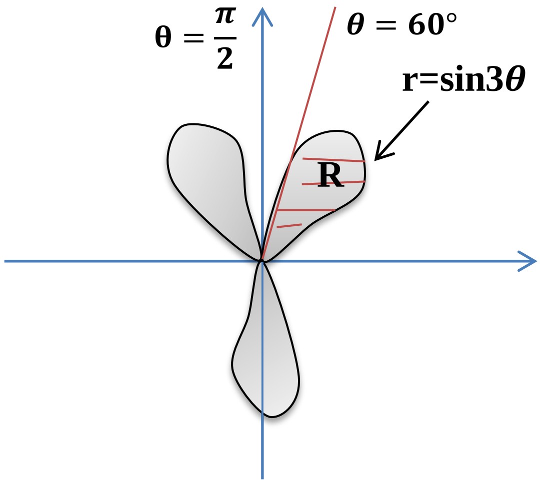
Finding Areas using polar double integrals

Area of $R = \iint_R 1\, dA = \iint_R dA = \iint_R r dr d\theta$

Example : use a polar double integral to find the area enclosed by the three – petated rose r=sin 3θ

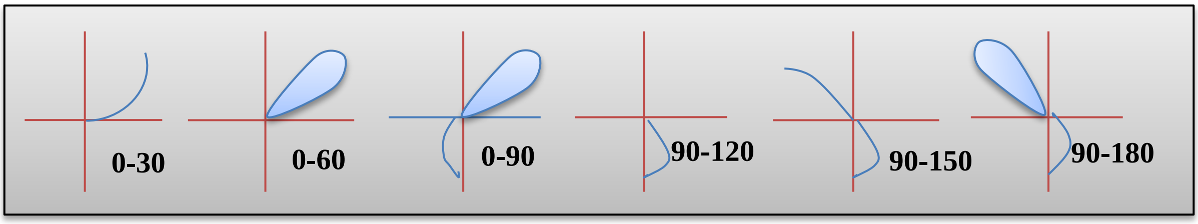


Solution :

$$A = \iint_R dA = 3 \int_0^{\pi/3} \int_0^{\sin 3\theta} r dr d\theta$$

$$= \frac{3}{2} \int_0^{\pi/3} sin^2 3\theta d\theta = \frac{3}{4} \int_0^{\pi/3} (1 - \cos 6\theta) d\theta$$

$$= \frac{3}{4} [\theta - \frac{\sin 6\theta}{6}]_{\theta=0}^{\pi/3} = \frac{1}{4}\pi$$

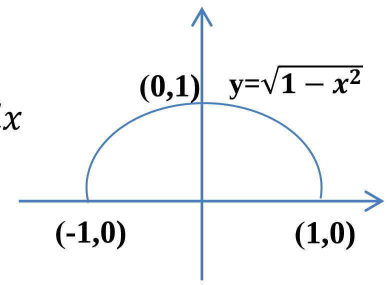


Converting Double Integrals from Rectangular to polar coordinates

Sometimes a double integral that is difficult to be evaluate in rectangular coordinates can be evaluate more easily in polar coor by making x = rcosθ, y= rsinθ

$$\iint_R f(x,y) dA = \iint_R f(r\cos\theta, r\sin\theta) dA = \iint_{\substack{appropriate \\ limits}} f(r\cos\theta, r\sin\theta) r dr d\theta$$

Example :use polar coor to evaluate $\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2} dy dx$

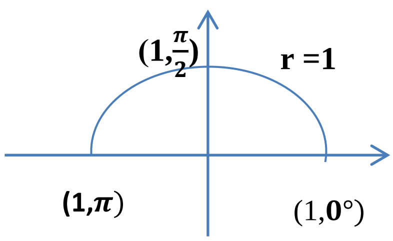


Solution : $\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2} dy dx = \iint_R (x^2+y^2)^{3/2} dA$

$$= \int_0^{\pi} \int_0^{1} (r^2 cos^2\theta + r^2 sin^2\theta)^{3/2} r dr d\theta$$

$$= \int_0^{\pi} \int_0^{1} (r^2)^{3/2} r dr d\theta = \int_0^{\pi} \int_0^{1} r^4 dr d\theta = \int_0^{\pi} \frac{r^5}{5}]_{r=0}^{1} d\theta$$

$$= \frac{1}{5}\theta]_{\theta=0}^{\pi} = \frac{\pi}{5}$$



3.4 Triple integrals

Triple integrals of f(x,y,z) is defined over a closed solid region (G) in an xyz – coordinates system

