int_2^4 \int_{\frac{x}{2}}^{\sqrt{x}} xydydx = \int_2^4 x \left.\frac{y^2}{2}\right]_{y=\frac{x}{2}}^{\sqrt{x}} dx = \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx$$

$$= \frac{x^3}{6} - \left.\frac{x^4}{32}\right]_2^4 = \left(\frac{64}{6} - \frac{256}{32}\right) - \left(\frac{8}{6} - \frac{16}{32}\right) = \frac{11}{6}$$

Example : evaluate $\iint_R (2x - y^2) dA$, over the triangular region R enclosed between the lines y= -x+1 , y = x+1 and y = 3

Solution :





V= $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$ Type II

V=$\int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dxdy$

=$\int_1^3 \frac{2x^2}{2} - y^2x]_{x=1-y}^{y-1}\, dy$

=$\int_1^3 [\{(y-1)^2 - y^3 + y^2)\} - \{(1-y)^2 - y^2 + y^3\}]\, dy$

=$\int_1^3 -2(y^3 - y^2) dy$

= $-2\frac{y^4}{4} + 2\frac{y^3}{3}]_1^3$

=$(-\frac{81}{2} + \frac{54}{3}) - (\frac{-1}{2} + \frac{2}{3})$

= $\frac{-68}{3}$

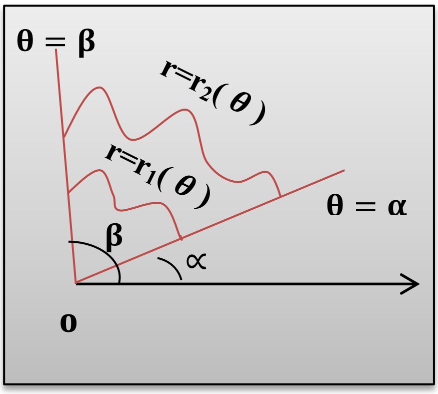
3.2.4 Double Integrals in Polar Coordinate

In this section we will study double integrals in which the integrand and the region of integration are expressed in polar coordinates

- **Simple polar Region**

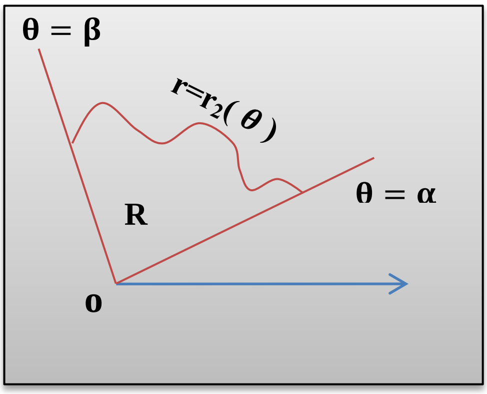
  A simple polar region in polar coordinates system is a region that is enclosed between two rays $\theta = \alpha \ and \ \theta = \beta$,and two continuous polar curves r = $r_1(\theta)$ and r = $r_2(\theta)$, where the equations of the rays and polar curves satisfy the following condition :

i) $\propto \leq \boldsymbol{\beta}$



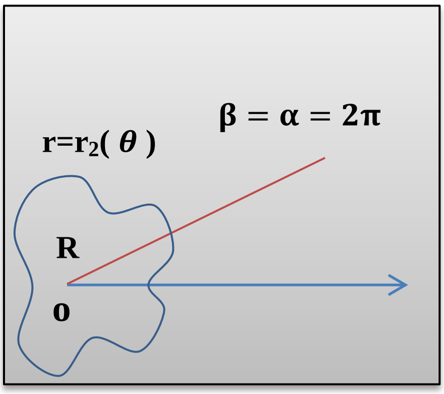
a

ii)$\boldsymbol{\beta - \alpha \leq 2\pi}$



b

iii)$0 \leq \boldsymbol{r_1(\theta) \leq r_2(\theta)}$

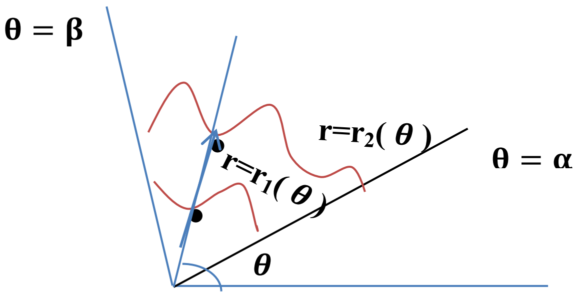


c

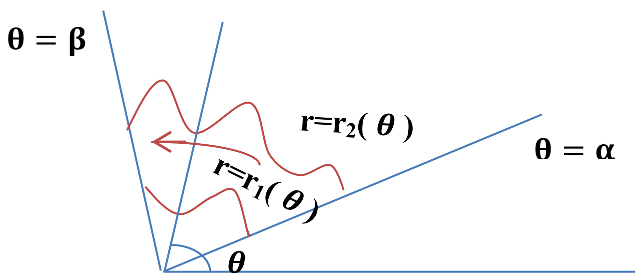
- **Evaluation polar Double integrals**

  If R is a simple polar region whose boundaries are the rays $\theta = \propto \ and \ \theta = \beta$ and the curves r = $r_1(\theta)$ and r = $r_2(\theta)$, and if $f(r,\theta)$ is continuous on R , then

  $$V = \iint_R f(r,\theta)dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)rdrd\theta \ \ , dA = rdrd\theta$$



