LECTURE NOTE IN DISCRETE MATHMATICS

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Contents

Fundamentals of Mathematical Logic

1.1 Propositions and Related Concepts

A **proposition** is any meaningful statement that is either true or false, but not
Both. We will use lowercase letters, such as *p, q, r, ...* , to represent propositions. We will also use the notation

$p : 1 + 1 = 3$

to define p to be the proposition $1 + 1 = 3$. The **truth value** of a proposition is true, denoted by T, if it is a true statement and false, denoted by F, if it is a false statement. Statements that are not propositions include questions and commands.

Exercise 1:
Which of the following are propositions? Give the truth value of the propositions.
a. $2 + 3 = 7$
b. What time is it?
c. $2 + 2 = 4$
d. How are you?

Solution:
a. A proposition with truth value (F).
b. Not a proposition since no truth value can be assigned to this statement.
c. A proposition with truth value (T).
d. Not a proposition.

New propositions called **compound propositions** or **propositional functions** can be obtained from old ones by using **symbolic connectives** which

We discuss next. The propositions that form a propositional function are called the **propositional variables.**
Let p and q be propositions. The **conjunction** of p and q, denoted $p \wedge q$, is the proposition: *p and q.* This proposition is defined to be true only when both p and q are true and it is false otherwise. The **disjunction** of p and q, denoted $p \vee q$, is the proposition: *p or q.* The 'or' is used in an inclusive way. This proposition is false only when both p and q are false, otherwise it is true.

Exercise 2:
Let $p : 5 < 9$
$q : 9 < 7.$
Construct the propositions $p \wedge q$ and $p \vee q$

Solution. :
The conjunction of the propositions p and q is the proposition
$p \wedge q : 5 < 9$ and $9 < 7.$
The disjunction of the propositions p and q is the proposition
$p \vee q : 5 < 9$ or $9 < 7.$

A **truth table** displays the relationships between the truth values of propositions.

Next, we display the truth tables of $p \wedge q$ and $p \vee q$

p	q	p $\wedge$q
T	T	T
T	F	F
F	T	F
F	F	F

P	q	p $\vee$q
T	T	T
T	F	T
F	T	T
F	F	F

Let p and q be two propositions. The **exclusive or** of p and q, denoted $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise. The truth table of the exclusive 'or' is displayed below

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Exercise 3:

a. Construct a truth table for $(p \oplus q) \oplus r$

b. Construct a truth table for $p \oplus p$

Solution :

a.

p	q	r	$p \oplus q$	(p ⊕q) ⊕ r
T	T	T	F	T
T	T	F	F	F
T	F	T	T	F
T	F	T	T	T
F	T	T	T	F
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

b.

p	p ⊕ p
T	F
F	F

The final operation on a proposition p that we discuss is the **negation** of p‹ the negation of p, denoted $\neg\ p$, is the proposition not p. The truth table of $\neg\ p$ is displayed below.

p	$\neg\ p$
T	F
F	T

Exercise 4:
Construct the truth table of $[\neg\ (p \wedge q)] \vee r$.
Solution :

p	q	r	$p \wedge q$	$\neg\ (p \wedge q)$	$[\neg\ (p \wedge q)] \vee r$
T	T	T	T	F	T
T	T	F	T	F	F
F	T	T	F	T	T
F	T	F	F	T	T

Exercise 5:
Find the negation of the proposition p : $\nwarrow 5 < x \leq 0$.

Solution :
The negation of p is the proposition $\neg\ p : x > 0$ *or* $x \leq\leq \nwarrow 5$

Definition 1 :
A compound proposition is called a **tautology** if it is always true, regardless of the truth values of the basic propositions which comprise it.

Exercise 6:
a.Construct the truth table of the proposition $(p \wedge q) \vee (\neg p \vee \neg q)$ Determine if this proposition is a tautology.

b. Show that $p \vee \neg\ p$ is a tautology.

Solution :

a.

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
T	T	F	F	F	T	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

Thus, the given proposition is a tautology.

b.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Again, this proposition is a tautology

Definition 2:

A compound proposition that has the value F for all possible values of the propositions in it is called a **contradiction**.

Exercise 7:

Show that the proposition $p \wedge \neg p$ is a contradiction.

Solution.

P	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Remark: if T is the symbol of tautology and F is the symbol of contradiction and let P any proposition then :

***p $\wedge$ T $\equiv$ P**
***P $\wedge$ F $\equiv$ F**
***P $\vee$ T $\equiv$ T**
***P $\vee$ F $\equiv$ P**

Definition 3 :
Two propositions are **equivalent** if they have exactly the same truth values
under all circumstances. We write $p \equiv q$

Exercise 8:
a. Show that $\neg (p \vee q) \equiv \neg p \wedge \neg q$
b. Show that $\neg (p \wedge q) \equiv \neg p \vee \neg q$
c. Show that $\neg (p) \equiv p$
a. and b. are known as DeMorgan's laws.

Solution :
a.

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg (p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

b.

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg (p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

c.

p	$\neg p$	$\neg (\neg p)$
T	F	T
F	T	F.

Exercise 9 (H.W):
a. Show that $p \wedge q \equiv q \wedge p$ and $p \vee q \equiv q \vee p$
b. Show that $(p \vee q) \vee r \equiv p \vee (q \vee r)$ and $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r$
c. Show that $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$ and $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$
d.Show that $\neg (p \wedge q) \equiv \neg p \wedge \neg q$

REMARK:
In propositional functions, the order of operations is that $\neg$ is performed first. The operations $\vee$ and $\wedge$ are executed in any order.

1.2 Law of algebra of proposition :

1: $P \equiv (p \wedge P)$, $P \equiv (P \vee P)$ **idempotent of** $\wedge$, $\vee$
2: $(P \vee q) \equiv (q \vee p)$, $(p \wedge q) \equiv (q \wedge p)$, **commutative of** $\wedge$, $\vee$
3: $[(P \vee q) \vee r] \equiv [P \vee (q \vee r)]$
, $[(P \wedge q) \wedge r] \equiv [P \wedge (q \wedge r)]$, **associative of** $\wedge$, $\vee$

4: $\neg(P \vee q) \equiv (\neg p \wedge \neg q)$, $\neg(P \wedge q) \equiv (\neg p \vee \neg q)$
Demorgans law

5: $[P \vee (q \wedge r)] \equiv [(P \vee q) \wedge (p \vee r)]$
$[P \wedge (q \vee r)] \equiv [(P \wedge q) \vee (p \wedge r)]$, **distributive law**

6: $(p \vee 1) \equiv 1$, $(p \wedge 0 \equiv 0)$, **domination law**
7: $(p \vee \neg p) \equiv 1$, $(p \wedge \neg p) \equiv 0$, **complement law**

8: $(p \vee 0) \equiv p$, $(p \wedge 1 \equiv p)$, **identity law**

9: $p \equiv \neg(\neg p)$, **double negation**

10: $p \to q \equiv (\neg p \vee q)$ **implication**

11: $p \Leftrightarrow q \equiv (p \to q) \wedge (q \to p)$, **equivalence**
12: $[(P \wedge q) \to r] \equiv [p \to (q \to r)]$, **exportation**

13: $(p \to q) \wedge (p \to \neg q)] \equiv \neg p$, **absurdity**

14: $(p \to q) \equiv (\neg q \to \neg p)$, **contra positive**

15: $p \vee (p \wedge q) \equiv p$, $p \wedge (p \vee q) \equiv p$, **absorption law**

Solve the Exercise by using law of algebra of proposition

Exercise 10 :

show that $p \to (q \to r) \equiv p \to (\neg q \vee r)$

solution:

$p \to (q \to r) \equiv p \to (\neg q \vee r)$ **, implication**

Exercise 11

: $[\neg p \wedge (\neg q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \equiv r$

solution :

$[\neg p \wedge (\neg q \wedge r)] \vee (q \wedge r) \vee (p \wedge r)$

$\equiv [\neg p \wedge (\neg q \wedge r)] \vee [(q \vee p) \wedge r]$ **by distributive law**

$\equiv [(\neg p \wedge \neg q) \wedge r] \vee [(q \vee p) \wedge r]$ **by associatively of** $\wedge$

$\equiv [(\neg p \wedge \neg q) \vee (q \vee p)] \wedge r$ **by distributive law**

$\equiv [\neg (p \vee q) \vee (q \vee p)] \wedge r$ **by demorgans law**

$\equiv [\neg (p \vee q) \vee (p \vee q)] \wedge r$ **by commutativity of** $\vee$

$\equiv 1 \wedge r$ **by complement law**

$\equiv r$ **by identity law**

Review Problems

Exercise 1:
Write the truth table for the proposition: $(p \vee (\neg p \vee q)) \wedge \neg (q \wedge \neg r)$
Exercise 2:
Let t be a tautology. Show that $p \vee t \equiv t$
Exercise 3:
Let c be a contradiction. Show that $p \vee c \equiv p$
Exercise 4:
Show that $(r \vee p) \wedge [(\neg r \vee (p \wedge q)) \wedge (r \vee q)] \equiv p \wedge q$

Exercise 5:
Show that the proposition $s = (p \wedge q) \vee (\neg p \vee (p \wedge \neg q))$ is a tautology.
Exercise 6:
Show that the proposition $s = (p \wedge \neg q) \wedge (\neg p \vee q)$ is a contradiction.
Exercise 7:
a. Find simpler proposition forms that are logically equivalent to $p \oplus p$ and $p \oplus (p \oplus p)$.
b. Is $(p \oplus q) \oplus r \equiv p \oplus (q \oplus r)$? Justify your answer.
c. Is $(p \oplus q) \wedge r \equiv (p \wedge r) \oplus (q \wedge r)$? Justify your answer.

1.3 Conditional and Biconditional Propositions

Let p and q be propositions. The implication $p \rightarrow q$ is the the proposition that is false only when p is true and q is false; otherwise it is true. p is called the **hypothesis** and q is called the **conclusion.** The connective $\rightarrow$ is called the **conditional** connective.

Exercise 12:

Construct the truth table of the implication $p \to q$

Solution.

The truth table is

P	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Exercise 13:

Show that $p \to q \equiv \neg p \vee q$.

Solution.

P	q	$\neg p$	$p \to q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

It follows from the previous exercise that the proposition $p \to q$ is always true if the hypothesis p is false, regardless of the truth value of q▷ We say that $p \to q$ is **true by default** or **vacuously true**. In terms of words the proposition $p \to q$ also reads:

(a) if p then q.
(b) p implies q.
(c) p is a sufficient condition for q
(d) q is a necessary condition for p
(e) p only if q

Exercise 14:

Use the if-then form to rewrite the statement "I am on time for work if I catch the 8:05 bus."

Solution.

If I catch the 8:05 bus then I am on time for work.

In propositional functions that involve the connectives $\neg$, $\wedge$, $\vee$, and $\rightarrow$ the order of operations is that $\neg$ is performed first and $\rightarrow$ is performed last.

Exercise 15:
a. Show that $\neg(p \rightarrow q) \equiv p \wedge \neg q$
b. Find the negation of the statement " If my car in the repair shop, then I cannot go to class.

Solution.
a. We use De Morgan's laws as follows.
$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$
$\equiv \neg(\neg p) \wedge \neg q$
$\equiv p \wedge \neg q$
b. "My car in the repair shop and I can get to class."

The **converse** of $p \rightarrow q$ is the proposition $q \rightarrow p$. The **opposite** or **inverse**
of $p \rightarrow q$ is the proposition $\neg p \rightarrow \neg q$. The **contrapositive** of $p \rightarrow q$ is the
proposition $\neg q \rightarrow \neg p$

Exercise 16:
Find the converse, opposite, and the contrapositive of the implication:
" If today is Thursday, then I have a test today."

Solution.
The converse: If I have a test today then today is Thursday.
The opposite: If today is not Thursday then I don't have a test today.
The contrapositive: If I don't have a test today then today in not Thursday

Exercise 17:
Show that $p \rightarrow q \equiv \neg q \rightarrow \neg p$

Solution:
We use De Morgan's laws as follows.
$p \to q \equiv \neg p \vee q$
$\equiv \neg (p \wedge \neg q)$
$\equiv \neg (\neg q \wedge p)$
$\equiv \neg\neg q \vee \neg p$
$\equiv q \vee \neg p$
$\equiv \neg q \to \neg p$

Exercise 18:
Show that $\neg q \to \neg p \equiv p \to q$

Solution:
We use De Morgan's laws as follows.
$\neg q \to \neg p \equiv q \vee \neg p$
$\equiv \neg (\neg q \wedge p)$
$\equiv \neg (p \wedge \neg q)$
$\equiv \neg p \vee \neg\neg$
q
$\equiv \neg p \vee q$
$\equiv p \to q$

Definition 4:
The **biconditional** proposition of p and q, denoted by $p \leftrightarrow q$, is the propositional function that is true when both p and q have the same truth values and false if p and q have opposite truth values. Also reads, "p if and only if q" or "p is a necessary and suffcient condition for q."

Exercise 19:
Construct the truth table for $p \leftrightarrow q$.

Solution.

P	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Exercise 20:
Show that the biconditional proposition of p and q is logically equivalent to the conjunction of the conditional propositions $p \to q$ and $q \to p$.

Solution.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

REMARK :The order of operations for the five logical connectives is as follows:
1. $\neg$ 2. $\wedge, \vee$ in any order.
3. $\to, \leftrightarrow$ in any order.

Definition 5 :

let p and q be proposition , p is said to be logical implies the proposition q , if and only $p \to q$ is a tautology , denoted by $p \Leftrightarrow q$, if $p \to q$ is not a tautology , then $p \not\Leftrightarrow q$.

Exercise 21 :
prove that $(p \wedge \neg q) \Leftrightarrow (\neg p \vee \neg q)$.

solution : we must prove that $(p \wedge \neg q) \to (\neg p \vee \neg q)$. is T

$$
\begin{aligned}
(p \wedge \neg q) \to (\neg p \vee \neg q) &\equiv \neg (p \wedge \neg q) \vee (\neg p \vee \neg q) \\
&\equiv (\neg p \vee q) \vee (\neg p \vee \neg q) \\
&\equiv (\neg p \vee \neg p) \vee (q \vee \neg q) \\
&\equiv \neg p \vee T \\
&\equiv T
\end{aligned}
$$

So, $(p \wedge \neg q) \Leftrightarrow (\neg p \vee \neg q)$.

Review Problems

Exercise 1:
Construct the truth table for the proposition: $\neg p \vee q \to r$
Exercise 2:
Construct the truth table for the proposition: $(p \to r) \leftrightarrow (q \to r)$

.
Exercise 3:
Write negations for each of the following propositions. (Assume that all variables represent fixed quantities or entities, as appropriate.)

1. If r is rational, then the decimal expansion of r is repeating.
2. If n is prime, then n is odd or n is 2.
3 If $x \geq 0$, then $x > 0$ or $x = 0$.

4. If n is divisible by 6, then n is divisible by 2 and n is divisible by 3.

Exercise 4 : Write the contra positives for the propositions of Exercise 3.
Exercise 5: Write the converse and inverse for the propositions of
Exercises 6 :
1 * construct a truth table for the following proposition :

$$[(p \to q) \wedge (q \to r)].$$

2* simplify the following expression by using the laws of algebra of proposition

a - $\neg(\neg p \wedge \neg q)$.
b - $\neg(p \vee q) \vee (\neg p \wedge q)$.
c - $\neg[(p \vee q) \wedge r] \vee q$.

3* prove the following equivalence by using the laws of algebra of proposition :

a- $p \wedge (p \vee q) \equiv p$
b- $p \wedge (\neg p \vee q) \equiv p \wedge q$
c- $(p \wedge q) \vee \neg p \equiv \neg p \vee q$

4* write each condition proposition symbolically , write the converse and contra positive of each statement and in words also , find the truth value of each condition proposition , its converse and its contraposition

1/ if 4 < 6 , then 9 > 12 .

2/ $|4| < 3$, if -3 < 4 < 3 .

1.4 Propositions and Quantifiers

Statements such as "$x > 3$" are often found in mathematical assertions and in computer programs. These statements are not propositions when the variables are not specified. However, one can produce propositions from such statements. A **predicate** is an expression involving one or more variables defined on some domain, called the **domain of discourse**. Substitution of a particular value for the variable(s) produces a proposition which is either true or false. For instance, $P(n)$: *n is prime* is a predicate on the natural numbers. Observe that $P(1)$ is false, $P(2)$ is true. In the expression $P(x)$, x is called a **free variable**. As x varies the truth value of $P(x)$ varies as well. The set of true values of a predicate $P(x)$ is called the **truth set** and will be denoted by TP

Definition 6:
If $P(x)$ and $Q(x)$ are two predicates with a common domain D then the notation $P(x) \Rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is also an element in the truth set of $Q(x)$

Exercise 22:
Consider the two predicates $P(x)$: x is a factor of 4 and $Q(x)$: x is a factor Of 8. Show that $P(x) \Rightarrow Q(x)$

Solution:
Finding the truth set of each predicate we have: $TP = \{1,2,4$ and $TQ = \{1, 2, 4, 8$. Since every number appearing in TP also appears in TQ

then $P(x) \Rightarrow Q(x)$
If two predicates $P(x)$ and $Q(x)$ with a common domain D are such that

$TP = TQ$ then we use the notation $P(x) \Leftrightarrow Q(x)$

Exercise 23:
Let $D =$ IR. Consider the two predicates $P(x) : 2 \leq x \leq 2$ and
$Q(x) : |x| \leq 2$. Show that $P(x) \Leftrightarrow Q(x)$.

Solution:
Indeed, if x in TP then the distance from x to the origin is at most 2. That is,
$|x| \geq 2$ and hence x belongs to TQ. Now, if x is an element in **then** $|x| \leq 2$,
i.e.$(x - 2)(x + 2) \leq 0$. Solving this inequality we find that $2 \leq x \leq 2$ That is, $x \in TP$

Another way to generate propositions is by means of **quantifiers**
For example $\forall x \in D, P(x)$ is a proposition which is true if $P(x)$ is true for all values of x in the domain D of P. For example, if k is an nonnegative integer, then the predicate $P(k)$: $2k$ *is even* is true for all $k \in$ IN. we write, $\forall k \in$ IN, ($2k$ *is even*)

Definition 7 :
The symbol $\forall$ is called the **universal quantifier.**
The proposition $\forall x \in D, P(x)$ is false if $P(x)$ is false for at least one value of x. In this case x is called a **counterexample**.

Exercise 24:

Show that the proposition $\forall x \in$ IR, $x > \frac{1}{x}$ is false.

Solution:

A counterexample is $x = \frac{1}{2}$ Clearly, $\frac{1}{2} < 2 = \frac{1}{\frac{1}{2}}$

Definition 8:
The notation $\exists x \in D, P(x)$ is a proposition that is true if there is at least
one value of $x \in D$ where $P(x)$ is true; otherwise it is false. The symbol $\exists$ is called the **existential** quantifier.

Exercise 25:
a. What is the negation of the proposition $\forall x \in D, P(x)$?
b. What is the negation of the proposition $\exists x \in D, P(x)$?
c. What is the negation of the proposition $\forall x \in D, P(x) \rightarrow Q(x)$?

Solution:

a. $\exists x \in D, \neg P(x)$
b. $\forall x \in D, \neg P(x)$
c. Since $P(x) \rightarrow Q(x) \equiv (\neg P(x)) \vee Q(x)$ then $\neg(\forall x \in D, P(x) \rightarrow Q(x)) \equiv \exists x \in D, P(x)$ *and* $\neg Q(x)$

Exercise 26:
Consider the universal conditional proposition
$\forall x \in D,$ *if* $P(x)$ *then* $Q(x)$.
a. Find the contra positive.
b. Find the converse.
c. Find the inverse.

Solution:
a. $\forall x \in D,$ *if* $\neg Q(x)$ *then* $\neg P(x)$
b. $\forall x \in D,$ *if* $Q(x)$ *then* $P(x)$
c. $\forall x \in D,$ *if* $\neg P(x)$ *then* $\neg Q(x)$

Exercise 27:
Find the negation of the following propositions:
a. $\forall x \exists y, P(x, y)$
b. $\exists x \forall y, P(x, y)$
Solution.
a. $\exists x \forall y, \neg P(x, y)$
b. $\forall x \exists y, \neg P(x, y)$

Review Problems

Exercise 1:

Consider the statement $\exists x \in \mathrm{IR}$ *such that* $x^2 = 2$
Which of the following are equivalent ways of expressing this statement?
a. The square of each real number is 2.
b. Some real numbers have square 2.
c. The number x has square 2, for some real number x
d. If x is a real number, then $x^2 = 2$.
e. Some real number has square 2.
f. There is at least one real number whose square is 2.