3.4.1 Evaluating of Triple integrals over rectangular Boxes

Be Let (G) be the rectangular box , where :

$$a \le x \le b, c \le y \le d, and\ k \le z \le l$$

If (f) is continuous on the region (G), then :

$$\iiint_G f(x,y,z)dv = \int_a^b \int_c^d \int_k^l f(x,y,z)dzdydx$$

Example :Evaluate the triple integral $\iiint_G 12xy^2 z^3 dv$ over the rectangular box (G), where

$$-1 \le x \le 2 , 0 \le y \le 3 , 0 \le z \le 2$$

Solution :

$$\iiint_G f(x,y,z)dv = \int_{-1}^{2} \int_{0}^{3} \int_{0}^{2} (12xy^2z^3)dzdydx$$

$=\int_{-1}^{2} \int_0^3 [12xy^2 \frac{z^4}{4}]_{z=0}^{2} dydx = \int_{-1}^{2} \int_0^3 (48xy^2)dydx$

$=\int_{-1}^{2} 48x \frac{y^3}{3}]_{y=0}^{3} dx = \int_{-1}^{2} 432dx$

$= 432\frac{x^2}{2}]_{x=-1}^{2}$

$= 432*2-\frac{432}{2}$

= 648

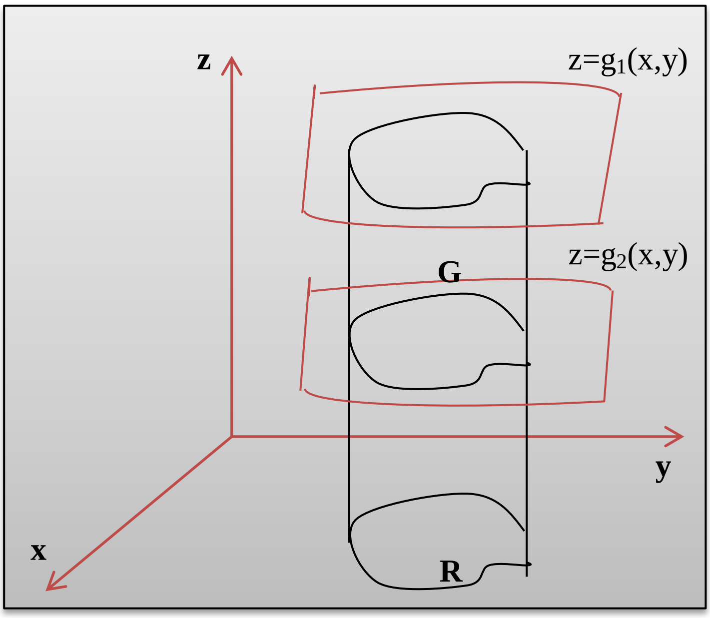
3.4.2 Evaluating of triple integrals over non-Rectangular Regions

We will assume that the solid (G) is bounded above by a surface $z = g_2(x,y)$ and below by a surface $z = g_1(x,y)$ and that the projection of the solid on the xy-plane is a type I or II region (R), we call a solid of this type a **simple xy –solid**

If f(x,y,z) is constant on G , then

$\iiint_G f(x,y,z)dv$

$= \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dz\right] dA$



Example: let G be the wedge in the first octant cut from the cylindrical solid

$y^2 + z^2 = 1$ by the plans y=x and x=0 , evaluate $\iiint_G zdv$

Solution :

$\iiint_G zdv = \iint_R \left[\int_0^{\sqrt{1-Y^2}} zdz\right] dA$

$= \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} zdzdxdy$

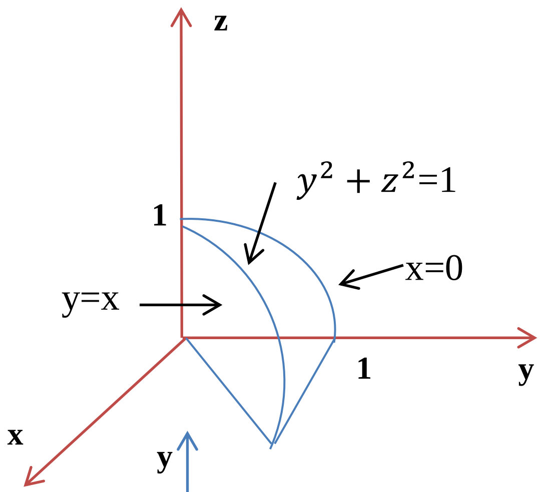
$= \int_0^1 \int_0^y \frac{1}{2} z^2]_{z=0}^{z=\sqrt{1-y^2}} dxdy$

