Consider the case of

$$f(x) = \frac{1}{x^2} \qquad \text{on} \qquad [-1, 1]$$

Here's the graph.

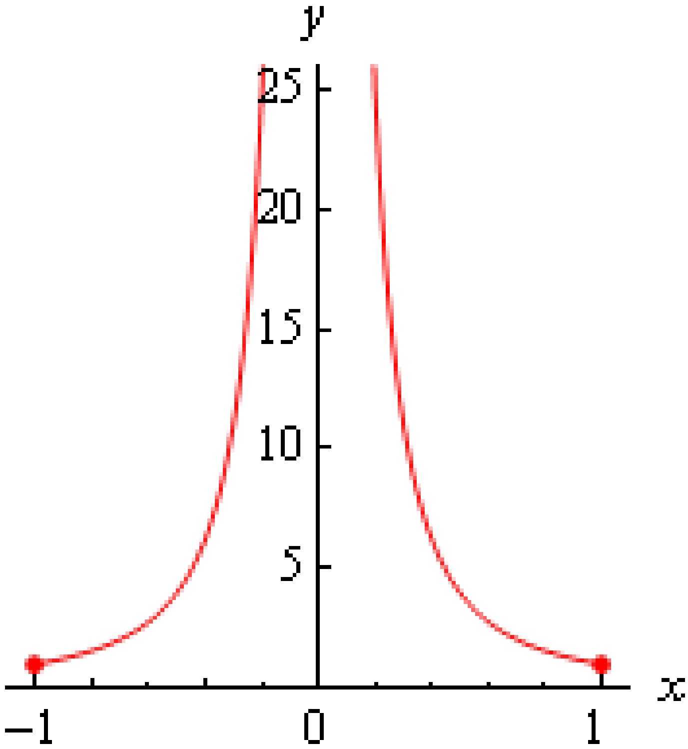

FIGURE 4.9

This function is not continuous at $x = 0$ as we move in towards zero the function approaching infinity. So, the function does not have an absolute maximum. Note that it does have an absolute minimum however. In fact the absolute minimum occurs twice at both $x = -1$ and $x = 1$.

If we changed the interval a little to say,

$$f(x) = \frac{1}{x^2} \qquad \text{on} \qquad [\frac{1}{2}, 1]$$

the function would now have both absolute extrema. We may only run into problems if the interval contains the point of discontinuity. If it doesn't then the theorem will hold.

We should also point out that just because a function is not continuous at a point that doesn't mean that it won't have both absolute extrema in an interval that contains that point. Below is the graph of a function that is not continuous at a point in the given interval and yet has both absolute extrema.

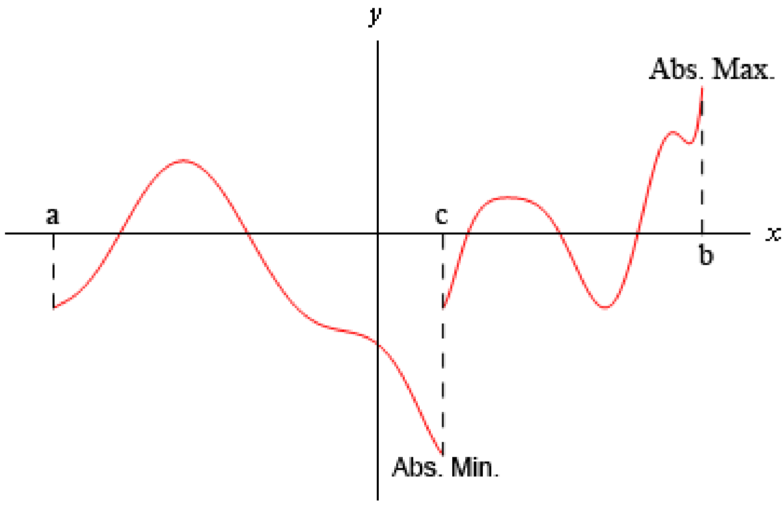

FIGURE 4.10

This graph is not continuous at $x = c$, yet it does have both an absolute maximum $(x = b)$ and an absolute minimum $(x = c)$. Also note that, in this case one of the absolute extrema occurred at the point of discontinuity, but it doesn't need to. The absolute minimum could just have easily been at the other end point or at some other point interior to the region. The point here is that this graph is not continuous and yet does have both absolute extrema.

The point of all this is that we need to be careful to only use the **Extreme Value Theorem** when the conditions of the theorem are met and not misinterpret the results if the conditions aren't met.

In order to use the **Extreme Value Theorem** we must have an interval and the function must be continuous on that interval. If we don't have an interval and/or the function isn't continuous on the interval then the function may or may not have absolute extrema.

We need to discuss one final topic in this section before moving on to the first major application of the derivative that we're going to be looking at in this chapter.

Format's Theorem

If $f(x)$ has a relative extrema at $x = c$ and $\dot{f}(c)$ exists then $x = c$ is a critical point of $f(x)$. In fact, it will be a critical point such that $\dot{f}(c) = 0$.