Exercise 2:
Determine whether the proposed negation is correct. If it is not, write a correct negation.
Proposition : For all integers n, if n^2 is even then n is even.
Proposed negation : For all integer n, if n^2 is even then n is not even.

Exercise 3:
Let $D = \{ 48 \Leftrightarrow 14 \Leftrightarrow 8\{ 0, 1, 3, 16, 23, 26, 32, 36\}$. Determine which

of the following propositions are true and which are false. Provide counterexamples for those propositions that are false.
a. $\forall x \in D$, if x is odd then $x > 0$.

b. $\forall x \in D$, if x is less than 0 then x is even.
c. $\forall x \in D$, if x is even then $x \leq 0$.

d. $\forall x \in D$, if the ones digit of x is 2, then the tens digit is 3 or 4.
e. $\forall x \in D$, if the ones digit of x is 6, then the tens digit is 1 or 2

Exercise 4:
Write the negation of the proposition: $\forall x \in$ IR, if $x(x+1) > 0$ then $x > 0$ or $x < 1$.

Exercise 5:
Write the negation of the proposition: If an integer is divisible by 2, then it is even.

Fundamentals of Set Theory

Set is the most basic term in mathematics and computer science. In this chapter we introduce the concept of sets and its various operations and then study the properties of these operations.

2.1 Basic Definitions

Definition 1 :

a set should be a well-defined collection of objects, these objects are called elements and we use capital letters, such as A,B,C,……..to represent sets and lower case letters to represent elements

Definition 2 :

In any application of the theory of sets, all the sets under investigation will likely be subsets of affixed set, well call this set the universal set.

There are two different ways to represent a set. The first one is to list, without repetition, the elements of the set. The other way is to describe a property that characterizes the elements of the set.

Exercise 1:

: for U={1,2,3,…..} the set of appositive integers

Let $A=\{1,4,9,\ldots.64,81\}=\{\, x^2 \mid \chi \in U, x^2 < 100\}$

$=\{x^2 \mid x \in U \wedge x^2 < 100\}$

Now, we introduce the following designation that appears frequently throughout the text.

We introduce her several sets that will be used through, this course

Z= the set of integers =$\{0,1,-1,2,-2,\ldots\ldots\}$

N= the set of nonnegative integer (Natural number) = $\{0,1,2,\ldots\}$

Z^+**= the set of positive integers =**$\{1,2,3,\ldots\}$

Q= the set of rational number={ a |b| a,b Є Z, b≠o}
Q^+= the set of positive rational number { r Є Q, r Z}
Q^*= the set of nonzero rational number
R= the set of real number
R^+= the set of positive real number
R^*= the set of nonzero real number
C= the set of complex number = {X + iy| x, y Є R, i^2 =-1}
For each n Є Z^+, Zn ={ 0,1,2,…n-1}

Definition 3 :
A set which contains no elements is called empty set (Null set)

Exercise 2:
: $\phi = \{ \chi \in Z : \chi \neq \chi \}$
$\phi = \{\chi \in \Re : \chi^2 < 0\}$
$\phi = \{\chi : \chi isarealumberand \chi^2 = -1\}$
Since the square of a real number χ is always nonnegative

Exercise 3 :
List the elements of the following sets.
a. *{ x: x* is a real number such that $x^2 = 1$*}*
b. *{ x: x* is an integer such that $x^2 - 3 = 0$*}*

Solution.
a. **[1,-1**]
b. ϕ

Exercise 4 :
Use a property to give a description of each of the following sets.
a. {*a, e, i, o,}* . b. { –1*,* 3*,* 5*,* 7*,* 9**}**

Solution.
a. *{x:x is a vowel}*
b.*{ n* $\in$ IN*: n is odd and less than* 10*}*

*Let A and B be two sets. We say that A is a **subset** of B, denoted by
$A \subseteq B$, if and only if every element of A is also an element of B Symbolically:

$$A \subseteq B \Leftrightarrow \Leftrightarrow \forall x, x \in A \text{ implies } x \in B$$

Exercise 5:

A= {1,2,3,4,5,6}, B={2,4,5}, C={1,2,3,4,5}

Then $B \subseteq A, B \subseteq C, C \subseteq B$.

Note: if A is any set, that $A \subseteq A$**. That is, every set is a subset of itself.**

*If there exists an element of A which is not in B then we write
$A \not\subset B$ *Since* the proposition $x \in \phi \emptyset$ is always false then for any

set
A we have $\phi \subseteq A \Leftrightarrow \Leftrightarrow \forall x, x \in \phi \ \emptyset \text{ implies } x \in A$

Exercise 6:

Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, and $C = \{4, 6\}$. Determine which of these sets are subsets of which other of these sets.

Solution :

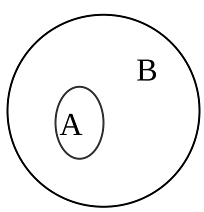
$B \subseteq A$ and $C \subseteq A$

*If sets A and $\subset B$ are represented as regions in the plane, relationships between A and B can be represented by pictures, called **Venn diagram.**

Exercise 7 :

Represent $A \subseteq B$ using Venn diagram.

Solution :



Remark:

We define C $\underline{\text{C}}$ D $\Longrightarrow$ **|C| ≤ |D|**

And CCD $\Longrightarrow$ **|C|<|D| . And id called proper subset of D.**

Theorem 1 :

Let A,B,C $\subseteq$ U , then

a) if $A\subseteq B$ and $B\subseteq C$ then $A\subseteq C$.

b) if $A\subset B$ and $B\subseteq C$ then $A\subset C$.

c) if $A\subseteq B$ and $B\subset C$, then $A\subset C$.

d) if $A\subset B$ and $B\subset C$, then $A\subset C$

Proof: **We shall prove a.,b and leave c.,d exercises**

to prove that $A\subseteq C$ we need to verify that for all $X\in U$, if $X \in C$.
we start with an element X from A.
since $A\subseteq B$, $X\in A$ implies $\rightarrow X\in B$, then with
$B \subseteq C$, $X\in B \rightarrow X \in C$
So, $X \in A \rightarrow X \in C$
Since $X \in A$, $X \in C$ and $A \subseteq C$

b-since $A\subset B$**, if** $X \in B$ **with** $B\subseteq C$**, it then follows that** $X \in C$**.**
so $A\subseteq C$**, However,** $A\subset B \Rightarrow$ **there exists an element** $b\in B$**, such that** $b\notin A$**, because** $B\subseteq C, b\in B \rightarrow b\in C$
thus, $A\subseteq C$ **and there exists an element** $b\in C$ **with** $b\notin A$ **so** $A\subset C$

Theorem 2 : for any universe υ **. let** $A\subseteq \upsilon$
Then $\phi \subseteq$ A. and if $A\neq\phi$ **then** $\phi\subset A$**,**

proof: if the first result is not true, then $\phi\subseteq A$**, so there is an element x from the universe with** $\times\in\phi$ **but** $\times\notin A$**.**
But $\times\in\phi$ **is impossible**
so, we reject the assumption $\phi\subseteq A$ **and find that** $\phi\subseteq A$**.In addition if** $A\neq\phi$ **,then there is an element** $a\in A$ **(and** $a\notin\phi$ **) , so** $\phi\subset A$

Definition 4 :
Two sets A and B are said to be **equal** if and only if $A \subseteq B$ and $B \subseteq A$ we write $A = B$ Thus, to show that $A = B$ it suffices to show the double inclusions mentioned in the definition. For non-equal sets we writ $A \neq B$

Exercise 8:
Determine whether each of the following pairs of sets are equal.
(a) $\{1,3,5\}$ **and** $\{5,3,1\}$
(b) $\{\{1\}\}$ **and** $\{1,\{1\}\}$

Solution :
(a)$\{1,3,5\} = \{5,3,1\}$
(b) $\{\{1\}\} \neq \{1,\{1\}\}$ ***since*** $1 \notin \{\{1\}\}$

Definition 5 :
Let A and B be two sets. We say that A is a **proper** subset of B, denoted by $A \subset B$, if $A \subseteq B$ and $A \neq B$. Thus, to show that A is a proper subset of B we must show that every element of A is an element of B and there is an element of B which is not in A

Exercise 9 :
Determine whether each of the following statements is true or false.

(a) $x \in \{x\}$ (b) $\{x\} \subseteq \{x\}$ (c) $\{x\} \in \{x\}$ (d) $\{x\} \in \{\{x\}\}$ (e) $\emptyset \subseteq \{x\}$

(f) $\phi \in \{x\}$

Solution:
(a)True (b) True (c) False (d) True (e) True (f) False

Definition 6 :
Two sets that have no common elements, called disjoint sets

Definition 7 :
If U is a given set whose subsets are under consideration, then we call U a **universal set.**

Let U be a universal set and A,B be two subsets of U. The **absolute complement** of A is the set $A^c = \{x \in U \neg x \notin A\}$

The **relative complement** of A with respect to B is the set

$$B - A = \{x \in U: \ x \in B \text{ and } x \notin A\}.$$

Exercise 10:

Let A= {X: X is an integer and $\chi \geq 4$}

Then $\overline{A}$ = { X: X is an integer and X<4}

Exercise 11:

Let $U =$ IR. Consider the sets $A = \{x \in \text{I}: \ x < -1 \text{ or } x > 1\}$ and $B = \{ x \in \text{IR}: \ x \leq \leq 0\}$

. Find a. A^c **b.** $B - A$

Solution:

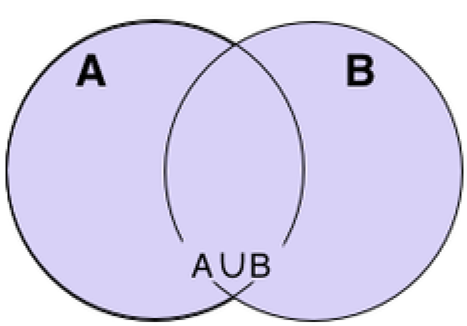
a. $A^c = [-1, 1]$. b. $B - A = [-1, 0]$

*Let A and B be two sets. The **union** of A and B is the set

$$A \cup B = \{x \neg x \in A \text{ or } x \in B\}$$

Where the 'or' is inclusive. This definition can be extended to more than two sets. More precisely, if $A_1, A_2, \ldots,$ are sets then

$$\cup_{n=1}^{\infty} A_n = \{x \neg x \in Ai \text{ for some } i\}$$



*Let A and B be two sets. The **intersection** of A and B is the set

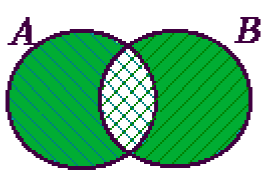
$$A\cap B=\{\,x.\ x\in A \text{ and } x\in B\}$$

If $A\cap B=\phi\,\emptyset$ we say that A and B are **disjoint** sets.

Note : The operation of union and intersection can be defined for three or more sets in the obvious manner.
In general if $A_1, A_2, \ldots A_n$ are subsets of $\cup$. then $A_1 \cup A_2 \cup \ldots \cup A_n$ will be denoted by $\cup_k^n A_k$ and $A_1 \cap A_2 \cap \ldots \cap A_n$ will be denoted by $\cap_k^n A_k$

Definition 8 .
The **symmetric difference** of A and B, denoted by $A\triangle B$, is the set containing those elements in either A or B but not both.

Exercise 12 :
Find $A\triangle B$ if $A=\{1, 3, 5\}$ and $B=\{1, 2, 3\}$
Solution. $A\triangle B=\{2,5\}$



The symmetric difference is equivalent to the union of both relative complements, that is $A\Delta B=(A-B)\cup(B-A),$

and it can also be expressed as the union of the two sets, minus their intersection:

$$A\Delta B=(A\cup B)-(A\cap B),$$

or with the XOR operation:

$A\Delta B = \{x : (x \in A) \text{ XOR } (x \in B)\}.$

The symmetric difference is commutative and associative:

$$A\Delta B = B\Delta A,$$
$$(A\Delta B)\Delta C = A\Delta(B\Delta C).$$

Thus, the repeated symmetric difference is an operation on a multiset of sets giving the set of elements which are in an odd number of sets.

The symmetric difference of two repeated symmetric differences is the repeated symmetric difference of the join of the two multi sets, where for

each double set both can be removed. In particular: $(A\Delta B)\Delta(B\Delta C) = A\Delta C.$

The empty set is neutral, and every set is its own inverse:

$A\Delta\varnothing = A,$
$A\Delta A = \varnothing.$

Given n sets $A_1, A_2, \ldots, A_n$ the **Cartesian product** of these sets is the set

$A_1 \times A_2 \times \ldots \times A_n = (a_1, a_2, \ldots, a_n): a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$"

Exercise 13:
Let $A = \{x, y\}, B = \{1, 2, 3\}$, and $C = \{a, b\}$. Find
a. $A \times B \times C$.
b. $(A \times B) \times c$

Solution:
a.
$A \times B \times C = \{(x, 1, a), (x, 2, a), (x, 3, a), (y, 1, a), (y, 2, a),$
$(y, 3, a), (x, 1, b), (x, 2, b), (x, 3, b), (y, 1, b), (y, 2, b), (y, 3, b)\}$

b.
$(A \times B) C \times =$
$\{((x, 1), a), ((x, 2), a), ((x, 3), a), ((y, 1), a), ((y, 2), a),$

$((y, 3), a), ((x, 1), b), ((x, 2), b), ((x, 3), b), ((y, 1), b)$
$((y, 2), b), ((y, 3), b)\}$

Definition 9 :
if A is a set, then the set of all subset of A is called the power set of A and denoted by p(A)= (2^k), where k is the number of elements Set A.

Exercise 14 :
Let A= {1,2,3), the power of A(p(A)) consists of the following subsets of A, P(A)= 2^3 = 8
P(A)= {ϕ, A, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}}

Review Problems:

Exercise 1:
Which of the following sets are equal?

a. $\{a, b, c, d\}$ b. $\{d, e, a, c\}$ c. $\{d, b, a, c\}$ d. $\{a, a, d, e, c, e\}$

Exercise2:

Let $A = \{c, d, f, g\}$,$B = \{f, j\}$, and $C = \{d, g\}$
Answer each of the following questions. Give reasons for your answers.

a. Is $B \subseteq A$?
b. Is $C \subseteq A$?
c. Is $C \subseteq C$?
d. Is C is a proper subset of A?

Exercise3:

a. Is $3 \in \{1, 2, 3\}$?
b. Is $1 \subseteq \{1\}$?
c. Is $\{2\} \in \{1, 2\}$?
d. Is $\{3\} \in \{1, \{2, \{3\}\}$?
e. Is $1 \in \{1\}$?
f. Is $\{2\} \subseteq \{1, \{2\}, \{3\}\}$?
g. Is $\{1\} \subseteq \{1, 2\}$?
h. Is $1 \in \{\{1\}, 2\}$?
i. Is $\{1\} \subseteq -1, \{2\}\}$?
j. Is $\{1\} \subseteq \{-1\}$?

Exercise4:

Let $A = \{b, c, d, f, g\}$and $B = \{a, b, c\}$ Find each of the following:

a. $A \cup B$
b. $A \cap B$
c. $A - B$
d. $B - A$

Exercise5:

Determine the following sets, i.e list their elements if they are nonempty, and write Φ if they are empty.

a\ $\{n \in N : n^2 = 9\}$
b\ $\{n \in Z : n^2 = 9\}$
c\ $\{n \in R : x^2 = 9\}$

d $\{n \in N : 3 < n < 7\}$

e $\{n \in Z : 3 < |n| < 7\}$

f $\{n \in R : x^2 < 0\}$

Exercise6:
Indicate which of the following relationships are true and which are false:

a. $Z^+ \subseteq Q$
b. $IR^- \subset Q$
c. $Q \subset Z$

d. $Z^+ \cup Z^- = Z$
e. $Q \cap IR = Q$
f. $IQ \cup Z = Z$
g. $Z^+ \cap IR = Z^+$
h. $Z \cup IQ = IQ$

Exercise 7:
Let $A =$
$\{x, y, z, w\}$ and $B = \{a, b\}$ List the elements of each of the following sets:
a. $A \times B$
b. $B \times A$
c. $A \times A$
d. $B \times B$

Exercise 8:

the veen diagram of this fig)shows sets A,B and C. Shad the following sets $A \cap B \cap C$, $A/(B \cup C)$, $A^c \cap (B \cup C)$, $A^c \cup B \cup C$, $(A \cup B)/C$, $(A^c \cap B)/C$.

Exercise 9:

prove the following $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

Exercise 10:

for U= {1,2,3,…,9,10}, Let A= {1,2,3,4,5}, B={1,2,3,4,8}, C={1,2,3,5,7} and D={2,4,6,8}. Determine each of the following: $\overline{C\cap D}$, $A\cup(B-C)$, $(B-C)-D$, $(A\cup B)-(C\cap D)$.

Exercise 11:

Determine all of the elements un each of the following

- **a- $\{1+(-1)^n \mid n \in N\}$**
- **b- $\{n+(1/n)^n (n \in\{1,2,3,5,7\}\}$**
- **c- $\{n^3+n^2 \mid n \in\{0,1,2,3,4\}\}$**

Exercise 12:

1/ Let U= { a, b, c, d, e, f, g, h, k}, A= {a, b, c, g}, B= {d,e,f,g}, C= { a,c,f}, D= { f, h, k}

compute AUB, BUC, A∩C, B∩D, A-B, $\overline{A}$, $A\oplus B$, $A\oplus C$, **A∩BUC, A∩B∩C,** $\overline{A\cup B}$, $\overline{A\cap B}$

2/ Let U= { 1,2,3,4,5,6,7,8,9}, A={1,2,4,6,8}, B={2,4,5,9}
C= {χ : χ is a positive integer and $\chi^2 \leq 16$}
D= { 7,8}
Compute: $\overline{A\cup B}$, $\overline{A\cap B}$, **A∩(B∩C), (AUB)UD** $\overline{A}$ **UA, A∩(**$\overline{C}$ **UD)**

2.2 Properties of Sets:

Theorem 3 :

- $A\cup B=B\cup A$

  $A\cap B=B\cap A$ **(Commutative properties)**

-

$\mathrm{A\,Y(B\,Y}\,C) = (\mathrm{A\,Y\,B})\mathrm{Y}\,C$

$\mathrm{A\,I\,(B\,I}\,C) = (\mathrm{A\,I\,B})\mathrm{I}\,C$ **(Associative properties)**

- $\mathrm{A\,I\,(B\,Y}\,C) = (\mathrm{A\,I\,B})\mathrm{Y}(\mathrm{A\,I}\,C)$

  $\mathrm{A\,Y(B\,I}\,C) = (\mathrm{A\,Y\,B})\mathrm{I}\,(C\,\mathrm{I\,A})$ **(Distributive properties)**

- $\begin{matrix}\mathrm{A\,Y\,A = A} \\ \mathrm{A\,I\,A = A}\end{matrix}$ **(Idempotent properties)**

- Properties of complement

1) $(\overline{\overline{\mathrm{A}}}) = \mathrm{A}$

2) $\mathrm{A} \cup \overline{\mathrm{A}} = \mathrm{Y}$

3) $\mathrm{A\,I}\,\overline{\mathrm{A}} = \Phi$

4) $\Phi = \mathrm{Y}$

5) $\overline{\mathrm{Y}} = \Phi$

6) $\overline{\mathrm{A\,Y\,B}} = \overline{\mathrm{A}}\,\mathrm{I}\,\overline{\mathrm{B}}$

7) $\overline{\mathrm{A\,I\,B}} = \overline{\mathrm{A}}\,\mathrm{Y}\,\overline{\mathrm{B}}$

-properties of universal set

1) $\mathrm{A} \cup \phi = \mathrm{A}$

2) $\mathrm{A} \cap \phi = \phi$

properties of the Null set.

1) $\mathrm{A} \cup \phi = \mathrm{A}$

2) $\mathrm{A} \cap \phi = \mathrm{A}$

The following exercise shows that the operation $\subseteq$ is reflexive and transitive, concepts that will be discussed in the next chapter.

