Second Derivative Test for Absolute Extrema

Let I be the range of all possible optimal values of $f(x)$ and further suppose that $f(x)$ is continuous on I , except possibly at the endpoints. Finally suppose that $x = c$ is a critical point of $f(x)$ and that c is in the interval I. Then,

1. If $\ddot{f}(x) > 0$ for all x in I then $f(c)$ will be the absolute minimum value of $f(x)$ on the interval I.
2. If $\ddot{f}(x) < 0$ for all x in I then $f(c)$ will be the absolute maximum value of $f(x)$ on the interval I.

Example We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost 10 dollars per m^2 and the material used to build the sides cost 6 dollars per m^2. If the box must have a volume of 50 m^3 determine the dimensions that will minimize the cost to build the box.

Solution

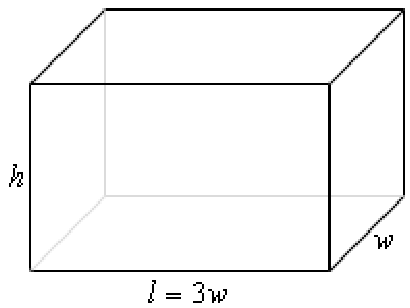

FIGURE 4.23

The two functions we'll be working with here this time are,

$$\begin{aligned} \text{Minimize :} \quad & C = 10(2lw) + 6(2wh + 2lh) = 60w^2 + 48wh \\ \text{Constraint :} \quad & 50 = lwh = 3w^2h \end{aligned}$$

we will solve the constraint for one of the variables and plug this into the cost. It will definitely be easier to solve the constraint for h.

$$h = \frac{50}{3w^2}$$

Plugging this into the cost gives,

$$C(w) = 60w^2 + 48w\frac{50}{3w^2} = 60w^2 + \frac{800}{w}$$

taking the first and the second derivatives,

$$\dot{C}(w) = 120w - 800w^{-2} = \frac{120w^3 - 800}{w^2} \qquad \ddot{C}(w) = 120 + 1600w^{-3}$$

So, it looks like we've got two critical points here. The first is obvious, $w = 0$, and it's also just as obvious that this will not be the correct value. We are building a box here and w is the box's width and so since it makes no sense to talk about a box with zero width we will ignore this critical point. This does not mean however that you should just get into the habit of ignoring zero when it occurs. There are other types of problems where it will be a valid point that we will need to consider.

The next critical point will come from determining where the numerator is zero.

$$120w^3 - 800 = 0 \qquad \Rightarrow \qquad w = \sqrt[3]{\frac{800}{120}} = \sqrt[3]{\frac{20}{3}} = 1.8821$$

So, once we throw out $w = 0$, we've got a single critical point and we now have to verify that this is in fact the value that will give the absolute minimum cost.

In this case we can't use **Method 1**. First, the function is not continuous at one of the endpoints, $w = 0$, of our interval of possible values. Secondly, there is no theoretical upper limit to the width that will give a box with volume of 50 m^3. If w is very large then we would just need to make h very small.

The second method listed above would work here, but that's going to involve some calculations, not difficult calculations, but more work nonetheless.

The third method however, will work quickly and simply here. First, we know that whatever the value of w that we get it will have to be positive and we can see second derivative above that provided $w > 0$ we will have $\ddot{C}(w) > 0$ and so in the interval of possible optimal values the cost function will always be concave up and so $w = 1.8821$ must give the absolute minimum cost.

All we need to do now is to find the remaining dimensions.

$$
\begin{aligned}
w &= 1.8821 \\
l &= 3w = 3(1.8821) = 5.6463 \\
h &= \frac{50}{3w^2} = \frac{50}{3(1.8821)^2} = 4.7050
\end{aligned}
$$

Also, even though it was not asked for, the minimum cost is : $C(1.8821) = 637.60$ dollars.

4.9 L'Hospital's Rule

Let us start with this definition,

L'Hospital's Rule

Suppose that we have one of the following cases,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \qquad \text{OR} \qquad \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

where a can be any real number, infinity or negative infinity. In these cases we have,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\dot{f}(x)}{\dot{g}(x)}$$

Example Evaluate each of the following limits.

(a) $\lim_{x\to 0} \frac{\sin x}{x}$

(b) $\lim_{t\to 1} \frac{5t^4-4t^2-1}{10-t-9t^3}$

(c) $\lim_{x\to \infty} \frac{\mathbf{e}^x}{x^2}$

Solution

(a) $\lim_{x\to 0} \frac{\sin x}{x}$

So, we have already established that this is a $0/0$ indeterminate form so let's just apply L'Hospital's Rule.