4.4 Finding Absolute Extrema

Here is the procedure for finding absolute extrema.

Finding Absolute Extrema of $f(x)$ ***on*** $[a, b]$

0. Verify that the function is continuous on the interval $[a, b]$.
1. Find all critical points of $f(x)$ that are in the interval $[a, b]$. This makes sense if you think about it. Since we are only interested in what the function is doing in this interval we don't care about critical points that fall outside the interval.
2. Evaluate the function at the critical points found in step 1 and the end points.
3. Identify the absolute extrema.

Example Determine the absolute extrema for $g(t) = 2t^3 + 3t^2 - 12t + 4$ and interval $[-4, 2]$ and interval.

Solution

$$\dot{g}(t) = 6t^2 + 6t - 12 = 6(t + 2)(t - 1)$$

then the critical points are $t = -2$ and $t = 1$.

Now we evaluate the function at the critical points and the end points of the interval.

$$g(-2) = 24 \qquad g(1) = -3$$
$$g(-4) = -28 \qquad g(2) = 8$$

Absolute extrema are the largest and smallest the function will ever be and these four points represent the only places in the interval where the absolute extrema can occur. So, from this list we see that the absolute maximum of $g(t)$ is 24 and it occurs at $t = -2$ (a critical point) and the absolute minimum of $g(t)$ is -28 which occurs at $t = -4$ (an endpoint).

In this example we saw that absolute extrema can and will occur at both endpoints and critical points. One of the biggest mistakes that students make with these problems is to forget to check the endpoints of the interval.

Example Determine the absolute extrema for $g(t) = 2t^3 + 3t^2 - 12t + 4$ and interval $[0, 2]$ and interval.

Solution

$$\dot{g}(t) = 6t^2 + 6t - 12 = 6(t+2)(t-1)$$

then the critical points are $t = -2$ and $t = 1$.
This means that we only want $t = 1$ since $t = -2$ falls outside the interval.

Now we evaluate the function at the critical points and the end points of the interval.

$$g(1) = -3 \qquad g(0) = 4 \qquad g(2) = 8$$

From this list of values we see that the absolute maximum is 8 and will occur at $t = 2$ and the absolute minimum is -3 which occurs at $t = 1$.

Example Suppose that the population (in thousands) of a certain kind of insect after t months is given by the following formula.

$$P(t) = 3t + \sin(4t) + 100$$

Determine the minimum and maximum population in the first 4 months.

Solution

$$\dot{P}(t) = 3 + 4\cos(4t)$$

and

$$\begin{aligned} 3 + 4\cos(4t) &= 0 \\ \cos(4t) &= -\frac{3}{4} \end{aligned}$$

The solutions to this are,

$$\begin{aligned} 4t &= 2.4189 + 2\pi n, \qquad & t &= 0.6047 + \frac{\pi n}{2}, \qquad & n &= 0, \pm 1, \pm 2, ... \\ 4t &= 3.8643 + 2\pi n, \qquad & t &= 0.9661 + \frac{\pi n}{2}, \qquad & n &= 0, \pm 1, \pm 2, ... \end{aligned}$$

$n = 0:$

$$t = 0.6047 \qquad t = 0.9661 \qquad \text{both are fall in the interval}[0,4]$$

$n = 1:$

$$t = 0.6047 + \frac{\pi}{2} = 2.1755 \qquad t = 0.9661 + \frac{\pi}{2} = 2.5369 \qquad \text{both are} \in [0,4]$$

$n = 2:$

$$t = 0.6047 + \pi = 3.7463 \qquad t = 0.9661 + \pi = 4.1077 \qquad \text{the first one only is} \in [0,4]$$

Then the critical points are,

$$0.6047, 0.9661, 2.1755, 2.5369, 3.7463$$

Finally, to determine the absolute minimum and maximum population we only need to plug these values into the function as well as the two end points.

$$P(0) = 100 \qquad P(4) = 111.7121 \qquad P(0.6047) = 102.4756$$
$$P(0.9661) = 102.2368 \qquad P(2.1755) = 107.1880 \qquad P(2.5369) = 106.9492$$
$$P(3.7463) = 111.9004$$

The minimum population is $100,000$ (remember that P is in thousands) which occurs at $t = 0$ and the maximum population is $111,900$ which occurs at $t = 3.7463$.

4.5 The Shape of a Graph, Part I

Let us write down the mathematical definition of increasing and decreasing.

Definition
1. Given any x_1 and x_2 from an interval I with $x_1 < x_2$ if $f(x_1) < f(x_2)$ then $f(x)$ is ***increasing*** on I.
2. Given any x_1 and x_2 from an interval I with $x_1 < x_2$ if $f(x_1) > f(x_2)$ then $f(x)$ is ***decreasing*** on I.

and let us summarizes the derivative ideas in the following fact.