Exercise 15 :

a. Suppose that A, B, C are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$

b. Find two sets A and B such that $A \in B$ and $A \subseteq B$

c. Show that $A \subseteq A$

Solution :

a. We need to show that every element of A is an element of C. Let $x \in A$.
Since $A \subseteq B$ then $x \in B$. But $B \subseteq C$ so that $x \in C$.
b. $A = \{x\}$ and $B = \{x, \{x\}\}$.
c. The proposition if $x \in A$ then $x \in A$ is always true. Thus, $A \subseteq A$

Theorem 4 : *Let A and B be two sets. Then*
a. $A \cap B \subseteq A$ *and* $A \cap B \subseteq B$
b. $A \subseteq A \cup B$ *and* $B \subseteq A \cup B$

Proof :
a. If $x \in A \cap B$ then $x \in A$ and $x \in B$. This still imply that $x \in A$ Hence, $A \cap B \subseteq A$. A similar argument holds for $A \cap B \subseteq B$
b. The proposition "if $x \in A$ then $x \in A \cup B$" is always true. Hence, $A \subseteq A \cup B$
A similar argument holds for $B \subseteq A \cup B$

Theorem 5 : *Let A be a subset of a universal set U. Then*
a. $\phi^c = U$
b. $U^c = \phi$
c. $(A^c)^c = A$
d. $A \cup A^c = U$
e. $A \cap A^c = \phi$

Theorem 6 :
If A and B are subsets of U then
a. $A \cup U = U$
b. $A \cup A = A$
c. $A \cup \phi = A$
d. $A \cup B = B \cup A$
e. $(A \cup B) \cup C = A \cup (B \cup C)$

Proof :

a. Clearly, $A \cup U \subseteq U$. Conversely, let $x \in U$. Then definitely, $x \in A \cup U$ That is, $U \subseteq A \cup U$.

b. If $x \in A$ then $x \in A$ or $x \in A$. That is, $x \in A \cup A$ and consequently $A \subseteq \leftharpoonup A \cup A \leftharpoondown$
Conversely, if $x \in \leftharpoonup A \cup A$)then $x \in A$. Hence,($A \cup A \Rightarrow \subseteq A$

c. If $x \in \leftharpoonup A \cup \phi \Rightarrow \emptyset$ then $x \in A$ since $x \notin \phi$ Thus,($A \cup \phi$) $\subseteq A$

Conversely, if $x \in A$ *then* $x \in A$ or $x \in \phi$ Hence, $A \subseteq \Leftarrow A \cup \phi \leftharpoondown \triangleright$

d. If $x \in A \cup B$ then $x \in A$ or $x \in B$ But this is the same thing as saying $x \in B$ or $x \in A$ That is, $x \in \leftharpoonup B \cup A \leftharpoondown \triangleright$ Now interchange the roles of A and B to show that $B \cup A \subseteq A \cup B \triangleright$

e. Let $x \in (A \cup B) \cup C$ Then $x \in (A \cup B)$ or $x \in C$. Thus, $(x \in A$ or $x \in B)$ or $x \in C$ This implies $x \in A$ or $(x \in B$ or $x \in C)$. Hence, $x \in A \ (B \cup C) \triangleright$ The converse is similar.

<u>Theorem 7 :</u> (H.W):

Let A and B be subsets of U. Then
a. $A \cap U = A$
b. $A \cap A = A$
c. $A \cap \phi = \phi$
d. $A \cap B = B \cap A$
e. $(A \cap B) \cap C = A \cap (B \cap C)$

<u>Theorem 8 :</u> *Suppose that $A \subseteq B$. Then*
a. $A \cap B = A$
b. $A \cup B = B$

Proof:

a. If $x \in A \cap B$ then by the definition of intersection of two sets we have $x \in A$ Hence, $A \cap B \subseteq A$ Conversely, if $x \in A$ then $x \in B$ as well since $A \subseteq B$ Hence, $x \in A \cap B$ This shows that $A \subseteq A \cap B$
b. If $x \in A \cup B$ then $x \in A$ or $x \in B$. Since $A \subseteq B$ then $x \in B$ Hence,
$A \cup B \subseteq B$ Conversely, if $x \in B$ then $x \in A \cup B$ This shows that $B \subseteq A \cup B$.

Theorem 9 : from any universe υ, and any set A,B$\subseteq \upsilon$ the following statement are equivalent:

$A \subseteq B, A \cup B = B, A \cap B = A, \overline{B} \subseteq \overline{A}$

Proof :

we proof a$\Rightarrow$b and b$\Rightarrow$c and left c$\Rightarrow$d and d$\Rightarrow$a as exercise.

(i) **a$\Rightarrow$b, if A,B are any set, then B $\subseteq$ A$\cup$B.**

for the opposite inclusion.

if $\chi \in A \cup B$, then $\chi \in A$ or $\chi \in B$, but since $A \subseteq B$ in either case we have $\chi \in B$.

so, A$\cup$B$\subseteq$ B and,

since we now have both in collusions, it follows A$\cup$B=B.

(ii) **b$\Rightarrow$c, given sets, A,B we always have $A \cap B \subseteq A$ let YЄA with A$\cup$B=B, YЄA$\Rightarrow$ YЄA $\cup$B$\Rightarrow$ YЄB (since A$\cup$B=B) $\Rightarrow$ YЄA⋂B.**

so, $A \subseteq A \cap B$.

and we conclude that A=A⋂B.

Exercise 16 :

Express $\overline{A - B} = \overline{A \cap \overline{B}}$

$= \overline{A} \cup \overline{\overline{B}}$ **Demoragan's law**

$= \overline{A} \cup B$ **Low of Double Complement**

Q: Proof that $(AUB)^c = A^c \cap B^c$

Solution:

- We first show that $(AUB)^c \subseteq A^c \cap B^c$

if $\chi \in (AUB)^c$, then $\chi \notin AUB$
thus, $\chi \notin A$ and $\chi \notin B$,
and so $\chi \notin A^c$ and $\chi \notin B^c$
Hence $\chi \in A^c \cap B^c$ $\Rightarrow (A\,Y\,B)^c C A^c\ I\ B^c$
- Next we show that $A^c\ I\ B^c \subset (A\,Y\,B)^c$
Let $\chi \in A^c \cap B^c$, then $\chi \in A^c$ and $\chi \in B^c$
So $\chi \notin A$ and $\chi \notin B$
Hence $\chi \notin AUB$
So $\chi \in (AUB)^c \Rightarrow A^c \cap B^c \subseteq (AUB)^c$
$\therefore (AUB)^c = A^c \cap B^c$

Q) prove that the following are equivalent:
$A \subset B$, $A \cap B = A$, $A \cup B = B$

proof:

Suppose $A \subset B$ **and let X∈A.**
Then X∈B, hence X∈A∩B,
and $A \subset A \cap B$, **By part (i)** $A \cup B \subset A$ **therefore** $A \cap B = A$.
on other hand, suppose $A \cap B = A$ **and let X∈A.**
Then X∈(A∩B), hence X∈A and X∈B therefore, $A \subset B$.
Both results show that $A \subset B$ **is equivalent to** $A \cap B = A$.
Suppose again that $A \subset B$ **and let X∈(A∩B)**
Then X∈A, or X∈B.
If X∈A, then X∈B, because $A \subset B$.
In either case X∈B, Therefore$(A \cup B) \subset B$.
By part (i), $B \subset (A \cup B)$.
Therefore $A \cup B = B$.
Now suppose $A \cup B = B$ **and let X∈A, then** $\chi \in A \cup B$ **by definition of union of sets.**
hence $\chi \in B = A \cup B$, **therefore** $A \subset B$.
Both results show that $A \subset B$ **is equivalent to** $A \cup B = B$.
Thus, $A \subset B$, $A \cap B = A$ **and** $A \cup B = B$ **are equivalent**

Chapter two

BY. MAKARIM A.

Q: Use the laws in operations of set to prove (U ∩ A) U (B ∩ A) = A

Solution:

(U ∩ A) U (B ∩ A) = (A∩U) U (A∩B)
= A ∩ (U Y B)
= A ∩ (B Y U)
= A ∩ U
= A

Q: Prove (If A and B are finite then $\eta(A \cup B) = \eta(A) + \eta(B) - \eta(A \cap B)$**.**

Proof: clearly $A \cup B$ **and** $A \cap B$ **are finite if A and B finite.**

$A \cup B = B \cup \left(A/B\right)$ **and** B **and** A/B **are disjoint.**

So, $\eta(A \cup B) = \eta(B) + \eta\left(A/B\right)$**.**

Also, $A = \left(A/B\right) \cup (A \cap B)$ **and** A/B **and** $A \cap B$ **are disjoint.**

So, $\eta(A) = \eta\left(A/B\right) + \eta(A \cap B)$

Or $\eta\left(A/B\right) = \eta(A) - \eta(A \cap B)$

By substituting for $\eta\left(A/B\right)$ **we get** $\eta(A \cup B) = \eta(B) + \eta(A) - \eta(A \cap B)$**.**

Review Problems:

Exercise 1:
Let $A,B,$ and C be sets. Prove that if $A \subseteq B$ then $A \cap C \subseteq B \cap C$

Exercise 2:
Find sets $A,B,$ and C such that $A \cap C = B \cap C$ but $A \neq B$

Exercise 3:
Find sets $A,B,$ and C such that $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$ but $A \neq B$

Exercise 4:
Let A and B be two sets. Prove that if $A \subseteq B$ then $B^C \subseteq A^C$

Exercise 5:
Let $A,B,$ and C be sets. Prove that if $A \subseteq C$ and B $C \subseteq$ then $A \cup B \subseteq C$

Exercise 6:
Let $A, B,$ and C be sets. Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Exercise 7:
Let $A, B,$ and C be sets. Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Exercises 8:

(1) prove that a: $A \subset B$ **if** $A \cap B^c = \Phi$
b: $A \subset B$ **if** $A^c \cup B = \cup$

(2) Simplify each of the following:
1- $\overline{A} \, Y \, \overline{B} \, Y (A \cap B \, I \, \overline{C})$
2- $(A - B) \cup (A \, I \, B)$

2.3 Sequence:

Definition 10 :
A sequence is simply a list of objects, one after another and numbered in natural increasing order by the positive integers.
*** The list may be finite, that is, it may stop after a certain number of items.**
*** The list may be infinite**

Exercise 17:

The list 1,4,9,16,25,…..n^2 is sequence of positive square integers, The first elements in this sequence is 1, the second is 4.

***In this example all the items listed are distinct; this need not be true for every sequence.**

Exercise 18 :
The sequence 1,0, 0, 1,0, 1, 0, 1, 2, 4. is a finite sequence with repeated items.

*** A general sequence. that is , one where we do not specify the entries, can be written as $a_1, a_2, a_3, \ldots\ldots a_n$ or sometimes as a_i, $1 \leq i < \infty$ where the sequence is finite, we may write it as $(a_i)\ 1,< i < n$.**

Definition 11 :

The set corresponding to a sequence is simply the set of all distinct elements in the sequence.

Note, that an essential feature of a sequence is the order in which the elements are listed in the sequence.

Exercise 19 :

Let a, b, a, b, a, b,……..an infinite sequence, Then the set corresponding to this sequence is simply {a, b}.

***The idea of a sequence is also important in computer science, where a sequence is some time called a linear array.**

Exercise 20 :

If we have the sequence $S_1, S_2, \ldots.$

we can represent this sequence as boxes Arrays S

S_1	S_2	S_3	.	.	.	.	S_n

and the sequence S(1), S(2), S(3), …..will be called the sequence of values of the arrays S

Exercises (H.W)

1) give the set corresponding to the sequence 2,, 1, 2, 1, 2, 1, 2, 1, 1, 3, 5, 7, 9, 11, 13

2) write out first four, beginning with n=1 terms of the squares whose general terms is given 2^n, $3n^2-2n-6$

3) write a formula for the nth term of the sequence 1,3, 5, 7, …..,

1, 4, 7, 10, 13, 16

1, 1, 1, 1, 1, 1, 1, ……

1, 1, 2, 6, 24

2.3.1 Computer Representation of sets and Subsets:

To represent a set in a computer, the elements of the set must be arranged in a sequence, when a universal set U is finite, Say U={x_1, x_2, …x_n} and A is a subset of U.
Then, the characteristic function f_A assigns 1 to elements X that belongs to A and O to elements X that does not belongs to A.
Thus, f_A can be represented by a sequence of 0's and 1's of length n.
Any set with n element can be arranged in a sequence of length n, So each of its subsets corresponds to a sequence of Zeros and ones of length n. representing the characteristic function of that subset.
This fact allows us to represent a universal set in a computer as an array A of length n assignment of a zero or one teach location A [n] of the array specifies a unique subset of U.

Exercise 21 :

Let U = { a, b, c, g, g, r, s, w}, the array length 8
Since A[k] = 1 for $1 \le \kappa \le 8$
If A= { a, c, r, w} then

$$\mathbf{A}[\kappa] = \begin{matrix} 1 if \kappa = 1,3,6,8 \\ 0 if k = 2,4,5,6,7 \end{matrix}$$

1	**0**	**1**	**0**	**0**	**1**	**0**	**1**

2.3.2 Characteristic Function

Definition 12 :
If A is a subset of a universal set U, the characteristic function f_A of A is defined as follows:

$$f_A(X) = \begin{cases} 1 if \chi \in A \\ 0 if \chi \notin A \end{cases}$$

Note,
We may add and multiply characteristic function, since their values are numbers.

Theorem 1 0 : :

f_A of subsets satisfy the following properties

(a) $f_{A\cap B} = f_A \cap f_B$ **: that is,** $f_{A\cap B(\chi)} = f_{A(\chi)} f_{B(\chi)}$ **for all** χ

(b) $f_{A\cup B} = f_A + f_B - f_A f_B$ **: That is,** $f_{A\cup B}{}^{(\chi)} = f_A{}^{(\chi)} + f_A{}^{(B)} - f_A{}^{(\chi)} f_B{}^{(\chi)}$ $\nu\chi$

(c) $f_{A\oplus B} = f_A + f_B - 2 f_A f_B$ **: That is** $f_{A\oplus B}{}^{(\chi)} = f_A^{(\chi)} + f_B^{(\chi)} - 2 f_A^{(\varphi)} f_B^{(\chi)}$ **for all**

Exercise 22 :

Let U={1, 2, 3, 4, 5, 6, }

A={1, 2, 4}

B={1`, 2, 3, 5}

find $f_{A\cap B}$, $f_{A\cup B}$, $f_{A\oplus B}$

solution:

f_A **= 1, 1, 0, 1, 0, 0**

f_B **= 1, 1, 1, 0, 1, 0**

$f_A + f_B$ **=2, 2, 1, 1, 1, ,0**

$f_{A\cap B}$ **=** $f_A * f_B$ **=1, 1, 1, 1, 1, 0**

$f_{A\cup B} = f_A + f_B - f_A * f_B$ **=1, 1, 1, 1, 1, 0**

$f_{A\oplus B} = f_A + f_B - 2 f_{A1} * f_B$ **= 2, 2, 1, 1, 1, 0 – (2, 2, 0, 0, 0, 0)**

=0, 0, 1, 1, 1, 0

2.4 Counting Sequences and Subset:

If two independent tasks (T_1) and (T_2). are to be performed in Sequence, and if (T_1) can be performed in (n_1) ways and (T2) in (n2) ways, the Sequence (T_1 T_2) can be performed in (n_1 n_2) ways.
In general if independent tasks (T_1, T_2, ……T_k) are to be performed in sequence and T1 can be performed in n1 ways , T_2 in n_2 ways,….T_k in n_k ways, then the sequence (T_1, T_2, ……..T_k) can be performed in exactly n1 n_2 …..n_k ways.

Exercise 23 :

Let S be a set with n elements. How many subsets does S have

Solution:

We know that each subset of S is determined by its characteristic function, and if S has n element, this function may be described as an array of 0's and 1's having length n.
The first element of the array can be filled in two ways (with a0 or a1), and this is true for all succeeding element as well, Thus, by the extended multiplication principle there are 2.2.2…….2=2^n ways of filling in array and therefore 2^n subset of S.

2.4.1 Finite sets, counting principle:

A set is said to be finite if it contains exactly m distinct elements here m denotes some nonnegative integer. other wise, a set is said to be infinite. for example, the empty set Φ and the set of letters of the English alphabet are finite hat, the set of even positive integers {2, 4, 6, …} is infinite. If a set A is finite, we let n (A) denote the number of elements of A.

If A and B are disjoint, therefore n(AUB)= n(A) + n(B).
we also have a formula for n(AUB) even when they are not disjoint

<u>Theorem 11 :</u>
If A and B are finite sets, then AUB and A∩B are finite and
n(AUB) = n(A) + n(B) – n(A∩B).

corollary:
If A and B and C are finite sets, then so is AUBUC, and n(AUBUC) = n(A) + n(B) + n(C) – n(A∩B) –n(A∩C) – n(B∩C) +n(A∩B∩C)

Exercise 24 :

Suppose that 100 of the 120 mathematics students at a college take at least one of the languages, French, German, and Russian. Also suppose
65 study French
45 = German
24 study Russian
20 = French and German
25 = French and Russian
15 = German and Russian
Find the number of students who study all three languages

Solution:
n(RUGUR) = n(F) + n(G) + n(R) – n(F∩G) – n(G∩R) + n(F∩G∩R)
n(FUGUR) = 100, because 100 of students study at lest one of languages.
100 = 65 + 45 +42 – 20- 25 – 15 + n(F∩G∩R)
so, n(F∩G∩R) = 8. Students study all three languages.
20-8 = 12 study F and G but not R
25-8 = 17 study F and R but not G
15-8 = 7 study G and R but not F
65-12-8-17 = 28 study F only
45-12-8-7 = 18 study G only
42-17-8-7 = 10 study R only
120-100 20 do not study any of the languages

Exercise 25 :

A survey is taken on methods of commuter travel each respondent is asked to check BUS, TRAIN, or AUTOMOBILE. as a major method of traveling to work move than one answer is permitted. The result reported were as follows
30 people checked BUS
35 = = TRAIN
100 = = AUTOMOBILE
15 = = BUS & TRAIN
15 = = = = AUTOMOBILE
20 = = TRW & =
5 = = all the methods
How many respondents completed their surveys?

Solution: Let A, B, and C be the set of people who checked BUS, TRW and AUTOMOBIL, respectively we know that:

$n(A) = 30$, $n(B) = 35$, $n(C) = 100$, $n(A\cap B) = 15$, $n(A\cap C) = 15$
$n(B\cap C) = 20$ and $n(A\cap B\cap C) = 5$
The total number of people responding is then
$n(AUBUC) = n(A) + n(B) + n(C) - n(A\cap C) - n(A\cap B) - n(B\cap C) + n(A\cap B\cap C) = (30 + 35 + 100) - (15 + 15 + 20) + 5 = 120$

Exercise (H.W):
4/ A survey of 500 television watchers produced the following information.
285watch football games
195 = basketball games
115 = hockey games
50 = hockey and basketball
45 = football and basketball
70 = football and hockey
50 do not watch any of three games
a/ How many people in the survey watch all three games
b/ How many people in the survey watch exactly one of the three games

2.4.2 Sigma notation:

Now , we need to introduce a concise way of writing the sum of a list of n+1 terms like
$a_m, a_{m+1}, a_{m+2}, \ldots.. a_{m+n}$, where m and n are integers and $n \geq 0$. This notation is called:- Sigma notation because it involves the capital Greek letters ∑, we use it to represent a summation by writing.

$$a_m, a_{m+1}, a_{m+2}, \ldots.. a_{m+n} = \sum_{i=1}^{m+n} ai$$

Here the letter i is called the index of the summation and this index account for all integers starting with the lower limit m and contriving on up to the upper limit m+n.

we may use this notation as follows.

$$\sum_{i=3}^{7} ai = a3 + a4 + a5 + a6 + a7$$

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

$$\sum_{i=11}^{100} i^3 = 11^3 + 12^3 + 13^3 + ... + 100^3 = \sum_{j=12}^{101} (j-1)^3 = \sum_{k=10}^{99} (k+1)^3$$

$$\sum_{i=3}^{3} ai == \sum_{i=4}^{4} a_{i-1} = \sum_{i=2}^{2} a_{i+1}$$

$$\sum_{i=1}^{5} a = a + a + a + a + a = 5a$$

<u>Theorem 12</u> :

(The Binomial Theorem) If X and Y are variable and n is appositive integers, then:

$$(X+Y)^n = \binom{n}{o} X^0 Y^n + \binom{n}{1} X^1 Y^{n-1} + \binom{n}{2} X^2 Y^{n-2} + ... + \binom{n}{n-1} X^{n-1} Y^1 + \binom{n}{n} X^n Y^0 = \sum_{k=0}^{n} \binom{n}{k} X^k Y^{n-k}$$

Exercise 26 :

1) from the binomial theorem it flows that the Coefficient of X^5Y^2 in the expansion of

$$(x+y)^7 is \binom{7}{5} = \binom{7}{2} = 21$$

2) to obtain the coefficient of a^5b^2 in the expansion of $(2a - 3b)^7$, replace 2a by x and -3b by y.

from the binomial theorem the coefficient of X^5Y^2 in $(X+Y)^7$ is $\binom{7}{5}$, and

$$\binom{7}{5} X^5 Y^2 = \binom{7}{5} (2a)^5 (-3b)^2 = \binom{7}{5} (2)^5 (-3)^2 a^5 b^2 = 6048 a^5 b^2$$

3) we need to know the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$.

