$$\begin{aligned} \text{Increase :} \quad & -\infty < x < -1 \quad \text{and} \quad 1 < x < \infty \\ \text{Decrease :} \quad & -1 < x < 1 \end{aligned}$$

Note that from the first derivative test we can also say that $x = -1$ is a relative maximum and that $x = 1$ is a relative minimum. Also $x = 0$ is neither a relative minimum or maximum.

Now let's get the intervals where the function is concave up and concave down. If you think about it this process is almost identical to the process we use to identify the intervals of increasing and decreasing. This only difference is that we will be using the second derivative instead of the first derivative.

The first thing that we need to do is identify the possible inflection points. These will be where the second derivative is zero or doesn't exist. The second derivative in this case is a polynomial and so will exist everywhere. It will be zero at the following points.

$$x = 0, \qquad x = \pm\frac{1}{\sqrt{2}} = \pm 0.7071$$

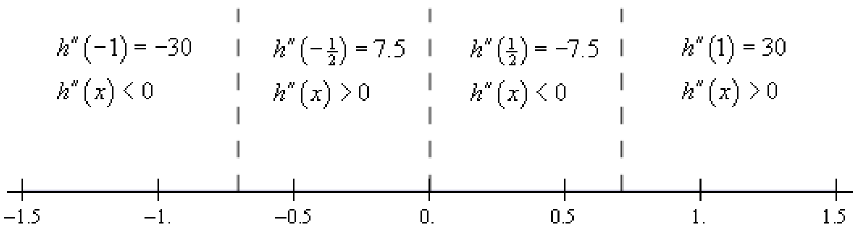

FIGURE 4.17

So, it looks like we've got the following intervals of concavity.

$$\begin{aligned} \text{Concave Up :} \quad & -\frac{1}{\sqrt{2}} < x < 0 \quad \text{and} \quad \frac{1}{\sqrt{2}} < x < \infty \\ \text{Concave Down :} \quad & -\infty < x < -\frac{1}{\sqrt{2}} \quad \text{and} \quad 0 < x < \frac{1}{\sqrt{2}} \end{aligned}$$

This also means that

$$x = 0, \qquad x = \pm\frac{1}{\sqrt{2}} = \pm 0.7071$$

are all inflection points.

Using all the above information to sketch the graph gives the following graph.

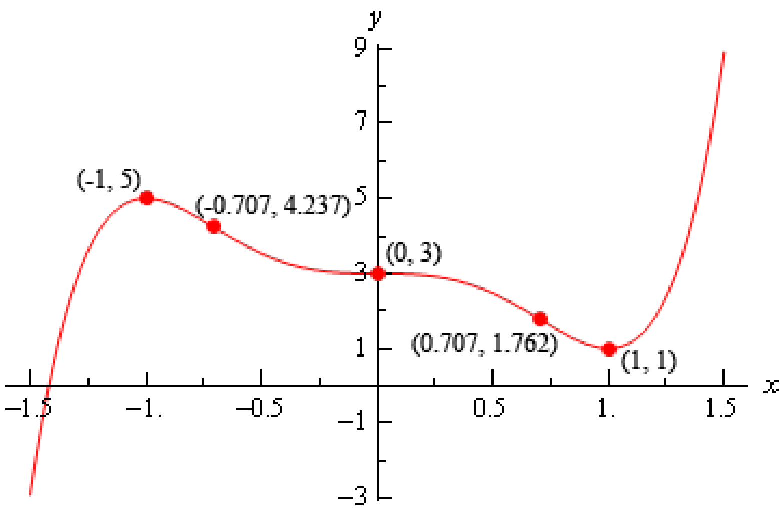

FIGURE 4.18

Here is the test that can be used to classify some of the critical points of a function.

Second Derivative Test

Suppose that $x = c$ is a critical point of $\dot{f}(c)$ such that $\dot{f}(c) = 0$ and that $\ddot{f}(x)$ is continuous in a region around $x = c$. Then,

1. If $\ddot{f}(c) < 0$ then $x = c$ is a relative maximum.
2. If $\ddot{f}(c) > 0$ then $x = c$ is a relative minimum.
3. If $\ddot{f}(c) = 0$ then $x = c$ can be a relative maximum, relative minimum or neither.

Example For the following function find the inflection points and use the second derivative test, if possible, to classify the critical points. Also, determine the intervals of increase/decrease and the intervals of concave up/concave down and sketch the graph of the function.

$$f(t) = t(6 - t)^{\frac{2}{3}}$$

Solution

$$\dot{f}(t) = \frac{18 - 5t}{3(6-t)^{\frac{1}{3}}} \qquad \ddot{f}(t) = \frac{10t - 72}{9(6-t)^{\frac{4}{3}}}$$

The critical points are,

$$t = \frac{18}{5} = 3.6 \qquad t = 6$$

Notice as well that we won't be able to use the second derivative test on $t = 6$ to classify this critical point since the derivative doesn't exist at this point. To classify this we'll need the increasing/decreasing information that we'll get to sketch the graph.

We can however, use the Second Derivative Test to classify the other critical point so let's do that before we proceed with the sketching work. Here is the value of the second derivative at $t = 3.6$.

$$\ddot{f}(3.6) = -1.245 < 0$$

So, according to the second derivative test $t = 3.6$ is a relative maximum.

