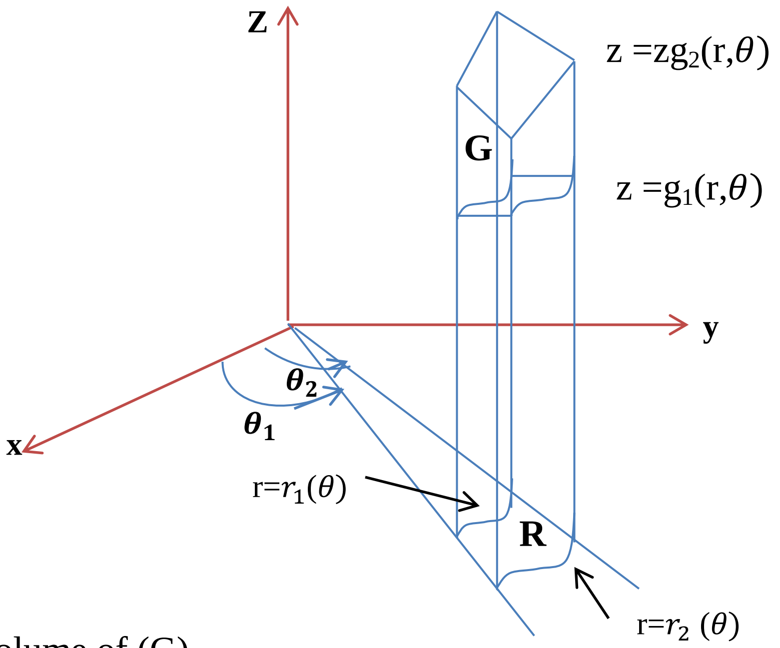
3.4.4 Triple integrals in cylindrical coordinates

Let (G) be a solid whose upper surface has the equation $z = zg_2(r,\theta)$ and lower surface has the equation $z = g_1(r,\theta)$ in cylindrical coordinates .

If the projection of the solid on the xy – plane is a simple polar region (R) , and if f (r, θ, z) is continues on (G) , then :

$$\iiint_G f(r,\theta,z)dv = \iint_R \left[\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)dz \right] dA$$

$$= \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z) r dz dr d\theta$$



Example : find the volume of (G)

$$V = \iiint_G dv = \iint_R \left[\int_0^{\sqrt{25-r^2}} dz \right] dA$$

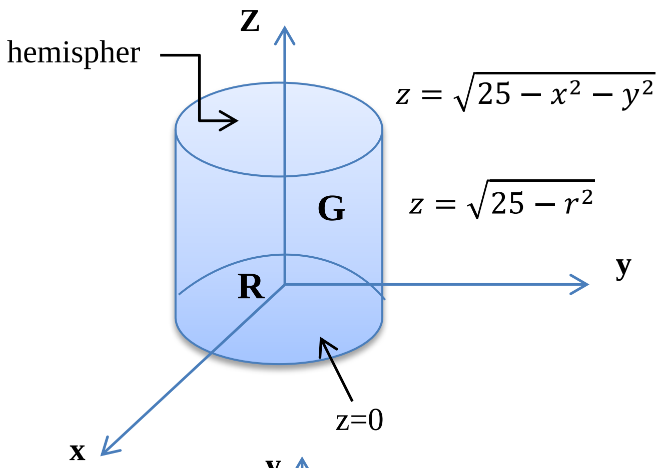
$$= \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{25-r^2}} r dz dr d\theta$$

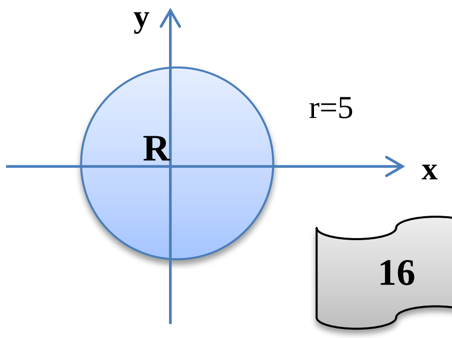
$$= \int_0^{2\pi} \int_0^3 [r^2]_{z=0}^{\sqrt{25-r^2}} dr d\theta$$

$$= \int_0^{2\pi} \int_0^3 r\sqrt{25-r^2}\, dr d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}(25-r^2)^{3/2}]_{r=0}^3 d\theta$$