If we replace a→u, 2b→w, -3c→x, 2d→y and 5→z, then we can apply the multinomial thermo $(n1,n2,n3,…,n_t)$ to $(u+w+x+y+z)^{16}$ and determine the coefficient of $u^2w^3x^2y^5z^4$ as (2,3,2,5,4)= 302,702,400.

But $(2,3,2,5,4)^{16}$ (a^2) $(2b)^3$ $(-3c)^2$ $(2d)^5$ $(5)^4=(2,3,2,5,4)^{16}(1)^2(2)^2(-3)^2(2)^5(5)^4(a^2b^3c^2d^5)$ = $435,891,455,000,000a^2b^3c^2d^5$

ملاحظة قمنا بفتح $(2,3,4,5,4)^{16}$ حسب القانون التالي:

$$=\frac{n!}{n1!n2!n3!\ldots n_t!}=\binom{n}{n1,n2,n3,\ldots,n_t}$$

Exercise: (H.W)

1- **determine the value of each of the following summation**

$\sum_{i=1}^{6}(i^2+1)$, $\sum_{i=0}^{10}[1+(-1)^i]$, $\sum_{k=n}^{2n}(-1)^k$ **where n is odd positive integer**

2- **Express each of the following using (Sigma notation)**

1) $\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots+\frac{1}{n!}, n\geq 2$

2)1+4+9+16+25+36+49

3) $\frac{1}{n}+\frac{2}{n+1}+\frac{3}{n+2}\ldots+\frac{n+1}{2n}$ **where n is positive integer**

3- **Determine the coefficient of X^9Y^3 in the expansions of $(x+y)^{12}$, $(X+2Y)^{12}$**

4-Determine the coefficient of (XYZ^2) in $(X+Y+Z)^4$ and (XYZ^2) in $(2X-Y-Z)^4$

Matrix and operation

3-1 Definitions and properties

Definition 1 :

a matrix is a rectangular array of numbers arranged in m horizontal Rows and n vertical columns.

$$\mathbf{A=}\begin{bmatrix} a_{11} & a_{12} & , & ., & a_{1n} \\ a_{21} & a_{22} & , & , & , \\ , & , & a_{33} & , & , \\ , & , & , & , & , \\ a_{m1} & a_{m2} & , & , & a_{mn} \end{bmatrix}$$

The ith row of A is $\begin{bmatrix} a_{i1} & a_{i2} & , & , & a_{in} \end{bmatrix}$ **,** $1 \le i \le m$

The ith column of A is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ , \\ , \\ a_{mj} \end{bmatrix}$ **,** $1 \le j \le n$

We can write the matrix A as [a_{ij}]

Definition 2:

***Size of matrix* : the dimension (size) of A is (m $\times$ n) read (m by n) , ***
if m=n we say that A is a *square matrix* , of order n .
*** if $a_{ij} = 0$ for $i \neq j$ we say that A is *diagonal matrix* .**
*** the elements a_{11} , a_{22} , a_{33}, …., a_{nn} form the main diagonal of A .**

Exercise 1 :

the following matrix is diagonal matrix,

$$\mathbf{A}=\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \quad , \quad \mathbf{A}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}_{3\times 3}$$

Definition 3:

A matrix all of whose entries are zero is called a zero matrix, for example $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Theorem 1:

a: A+B = B+A

b: A+0 = 0+A=A

c (A+B)+C=A+(B+C)

Definition 4:

- **if the diagonal , elements are equal 1 , then the matrix is called**
- **identity *matrix* of order n ,** $\mathbf{I}_n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.
- ***a Boolean matrix* is an $m \times n$ whose elements are either ; zero or one**

$$\mathbf{A}=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad , \quad \mathbf{B}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 2 :

let $A = \begin{bmatrix} 3 & -2 & 5 \\ 4 & 1 & 2 \end{bmatrix}$ **,** $\quad$ **B=** $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ $\quad$ **, C=** $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & -1 \\ 2 & 0 & 3 \end{bmatrix}$

The value of $a_{12}= -2,\ a_{22}= 1,\quad a_{23} = 2$ **,**
The value of $b_{11}= 3,\quad b_{31}= 4$ **,**
The value of $c_{11}= 2,\ c_{13} = 4,\ c_{33}= 3$ **.**

Definition 5:

Two matrix A= [a_{ij} **] , B = [** b_{ij} **] are said to be *equal* if** $a_{ij} = b_{ij}$

Exercise 3:

if A= $\begin{bmatrix} 2 & -3 & -1 \\ 0 & 5 & 2 \\ 4 & -4 & 6 \end{bmatrix}$ and B= $\begin{bmatrix} 2 & y & -1 \\ x & 5 & 2 \\ z & -4 & 6 \end{bmatrix}$ find x, y, z

solution:

x=0, y=3, z=4

Exercise 4 :

let $\quad$ **A =** $\begin{bmatrix} 3 & -2 & 5 \\ 4 & 1 & 2 \end{bmatrix}$ **, B =** $\begin{bmatrix} x & -2 & 5 \\ 4 & y & z \end{bmatrix}$ **,**

Solution:

X= 3 , y= 1 , z= 2

3-2 operation of matrices :

Definition 6 :

***Sum of matrix* : the sum of two matrices A= [** a_{ij} **] , B = [** b_{ij} **] is the matrix C=[** c_{ij} **] , such that** $c_{ij} = a_{ij} + b_{ij}$,

Exercise 5:

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & -1 \\ 2 & 0 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & -2 & 4 \\ 5 & 1 & -1 \\ 2 & 0 & 6 \end{bmatrix}, \quad \mathbf{C} = \mathbf{A+B} = \begin{bmatrix} 0 & 1 & 8 \\ 10 & 7 & -2 \\ 4 & 0 & 9 \end{bmatrix}$$

Exercise 6 :

Let $A = \begin{bmatrix} 3 & 4 & -1 \\ 5 & 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 & 1 \\ 0 & 2 & 3 \end{bmatrix}$ then

$$A+B = \begin{bmatrix} 3+4 & 4+5 & -1+1 \\ 5+0 & 0+2 & -1+3 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 0 \\ 5 & 2 & 1 \end{bmatrix}$$

Theorem 2:

if A , B , c are three matrices , and O is zero matrix , then

(1) A+B=B+A

(2) A+(B+C)=(A+B)+C

(3) A(BC)=(AB)C

(4) A(B+C)=AB+AC

(5) (B+C)A=BA+CA

(6) A(B-C)=AB-AC

(7) (B-C)A=BA-CA

(8) a(B+C)=aB+aC

(9) a(B-C)=aB-aC

(10) (a+b)C=aC+bC

(11) (a-b)C=aC-bC

(12) (ab)C=a(bC)

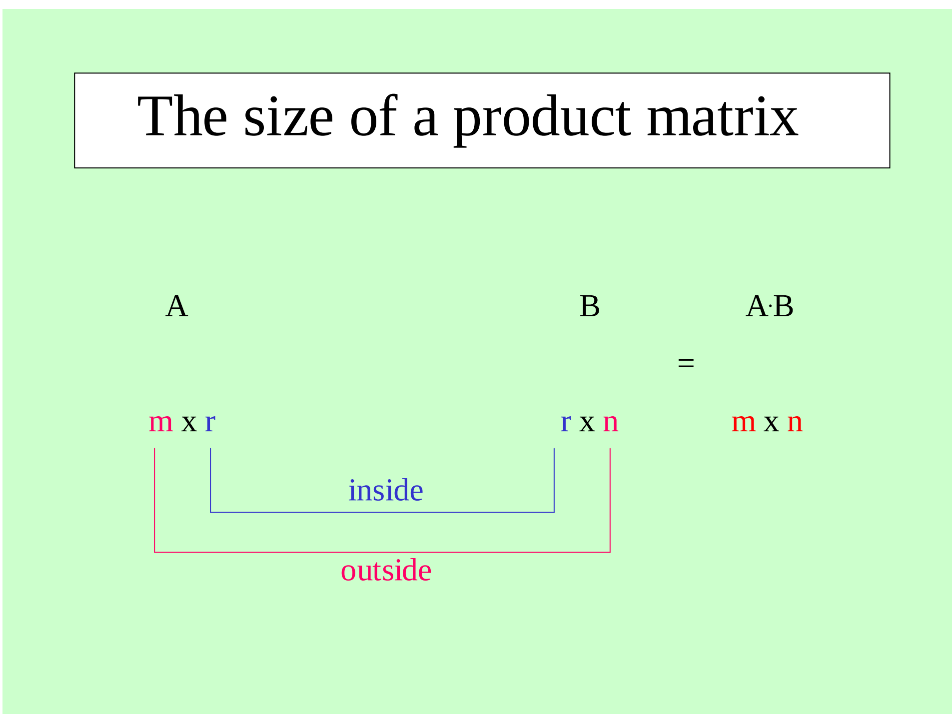
(13) a(BC)=(aB)C=B(aC)

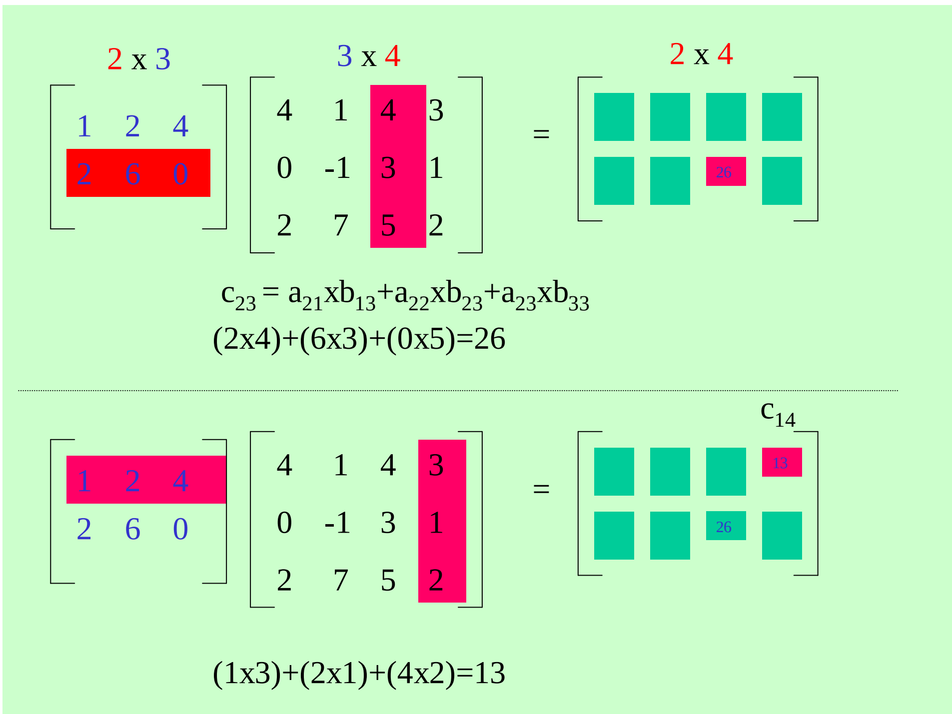
Definition 7:

***Matrix multiplication*: the product of A and B is the matrix**
$C = [\, c_{ij}]$, such that , $C_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{ip} b_{pj}$

$$\mathbf{A}_{m\times p} \cdot \mathbf{B}_{P\times n} = \mathbf{C}_{m\times n},$$

$$\begin{bmatrix} a_{11} & a_{12} & , & , & a_{1p} \\ a_{21} & a_{22} & , & , & a_{2p} \\ , & , & , & , & , \\ , & , & , & , & , \\ ai1 & a_{i2} & , & , & a_{ip} \\ , & , & , & , & , \\ a_{m1} & a_{m2} & , & , & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & , & b_{ij} \\ b_{21} & b_{22} & , & b_{2j} \\ , & , & , & , \\ , & , & , & , \\ , & b_{p2} & , & , \\ , & , & , & , \\ b_{p1} & b_{p2} & , & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & , & c_{1n} \\ c_{21} & c_{22} & , & c_{2n} \\ , & , & , & , \\ , & , & , & , \\ , & c_{i2} & , & , \\ , & , & , & , \\ c_{m1} & c_{m2} & , & c_{mn} \end{bmatrix}$$





Exercise 7 :

let $\mathbf{A}=\underset{\mathbf{2\times 3}}{\begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \end{bmatrix}}$, $\mathbf{B}=\underset{\mathbf{3\times 2}}{\begin{bmatrix} 3 & 1 \\ -2 & 2 \\ 5 & -3 \end{bmatrix}}$

$$\mathbf{C= A\times B} = \begin{bmatrix} 2(3)+3(-2)+(-4)(5) & 2(1)+3(2)+(-4)(-3) \\ 1(3)+2(-2)+3(5) & 1(1)+2(2)+3(-3) \end{bmatrix}$$

$$= \underset{\mathbf{2\times 2}}{\begin{bmatrix} -20 & 20 \\ 14 & -4 \end{bmatrix}}$$

Rules of Matrix Arithmetic

$A=\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$ $B=\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

Multiply A·B: Multiply B·A

$A\cdot B=\begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$ $B\cdot A=\begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$

Therefore: $A\cdot B \neq B\cdot A$

Theorem 3 :

if A , B and C are three matrices , then ,

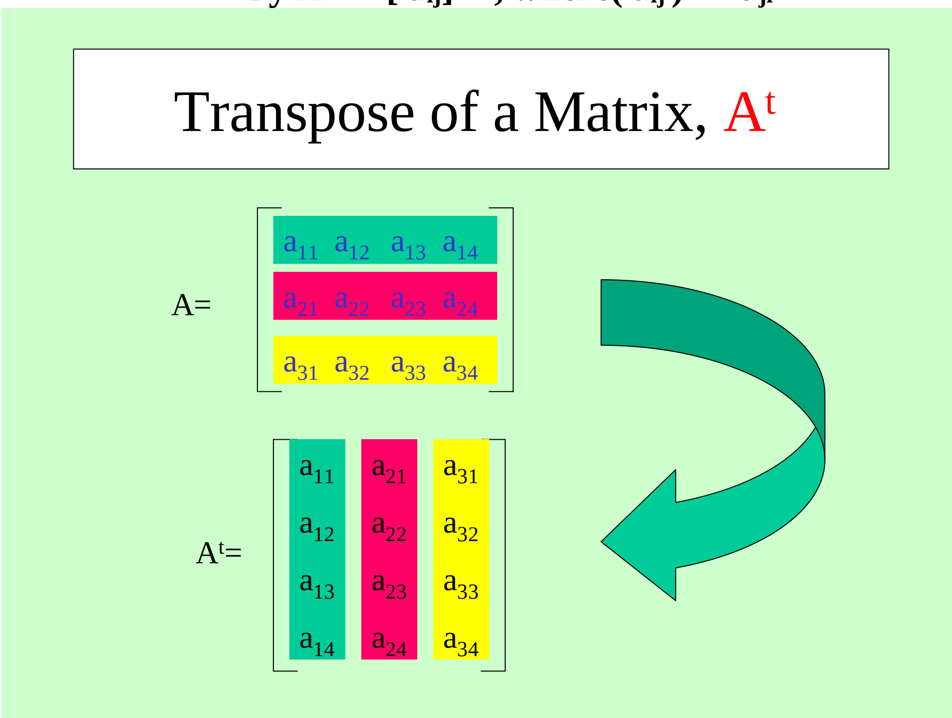
- $A\times(B\times C)=(A\times B)\times C$
- $A\times(B+C)=(A\times B)+(A\times C)$
- $(A+B)\times C=(A\times C)+(B\times C)$

Remark: **if A is an $n\times n$ matrix and p , q are a positive integer s , we define**

- $A^p= A.A.A\ldots\ldots A$, and $A^0 = I_n$
- $A^P\times A^q=A^{p+q}$
- $(A^P)^q=A^{pq}$
- $(AB)^p=A^pB^p$

Definition 8:

if A= [a_{ij}] is matrix of size m $\times$ n , we define the *transpose* of A By A^T = [$a_{ij}]^T$, where(a_{ij} $)^T$ = a_{ji}



i.e the transpose of A is obtained by interchanging the rows and columns of A .

Exercise 8:

let $A=\begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \end{bmatrix}_{2\times 3}$, $A^T=\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -4 & 3 \end{bmatrix}_{3\times 2}$

Definition9:
a matrix A is called symmetric if $A^T = A$, and this matrix must be a square , $a_{ij} = a_{ji}$

Exercise 9:

if A = $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 5 \\ -3 & 5 & 6 \end{bmatrix}$ **, and B=** $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

, then A is symmetric , since $a_{12} = a_{21} = 2$ …. $a_{ij} = a_{ji}$

But , B is not symmetric , since $a_{13} \neq a_{31}$

Theorem 4 :
***($A^T)^T = A$**
*** $(A+B)^T = A^T + B^T$**
*** $(AB)^T = B^T A^T$**
*** $(rA)^T = r A^T$, where r is constant .**

Review problem :

Exercise 1:

: let A= $\begin{bmatrix} 3 & -2 & 5 \\ 4 & 1 & 2 \end{bmatrix}$, B = $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$, C= $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & -1 \\ 2 & 0 & 3 \end{bmatrix}$

(a) what is a_{12}, a_{22}, a_{23}?
(b) what is b_{11}, b_{31}?
(c) what is c_{13}, c_{23}, c_{33}?
(d) list the elements of the main diagonal?

Exercise 2:

which of the following are diagonal matrices ?

A = $\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$ **, B=** $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ **, C=** $\begin{bmatrix} 2 & 6 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Exercise 3:

$\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix}$ **, find a, b, c and , d**

Exercise 4:

let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & -2 \end{bmatrix}$ **,** $B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}$ **,** $C = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 5 \\ 3 & 1 & 2 \end{bmatrix}$**,** $E = \begin{bmatrix} 3 & 1 & -2 \\ 5 & 4 & -3 \\ 0 & 1 & 2 \end{bmatrix}$

IF possible, compute :

- $(2C-3E)^T \times B$
- $(B^T + A) \times C$
- $(BC)^T$ **, and** $C^T \times B^T$
- $(3E)A^T$

Exercise 5:

consider the matrix $A = \begin{bmatrix} 1 & -2 & 5 \\ 3 & -2 & 3 \\ 32 & 0 & 1 \end{bmatrix}$ **,** $B = \begin{bmatrix} 1 & 2 & -2 & 2 \\ 3 & 0 & 1 & 2 \\ 2 & -1 & 4 & 1 \\ 0 & -3 & 1 & 3 \end{bmatrix}$,

Evaluate 1* $\sum_{i=1}^{3} a_{ii}$ **, 2*** $\sum_{i=1}^{4} b_{ii}$ **.**

Exercise 6:

for each n $\in$ **N, let** $A_n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ **, and , B =** $\begin{bmatrix} 1 & (-1)^n \\ -1 & 1 \end{bmatrix}$,

*** give** A_n^T **for all n** $\in$ **N ,**

*** find { n** $\in$ **N , :** $B_n^T = B_n$ **}.**

3-3 A procedure (an algorithm) for finding the inverse of an invertible matrix.

Definition 10:

Row Equivalent : Matrices that can be obtained from one another by a finite sequence of elementary row operations are said to be *row equivalent*.

Theorem 5 :

If A is an nxn matrix, then the following statements are equivalent, that is, all are true:

(a) A is invertible

(b) AX=0 has only the trivial solution (the only solution is $x_1=0$, $x_2=0 \ldots x_n=0$)

(c) A is row equivalent to In.

3-3-1 Why study determinants?:

They have important applications to system of linear equations and can be used to produce formula for the inverse of an invertible matrix.

