Computing rank using determinants

Definition

Let A be an $m \times n$ matrix. A minor of A of order k is a determinant of a $k \times k$ sub-matrix of A.

We obtain the minors of order k from A by first deleting $m - k$ rows and $n - k$ columns, and then computing the determinant. There are usually many minors of A of a given order.

Example

Find the minors of order 3 of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Computing minors

Solution

We obtain the determinants of order 3 by keeping all the rows and deleting one column from A. So there are four different minors of order 3. We compute one of them to illustrate:

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1 \cdot (-4) + 2 \cdot 0 = -4$$

The minors of order 3 are called the maximal minors of A, since there are no 4×4 sub-matrices of A. There are $3 \cdot 6 = 18$ minors of order 2 and $3 \cdot 4 = 12$ minors of order 1

Computing rank using minors

Proposition

Let A be an $m \times n$ matrix. The rank of A is the maximal order of a non-zero minor of A.

Idea of proof: If a minor of order k is non-zero, then the corresponding columns of A are linearly independent.

Computing the rank

Start with the minors of maximal order k. If there is one that is non-zero, then $\text{rk}(A) = k$. If all maximal minors are zero, then $\text{rk}(A) < k$, and we continue with the minors of order $k-1$ and so on, until we find a minor that is non-zero. If all minors of order 1 (i.e. all entries in A) are zero, then $\text{rk}(A) = 0$.

Rank: Examples using minors

Example

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Solution

The maximal minors have order 3, and we found that the one obtained by deleting the last column is $-4 \neq 0$. Hence $\text{rk}(A) = 3$.

Rank: Examples using minors

Example

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix}$$

Rank: Examples using minors

Solution

The maximal minors have order 3*, so we compute the* 4 *minors of order* 3*. The first one is*

$$\begin{vmatrix} 1 & 2 & 1 \\ 9 & 5 & 2 \\ 7 & 1 & 0 \end{vmatrix} = 7 \cdot (-1) + (-1) \cdot (-7) = 0$$

The other three are also zero. Since all minors of order 3 *are zero, the rank must be* $\mathrm{rk}(A) < 3$*. We continue to look at the minors of order two. The first one is*

$$\begin{vmatrix} 1 & 2 \\ 9 & 5 \end{vmatrix} = 5 - 18 = -13 \neq 0$$

It is not necessary to compute any more minors, and we conclude that $\mathrm{rk}(A) = 2$*. In fact, the first two columns of* A *are linearly independent.*