$$= \int_0^{2\pi} \frac{61}{3} d\theta = \frac{122}{3}\pi$$



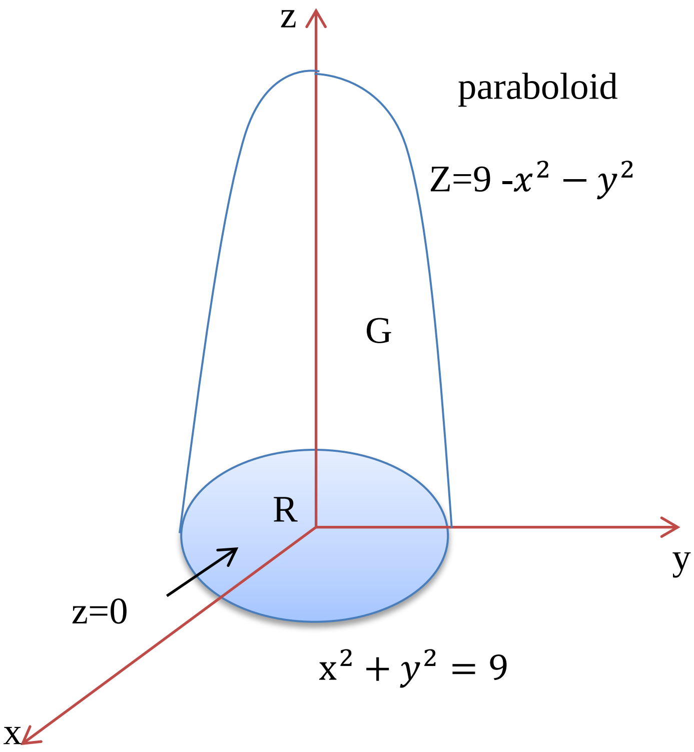


3.4.5 Converting travel integrals from rectangular to cylindrical coordinates

$$\iint_{G} \boldsymbol{f(x,y,z)dv} = \iiint_{\substack{appropirate\\ limits}} \boldsymbol{f(r\cos\theta, r\sin\theta, z)rdzdrd\theta}$$

Example : use cylindrical coordinates to evaluate

$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2 dzdydx$



Solution :

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2 dzdydx$$

$=\iiint_G \ x^2 dv$

$\int_0^{2\pi}\int_0^3\left[\int_0^{9-r^2} r^2\cos\ ^2\theta dz\right]dA$

$=\int_0^{2\pi}\int_0^3\int_0^{9-r^2}(r^2\cos\ ^2\theta)rdzdrd\,\theta$

$=\int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^3\cos\ ^2\theta dzdrd\theta$