***The determinant of a square matrix A is denoted by det(A) or |A|.**
*** If A is 1x1 matrix $A=[a_{11}]$, then: $\det(A)= a_{11}$**

Determinant of a 2x2 matrix

If A is a 2x2 matrix

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then we define

$$\det(A)=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \det(A)= a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example 1

$$A=\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

Then we define

$$\det(A)=\begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix} = \det(A)= 5.2 - 4.3=-2$$

$$\begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix}$$

Example 2:

if $A=\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ then det A= |A|= 3-2=1

If A is a 3x3 matrix $A=\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Then we define

$$\det(A)=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 2

$$\det(A)=\begin{vmatrix} 1 & 5 & -3 \\ 1 & 0 & 2 \\ 3 & -1 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} -5\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} -3\begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix}$$

$$=1[0*2-(-1*2)]-5[1*2-2*3]-3[-1*1-0*3]=25$$

If $A=\begin{pmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{pmatrix}_{3\times 2}$ then there are two methods to compute the det A

Method 1:

det A=a1 det $\begin{pmatrix} b2 & c2 \\ b3 & c3 \end{pmatrix}$- b1 det $\begin{pmatrix} b1 & c1 \\ b3 & c3 \end{pmatrix}$+ c1 det $\begin{pmatrix} b2 & c1 \\ b2 & c2 \end{pmatrix}$

Exercise 10:

$$A=\begin{pmatrix} 2 & -1 & 5 \\ 5 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$$

det A = 2det $\begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}$- (-1) det$\begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix}$ + 5det $\begin{pmatrix} 5 & 2 \\ 1 & 2 \end{pmatrix}$

= 2 (4 - 6) + 1(10-3) + 5 (10-2)

= -4 +7+40=43

Exercise 11:

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 3 & 1 & 4 & 2 \\ 5 & 2 & 1 & 10 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

$$\det A = 1\det\begin{pmatrix} 1 & 4 & 2 \\ 2 & 1 & 10 \\ 3 & 1 & 0 \end{pmatrix} - 2\det\begin{pmatrix} 3 & 4 & 2 \\ 5 & 1 & 10 \\ 2 & 1 & 0 \end{pmatrix} + 3\det\begin{pmatrix} 3 & 1 & 2 \\ 5 & 2 & 10 \\ 2 & 3 & 0 \end{pmatrix} - 0\det\begin{pmatrix} 3 & 1 & 4 \\ 5 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}$$

Complete the solution?

<u>**Method 2**</u>:

Duplicate Column Method – for 3x3

From the previous slide:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$\left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right.$$

Example 3: $\det(A) = \left|\begin{array}{ccc|cc} 1 & 5 & -3 & 1 & 5 \\ 1 & 0 & 2 & 1 & 0 \\ 3 & -1 & 2 & 3 & -1 \end{array}\right.$ $=0+30+3-0-(-2)-10=25$

Remark: we can not use method 2 for matrix higher than 3×3, for matrix 4x4 and higher, we use method 1.

Definition 11:

If det A=0, then A is called Singular matrix

3-3-2 Properties of Determinants:

1. **if all element of any row (column) of matrix A is Zero then det A = |A |= 0**
2. **if two rows (columns) ae equal, then the determinant is zero**
3. **if A=** $\begin{pmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{pmatrix}$ **and B=** $\begin{pmatrix} k1a1 & k2b2 & k3c3 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{pmatrix}$ **Then |A|=K|B| this is true if any row (column) in B is K times the corresponding row (column) in A.**

   There for if B= $\begin{pmatrix} ka1 & kb1 & kc1 \\ fa2 & fb2 & fc2 \\ ga3 & gb3 & gc3 \end{pmatrix}$**, then |B| = kgf |A|**
4. **if tow rows (column) are proportional, then the determinant is Zero.**
5. **If B is obtained from B by interchanging two adjacent rows (column), then |B| = -|A|**

Minor and Cofactor of a Matrix Entry

Definition:

If A is a square matrix, then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the sub-matrix that remains after the i^{th} row and j^{th} column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the cofactor of entry a_{ij}.

For 3x3 matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \qquad C_{11}=(-1)^{1+1}M_{11}=M_{11}$$

Example 4

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

The minor of a_{11} is:

$$M_{11}=\begin{vmatrix} 5 & 3 \\ 0 & 8 \end{vmatrix}=5X8-3X0=40$$

The cofactor of a_{11} is:

$$C_{11}=(-1)^{i+j}M_{ij}=(-1)^{2}M_{11}=M_{11}=40$$

The minor of a_{23} is:

$$M_{23}=\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}=1X0-2X1=-2$$

The cofactor of a_{23} is:

$$C_{23}=(-1)^{i+j}M_{ij}=(-1)^{5}M_{23}=-M_{23}=2$$

Finding determinant using the cofact

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$C_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \qquad C_{12}=-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \qquad C_{13}=\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

And for nxn matrix: $= a_{11}C_{11} + a_{12}C_{12} + \ldots\ldots + a_{1n}C_{1n}$

Example: evaluate det(A) for:

$$A=\begin{bmatrix} 1 & 0 & 2 & -3 \\ 3 & 4 & 0 & 1 \\ -1 & 5 & 2 & -2 \\ 0 & 1 & 1 & 3 \end{bmatrix} \qquad \det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

$$\det(A)=(1)\begin{vmatrix} 4 & 0 & 1 \\ 5 & 2 & -2 \\ 1 & 1 & 3 \end{vmatrix} - (0)\begin{vmatrix} 3 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 & 1 \\ -1 & 5 & -2 \\ 0 & 1 & 3 \end{vmatrix}$$

$$-(-3)\begin{vmatrix} 3 & 4 & 0 \\ -1 & 5 & 2 \\ 0 & 1 & 1 \end{vmatrix} = (1)(35)-0+(2)(62)-(-3)(13)=198$$

Why it is useful to find the inverse of a matrix?

- Many problems in engineering and science involve systems of n linear equation with n unknown (that is square matrix).
- The method is particularly useful when it is necessary to solve a series of systems:

  $AX=B_1$, $AX=B_2$……….. $AX=B_k$,

In this case each has the same square matrix A and the solutions are:

$$X=A^{-1}B_1,\ X=A^{-1}B_2\dots\dots\ X=A^{-1}B_k$$

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In this case each has the same square matrix A and the solutions are:

$$X=A^{-1}B_1,\ X=A^{-1}B_2\dots\dots\ X=A^{-1}B_k$$

$$AC=V$$

voltage

current

$$\begin{bmatrix} a_{11} & a_{12}\dots\dots & a_{1n} \\ a_{21} & a_{22}\dots\dots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2}\dots\dots & a_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ : & : & : \\ a_{n1} & a_{n2} \ldots\ldots & a_{nn} \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \\ : \\ V_n \end{bmatrix}$$

3-3-3 Inverse of matrix

How to find the inverse matrix?

For a 2x2 matrix

$$A = \begin{bmatrix} a & b^{x(-1)} \\ {}^{x(-1)}c & d \end{bmatrix}$$

If $ad\text{-}bc \neq 0$ then

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Find the inverse of: $A=\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Using the formula:

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}=\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

a=1; b=2;

c=1; d=3

Since: ad-bc=3-2=1

$$A^{-1}=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}=\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$B=A^{-1}=\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ is the inverse of $A=\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

$$A\cdot B=\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

$$B\cdot A=\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

3-4 using inverse matrix to solve to system of linear Equation.

Example: consider the system of linear equations:

$x_1+2x_2+3x_3=5$

$2x_1+5x_2+3x_3=3$

$x_1 \quad +8x_3=17$

We can write the this system as AX=B

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \qquad X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad B=\begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

Theorem 1

- If A is an invertible nxn matrix, then for each nx1 matrix B, the system of equation AX=B has exactly one solution, namely $X= A^{-1}B$.

$$\begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ : & : & : \\ a_{n1} & a_{n2} \ldots\ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ : \\ B_n \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ : & : & : \\ a_{n1} & a_{n2} \ldots\ldots & a_{nn} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \\ : \\ B_n \end{bmatrix}$$

Example: consider the system of linear equations:

$$x_1+2x_2+3x_3=5$$
$$2x_1+5x_2+3x_3=3$$
$$x_1 \qquad +8x_3=17$$

We can write the this system as AX=B

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \qquad X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad B=\begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{adjA}{\det A}$$

In example 1 we found that the inverse of A^{-1} is :

$$A^{-1}=\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By theorem 1 the solution of the system is:

$$X=A^{-1}B=\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or: $x_1=1$, $x_2=-1$, $x_3=2$

Why it is useful to find the inverse of a matrix?

- Many problems in engineering and science involve systems of n linear equation with n unknown (that is square matrix).
- The method is particularly useful when it is necessary to solve a series of systems:

$$AX=B_1,\ AX=B_2 \ldots\ldots\ AX=B_k,$$

In this case each has the same square matrix A and the solutions are:

$$X=A^{-1}B_1,\ X=A^{-1}B_2 \ldots\ldots\ X=A^{-1}B_k$$

Electronic Circuitry

$$\underset{\text{current}}{A\,C} = \underset{\text{voltage}}{V}$$

$$\begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ : & : & : \\ a_{n1} & a_{n2} \ldots\ldots & a_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ : \\ v_n \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ : & : & : \\ a_{n1} & a_{n2} \ldots\ldots & a_{nn} \end{bmatrix}^{-1} \begin{bmatrix} v_1 \\ v_2 \\ : \\ v_n \end{bmatrix}$$

Exercise 12 :

solve the equation by inversion method

X + 2y + 3 Z = 4

X + 3y + 4 Z = 0

X + 4y + 3 Z = 2

Solution:-

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

$|A| = {}^{(1)} (9-16) - 2 (3-4) + 3 (4-3) = -7+2+3=-2$

$C_{11} = (-1)^2 \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 9-16 = -7$

$C_{12} = (-1)^3 \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -(3-4) = 1$

$C_{13} = (-1)^4 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} =4-3 = 1$

$C_{21} = - \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} =- (6-12) = 6$

$C_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = (3-31) = 0$

$C_{23} = - \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = - (4-2) = -2$

$C_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 8-9 = -1$

$C_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$

$C_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = (3-2) = 1$

$$\text{adj } A = \begin{pmatrix} -7 & 1 & 1 \\ 6 & 0 & -2 \\ -1 & -1 & 1 \end{pmatrix}^T = \begin{pmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} +7/2 & -3 & 1/2 \\ -1/2 & 0 & +1/2 \\ -1/2 & 1 & -1/2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7/2 & -3 & 1/2 \\ -1/2 & 0 & 1/2 \\ -1/2 & 1 & 1/2 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 14-0+1 \\ -2+0+1 \\ -2+0-1 \end{pmatrix} = \begin{pmatrix} 15 \\ -1 \\ -3 \end{pmatrix}$$

$\therefore$ x=15, y=-1, z=-3

Review problem:

(1)Write which of the following system of linear equation as a single Matrix equation AX=B .

(1) $X_1+8X_2-2X_3=3$, $4X_1-7X_2+X_3=-3$, $-2X_1-5X_2-2X_3=1$

(2) $X_1-3X_2+6X_3=2$, $7X_1+5X_2+X_3=-9$

(3) $5X_1+2X_2=6$, $4X_1-3X_2=-2$, $3X_1+X_2=9$

(4) $2X_1+5X_2-3X_3+4X_4=4$, $X_1+9X_3+5X_4=12$,
$3X_1-3X_2-8X_3+5X_4=2$

(2) solve the following system by determine the inverse of the three variable (a) $X_1+2X_2-X_3=2$, $X_1+X_2+2X_3=0$, $X_1-X_2-X_3=1$
$X_1-X_2 = 1$.

(b) $X_1+X_2+2X_3=2$, $X_1+2X_2+X_3=0$, $X_1+2X_2+3X_3=1$

(c) $2X_1+5X_2+3X_3=3$, $X_1+8X_3=15$, X_1-2X_2+2

Relations and Functions

4-1 some basic definition :

A "relation" is a fundamental mathematical notion expressing a relationship between elements of sets.

Definition 1:
A binary relation from a set A to a set B is a subset $R \subseteq A \times B$.

So, R is a set of ordered pairs. We often write $a \sim_R b$ or aRb to mean that $(a, b) \in R$.
Many times, we will talk about a "relation on the set A", which means that the relation is a subset of $A \times A$. We can also define a ternary relation on A as a subset $R \times A^3$ or, in general, an n-ary relation as a subset $R \times A^n$, or $R \times A_1 \times A_2 \times \times \times A^n$ if the sets A^i are different. In this class, we will focus only on binary relations. Here are some

Exercise 1:
1. Let $A = \mathbb{N}$ and define $a \sim_R b$ iff $a \leq b$.
2. Let $A = \mathcal{P}(\mathbb{N})$ and define $a \sim_R b$ iff $a \cap b$ is finite.
3. Let $A = \mathbb{R}^2$ and define $a \sim_R b$ iff $d(a, b) = 1$.
4. Let $A = \mathcal{P}(\{1, \ldots, n\})$ and define $a \sim_R b$ iff $a \subseteq b$.

4-2 Properties of Relations :

*A **binary relation** R from a set A to a set B is a subset of $A \times B$. If $(a, b) \in R$ we write aRb and we say that a is related to b. If a is not related to B we write $a \; R \; b$
In case $A = B$ we call R a **binary relation** on A

The set $Dom(R) = \{a \in A \,.\, (a, b) \in R \text{ for some } b \in B\}$ is called the ***domain*** of R
The set $Range(R) = \{b \in B \neg \, (a, b) \in R \text{ for some } a \in A\}$ is called the ***range*** of R

Exercise2:

a. Let $A = \{2, 3, 4\}$ and $B = \{3, 4, 5, 6, 7\}$. Define the relation R by aRb if and
only if *a divides b*. Find, R ,$Dom(R)$, $Range(R)$

b. Let $A = \{1, 2, 3, 4\}$. Define the relation R by aRb if and only if $a \le b$. Find, R ,$Dom(R)$,$Range(R)$

Solution:

a. $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$
,$Dom(R) = \{2, 3, 4\}$, and $Range(R) = \{3, 4, 6\}$

b. $R = \{(1, 1),(1, 2), (1, 3), (1, 4), (2, 2), (2, 3),(2, 4), (3, 3),(3, 4)$
$(4, 4)\}$
,$Dom(R) = A$, $Range(R) = A$

To draw a digraph of a relation on a set A, we first draw *dots or* ***vertices*** to represent the elements of A. Next, if $(a, b) \in R$ we draw an arrow (called a **directed edge**) from a to b.
Finally, if $(a, a) \in R$ then the directed edge is simply ***a loop***.

*Next we discuss three ways of building new relations from given ones. Let R be a relation from a set A to a set B. The **inverse** of R is the relation R^{-1} from $Range(R)$ to $Dom(R)$ such that:
$R^{-1} = \{(b, a) \in B \times A , (a, b) \in R\}$

Exercise 3:

Let $R = \{(1, y), (1, z), (3, y)\}$ ***be*** a relation from $A = \{1, 2, 3\}$ to $B = \{x, y, z\}$

a. Find. R^{-1}

b. Compare R^{-1} and R

Solution:

a. $R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$

b. $R^{-1} = R$

* Let R and S be two relations from a set A to a set B Then we define the relations $R \cup S$ and $R \cap S$ by:

$$R \cup S = \{(a, b) \in A \times B : (a, b) \in R \text{ or } (a, b) \in S\},$$

Exercise 4:

Given the following two relations from $A = \{1, 2, 4\}$ to $B = \{2, 6, 8, 10\}$:

aRb if and only if a □ b

aSb if and only if $b - 4 = a$

List the elements of R, S, $R \cup S$, and $R \cap S$

Solution:

$R = \{(1, 2), (1, 6), (1, 8), (1, 10), (2, 2), (2, 6), (2, 8), (2, 10), (4, 8)\}$

$S =$ ¶(2, 6), (4, 8)◇

$R \cup S = R$ ، $R \cap S = S$

Definition 2:

Now, If we have a relation R from A to B and a relation S from B to C we can define the relation $S \circ R$, called the **composition** relation, to be the relation from A to C defined by:

$$S \circ R = \{(a, b) \neg \ (b, c) \in R \text{ and } (a, c) \in S \text{ for some } b \in B\}$$

Exercise 5:

Let $R = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$ ⇔ $S = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$, Find $S \circ R$

Solution.

$S \circ R = \{(1, u), (1, t), (2, s), (2, t), (3, s), (3, t), (3, u)\}$

Note: RoS ≠ SoR

Definition 3 :

let R and S be two relations from A to B , then

$R = \{ (a, b) \in A \times B: (a,b) \notin R \}$.

$R \cap S = \{(a,b) : (a,b) \in R \text{ and } (a,b) \in S\}$

$R \cup S = \{ (a,b): (a,b) \in R \text{ or } (a,b) \in S \}$

$R - S = \{ (a,b) : (a,b) \in R \text{ but } (a,b) \notin S \}$

$R \oplus S = \{ (R - S) \cup (S - R)\}$

Definition 4:
We next define four types of binary relations:
A relation R on a set A is called **reflexive** if $(a, a) \in R$ for all $a \in A$. In this case, the digraph of R has a loop at each vertex.

Exercise 6:
a. Show that the relation $a \leq b$ on the set $A = \{1, 2, 3, 4\}$ is reflexive.
b. Show that the relation on IR defined by aRb if and only if $a < b$ is not reflexive.

Solution:
a., each vertex has a loop.
b. Indeed, for any real number a we have $a - a = 0$ and not $a - a < 0$.

Definition 5 :
A relation R on A is called **symmetric** if whenever $(a, b) \in R$ then we must have $(b, a) \in R$. The digraph of a symmetric relation has the property that whenever there is a directed edge from a to b, there is also a directed edge from b to a

Exercise 7:
a. Let $A = \{a, b, c, d\}$ and $R = \{(a, a), (b, c), (c, b), (d, d)\}$ Show that R is symmetric.
b. Let IR be the set of real numbers and R be the relation aRb if and only if $a < b$. Show that R is not symmetric.

Solution:
a. bRc and cRb so R is symmetric.
b. $2 < 4$ but $4 < 2$

Definition 6 :
A relation R on a set A is called **anti symmetric** if whenever $(a, b) \in R$ and $a \neq b$ then $(b, a) \notin R$. The digraph of an anti symmetric relation has the property that between any two vertices there is at most one directed edge.

Exercise 8:

a. Let IN be the set of nonnegative integers and R the relation aRb if and only if a divides b. Show that R is anti symmetric.

b. Let $A = \{a, b, c, d\}$ and $R = \{(a, a), (b, c), (c, b), (d, d)\}$. Show that R is not anti symmetric.

Solution.

a. Suppose that $a \mid b$ and $b \mid a$. We must show that $a = b$. Indeed, by the definition of division, there exist positive integers k_1 and k_2 such that $b = k_1 a$ and $a = k_2 b$ This implies that $a = k_2 k_1 a$ and hence $k_1 \ k_2 = 1$. Since k_1 and k_2 are positive integers then we must have $k_1 = k_2 = 1$ Hence, $a = b$

b. bRc and cRb with $b \neq c$

Definition 7 :

A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$ *then* $(a, c) \in R$. The digraph of a transitive relation has the property that whenever there are directed edges from a to b and from b to c then there is also a *directed* edge from a to c

Exercise 9 :

a. Let $A = \{a, b, c, d\}$ and $R = \{(a, a), (b, c), (c, b), (d, d)\}$ Show that R is not transitive.

b. Let Z be the set of integers and R the relation aRb if a divides b. Show that R is transitive.

Solution:

a. $(b, c) \in R$ and $(c, b) \in R$ but $(b, b) \notin R$.

b. Suppose that $a \mid b$ and $b \mid c$. Then there exist integers k_1 and k_2 such that $b = k_1 a$ and $c = k_2 b$. Thus, $c = (k_1 \ k_2)a$ which means that $a \mid c$

Remark.
⇀A relation that is reflexive, symmetric, and transitive on a set A is called an **equivalence relation on A.** For example, the relation"=" is an equivalence relation on IR.

Exercise 10 :
Let Z be the set of integers and $n \in$ Z. Let R be the relation on Z defined by aRb if $a \setminus b$ is a multiple of n. We denote this relation

by $a \equiv b \ (mod\ n)$ read "a congruent to b modulo n." Show that R is an equivalence relation on Z.

Solution:
$\equiv\equiv$ is reflexive: For all $a \in$ Z, $a - a = 0 \bullet n$ That is, $a \equiv a \ (mod\ n)$

$\equiv$is symmetric: Let $a, b \in$ Z such that $a \equiv b \ (mod\ n)$. Then there is an integer k such that $a\text{-}b = kn$ Multiply both sides of this equality by (-1) and letting $k' = -k$ we find that $b - a = k' n$ That is $b \equiv a \ (mod\ n)$
$\equiv$ is transitive Let $a, b, c \in$ Z be such that $a \equiv\equiv b \ (mod\ n)$ and

$b \equiv\equiv c \ (mod\ n)$

Then there exist integers k_1 and k_2 such that $a - b = k1n$ and $b - c = k_2n$
Adding these equalities together we find $a - c = kn$ where
$k = k_1 + k_2 \in$ Z which shows that $a \equiv c \ (mod\ n)$

Definition 8 :
let R be an equivalence relation on the set A ,
If a $\in$ A then [a] = { x$\in$A : x Ra} , [a] is called the *equivalence class* of R .

Exercise 11 :
let A= { 1,2,3}
, R = (1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)}, find [1] ,[2]

Solution:
[1]= { 1,2} , [2] = { 1,2}

نلاحظ من خلال هذا المثال ان صف التكافؤ ل (1) هو العناصر التي ترتبط مع المسقط الاول (1) , كذالك صف التكافؤ ل (2) هي العناصر التي ترتبط مع المسقط الاول (2) , وهذ ايعني ان عناصر التكافؤ هي قيم المسقط الثاني للازواج المرتبة.