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=\frac{1}{1}=1$$

(b) $\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^3}$

In this case we also have a 0/0 indeterminate form and if we were really good at factoring we could factor the numerator and denominator, simplify and take the limit. However, that's going to be more work than just using L'Hospital's Rule.

$$\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^3}=\lim_{t\to 1}\frac{20t^3-8t}{-1-27t^2}=\frac{20-8}{-1-27}=-\frac{3}{7}$$

(c) $\lim_{x\to\infty}\frac{\mathbf{e}^x}{x^2}$

This was the other limit that we started off looking at and we know that it's the indeterminate form ∞/∞ so let's apply L'Hospital's Rule.

$$\lim_{x\to\infty}\frac{\mathbf{e}^x}{x^2}=\lim_{x\to\infty}\frac{\mathbf{e}^x}{2x}=\lim_{x\to\infty}\frac{\mathbf{e}^x}{2}=\infty$$

Sometimes we will need to apply L'Hospital's Rule more than once.

4.10 General Method of Sketching the Graph of a Function

4.11 Convexity, Concavity and the Second Derivative

4.12 Exercises

1- Suppose that the amount of money in a bank account after t years is given by, $A(t)=2000-10t\mathbf{e}^{5-\frac{t^2}{8}}$ Determine the minimum and maximum amount of money in the account during the first 10 years that it is open.

2- Determine the absolute extrema for the following function and interval, $Q(y) = 3y(y+4)^{\frac{2}{3}}$ on $[-5,-1]$.

3- Suppose that the elevation above sea level of a road is given by the following function, $E(x) = 500 + \cos\frac{x}{4} + \sqrt{3}\sin\frac{x}{4}$ where x is in miles. Assume that if x is positive we are to the east of the initial point of measurement and if x is negative we are to the west of the initial point of measurement.

If we start 25 miles to the west of the initial point of measurement and drive until we are 25 miles east of the initial point, how many miles of our drive were we driving up an incline?

4- The population of rabbits (in hundreds) after t years in a certain area is given by the following function, $P(t) = t^2 ln(3t) + 6$ Determine if the population ever decreases in the first two years.

5- Use the second derivative test to classify the critical points of the function, $h(x) = 3x^5 - 5x^3 + 3$.

6- Suppose that we know that $f(x)$ is continuous and differentiable everywhere. Let's also suppose that we know that $f(x)$ has two roots. Show that $f'(x)$ must have at least one root.

7- We want to construct a box with a square base and we only have $10m^2$ of material to use in construction of the box. Assuming that all the material is used in the construction process determine the maximum volume that the box can have.

8- A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.

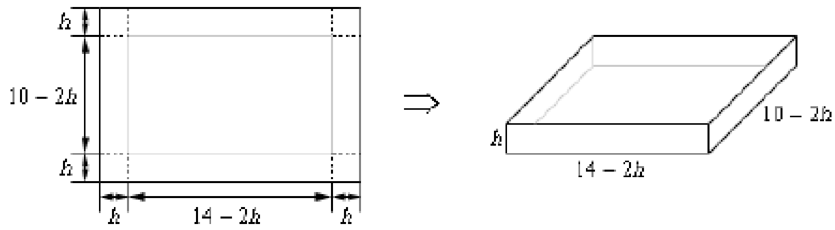

FIGURE 4.24

9- We have a piece of cardboard that is 14 inches by 10 inches and we're going to cut out the corners as shown below and fold up the sides to form a box, also shown below. Determine the height of the box that will give a maximum volume.

10- A printer need to make a poster that will have a total area of 200 $inch^2$ and will have 1 inch margins on the sides, a 2 inch margin on the top and a 1.5 inch margin on the bottom. What dimensions will give the largest printed area?

11- A window is being built and the bottom is a rectangle and the top is a semi-circle. If there is 12 meters of framing materials what must the dimensions of the window be to let in the most light?

12- the area of the largest rectangle that can be inscribed in a circle of radius 4.

13- Determine the point(s) on $y = x^2 + 1$ that are closest to $(0, 2)$.

14- A 2 feet piece of wire is cut into two pieces and once piece is bent into a square and the other is bent into an equilateral triangle. Where should the wire cut so that the total area enclosed by both is minimum and maximum?

15- A piece of pipe is being carried down a hallway that is 10 feet wide. At the end of the hallway the there is a right-angled turn and the hallway narrows down to 8 feet wide. What is the longest pipe that can be carried (always keeping it horizontal) around the turn in the hallway?

16- Two poles, one 6 meters tall and one 15 meters tall, are 20 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground

FIGURE 4.25

somewhere between the two poles. Where should the wire be staked so that the minimum amount of wire is used?

17- Two poles, one 6 meters tall and one 15 meters tall, are 20 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground somewhere between the two poles. Where should the wire be staked so that the angle formed by the two pieces of wire at the stake is a maximum?

18- A trough for holding water is be formed by taking a piece of sheet metal 60 cm wide and folding the 20 cm on either end up as shown below. Determine the angle θ that will maximize the amount of water that the trough can hold.