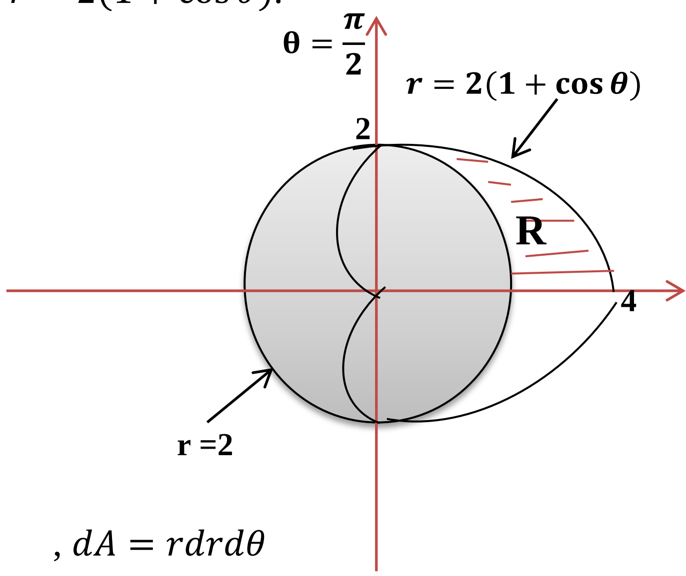
a



b

Example : Evaluate $\iint_R \sin\theta dA$ where R is the region in the first quadrant that is outside the circle r=2 and inside the cardioid $r = 2(1+\cos\theta)$.

Solution :



$$\iint_R \sin\theta dA = \int_0^{\frac{\pi}{2}}\int_2^{2(1+\cos\theta)}(\sin\theta) r dr d\theta \qquad , dA = rdrd\theta$$

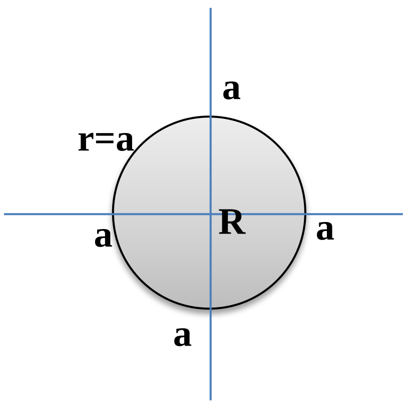
$$=\int_0^{\pi/2}\frac{1}{2}r^2(\sin\theta)]_{r=2}^{2(1+\cos\theta)}d\theta = \int_0^{\frac{\pi}{2}}[2(1+\cos\theta)^2\sin\theta - 2\sin\theta]\,d\theta$$

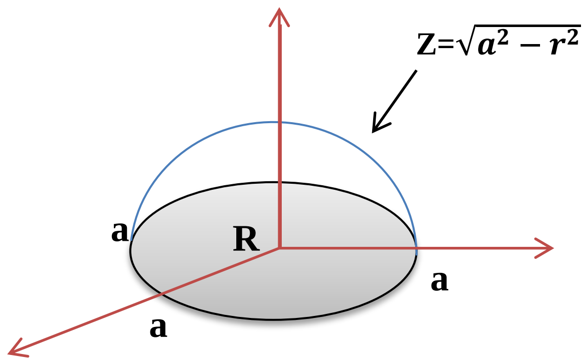
$$=2\int_0^{\pi/2}[(1+\cos\theta)^2\sin\theta-\sin\theta]d\theta = 2[-\frac{1}{3}(1+\cos\theta)^3+\cos\theta]_{\theta=0}^{\pi/2}$$

$$=2\left[-\frac{1}{3}-\left(-\frac{8}{3}+1\right)\right]=2\left(-\frac{1}{3}+\frac{5}{3}\right)=\frac{8}{3}$$

Example :the sphere of radius (a) centered at the origin is expressed in rectangular coordinates as $x^2+y^2+z^2=a^2$, and its equation in cylindrical coordinates is $r^2+z^2=a^2$. Use this equation and a polar integral to find the volume of the sphere

Solution :





$$v = 2\iint_R (\sqrt{a^2-r^2}\,d\theta = \int_0^{2\pi}\int_0^{a}\sqrt{a^2-r^2}\,(2r)drd\theta$$

$$=\int_0^{2\pi}[-\frac{2}{3}(a^2-r^2)^{3/2}]_{r=0}^{a}\,d\theta = \int_0^{2\pi}\frac{2}{3}a^3 d\theta = \frac{2}{3}a^3\theta]_{\theta=0}^{2\pi} = \frac{2}{3}a^3 2\pi$$

$$v = \frac{4}{3}\pi a^3$$