Remark :

* $[a] \neq \phi$, $\exists\, a \in [a]$,
* $[a] \subseteq A$, $\forall\, a \in [a]$

Theorem :
$[a] = [b]$ iff $a\ R\ b$ and if $[a] \neq [b]$, then $[a] \cap [b] = \phi$.

Definition 9 :
if R is an equivalence on a set A , then the collection of all equivalence class of the elements of A gives o partition of A , denoted by A/R .

Exercise 12 :
let S = { 1,2,3} and the relation
R = { (1,1),(1,2),(2,1),(2,2),(3,3) } is equivalence relation on S under the Relation R , find the partition of S (S/R) .

Solution:
$[1] = \{ 1,2\}$, $[2] = \{ 1,2\}$, $[3] = \{ 3\}$,

$P = (S/R) = \{ \{ 1,2\} ,\{ 3\} \}$.

Remark :
let p = { A_i } be a partition of a set A , we defined a relation R on A by aRb iff a,b belong to the some A_i , then R is equivalence relation and A/R . = p.

Exercise 13 :

let A ={ 1,2,3,4} , find R of the following partition

p_1 **= { { 1,2},{3,4} } ,** p_2 **= {{1,2,3} , {4}}**

solution :

R_1 **= { (1,1),(2,1),(1,2),(2,2),(3,3),(4,3),(3,4),(4,4),},**

R_2 **= {(1,1),(2,1),(3,1),(1,2),(2,2),(3,2),(1,3),(2,3),(3,3),(4,4)}**

4-3 Partial Orders :

Partial orders are another type of binary relation that is very important in computer science. They have applications to task scheduling, database concurrency control, and logical time in distributed computing,

Definition 10 :

A binary relation $R \subseteq A \times A$ is a partial order if it is reflexive, transitive, and ant symmetric.

A partial order is always defined on some set A. The set together with the partial order is called apposed"

Definition 11 :

A set A together with a partial order $\leq$ is called a posit $(A, \leq)$.

Exercise 14 :

Consider the following relations:

- $A = \mathbb{N}, R = \leq$ **easy to check reflexive, transitive, anti symmetric**
- $A = \mathbb{N}, R = \geq$ **, same.**
- $A = \mathbb{N}, R = <$**, not because not reflexive**
- $A = \mathbb{N}, R = |$ **(divides), easy to check reflexive, transitive, anti**

symmetric

• $A = \mathcal{P}(\mathbb{N}), R = \subseteq$, **check reflexive:** $S \subseteq S$, **transitive:** $S \subseteq S' \wedge S' \subseteq S''$ $\rightarrow S \subseteq S''$ **anti symmetric** $S \subseteq S' \wedge S' \subseteq S \rightarrow S = S'$

• $A =$ **"set of all computers in the world",** $R =$ **"is (directly or indirectly) connected to". Not a partial order because it is not true that** $aRb \wedge bRa \;\; a = b$. **In fact, it is symmetric and transitive. Equivalence relation.**

Definition 12 :

**Now, let $A_1, A_2, \ldots, A_n$ be a partition of a set A. That is, the A_{is} are subsets of A that satisfy

(I $\bigcup_{i=1}^{n} A_i = A$

(ii) $A_i \cap A_j = \phi \emptyset$ for $i \neq j$

Define on A the binary relation $x\,R\,y$ if and only if x and y belongs to the same set Ai for some $1 \leq i \leq n$

Remark.
Let R be an equivalence relation on A. For each $a \in A$ let $[a] = \{x \in A \neg \;\; xRa\}$ $\qquad A/R = \{[a] \neg a \in A\}$.

Then the union of all the elements of A/R is equal to A and the intersection of any two distinct members of A/R is the empty set. That is, the family A/R forms a partition of A.

Review Problems:

Exercise 1:
Let $X = \{a, b, c\}$. Recall that $\mathcal{P}(X)$ is the power set of X. Define a binary relation $\mathcal{R}$ on $\mathcal{P}(X)$ as follow $|A|, \; |A| = B$

a. Is $\{a, b\}\ \mathcal{R}\ \{b, c\}$?

b. Is $\{a\}\ \mathcal{R}\ \{a, b\}$?

c. Is $\{\ c\ \}\ \mathcal{R}\ \{b\}$?

Exercise 2:
Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ and define the binary relation R as follows: $(x, y) \in A \times B, (x, y) \in R \Leftrightarrow\Rightarrow x < y$ List the elements of the sets R and R^{-1}

Exercise 3:
Let $A = \{2, 4\}$ and $B = \{6, 8, 10\}$ and define the binary relations R, S
from A to B as follows:

$(x, y) \in A \times B, (x, y) \in R \Leftrightarrow\Leftrightarrow Y/X$

$(x, y) \in A \times B, x\,S\,y \Rightarrow y \setminus 4 = x$

List the elements of $A \times B$,R , S ,$R \cup S$, and $R \cap S$

Exercise 4:
Consider the binary relation on IR defined as follows:

1- $x, y \in R, x\,R\,y \Leftrightarrow x \geq y$

2- $x, y \in R, x\,R\,y \Leftrightarrow xy \geq 0$

Is R reflexive? Symmetric? Transitive?

Exercise 5:
Let $A = \mathrm{IN} \times \mathrm{IN}$. Define the binary relation R on A as follows:
$(a, b)\,R\,(c, d) \Leftrightarrow\Leftrightarrow a + d = b + c$

a. Show that R is reflexive.
b. Show that R is symmetric.
c. Show that R is transitive.
d. List five elements in $[(1, 1)]$
e. List five elements in $[(3, 1)]$
f. List five elements in $[(1, 2)]$
g. Describe the distinct equivalence classes of R

4-4 Functions:
4-4-1 Definitions and Examples

A function is a special case of a relation. A **function** f from a set A to a set B is a relation from A to B such that for every $x \in A$ there is a unique $y \in B$. such that $(x, y) \in f$. For $(x, y) \in f$ we use the notation $y = f(x)$ We call y the **image** of x under f. The set A is called the **domain** of f whereas B is called the **co domain**. The collection of all images of f is called the **range** of f.

Exercise 15 :
Show that the relation $f = \{(1, a), (2, b), (3, a)\}$ ***defines*** a function from $A = \{1, 2, 3\}$ to $B = \{a, b, c\}$ Find its range.

Solution.
Since every element of A has a unique image then f is a function. Its range consists of the elements a and b

Exercise 16 :
Show that the relation $f = \{(1, a), (2, b), (3, c), (1, b)\}$ does not define a function from $A = \{1, 2, 3\}$ to $B = \{a, b, c\}$

Solution.
Indeed, since 1 has two images in B then f is not a function.

* A **sequence** of elements of a set A is a function from $\mathbb{N}^*$ to A. We write (a_n) and we call a_n the nth term of the sequence.

Exercise 17:

a. Define the sequence $a_n = n, \quad n \geq 1$. Compute $\sum_{k=1}^{n} a_k$

b. Define the sequence $a = n^2$ Compute the sum $\sum_{k=1}^{n} a_k$

Solution:

a. Let $S_n = \sum_{k=1}^{n} a_k$ Then write Sn in two different ways, namely,

$S_n = 1 + 2 + \ldots + n$ and $S_n = n + (n-1) + \ldots\ldots +$

1 Adding, we obtain

$2S_n = (n+1) + (n+1) + \ldots + (n+1) = n(n+1)$ Thus,

$S_n = \frac{n(n+1)}{2}$

b. First note that $(n+1)^3 - n^3 = 3n^2 + 3n + 1$. From this we obtain the following chain of equalities:

$2^3 - 1^3 = 3(1)^2 + 3(1) + 1$

$3^3 - 2^3 = 3(2)^2 + 3(2) + 1$

.....

$(n+1)^3 - n^3 = 3n^2 + 3n + 1$

Adding these equalities we find

$3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n = (n+1)^3 - 1$, Using a. we find

$3\sum_{k=1}^{n} k^2 + \frac{3n(n+1)}{2} + n = n^3 + 3n^2 + 3n$

A simple arithmetic shows that

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Exercise18:

Let $A = \{a, b, c\}$ Define the function f: $P(A) \to$IN by $f(|X|) = X$

Find the range of f

Solution:

By applying f to each member of $P(A)$ we find $Range(f) =$

$\{0, 1, 2, 3\}$

Exercise19:

Consider the alphabet $\Sigma = \{a, b\}$ and the function f: $\Sigma^* \to$ Defined as follows: for any string $s \in \Sigma^*$

$f(s) = the\ number\ of\ a_s\ in\ s$ Find $f(\varepsilon)$, $f(ababb)$, and $f(bbbaa)$

Solution.
$f(\varepsilon) = 0$, $f(ababb) = 2$, and $f(bbbaa) = 2$

4-4-2 Definition 13 : (Hamming distance function):

Let $\Sigma = \{0, 1\}$ and Σ^n be the set of all strings of 0's and 1's of length n.

Define the function $H: \Sigma^n \times \Sigma^n \to \mathbb{N}$ as follows: for any $(s, t) \in \Sigma^n \times \Sigma^n$

$H(s, t) =$
number of positions in which s and t have different values

Exercise 20 :
For the case $n = 5$, find $H(00101, 01110)$ and $H(10001, 01111)$

Solution:
$H(00101, 01110) = 3$ and $H(10001, 01111) = 4$

4-4-3Definition 14 : (Boolean functions)

An **n-place Boolean function** f is a function from the Cartesian product $\{0, 1\}^n$ to $\{0, 1\}$ Consider the 3-place Boolean function $f: \{0, 1\}^3 \to \{0, 1\}$ defined by $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3) \bmod 2$

Describe f using an input/output table.

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

4.4.4 *Definition* 15 .— Encoding and Decoding functions)

Let $\Sigma = \{0, 1\}$and Σ^* be the set of all strings of 0's and 1's. Let L be the set of all strings over Σ that consist of consecutive triples of identical bits. Thus, $111000 \in L$. A message consisting of 0's and 1's is encoded by writing each bit in it three times. The encoded message is decoded by replacing each section of three identical bits by the one bit to which all three are equal.

We define the encoding function $E: \Sigma^* \to L$ by

$E(s) =$ *the string obtained from s by replacing each bit of s by the same bit written three times*

and we define the decoding function $D: L \to \Sigma^*$ by

$D(s)=$ *the string obtained from s by replacing consecutive triple of bits of s by a single copy of that bit*▷

Exercise 21 :

Find $E(0110)$ and $D(111111000111)$

Solution:

We have $E(0110) = 000111111000$ and $D(111111000111) = 1101$.

Review Problems:

Exercise 1:

Let $A = \{1, 2, 3, 4, 5\}$ and let $F: \mathcal{P}(A) \to \mathbb{Z}$ be defined as follows:

$$F(X) = \begin{cases} 0 & if has even number of element \\ 1 & if has odd number of element \end{cases}.$$

Find the following

a. $F(\{1, 3, 4\})$

b. $F(\phi)$

c. $F(\{2, 3\})$

d. $F(\{2, 3, 4, 5\})$

Exercise 2:
Let $\Sigma = \{a, b\}$ and Σ^* be the set of all strings over Σ

a. Define $f: \Sigma^* \rightarrow Z$ as follows:

$$f(s) = \begin{cases} the\ number\ of\ b_s\ to\ the\ left\ of\ the\ left & most\ a\ in\ s \\ 0 & if\ s\ contains\ no\ a_s \end{cases}$$

Find $f(aba)$, $f(bbab)$, and $f(b)$. What is the range of f?

b. Define $g: \Sigma^* \rightarrow \Sigma^*$ as follows:
$g(s) =$
the string obtained by writing the characters of s in reverse order
Find $g(aba)$, $g(bbab)$, and $g(b)$. What is the range of g?

Exercise 3:
Let E and D be the encoding and decoding functions.
a. Find $E(0110)$ and $D(111111000111)$.
b. Find $E(1010)$ and $D(000000111111)$

Exercise 4:
Let H denote the Hamming distance function on Σ^5
a. Find $H(10101, 00011)$
b. Find $H(00110, 10111)$

Exercise 5:
Consider the three-place Boolean function $f: \{0, 1\}^3 \rightarrow \{0, 1\}$ defined as follows: $f(x_1, x_2, x_3) = (3x_1 + x_2 + 2x_3) \bmod 2$
a. Find $f(1, 1, 1)$ and $f(0, 1, 1)$
b. Describe f using an input/output table.

4-5 Recursion:

A **recurrence relation** for a sequence $a_0, a_1, \ldots$ is a relation that defines a_n in terms of $a_0, a_1, \ldots, a_{n-1}$. The formula relating a_n to earlier values in the sequence is called the **generating rule**. The assignment of a value to one of *the* a's is called an **initial condition.**

Exercise 22 :

The Fibonacci sequence 1, 1, 2, 3, 5, …… is a sequence in which every number after the first two is the sum of the preceding two numbers. Find the generating rule and the initial conditions.

Solution:

The initial conditions are $a_0 = a_1 = 1$ and the generating rule is

$$a_n = a_{n-1} + a_{n-2} \quad , \qquad n \geq 2$$

Exercise 23 :

Let $n \geq 0$ and find the number sn of words from the alphabet $\Sigma = \{0, 1\}$ of
length n not containing the pattern 11 as a sub word.

Solution:

Clearly, $s_0 = 1$(empty word) and $s_1 = 2$. We will find a recurrence relation for s_n,
$n \geq 2$▷ Any word of length n with letters from Σ begins with either 0 or 1.

If the word begins with 0, then the remaining n-1 letters can be any sequence of 0's or 1's except that 11 cannot happen. If the word begins with 1 then the next letter must be 0 since 11 can not happen; the remaining n - 2 letters can be any sequence of 0's and 1's with the exception that 11 is not allowed. Thus
the above two categories form a partition of the set of all words of length n with letters from Σ and that do not contain 11. This implies the recurrence relation

$$s_n = s_{n-1} + s_{n-2} \qquad n \geq 2$$

*The most basic method for finding the solution of a sequence defined recursively is by using **iteration.** The iteration method consists of starting with the initial values of the sequence and then calculate successive terms of the sequence until a pattern is observed. At that point one guesses an explicit formula for the
Sequence and then uses mathematical induction to prove its validity.

Exercise24 :
Find a solution for the recurrence relation

$$\begin{cases} a_0 & =1 \\ a_{n-1}+2 & n\geq 1 \end{cases}$$

Solution :
Listing the first five terms of the sequence one finds
$a_0 = 1$
$a_1 = 1 + 2$
$a_2 = 1 + 4$
$a_3 = 1 + 6$
$a_4 = 1 + 8$
Hence, a guess is $a_n = 2n + 1$, $n \geq 0$. It remains to show that this formula is valid by using mathematical induction. Basis of induction:
For $n = 0$, $a_0 = 1 = 2(0) + 1$
Induction hypothesis: Suppose that $a_n = 2n + 1$
Induction step: We must show that $a_{n+1} = 2(n + 1) + 1$
By the definition of a_{n+1} we have $a_{n+1} = a_n + 2 = 2n + 1 + 2 = 2(n + 1) + 1$

Exercise 25 :
Consider the arithmetic sequence
$a_n = a_{n-1} + d$, $n \geq 1$
Where a_0 is the initial value. Find an explicit formula for a_n

Solution:
Listing the first four terms of the sequence after a_0 we find
$a_1 = a_0 + d$
$a_2 = a_0 + 2d$
$a_3 = a_0 + 3d$
$a_4 = a_0 + 4d$
Hence, a guess is $a_n = a_0 + nd$

Next, we prove the validity of this formula by induction. Basis of induction: For $n = 0$, $a_0 = a_0 + (0)d$

Induction hypothesis: Suppose that $a_n = a_0 + nd$

Induction step: We must show that $a_{n+1} = a_0 + (n + 1)d$

By the definition of a_{n+1} we have $a_{n+1} = a_n + d = a_0 + nd + d = a_0 + (n + 1)d$

Exercise 26 :

Find a solution to the recurrence relation

$$\begin{cases} a_0 & = 0 \\ a_n = & a_{n-1} + (n-1) n \geq 1 \end{cases}$$

Solution :

Writing the first five terms of the sequence we find

$a_0 = 0$

$a_1 = 0$

$a_2 = 0 + 1$

$a_3 = 0 + 1 + 2$

$a_4 = 0 + 1 + 2 + 3$

A guessing formula is that $a_n = 0 + 1 + 2 + \ldots\ldots + (n - 1)/ \frac{n(n-1)}{2}$

We next show that the formula is valid by using induction on $n \geq 0$

Basis of induction: $a_0 = 0 = \frac{0(0-1)}{2}$

Induction hypothesis: Suppose that $a_n = \frac{n(n-1)}{2}$

Induction step: We must show that $a_{n+1} = \frac{n(n+1)}{2}$. Indeed,

$a_{n+1} = a_n + n$ □ $\frac{n(n-1)}{2} \rightarrow n$ .. $\frac{n(n+1)}{2}$

Review Problems:

Exercise 1:
Find the first four terms of the following recursively defined sequence:

$$\begin{cases} a_1 = 1 \qquad a_2 = 2 \\ a_n = a_{n-1} + a_{n-2} + 1 \quad n \geq 1 = 3 \end{cases}$$

Exercise 2:
Find a formula for each of the following sums:

a. $1 + 2 + \ldots + (n - 1)$, $n \geq \geq 2$

b. $0 + 2 + 4 + 6 + 8 + \ldots\ldots + 2n$, $n \geq 1$

c. $3 . 1 + 3 . 2 + 3 . 3 + \ldots\ldots . 3 . n$, $n \geq \geq 1$

Exercise 3:
Find a formula for each of the following sums:

a. $1 + 2 + 2^2 + \ldots\ldots . + 2^{n-1}$, $n \geq 1$

b. $3^{n-1} + 3^{n-2} + \ldots\ldots . + 3^2 + 3 + 1$, $n \geq 1$

c. 3. $2^n + 3 . 2^{n-1} + 3 . 2^{n-2} + \ldots . + 3 . 2 + 3 . 2 + 3$, $n \geq 1$

d. $2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + \ldots\ldots + (-1)^{n-1} . 2 + (-1)^n$, $n \geq 1$

Exercise 4:
Use iteration to guess a formula for the following recusively defined sequence
and then use mathematical induction to prove the validity of your formula:

$c_1 = 1$, $c_n = 3c_{n-1} + 1$, for all $n \geq 2$

Exercise 5:
Use iteration to guess a formula for the following recursively defined sequence
And then use mathematical induction to prove the validity of your formula:

$a_0 = 1$, $a_n = 2^n - a_{n-1}$, for all $n \geq 2$

4-6 Representing a Relation by a Matrix:

Let A be a set with n elements and R be a binary relation on A▷
Define the $n \times n$ matrix $M(R) = (m_{ij})$ as follows:

$$m_{ij} = \begin{cases} 1 & if\,(a_i, b_j) \in R \\ 0 & if\,(a_i, b_j) \notin R \end{cases}$$

If the numbers on the main diagonal of $M(R)$ are all equal to 1 then R is re- flexive. If $M(R)^T = M(R)$, where $M(R)^T$ is the transpose of $M(R)$, then the relation R is symmetric. If $m_{ij} = 0$ or $m_{ij} = 0$ for $i \neq j$ then R is anti symmetric.

Exercise 27 :
let A = { 2,4,6} , B = { r, s } , and
R = { (2 , r) , (2,s) , (6,s) }

Solution:

$$\mathbf{MR_{m \times n}} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

هنا **m** تمثل عدد صفوف المصفوفة (عدد عناصر المجموعة **A**)
و **n** تمثل عدد اعمدة المصفوفة (عدد عناصر المجموعة **B**)

Exercise 28 :
let A=B= { 1,2,3,4}, { (a,b) : a=b}

A×B = { (1,1) ,(1,2), (1,3) ,(1,4) ,(2,1) ,(2,2) ,(2,3), (2,4) ,(3,1) ,(3,2) ,(3,3) ,(3,4) ,(4,1), (4,2), (4,3) ,(4,4) }.
R= { (1,1) , (2,2) ,(3,3) ,(4,4) }
Solution:

$$\mathbf{MR_{m \times n}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 29 :

Consider the matrix MR = $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ **write the set A and B , and determine the element of relation R .**

Exercise 30 :

The relation from {0, 1, 2, 3} to {a, b, c} defined by the list:
{(0, a), (0, c), (1, c), (2, b), (1, a)}. is represented by the matrix

	a	b	c
0	1	0	1
1	1	0	1
2	0	1	0
3	0	0	0

Exercise 31 :

The divisibility relation over {1, 2, …. , 12} is represented by the enormous matrix

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	0	1	0	1	0	1	0	1	0	1
3	0	0	1	0	0	1	0	0	1	0	0	1
4	0	0	0	1	0	0	0	1	0	0	0	1
5	0	0	0	0	1	0	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	0	0	0	1
7	0	0	0	0	0	0	1	0	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0
9	0	0	0	0	0	0	0	0	1	0	0	0
10	0	0	0	0	0	0	0	0	0	1	0	0
11	0	0	0	0	0	0	0	0	0	0	1	0
12	0	0	0	0	0	0	0	0	0	0	0	1

Again, properties can by recognized by examining the representation:
Reflexivity the major diagonal is all 1.
Symmetry: the matrix is clearly not symmetric across the major diagonal.
Transitivity: not so obvious

Exercise 32 :
Let $A = \{1, 2, 3\}$ and
$R = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$. Find
$M(R)$ and use it to determine if the relation R is reflexive, symmetric or anti symmetric.