$=\int_2^{2\pi}\int_0^3[zr^3\cos\ ^2\theta]_{z=0}^{9-r}drd\theta$

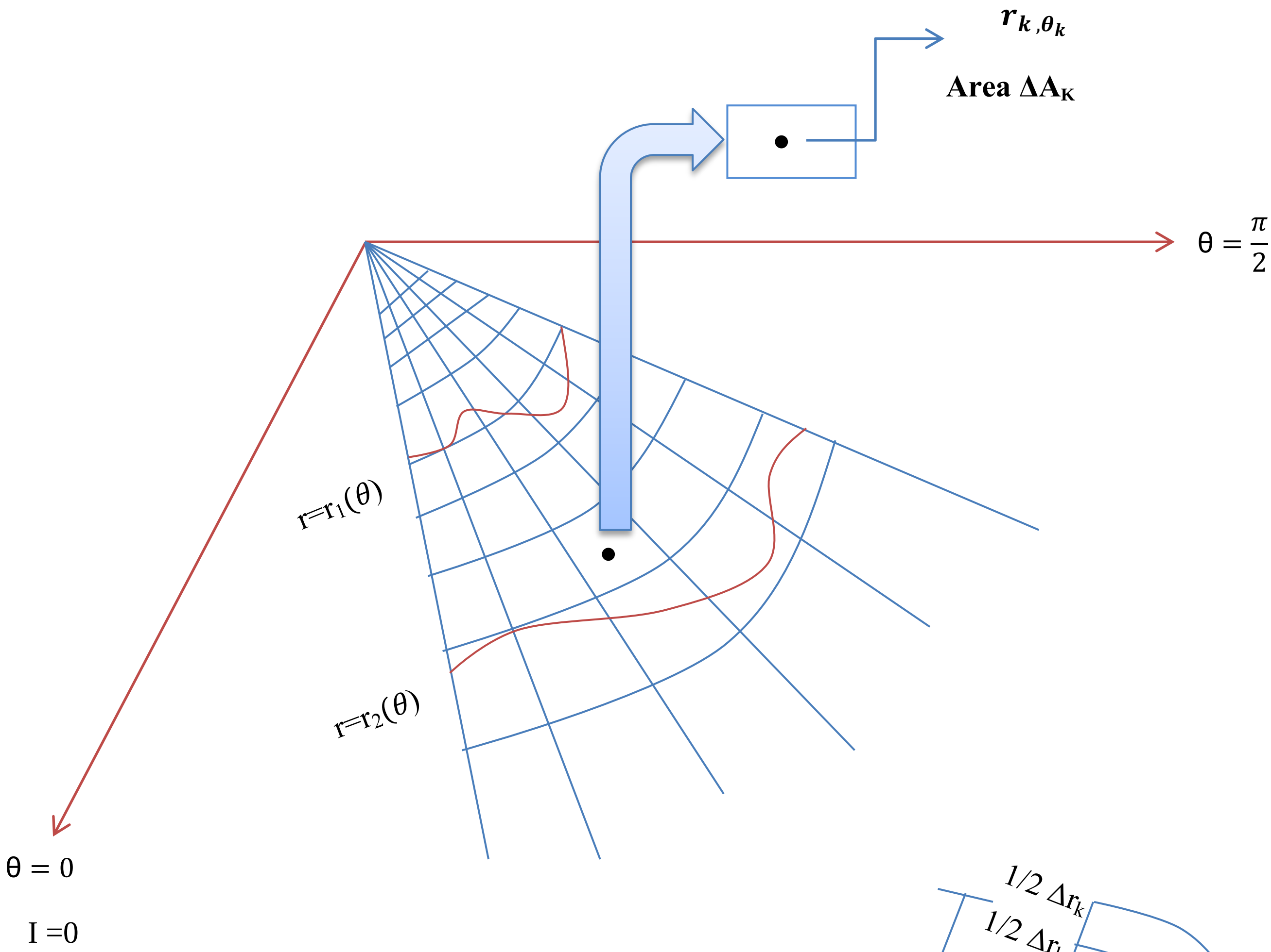
$=\int_0^{2\pi}\int_0^3(9r^3-r^5)\cos\ ^2\theta drd\theta$

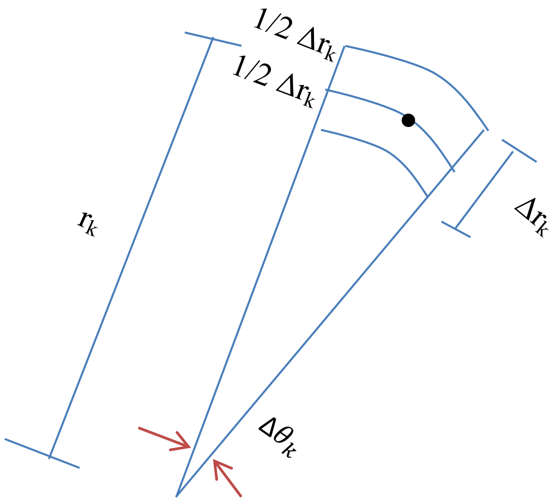
$=\int_0^{2\pi}[(\frac{9r^4}{4}-\frac{r^6}{6})\cos\ ^2\theta]_{r=0}^3 d\theta$

$=\frac{243}{4}\int_0^{2\pi}\cos\ ^2\theta d\theta$

$=\frac{243}{4}\int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)d\theta$

$=\frac{243}{4}\pi$





The polar rectangular has

- Center angle $\Delta\theta_k$
- Radial thickness Δr_k
- Liner radius $r_k - \frac{1}{2}\Delta r_k$
- Outer radius $r_k + \frac{1}{2}\Delta r_k$

The area ΔA_k of this polar rectangular is the difference in area of two sectors

$$\Delta A_K = \frac{1}{2}(r + \frac{1}{2}\Delta r_k)^2 \Delta\theta_k - \frac{1}{2}(r - \frac{1}{2}\Delta r_k)^2 \Delta\theta_k$$

$$= r_k \Delta r_k \Delta\theta_k$$