Rings

- If we can define ***one more operation*** on an ***abelian group***, we have a ***ring***, provided the elements of the set satisfy some properties with respect to this new operation also.

- Just to set it apart from the operation defined for the abelian group, we will refer to the new operation as ***multiplication***. Note that the use of the name 'multiplication' for the new operation is <u>merely a notational convenience</u>.

- A ring is typically denoted **{*R*, +, ×}** where R denotes the set of objects, '+' the operator with respect to which *R* is an abelian group, the '×' the additional operator needed for *R* to form a ring.

Rings: Properties of the elements with respect to the other operator

1. R must be ***closed*** with respect to the additional operator '×'.
2. R must exhibit ***associativity*** with respect to the additional operator '×'.
3. The additional operator (that is, the "multiplication operator") must ***distribute*** over the group addition operator. That is

$$a \times (b + c) = a \times b + a \times c$$

$$(a + b) \times c = a \times c + b \times c$$

✓ The "multiplication" operation is frequently shown by just concatenation in such equations:

$$a (b + c) = a b + a c$$

$$(a + b) c = a c + b c$$

Examples of a Ring

- ✓ For a given value of N , the set of all $N \times N$ square matrices over the real numbers under the operations of *matrix addition* and *matrix multiplication* constitutes a ***ring***.
- ✓ The set of *all even integers*, positive, negative, and zero, under the operations arithmetic addition and multiplication is a ***ring***.
- ✓ The set of *all integers* under the operations of arithmetic addition and multiplication is a ***ring***.
- ✓ The set of *all real numbers* under the operations of arithmetic addition and multiplication is a ***ring***.

Commutative Rings

- A ring is commutative if the ***multiplication operation is commutative*** for all elements in the ring. That is, if all a and b in R satisfy the property

$$a\,b = b\,a$$

- Examples of a commutative ring:

1. The set of ***all even integers***, positive, negative, and zero, under the operations arithmetic addition and multiplication.
2. The set of ***all integers*** under the operations of arithmetic addition and multiplication.
3. The set of ***all real numbers*** under the operations of arithmetic addition and multiplication.

Integral Domain

An ***integral domain*** $\{R, +, \times\}$ is a commutative ring that obeys the following two additional properties:

1. ***Additional Property 1***: The set R must include an *identity element* for the *multiplicative operation*. That is, it should be possible to symbolically designate an element of the set R as '1' so that for every element a of the set we can say $a\ 1 = 1\ a = a$

2. ***Additional Property 2***: Let 0 denote the identity element for the *addition operation*. If a multiplication of any two elements a and b of R results in 0, that is if $a\,b = 0$ then either a or b ***must be*** 0.

Examples of Integral Domain

1. The set of ***all integers*** under the operations of arithmetic addition and multiplication.

2. The set of ***all real numbers*** under the operations of arithmetic addition and multiplication.

Fields

A ***field***, denoted $\{F, +, \times\}$, is an *integral domain* whose elements satisfy the following *additional property*:

- For *every element* a in F, except the element designated 0 (the identity element for the '+' operator), there must also exist in F its ***multiplicative inverse***. That is, if $a \in F$ and $a \neq 0$, then there must exist an element $b \in F$ such that

$$a\,b = b\,a = 1$$

where '1' symbolically denotes the element which serves as the identity element for the multiplication operation. For a given a, such a b is often designated a^{-1}.

Positive and Negative Examples of Fields

- ✓ The set of *all real numbers* under the operations of arithmetic addition and multiplication is a ***field***.
- ✓ The set of *all rational numbers* under the operations of arithmetic addition and multiplication is a ***field***.
- ✓ The set of *all complex numbers* under the operations of complex arithmetic addition and multiplication is a ***field***.
- ❑ The set of *all even integers*, positive, negative, and zero, under the operations arithmetic addition and multiplication is ***NOT a field*** .
- ❑ The set of *all integers* under the operations of arithmetic addition and multiplication is ***NOT a field***.

Finally . . .

- **Acknowledgment:** These lecture notes are based on the textbook by William Stallings and notes prepared by Avinash Kak, Purdue University. My sincere thanks are devoted to them and to all other people who offered the material on the web.

- Students are advised to study and solve the problems and answer the questions in **Assignment-5**.