***If we represent the relations as matrices, then we can compute the composition by a form of "boolean" matrix multiplication, where $+$ is replaced by $\vee$ (Boolean OR) and $\times$ is replaced by $\wedge$ (Boolean AND).**

Let R_1 be a relation in Exercise 30 and Let R_2 be the relation from $\{a, b, c\}$ to $\{d, e, f\}$ given by:

$$\begin{array}{c|ccc} & d & e & f \\ \hline a & 1 & 1 & 1 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{array}$$

Then $R_2 \circ R_1 = \{(0, d), (0, e), (0, f), (1, d), (1, e), (1, f), (2, e)\}$, that is,

$$\begin{array}{c|ccc} & d & e & f \\ \hline 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 \end{array}$$

***A relation on a set A can be composed with itself. The composition $R \circ R$ of R with itself is written $R2$. Similarly R^n denotes R composed with itself n times. R^n can be recursively defined: $R^1 = R$,**
$R^n = R \circ R^{n-1}$.

Definition 16 :
Boolean matrix operation:

Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be $m \times n$ Boolean matrices , we define
$A\vee B = [C_{ij}]$, $C_{ij} = a_{ij} \vee b_{ij}$ (OR) .
$A\wedge B = [d_{ij}]$, $d_{ij} = a_{ij} \wedge b_{ij}$ (and) .

Finally, if A is an $m \times p$ and B is an $p \times n$ Boolean matrix . we define $A\otimes B = [e_{ij}]$, the Boolean product of A and B by :

$$e_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots\ldots \vee (a_{ip} \wedge b_{pj})$$

هنا تجرى عملية الضرب االبولوني نفس عملية الضرب الاعتيادية للمصفوفة ماعدا تستبدل الضرب والجمع ب ($\wedge$) و ($\vee$)

Exercise 33 :

if $A=\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\\0&0&0\end{pmatrix}$, $B=\begin{pmatrix}1&1&0\\1&0&1\\0&0&1\\1&1&0\end{pmatrix}$, $C=\begin{pmatrix}1&0&0\\0&1&1\\1&0&1\end{pmatrix}$

Compute: $A\vee B$, $A\wedge B$, $A\otimes B$

Solution:

$$A\vee B = \begin{pmatrix}1\vee1 & 0\vee1 & 1\vee0\\0\vee1 & 1\vee0 & 1\vee1\\1\vee0 & 1\vee0 & 0\vee1\\0\vee1 & 0\vee1 & 0\vee0\end{pmatrix} = \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&0\end{pmatrix}$$

$$A\wedge B = \begin{pmatrix}1\wedge1 & 0\wedge1 & 1\wedge0\\0\wedge1 & 1\wedge0 & 1\wedge1\\1\wedge0 & 1\wedge1 & 0\wedge1\\0\wedge1 & 0\wedge1 & 0\wedge0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

A⊗ B =

$$\begin{pmatrix} (1\wedge 1)\vee(1\wedge 0)\vee(0\wedge 1) & (1\wedge 0)\vee(1\wedge 1)\vee(0\wedge 0) & (1\wedge 0)\vee(1\wedge 1)\vee(0\wedge 1) \\ (1\wedge 1)\vee(0\wedge 0)\vee(1\wedge 1) & (1\wedge 0)\vee(0\wedge 1)\vee(1\wedge 0) & (1\wedge 0)\vee(0\wedge 1)\vee(1\wedge 1) \\ (0\wedge 1)\vee(0\wedge 0)\vee(1\wedge 1) & (0\wedge 0\vee(0\wedge 1)\vee)(1\wedge 0) & (0\wedge 0)\vee(0\wedge 1)\vee(1\wedge 1) \\ (1\wedge 1)\vee(1\wedge 0)\vee(0\wedge 1) & (1\wedge 0)\vee(1\wedge 1)\vee(0\wedge 0) & (1\wedge 0)\vee(1\wedge 1)\vee(0\wedge 1) \end{pmatrix}$$

Theorem:
if A , B and C are Boolean matrices , then

* $A\vee B = B\vee A$, $A\wedge B = B\wedge A$
* $(A\vee B)\vee C = A\vee(B\vee C)$, $(A\wedge B)\wedge C = A\wedge(B\wedge C)$.
* $A\vee(B\wedge C) = (A\vee B)\wedge(A\vee C)$,
  $A\wedge(B\vee C) = (A\wedge B)\vee(A\wedge C)$

*$A\otimes(B\otimes C) = (A\otimes B)\otimes C$

Elements of Graph Theory:

5.1 : Vertex, vertices

A *vertex* is a connection point. A *Graph* has a set of vertices, usually shown as

$V = \{v_0, v_2, \ldots, v_n\}$, $V = \{A, B, C\}$ or $V = \{1, 2, \ldots N\}$.

A vertex may have no connections, one connection or many connections.

An *edge* is a connection between vertices. Given vertices v_i and v_j in a Graph,

Definition 1 :

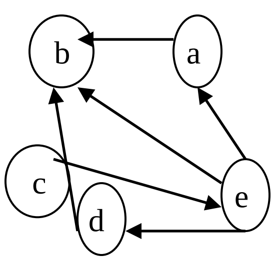
- If A is a finite set and R is a relation on A, then we can represent R by directed graph or digraph of circles and lines, these circles is called vertices and the lines is called edge

Note:-

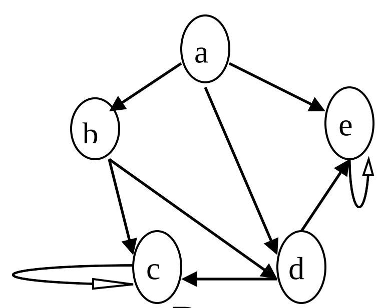
1. **the number of vertices = the number of elements of A**
2. **the number of edge = the number of order pair of R.**

Exercise 1:

:**



S



R

Let A= {a,b,c,d,e} and Let R & S be two relations represent by the following digraph, then find $\overline{R}, R^{-1}, R \cap S$

Definition 2 :

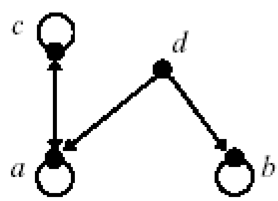
A Graph is called undirected *if the edges have no implied direction, i.e., (v_i , v_j) is the same as (v_j , v_i), the edge just connects v_i to v_j .*

Definition 3 :

A Graph is called directed *if the edges have a direction, i.e., (vi, vj) means an edge starting at vi and going to v_j , i.e, (v_i , v_j) is not the same as (v_j , v_i).*

Exercise 2:

A relation on the set {a,b,c,d} is defined by the following list: {(*a*,*c*), (*c*,*c*), (*a*,*a*), (*b*,*b*), (*c*,*a*), (*d*,*b*), (*d*,*a*)}. Draw the directed graph representation of this relation and write its logical matrix.



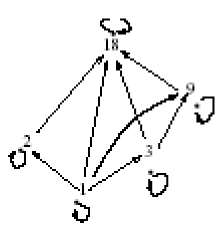
$$\begin{pmatrix} T & F & T & F \\ F & T & F & F \\ T & F & T & F \\ T & T & F & F \end{pmatrix},$$ where it is assumed that a,b,c,d $\rightarrow$ 1,2,3,4.

Exercise 3:

Let $A = \{1, 2, 3, 9, 18\}$ and the "divides" relation on A. Draw directed graph of this relation.

Solution:

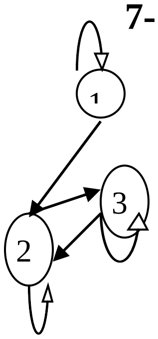
The directed graph of the given relation is

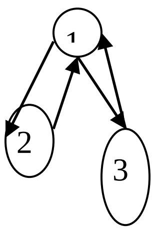


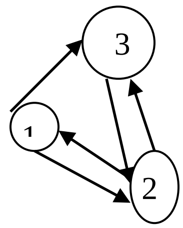
5-2 Properties of Relations

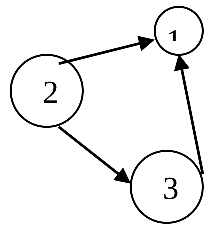
Let R be a relation on a set A, we list four types of relations:-

1- **R is reflexive if aRa** $\forall a \in A$
2- **R is irreflexive if** $a \not R a \quad \forall a \in A$.
3- **R is symmetric if aRb →bRa.**
4- **R is not symmetric if aRb and** $b \not R a$
5- **R is anti – symmetric if aRb and bRa** $\Rightarrow a = b$
6- **R is transitive if aRb and bRc** $\Rightarrow aRc$
7- **R is equivalence relation if it is reflexive + symmetric +transitive**









Reflexive symmetric not symmetric anti-
symmetric

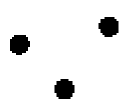
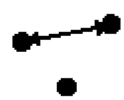
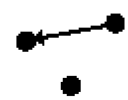
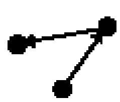
Exercise 4:

:**

let A={1,2,3,4}, R={(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)}.
Check that.

Exercise 5:

Give examples (in the form of directeD graph) of relations that are:
@) symmetric and transitive;
b) Symmetric but not transitive;
c) Transitive "ut not symmetric;
d) Neither symmetric nor transitive;

a) 
b) 
c) 
d) 

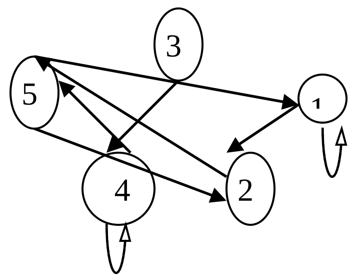
Definition 4 :

If R is a relation on a set A and a Є A, then the in – degree of a is the number of b Є A such that (b,a) ЄR, the out – degree of a is the number b Є A such that (a,b) ЄR

Note: from the digraph we have:

1- **the in – degree of a vertex a= the number of edges terminating at a** تعني عدد الاسهم الداخلة الى الرأس.
2- **The out – degree of a vertex a= the number of edges leaving a .**

Exercise 6:



find the in – degree and out – degree of a vertices 1, 4, 2 when A={1,2,3,4,5}, R= {(1,1), (1,2), (3,1), (4,4), (2,5), (3,4), (4,5), (5,1)}

solution:

1- **in – degree of 1=3.**
   Out – degree of 1= 2.
   Out – degree of 4= 2.
2- **in – degree of 2 = 1.**
   Out – degree of 2 = 1.

Exercise 7:

find the digraph $R \cap \mathrm{B}\times\mathrm{B}$ **where A= {a,b,c,d,e,f}, B={a,b,c}, R={(a,a), (a,c), (b,c), (a,e), (b,e), (c,e)}**

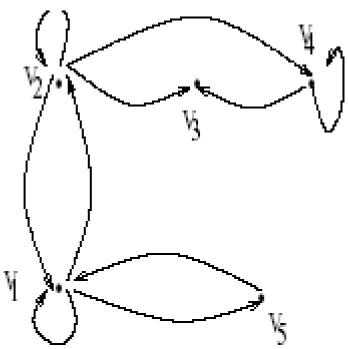
Solution :

B×B= {(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)}.

$R \cap (\mathrm{B}\times\mathrm{B})$ **= {(a,a), (a,c), (b,c)}**

Exercise 8:

Find the in-degree and out-degree of each of the vertex in the graph G with directed edges



Definition 5 :

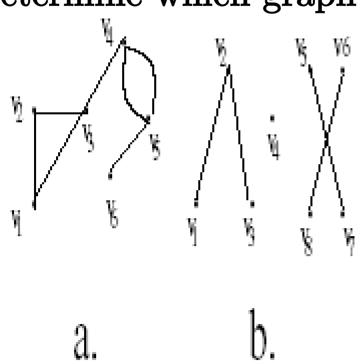
*Two edges associated to the same vertices are called **parallel**.
*An edge incident to a single vertex is called a **loop.**
*A graph with neither loops nor parallel edges is called **simple** graph.

Definition 6 :

An undirected graph is called **connected** if there is a path between every pair of distinct vertices of the graph. A graph that is not connected is said to be **disconnected**.

Exercise9:

Determine which graph is connected and which one is disconnected.



Solution:

a. Connected.

b. Disconnected since there is no path connecting the vertices $v1$ and $v4$.

Definition 7 :

A path in a relation R is a sequence a_0, …., a_k with $k \geq 0$ such that $(a_i, a_{i+1}) \in R$ for every $i < k$. We call k the length of the path.

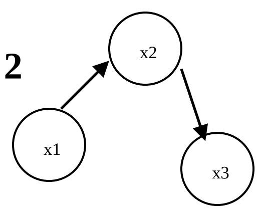
$R^n =$ ¶(a, b) ¬ there is a length n path from a to b in R◊

5-3 Path in Relation and Digraph

Definition 8 :

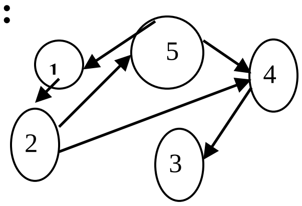
Suppose that R is a relation on a set A, a path of length n from a vertex a to a vertex b is finite sequence.
Π= a_1, x_1, x_2, ….., x_{n-1},b beginning with a and ending with b, such that aRx_1, x_1Rx_2,…., x_{n-1}Rb

Exercise 10:
from the digraph X_3, X_1, X_2 is the path of length 2



Note: A path of length n has n+1 element of A and they are not necessarily distinct.

Exercise 11:



: In this fig we have
π 1= 1,2,5,4,3 is path of length 4
π 2= 1,2,5,1 is path of length 3 from 1 to it self
π 3= 2, 2 is path of length 1 from a vertex 2 to it self

Definition 9 :

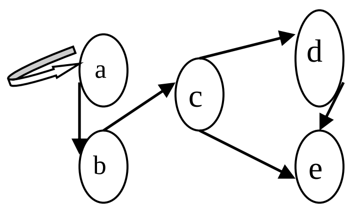
A cycle is a path that begins and ends at the same vertex.

Exercise 12:
in the last example, we have π_2= 1,2,5,1 and π_3= 2, 2 are cycles.

Definition 10 :

1- a R^n b is mean that there is a path of length n from a to b
2- a R^{α} b is mean that there is some path in R from a to b. R^{∞} is some time called connectivity relation for R.

Exercise 13:



Let A= {a,b,c,d,e}, R= {(a,a), (a,b), (b,c), (c,e), (c,d), (d,e)}.
Compute R^2, R^{∞}
Solution :
R= {(a,a), (a,b), (b,c), (c,e), (c,d), (d,e)}.
aR^2a since aRa and aRa
aR^2b since aRa and aRb
aR^2c since aRb and bRc
bR^2e since bRc and cRe
bR^2d since cRe and cRd
cR^2e since cRd and dRe
$\therefore$ R2= {(a,a), (a,b), (a,c), (b,e), (b,d), (c,e)}.

To compute R^{∞}, we need all ordered pairs of vertices for which there is a path of any length from the first vertex to the second one.
We can find R^{∞}= {(a,a), (a,b), (a,c), (a,d), (a,e), (b,c), (b,d), (b,e), (c,d), (c,e), (d,e)}.

Review Problems:

1- Let A={1,2,3,4} ,R_1= {(1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4), (4,1)}
And R_2= {(1,1), (1,3), (2,3), (3,3), (3,2), (4,3)}
Draw the digraphs of R1 and R2 and determine the in – degree and out – degree of the vertices 2, 3, 4.
2- For the following relation on S= {0,1,2,3} give a matrixes for them:

{(m, n) ЄR_1 if m+n=3}

1. **{(m, n) ЄR_2 if m ≤ n}**

2. **{(m, n) ЄR_3 if Max {m,n}}=3**

3- Let A= {1,2,3,4,5}, Determine whether the relation R whose digraph is given is reflexive, symmetric.

4- Let R be the following symmetric relation on the Set A={1,2,3,4,5}
R= {(1,2), (2,1), (3,4), (4,3), (3,5), (5,3), (4,5), (5,4), (5,5)}
Draw the graph of R and find [1], [3], [5].

5- If A={1,2,3,4}, R= A×A find A/R

6- Let A= {1,2,3,4} and B= {a, b, c} and
R= {(1,a), (1,b), (2,b), (2,c), (3,b), (4,a)}
S= {(1,b), (2,c), (3,b), (4,b)}, then complete:
$\overline{R}, R \cap C, R \cup C, R \oplus S, R^{-1}, DomR^{-1}, RomR^{-1}$

7- Let A= {1,2,3,4} Let R={(1,1), (1,2), (2,3), (2,4), (3,4), (4,1), (4,2)}
S= {(3,1), (4,4), (2,3), (2,4), (1,1), (1,4)}
Find 1- RoR 2- SoR 3-SoS 4- Is (1,1) Є SoR?

5-4 Reflexive and symmetric closures:

Definition 10 :

The converse of relation r is r^c = {(a,b)|(b,a) ∈ r}

Definition 11 :
Reflexive closure of any r exists and equals r ∪ e.

Theorem 1 :

Symmetric closure of any r exists and equals r ∪ r^c
***The transitive closure r * of relation r on S contains only such pairs**
(a,b) ∈ S×S that there exists a path from a to b in relation r.

Connectively and Wars hall's Algorithm :

***Warshall's algorithm**

Th7 (Warshall): Let $R = \{R_{km}\}$ be the logical matrix of relation r on set A. Define matrices $W^{[i]} = W^{[i]}_{km}$ as follows:

$W^{[0]} = R$

$W^{[j+1]}_{km} = W^{[j]}_{km}$ or $\left(W^{[j]}_{kj} \text{ and } W^{[j]}_{jm}\right)$

Then the matrix of the transitive closure $r^* = W^{[|A|]}$

$$\begin{pmatrix} T & T & . & . \\ . & . & T & . \\ . & T & . & T \\ T & . & . & T \end{pmatrix}$$

Exercise 14 :

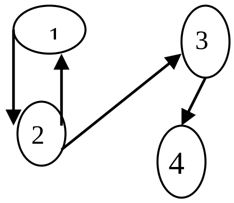
Using Wars hall's algorithm, find the transitive closure of the following relations on {*a*,*b*,*c*,*d*,*e*}:
a) {(*ac*),(*bd*),(*ca*),(*db*),(*ed*)}
b) {(*ab*),(*ac*),(*ae*), (*ba*),(*bc*),(*ca*), (*cb*),(*da*),(*ed*

$$\text{a)} \begin{pmatrix} T & . & T & . & . \\ . & T & . & T & . \\ T & . & T & . & . \\ . & T & . & T & . \\ . & T & . & T & . \end{pmatrix}$$

$$\text{b)} \begin{pmatrix} T & T & T & T & T \\ T & T & T & T & T \\ T & T & T & T & T \\ T & T & T & T & T \\ T & T & T & T & T \end{pmatrix}$$

Theorem 2 :
Let R be a relation on a set A. then R^{∞} us the transitive closure on R.

Exercise 15:
Let A= {1,2,3,4} and Let R={(1,2), (2,3), (3,4), (2,1)}.
Find the transitive closure R.
Solution :
the digraph of R is:



Since R^∞ is transitive closure, we can find R^∞ by computing all paths.
So/ R^∞ = {(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4)}

Theorem 3 :
Let A be a set with |A| = n and Let R be a relation on A. Then: $R^\infty = R \cup R^2 \cup \ldots R^n$ **.**

Remark:
method in theorem (1) has certain difficult because graphical method is impractical for large Sets and relation and is not systematic.
Let R be a relation on a set A={a1,a2,….,an}

تتلخص طريقة **warshal's** كالآتي:

Step (1): نقوم بتمثيل العلاقة على شكل مصفوفة**W_o= MR**

Such that W_o= MR …… Wn = MR^α

Step(2): نقوم بايجاد المصفوفة **wk** من التي تسبقها **WK-1**.

Step(3): نقوم بتحويل كل **(1's)** من **WK-1** الى **WK**.