Here is the number line for the first derivative.

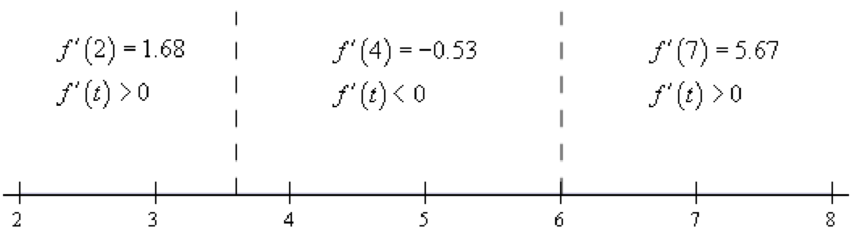

Figure 4.19

So, according to the first derivative test we can verify that $t = 3.6$ is in fact a relative maximum.

We can also see that $t = 6$ is a relative minimum.

Okay, let's finish the problem out. We will need the list of possible inflection points. These are,

$$t = 6 \qquad\qquad t = \frac{72}{10} = 7.2$$

Here is the number line for the second derivative. Note that we will need this to see if the two points above are in fact inflection points.

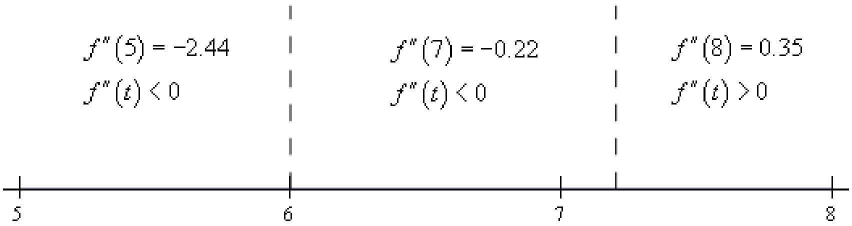

FIGURE 4.20

So, the concavity only changes at $t = 7.2$ and so this is the only inflection point for this function.

Here is the sketch of the graph.

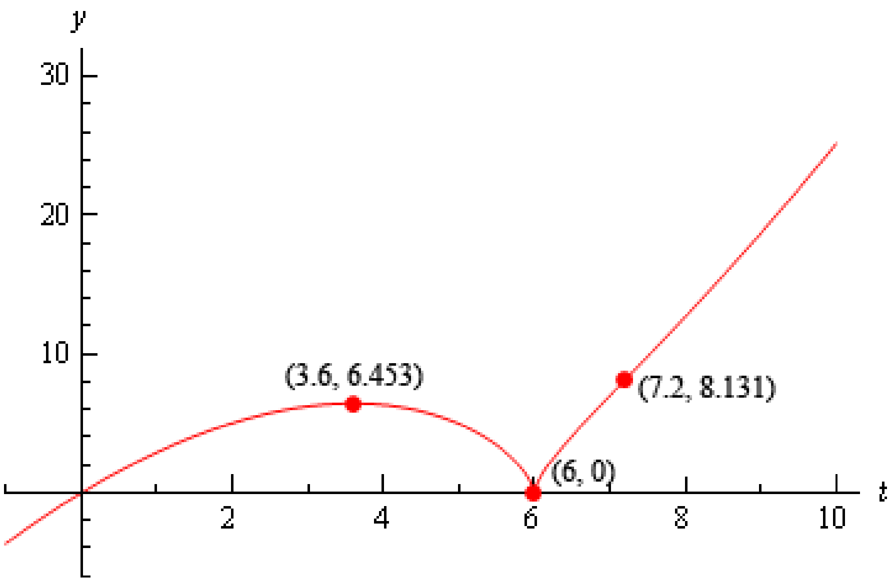

FIGURE 4.21

The change of concavity at $t = 7.2$ is hard to see, but it is there it's just a very subtle change in concavity.

4.7 The Mean Value Theorem

Before we get to the Mean Value Theorem we need to cover the following theorem.

Rolle's Theorem

Suppose that $f(x)$ is a function that satisfies all of the following.

1. $f(x)$ is continuous on the closed interval $[a, b]$.
2. $f(x)$ is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then there is a number c such that $a < c < b$ and $\dot{f}(c) = 0$. Or, in other words $f(x)$ has a critical point in (a, b).

and the **Mean Value Theorem** is

Mean Value Theorem

Suppose $f(x)$ is a function that satisfies both of the following.

1. $f(x)$ is continuous on the closed interval $[a, b]$.
2. $f(x)$ is differentiable on the open interval (a, b).

Then there is a number c such that $a < c < b$ and

$$\dot{f}(c) = \frac{f(b) - f(a)}{b - a}$$
$$f(b) - f(a) = \dot{f}(c)(b - a)$$

According to the above theorem, if we draw in the tangent line to $f(x)$ at $x = c$ we know that its slope is $\dot{f}(c)$.

What the Mean Value Theorem tells us is that these two slopes must be equal or in other words the secant line connecting $\boldsymbol{A}$ and $\boldsymbol{B}$ and the tangent line at $x = c$ must be parallel. We can see this in the following sketch.

Example Determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the following function.

$$f(x) = x^3 + 2x^2 - x \qquad \text{on} \qquad [-1, 2]$$

Solution