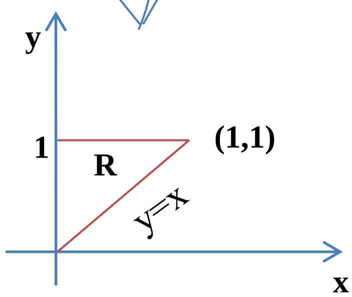
$= \int_0^1 \int_0^y \frac{1}{2}(1-y^2)dxdy$

$= \frac{1}{2}\int_0^1 (1-y^2)x]_{x=0}^{y} dy$

$= \frac{1}{2}\int_0^1 (y-y^3)dy$

$= \frac{1}{2}[\frac{1}{2}y^2 - \frac{1}{4}y^4]_{y=0}^{1} = \frac{1}{8}$



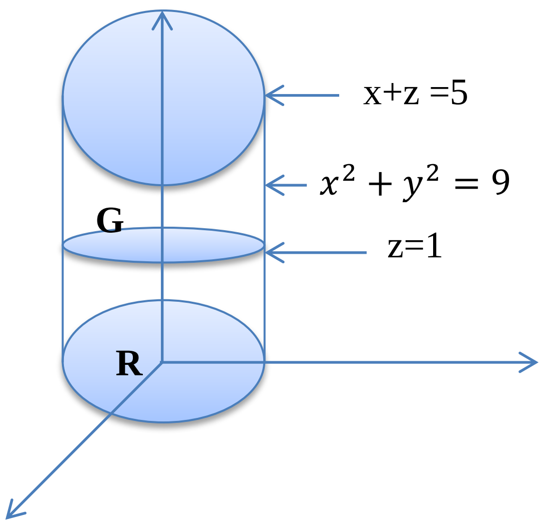


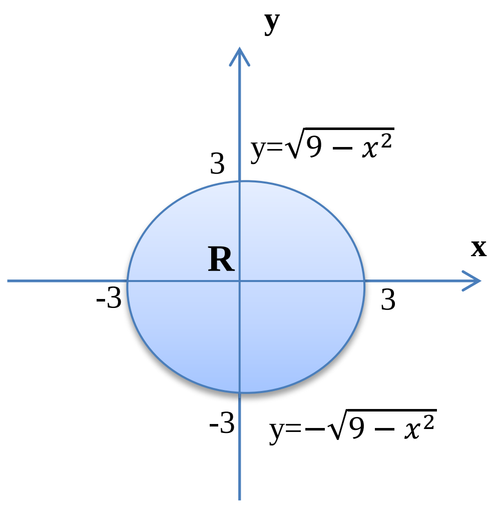
3.4.3 Volume calculated As a Triple Integral

In special case where f(x,y,z) =1

Volume of G $=\iiint_G \; dv$

Example : use a triple integral to find the volume of the solid with in the cylinder $x^2+y^2=9$ and between the planes z=1 and x+z =5





$$\text{Volume of G}\iiint_G \; dv = \iint_R \left[\int_1^{5-X} 1dz\right] dA$$

$$=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-x} dzdydx$$

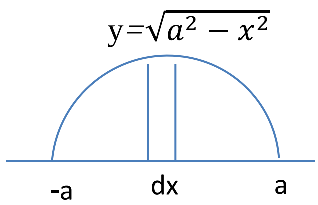
$$=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z]_1^{5-x}dydx$$

$$=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(4-x)dydx$$

$$=\int_{-3}^{3}(8-2x)\sqrt{9-x^2}dx$$

$$=8\int_{-3}^{3}\sqrt{9-x^2}dx-\int_{-3}^{3}2x\sqrt{9-x^2}dx$$

$$=\int_{-a}^{a}\sqrt{a^2-x^2}\,dx \;=\frac{1}{2}\pi a^2 =8(\frac{9}{2}\pi)-0$$



$$=36\pi$$

Note: the second integral is (zero) because the integral is an odd function