Mathematical consideration of radioactive decay

Consider the number of nuclei (dN) decaying in a short time dt.
dN is proportional to:-
N - the number of radioactive nuclei present at that moment
dt - the time over which the measurement is made
the element, represented by a constant (λ) called
the **disintegration (or decay) constant**.
So:

$dN = - \lambda N dt$ Activity: $dN/dt = - \lambda N$

the minus sign is there because the number of radioactive nuclei decreases as time increases.
The quantity dN/dt is the rate of decay of the source or the activity of the source and is the number of disintegrations per second.

This is measured in units called Becquerels (Bq) where 1 Bq = 1 disintegration per second.
A larger and more traditional unit is the Curie (Ci)
1 Ci = 3.7×10^{10} Bq.

The disintegration constant or decay constant (λ) can be defined as the probability of a nucleon decaying in the next second.

The number of nuclei in a sample(N) can be related to the mass of the source (m) using the molar mass (M) and Avogadro's number (L) by the formula:-

Number of moles = mass (g.) /m wt (g./mole)

N = Number of moles X Avogadro's number

or

m = MN/L

Where :

m= mass (gram)

N = Number of atoms (radioactive)

M = Molar mass or Number of moles (mole)

L = Avogadro's number

M wt = molecular weight (gram/mole)

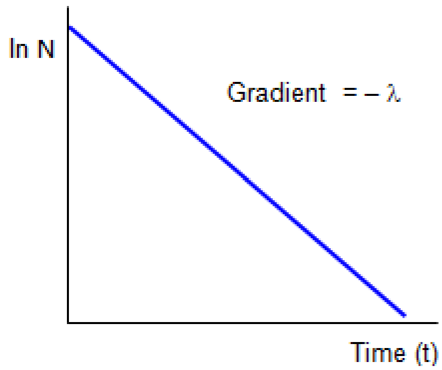
and we can use it to find out the mass of a given source if we know its activity.

Example problem

A school has a radium 226 source with an activity of 5 μCi = ($5x10^{-6}x3.7x10^{10}$ Bq).
What is the mass of the source?
Avogadro constant (L) = $6.02x10^{23}$) and the disintegration constant for radium 226 is $1.35x10^{-11}$ s^{-1}
(see later for an explanation of this term)
Using the formula $dN/dt = -\lambda N = -\lambda(m/M)L$
so $5x10^{-6}x3.7x10^{10} = 1.35x10^{-11}(m/226)6.02x10^{23}$

Therefore $m = (5x10^{-6}x3.7x10^{10}x\ 226)/\ 1.35x10^{-11}\ x\ 6.02x10^{23}$ = $5.14x10^{-6}$ grams = 5.14 μg

Returning to the formula
$dN/dt = -\lambda N$ and rearranging gives:
$dN/N = -\lambda dt$ which when integrated between the limits N = N_o and N = N for the number of nuclei at time 0 and t gives:



$\ln N_o - \ln N = \lambda t$

If we plot ln(N) against t we have a straight line graph with gradient -λ) and an intercept on the ln(N) axis of ln(N_o). It is this sort of graph that would be most helpful in finding the half life (T) by measuring the gradient and then using the relation between the half-life and the disintegration constant (see below).

Returning to the equation and taking antilogs of both sides gives:

Radioactive decay law: $N = N_o e^{-\lambda t}$

A graph of N against t would give an exponential decay graph, and if background radiation were ignored the line would tend towards N = 0 as time goes by. Since N is directly proportional to the activity (A) and the mass (m) of the sample we have three alternative forms of this formula.
It can be expressed as:

(Number of nuclei) $N = N_o e^{-\lambda t}$
(Activity) $A = A_o e^{-\lambda t}$
(Mass) $m = m_o e^{-\lambda t}$

Half life and the radioactive decay constant

We can now get a much better idea of the meaning of not only the half life (T) but also of the decay constant (λ).
When $N = N_o/2$ the number of radioactive nuclei will have halved and so one half life will have passed.
Therefore when t = T $N = N_o/2 = N_o e^{-\lambda T}$ and so $1/2 = e^{-\lambda T}$.
Taking the inverse gives $2 = e^{\lambda T}$ and so:

$\ln(2) = 0.693 = \lambda T$ and so
Decay constant $(\lambda) = 0.693/T$

Example problem

The decay constant (λ) of a particular isotope (radon 220) is $1.33\times10^{-2}\ s^{-1}$.

How long will it take for the activity of a sample of this isotope to decay to one eighth of its original value?

Half life = 0.693/λ = 0.693/1.33x10-2 = 52 s.

Number of half lives required to reduce the activity to one eighth = 3

Therefore time needed = 3 x 52 = 156 s = 2m 36s

Example problems

1. A sample of material is found to contain 2 g of the isotope gold 199.

How much will remain 10 days later?

The half life of gold 199 is 3.15 days and so the decay constant is therefore $0.693/3.15\times86400 = 2.55\times10^{-6}\ s^{-1}$. However since the half life and the time over which the decay takes place are both given in days we do not need to change both into seconds.

Using the formula:- $m = m_o e^{-\lambda t}$ we have $m = 2xe^{-(0.693/3.15)10}$ = 0.22 g

2. A sample of carbon 14 has an activity of 2.5 Bq when corrected for background radiation.

If the initial count rate of a sample of the same mass was 3.7 Bq how old is the sample?

(Half life of carbon 14 = 5570 years).

Using the formula $A = A_o e^{-\lambda t}$ we have $2.5 = 3.7e^{-(0.693/5570)t}$ where t is in years.

Taking logs gives: ln(2.5/3.7) = -(0.693/5570)t this gives t = 3151 years.

Problem

1. 0.200g of polonium ($^{210}_{84}Po$) half –life 138.4 days is kept in a vessel. How much of it will remain after 21.0days. (N =?)(Ans = 0.184g)

2. What is the t½ of an isotope if a sample of it gives 10,000 Activity and 3.50 hrs later it gives 8335 Activity? (t½ =?) (Ans = 13.3hrs)

3. How much of a 1.00g sample of $^{238}_{92}U$ will disintegrate in a period of 10yrs. t½ = 4.51×10^9 yrs? (N or mass =?) (Ans =1.54×10^{-9}g)