

Numerical Solutions of Ordinary Differential Equations (ODEs)

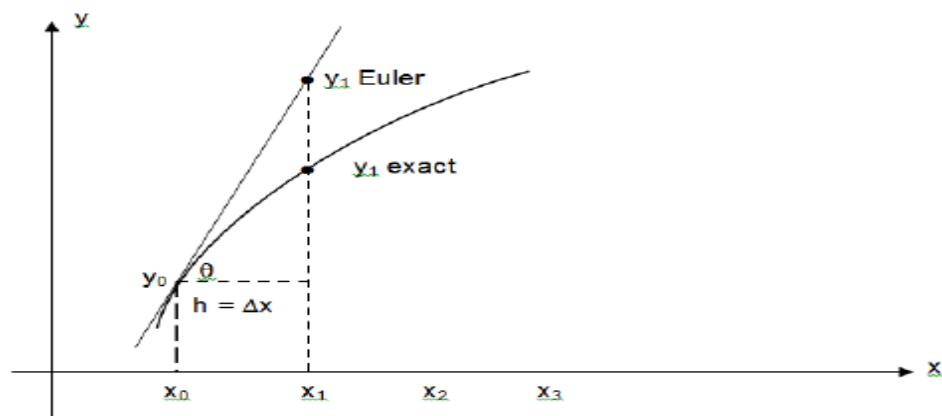
First Order Ordinary Differential Equation

Given: $\frac{dy}{dx} = f(x, y)$ with initial condition (IC) $y_{(x=x_0)} = y_0$

Required: y value at any $x > x_0$.

Euler Method

The slope of the curve $y = f(x)$ at $x = x_0$ = the slope of the tangent to that curve at $x = x_0$.



$$\left. \frac{dy}{dx} \right|_{(x_0, y_0)} = \text{slope of the tangent}|_{(x_0, y_0)} = \tan \theta = \frac{y_1 \text{ Euler} - y_0}{h}$$

$$y_1 = y_0 + h \left. \frac{dy}{dx} \right|_{(x_0, y_0)}$$

$$y_2 = y_1 + h \left. \frac{dy}{dx} \right|_{(x_1, y_1)}$$

and so on.

Example: Solve $\frac{dy}{dx} = x + y$ with IC $y_{(x=0)} = 1$ to find y at $x = 0.1$. Use $h = 0.02$ and work to 4dp.

Solution

$$y_{i+1} = y_i + h \left. \frac{dy}{dx} \right|_i$$

$$\boxed{y_{i+1} = y_i + 0.02 (x_i + y_i)}$$

$$\begin{aligned} y_{0.02} = y_1 &= y_0 + 0.02 (x_0 + y_0) \\ &= 1 + 0.02 (0 + 1) = 1.0200 \end{aligned}$$

$$\begin{aligned} y_{0.04} = y_2 &= y_1 + 0.02 (x_1 + y_1) \\ &= 1.0200 + 0.02 (0.02 + 1.0200) = 1.0408 \end{aligned}$$

and so on to finally get $y_{(0.10)} = y_5 = 1.1081$

Simultaneous First Order Ordinary Differential Equations

For (2) ODEs of 1st order

Given: $\frac{dy}{dx} = f(x, y, z)$ and $\frac{dz}{dx} = g(x, y, z)$ with ICs $y_{(x=x_0)} = y_0$;

$z_{(x=x_0)} = z_0$.

Required: y and z values at any $x > x_0$.

Use **Euler Method** for y and z .

Example: Solve $\frac{dy}{dx} = 2x^2 - y + 3z + 1$, $\frac{dz}{dx} = x + y^2 - z^2$ with ICs $y_{(x=0)} = 0$ and $z_{(x=0)} = 1$ to find y and z values at $x = 1$ by using **Euler Method**. Use $h = 0.2$ and work to 3dp.

Solution

$$y_{i+1} = y_i + h \left. \frac{dy}{dx} \right|_i = y_i + 0.2 (2x_i^2 - y_i + 3z_i + 1)$$

$$z_{i+1} = z_i + h \left. \frac{dz}{dx} \right|_i = z_i + 0.2 (x_i + y_i^2 - z_i^2)$$

$$y_{0.2} = y_1 = y_0 + 0.2 (2x_0^2 - y_0 + 3z_0 + 1)$$

$$= 0 + 0.2 [2(0)^2 - 0 + 3(1) + 1] = 0.800$$

$$z_{0.2} = z_1 = z_0 + 0.2 (x_0 + y_0^2 - z_0^2) = 1 + 0.2 [0 + (0)^2 - (1)^2] = 0.800$$

$$y_{0.4} = y_2 = y_1 + 0.2 (2x_1^2 - y_1 + 3z_1 + 1)$$

$$= 0.8 + 0.2 [2(0.2)^2 - 0.8 + 3(0.8) + 1] = 1.336$$

$$z_{0.4} = z_2 = z_1 + 0.2 (x_1 + y_1^2 - z_1^2) = 0.8 + 0.2 [0.2 + (0.8)^2 - (0.8)^2] = 0.840$$

and so on to finally get $y_{(1)} = y_5 = 3.456$ and $z_{(1)} = z_5 = 2.518$

Second Order Ordinary Differential Equations

Given: $\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$ with ICs $y_{(x=x_0)} = y_0$ and $y'_{(x=x_0)} = y'_0$.

Required: y value at any $x > x_0$.

Convert the 2nd order ODE to (2) 1st order ODEs as follows:

$$\text{Let } z = \frac{dy}{dx}$$

$$\text{Then } \frac{dz}{dx} = \frac{d^2y}{dx^2} = f(x, y, z) \text{ and } z_{(x=x_0)} = y'_{(x=x_0)} = y'_0$$

Thus we have 2 ODEs of 1st order:

$$\frac{dy}{dx} = z \quad 1^{\text{st}} \text{ ODE}$$

$$\frac{dz}{dx} = \frac{d^2y}{dx^2} = f(x, y, z) \quad 2^{\text{nd}} \text{ ODE}$$

$$\text{with ICs } y_{(x=x_0)} = y_0 \text{ and } z_{(x=x_0)} = y'_0$$

Example: Use Euler Method to find $y_{(x=1)}$ for $y'' - y = x$ with ICs $y_{(0)} = 0$ and $y'_{(0)} = 1$. Use $h = 0.1$ and work to 4dp.

Solution

$$\frac{dy}{dx} = z$$

$$\frac{dz}{dx} = x + y$$

$$y_{(x=0)} = 0 \text{ and } z_{(x=0)} = 1$$

$$y_{i+1} = y_i + h \left. \frac{dy}{dx} \right|_i = y_i + 0.1z_i$$

$$z_{i+1} = z_i + h \left. \frac{dz}{dx} \right|_i = z_i + 0.1(x_i + y_i)$$

$$y_{(0.1)} = y_1 = y_0 + 0.1 z_0 = 0 + 0.1(1) = 0.1000$$

$$z_{(0.1)} = z_1 = z_0 + 0.1(x_0 + y_0) = 1 + 0.1(0 + 0) = 1.0000$$

$$y_{(0.2)} = y_2 = y_1 + 0.1 z_1 = 0.1 + 0.1(1) = 0.2000$$

$$z_{(0.2)} = z_2 = z_1 + 0.1(x_1 + y_1) = 1 + 0.1(0.1 + 0.1) = 1.0200$$

Continue to find $y_{(x=1)} = y_{10} = 1.3500$