So, we need to determine if the derivative is ever zero.

$$(1+t)\mathbf{e}^t = 0$$

Now, we know that exponential functions are never zero and so this will only be zero at $t = -1$.
So, if we are going to allow negative values of t then the object will stop moving once at $t = -1$.
If we aren't going to allow negative values of t then the object will never stop moving.

3.5 Derivatives of Inverse Trig Functions

If $f(x)$ and $g(x)$ are inverses of each other then,

$$\dot{g}(x) = \frac{1}{\dot{f}\left(g(x)\right)}$$

and let us recall as well that two functions are inverses if $f(g(x)) = x$ and $g(f(x)) = x$.

Inverse Sine

The definition of the inverse sine is,

$$y = \sin^{-1} x \qquad \Leftrightarrow \qquad \sin y = x \qquad \text{for} \qquad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Example Evaluate $\sin^{-1} \frac{1}{2}$.

Solution

$$\sin y = \frac{1}{2}$$

Then

$$y = \frac{\pi}{6}$$

We have the following relationship between the inverse sine function and the sine function.

$$\sin\left(\sin^{-1} x\right) = x \qquad\qquad \sin^{-1}(\sin x) = x$$

In other words they are inverses of each other. This means that we can use the fact above to find the derivative of inverse sine. Let's start with,

$$f(x) = \sin x \qquad\qquad g(x) = \sin^{-1} x$$

Then,

$$\dot{g}(x) = \frac{1}{\dot{f}\left(g(x)\right)} = \frac{1}{\cos\left(\sin^{-1} x\right)}$$

This is not a very useful formula. Let's see if we can get a better formula. Let's start by recalling the definition of the inverse sine function.

$$y = \sin^{-1} x \qquad \Rightarrow \qquad x = \sin y$$

then,

$$\cos\left(\sin^{-1} x\right) = \cos y$$

Now, recall that

$$\cos^2 y + \sin^2 y = 1 \qquad \Rightarrow \qquad \cos y = \sqrt{1 - \sin^2 y}$$

Now,

$$\cos\left(\sin^{-1} x\right) = \cos y = \sqrt{1 - \sin^2 y}$$

Then

$$\cos\left(\sin^{-1} x\right) = \cos y = \sqrt{1 - x^2}$$

Putting all of this together gives the following derivative.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

Inverse Cosine

Now let's take a look at the inverse cosine. Here is the definition for the inverse cosine.

$$y = \cos^{-1} x \qquad \Leftrightarrow \qquad \cos y = x \qquad for \qquad 0 \leq y \leq \pi$$

Example Evaluate $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
Solution

$$\cos y = -\frac{\sqrt{2}}{2}$$

From a unit circle we can see that we must have $y = \frac{3\pi}{4}$.

The inverse cosine and cosine functions are also inverses of each other and so we have,

$$\cos\left(\cos^{-1} x\right) = x \qquad\qquad \cos^{-1}\left(\cos x\right) = x$$

To find the derivative we'll do the same kind of work that we did with the inverse sine above.

$$f(x) = \cos x \qquad\qquad g(x) = \cos^{-1} x$$

then,

$$\dot{g}(x) = \frac{1}{\dot{f}\left(g(x)\right)} = \frac{1}{-\sin\left(\cos^{-1} x\right)}$$

Upon simplifying we get the following derivative.

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

Inverse Tangent

Here is the definition of the inverse tangent.

$$y = \tan^{-1} x \qquad \Leftrightarrow \qquad \tan y = x \qquad for \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Example Evaluate $\tan^{-1}$

Solution

$$\tan y = 1$$

From a unit circle we can see that $y = \frac{\pi}{4}$.

Because there is no restriction on x we can ask for the limits of the inverse tangent function as x goes to plus or minus infinity. To do this we'll need the graph of the inverse tangent function. This is shown below.

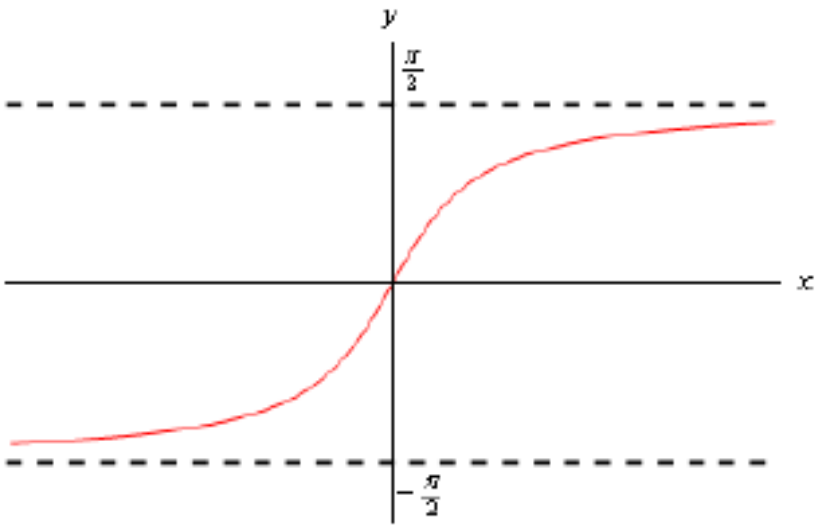

FIGURE 3.2

From this graph we can see that

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \qquad\qquad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

The tangent and inverse tangent functions are inverse functions so,

$$\tan\left(\tan^{-1} x\right) = x \qquad\qquad \tan^{-1}\left(\tan x\right) = x$$

then,

$$\dot{g}(x) = \frac{1}{\dot{f}\left(g(x)\right)} = \frac{1}{\sec^2\left(\tan^{-1} x\right)}$$

The denominator is then,

$$\sec^2\left(\tan^{-1} x\right) = \sec^2 y$$

Using

$$\cos^2 y + \sin^2 y = 1$$

divide by $\cos^2 y$,

$$1 + \tan^2 y = \sec^2 y$$

and the denominator is

$$\sec^2\left(\tan^{-1} x\right) = \sec^2 y = 1 + \tan^2 y$$

then

$$\sec^2\left(\tan^{-1} x\right) = 1 + \tan^2 y = 1 + x^2$$

The derivative of the inverse tangent is then,

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

The derivatives of all six inverse trig functions.

$$\begin{aligned}
\frac{d}{dx}\left(\sin^{-1} x\right) &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}\left(\cos^{-1} x\right) &= -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\left(\tan^{-1} x\right) &= \frac{1}{1+x^2} & \frac{d}{dx}\left(\cot^{-1} x\right) &= -\frac{1}{1+x^2} \\
\frac{d}{dx}\left(\sec^{-1} x\right) &= \frac{1}{x\sqrt{x^2-1}} & \frac{d}{dx}\left(\csc^{-1} x\right) &= -\frac{1}{x\sqrt{x^2-1}}
\end{aligned}$$

Example Differentiate the following functions.

(a) $f(t) = 4\cos^{-1}(t) - 10\tan^{-1}(t)$