Step(4): نقوم بتعيين المواقع التي تحتوي على **(1)** من العمود **K** والصف **K**.

Step(5): نقوم بوضع **(1)** في المواقع التي حددناها من المصفوفة **WK-1** ونضعها في **WK**.

والمثال التالي يوضح الخطوات السابقة:-

Exercise: 16 :
Let A= {1,2,3,4} and Let R={(1,2), (2,3), (3,4), (2,1)}
Find the transitive closure of R.

Solution:

Step 1: $W_0 = MR = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **column 1 = location 2 ,**

row 1 = location 2 $\Rightarrow$ $a_{11} = 1$

$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **, column 2 = location 1 ,2 , row 2= location 1,2,3**

$\Rightarrow$ $a_{11} = 1$, $a_{12} = 1$, $a_{13}=1$, $a_{21} = 1$, $a_{22}=1$, $a_{23}= 1$

$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **, column 3 = location 1,2 , row 3 = location 4**

$\Rightarrow$ $a_{14}=1$, $a_{24} = 1$

$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R^{\infty} = W_3 = \{ (1,1) ,(1,2) ,(1,3) ,(1,4),(2,1),(2,2),(2,3),(2,4),(3,4)\}$

Exercise 17: H.W

(1) Let A= {1,2,3} and Let R={(1,1), (1,2), (2,3), (1,3), (3,1), (3,2)}

Compute the transitive Closure . by using Wars hall's algorithm.

(2)Let A={1,2,3,4} for the relation R whose matrix is given, find the matrix of the transitive closure by using wars hall's algorithm.

$$\mathbf{MR} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \qquad \mathbf{MR} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

5-5 Trees:

An undirected graph is called a **tree** if each pair of distinct vertices has exactly one path. Thus, a tree has no parallel edges and no loops.

Definition 12 :

A **binary tree:** is a rooted tree such that each vertex has at most two children. Moreover, each child is designated as either a **left child** or a **right child**.

Definition 13 :

1-(node level) :is the number of paths from node to root .

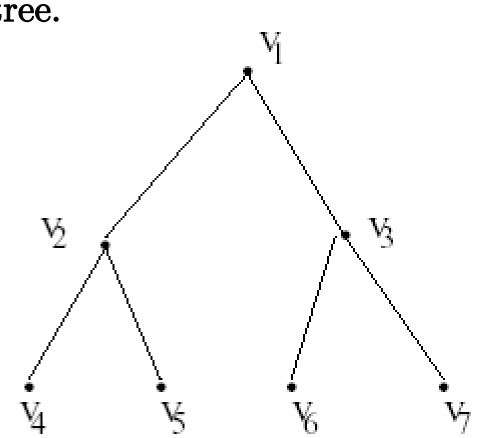
2-(node degree) : is the number of paths that out from it .

3-(tree degree) : is the high degree of node degree that includes in tree.

4- (Tree height): is the bigger level to any node in tree.

Exercise 18 :

Find the level of each vertex and the height of the following rooted tree.

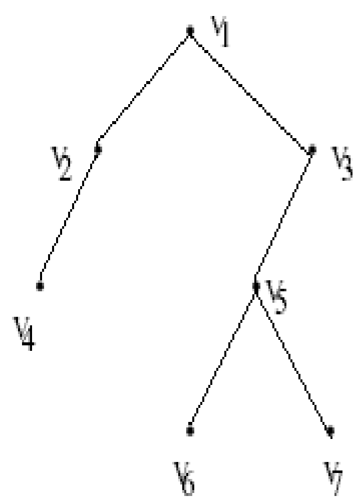


Solution:
$v1$ is the root of the given tree.

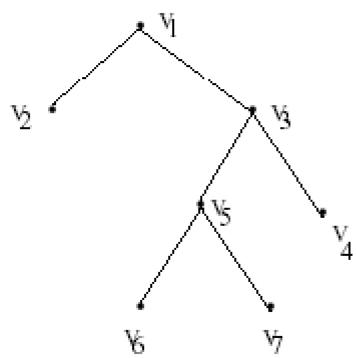
Vertex	*level*
v_2	1
v_3	1
v_4	2
v_5	2
v_6	2
v_7	2

Definition 12 :

Exercise 19 :
a. Show that the following tree is a binary tree.
b. Find the left child and the right child of vertex v_5.
c. A **full binary tree** is a binary tree in which each vertex has either two

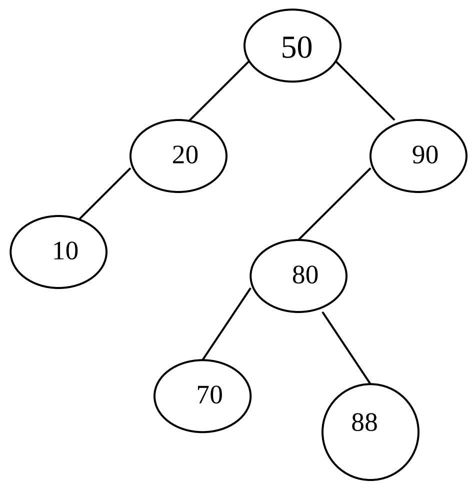


Solution:
a. Follows from the definition of a binary tree.
b. The left child is v_6 and the right child is v_7.



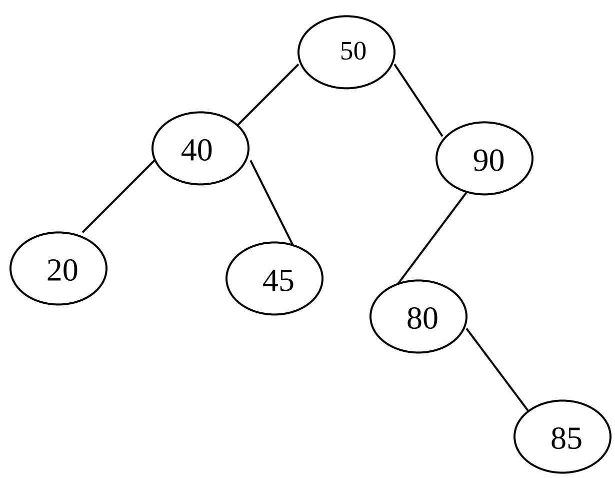
5-5-1 Binary search tree

Binary tree that ordered from left node to right node, such that the left node have the small number and the right node have the bigger number.



Exercise20: Built the binary tree to the following system number
50 ,90 , 40 , 45 , 20 , 80 , 85 .

Solution:

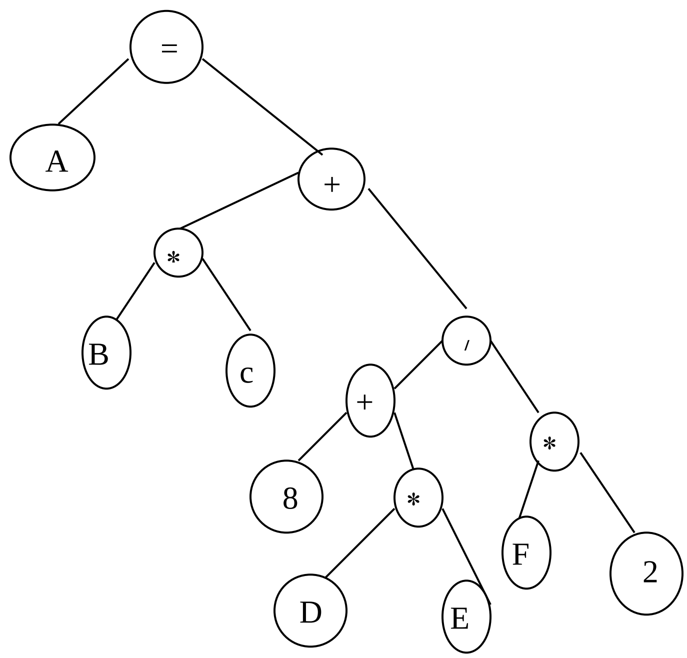


5-5-2 Representation of Arithmetic Expression using Binary Tree

Some of important application for binary tree is representation the arithmetic expression such that (*, + , - , ….) is represent the node
But number is represent the leaf .

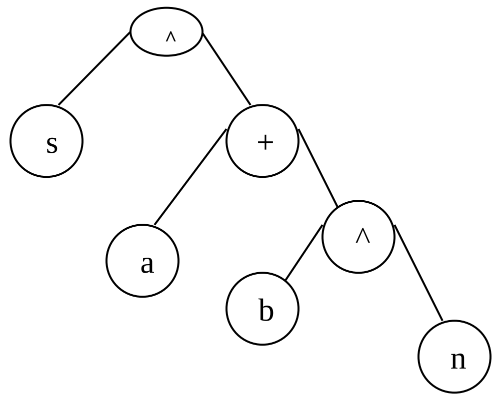
Exercise 21 : Built the binary tree to the following:

A= B*C+(8+D*E)/(F*2) .
Solution:



Exercise 22 : Built the binary tree to the following: $S^{a+b^{n}}$

Solution:



Review Problems:

Exercise 1: Built the binary tree to the following system number

- **(1) 20, 4,6,12,,8,3,7,9,5**
- **(2) 50,85,10,75,38,90,30,70,40,95**
- **(3) 18,23,50,42,63,20,28,33,47,3**

Exercise 2: Built the binary tree to the following expression

- **(1) X=2*(a-b/c)**
- **(2) A+b*(c+d)**
- **(3) a^5+8*b^3-2*c-5**
- **(4) a+b-[(c+d)*e]**

Introduction to Cryptograph

6-1 Group

Definition 1 :

A group (G, *) is a set G on which a binary operation * is defined which satisfies the following axioms:

Closure: For all $a, b \in G$, $a * b \in G$.
Associative: For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.
Identity: $\exists\, e \in G$ s.t. for all $a \in G$, $a * e = a = e * a$.
Inverse: For all $a \in G$, $\exists\, a$-1 $\in$ G s. t. a* a-1 = a-1 *a = e.

Definition 2 :

A group (G, *) is called an abelian group if * is a commutative Operation:

Commutative: For all $a, b \in G$, $a * b = b * a$.

Example 1:

. The following are examples of groups.
(1) G = Z, * = +, I = 0 and x^{-1}=-x.
(2) G = Q, *= +, I = 0 and x_{-1} =-x.
(3) G = R, * = +, I = 0 and x^{-1} =-x

Rmarke :

1* (R-0 , .) is abelian group but (Z, .) is not group , since 0 is at least one integer which dose not have its multiplicative inverse

2* the algebric system (Z, -) is not a group , since substraction on Z is not associative as for example (3-4) -5 $\neq$ 3- (4-5) .
Similarly , non of the system (Q, -) , (R, -) and (C,-) .

Exercise 1:
Show that the set G={ 1,-1,i,-i } is abelian group with respect to multiplication composition .

Solution: we taking the multiplication table

0	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Closure: For all $a, b \in G$, $(a \cdot b) \in G$.

Associative: the element of G are complex number and the multiplication of complex number being associative, it follows that multiplication on G is associative .

Identity: the element 1 is the identity , since $1*a = a*1 = a \quad \forall\ a \in G$

Inverse: the inverse element of 1,-1,i,-i, are 1 , -1 , -i and i respectively

Now, the commutative law, $\forall\ a, b \in G$, $a \cdot b = b \cdot a$

$1 \cdot i = i \cdot 1 = i$

6-2 New composition on Integers:

Let m be an arbitrary fixed positive integer, then for any integer a and b , we have the following compositions :

1* Additive modulo m , denoted by $+_m$ is defined as $a +_m b = r$ whew r is the least positive remainder obtained by dividing (a+b) by m , $0 \le x < m$.

2* multiplicative modulo m, denoted by x_m is defined as a x_mb = r where r is the least positive remainder obtained by dividing ab by m .

Exercise 2 :

$2 +_5 3=0$ …… i.e 5 by 5 is 0

$2 \times_5 3 = 1$, $6 \times_8 3 = 2$ and $4 \times_8 6 = 0$

Definition 3 :

(additive group) let m be an arbitrary but fixed positive integer , then the set Z_m= { [0],[1],[2],,,,,[m-1]}of all disjoint residue class modulo m .

Definition 4 :

(multiplication group) let p be an arbitrary but fixed prime integer , then the set G= { [1], [2], ……[p-1]} of all distinct non- zero residue class modulo p is a finite abelian group with respect to multiplication of residue class as the composition .

Exercise 3 :

Show that the set G= { 0,1,2,3 } is a finite abelian group of order four with respect to additive modulo 4 ,

Solution:

$+4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

1* closure: $\forall\, 1,2 \in G$, $1 +_4 2 = 3 \in G$
2* Associative: $\forall\, 1,2,3 \in G$, $(1 +_4 2) +_4 3 = 1 +_4 (2 +_4 3) = 2$
3* identity element: is o , since $0 +_4 a = a$, $\forall\, a \in G$
4* the inverse element: $0^{-1} = 0$, $1^{-1} = 3$, $2^{-1} = 2$, $3^{-1} = 1$
5* commutative law: since + is commutative, so $+_4$ is commutative .

Definition 5 :
a group G is said to be a cyclic group , if there exists an element , a $\in$ G , such that every element of G is expressible as some integer power of a .

Exercise 4 :
the group ({ 0,1,2,3,}, $+_4$) is cyclic group generated by 1 , Since
$1 +_4 1 = 2$, $1+_4 1 +_4 1 = 3$, $1 +_4 1 +_4 1 +_4 1 = 0$

Theorem 1 :
every cyclic group is necessarily abelian .

Proof:
let G= { a} be cyclic group generated by a .

Let x and y be any two arbitrary element of g then $x= a^m$ and $y =a^n$ for some integer m and n

$Xy = a^m. a^n = a^{m+n} = a^{n+m} = a^n. a^m = yx$

Thus, xy = yx , $\forall$ x,y $\in$ G
Hence , G is an abelian group .

Remark:
an abelian group is not always cyclic group .

Exercise 5 :
Find the number of generator of acyclic group of order 10

Solution:
let G = { a } be a cyclic group of order 10 , generated by an element a , then o(a) = o(G) = 10

SO , G={ $a,a^2,a^3 , a^4 , a^5 , a^6, a^7 , a^8 , a^9 , a^{10} = e$ }

Now the numbers less than 10 and relatively prime to 10 are 1,3,7,9
So , a, a^3, a^7, a^9, are generators of G .

Definition 6 :
the multiplication table of a group is called cayley table .

Definition 7 :
Latin square is formed from clayey table , where each row and column has the number of elements of the group and no two elements can be repeated in arrow or column .

Exercise 6 :
Let Z_5 be a group of integer modulo 5 ,
The Latin square can be formed by the clayey table of Z_5.

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Definition 8 :
tow Latin square $L_1 = [\ a_{ij}\ ]$, and $L_2 = [\ b_{ij}\ ]$ on n symbols 1,2,3,,,,n are said to be orthogonal if every ordered pair of s symbols occurs exactly once among the n^2 pairs (a_{ij}, b_{ij}) . , i= 1,2,,,,,n , j=1,2,,,n

Exercise 7 :
a pair of orthogonal order 3 Latin squares and the 9 distinct order pairs that they form

$$\begin{matrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{matrix} \quad , \quad \begin{matrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{matrix}$$

Definition 9 :
a Latin square is said to be in the standard form if the symbols are in the initial row in the natural order 0,1,2……s-1

Exercise8:

Two orthogonal Latin squares of order 5

$$\begin{matrix} 2 & 4 & 1 & 0 & 3 \\ 4 & 1 & 0 & 3 & 2 \\ 1 & 0 & 3 & 2 & 4 \\ 0 & 3 & 2 & 4 & 1 \\ 3 & 2 & 4 & 1 & 0 \end{matrix} \quad , \quad \begin{matrix} 4 & 0 & 2 & 1 & 3 \\ 2 & 1 & 3 & 4 & 0 \\ 3 & 4 & 0 & 2 & 1 \\ 0 & 2 & 1 & 3 & 4 \\ 1 & 3 & 4 & 0 & 2 \end{matrix}$$

To standardize the first square in we make the transformation

$$2\to4\ ,\ 4\to1\ ,\ 1\to2,\ 0\to3\ ,\ 3\to4$$

Thus, the transformed square orthogonal to one another,

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{matrix} \quad , \quad \begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 4 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \end{matrix}$$

Theorem 2 :

if there is an orthogonal family of r Latin square of order n , then r $\leq$ n-1 .

Exercise 9 :

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{matrix} \quad , \quad \begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 1 & 2 \end{matrix}$$

Latin square of order 3

$$\begin{matrix}1&2&3&4\\2&1&4&3\\3&4&1&2\\4&3&2&1\end{matrix},\quad\begin{matrix}1&2&3&4\\3&4&1&2\\4&3&2&1\\2&1&4&3\end{matrix},\quad\begin{matrix}1&2&3&4\\4&3&2&1\\2&1&4&3\\3&4&1&2\end{matrix}$$

Latin square of order 4

$$,\begin{matrix}1&2&3&4&5\\3&4&5&1&2\\5&1&2&3&4\\2&3&4&5&1\\4&5&1&2&3\end{matrix},\quad\begin{matrix}1&2&3&4&5\\4&5&1&2&3\\2&3&4&5&1\\5&1&2&3&4\\3&4&5&1&2\end{matrix},\quad\begin{matrix}1&2&3&4&5\\5&1&2&3&4\\4&5&1&2&3\\3&4&5&1&2\\2&3&4&5&1\end{matrix}$$

Latin square of order 5

6-3 Codes and Latin square :

What we have called coding theory , shod more properly be called the theory of error – correcting codes , since there is another aspect of coding theory which is older and deals with the creation and decoding of secret messages , this field is called cryptography.

Definition 10:

an n- ary code is a subset c $\subset$ Z_2^n , the elements of C are called code – words . Given a code C , an encoding function is any bisection E : Z_2^m $\to$ C , but a decoding function is any function
D: $Z_2^n \to : Z_2^m$

i.e E(s) = the by string obtained form replacing each bit of s by the same bit written three times .

D(s) = the string obtained from by replacing consecutive triple of bits of s by a single copy of that bit.

Exercise10:
Find E(0110) and , D(111 111 000 111)

Solution: E(0110) = 000 111 111 000
D(111 111 000 111) = 1101

Definition 11 : (Hamming distance function)

Let X= x_1 x_2 ….x_n , Y= y_1 y_2 ….y_n be two words from Z_2^n, the Hamming distance d(x,y) between x and y is the number of places in which they differ.

i.e d(x,y) = { i: 1 $\le$ i $\le$ n , $x_i \neq y_i$ } .

Exercise 11 :
for n= 5 , find H(00101, 01110) and H(10001, 01111)

solution : H(00101 , 0110) = 3 , H(10001 , 01111) = 4

Definition 12 :
a code is said to be t- error correcting if when no more than t-error has occurred in the transmissions of code word .

Remark:
block of repeated symbols is called a code word .
i.e
a code word is what is transmitted in place of one piece of information in the original message.

We note that if we have n $\times$n Latin square we can build n^2 code words by using ordered triplets (i,j,a_{ij}) .

.

Exercise 12 :

Let the Latin square of group Z_3 ,

0	1	2
1	2	0
2	0	1

The code words are ,

(0,0,0) , (1,0,1) , (2,0,2)
(0,1,1) , (1,1,2) , (2,1,0)
(0,2,2,) , (1,2,0) , (2,2,1)

Theses triplets are of Hamming distance at least 2 apart because of construction Latin square .

Theorem 3 :
any pair of orthogonal Latin square of order n yields a $_1$- error correcting code with n^2 code words .

Exercise 13 :
the clayey table of Z_5 and of its orthogonal

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

,

1	2	3	4	0
0	1	2	3	4
4	0	1	2	3
3	4	0	1	2
2	3	4	0	1

Then the cod words is

(0,0,0,1) , (1,0,1,2) , (2,0,2,3) , (3,0,3,4) , (4,0,4,0)
(0,1,1,0) , (1,1,2,1) , (2,1,3,2) , (3,1,4,3) , (4,1,0,4)
(1,2,2,4) , (1,2,3,0) , (2,2,4,1) , (3,2,0,2) , (4,2,1,3)
(0,3,3,3) , (1,3,4,4,) , (2,3,0,0) , (3,3,1,1) , (4,3,2,2)
(0,4,4,2) , (1,4,0,3) , (2,4,1,4) , (3,4,2,0) , (4,4,3,1

Review Problems:

Exercise 1:

show that the set of four matrices $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ **forms an abelian group under multiplication of matrices .**

Exercise 2 :

let G= { 1,-1,i,-i } , where $i^2 = -1$, prove that (G, *) a cyclic group.

Exercise 3 :

Given an example of a finite abelian group which is not cyclic .

Exercise 4 :

let Q be the set of rational number defined on the operation (*) is abelian group .

Exercise 5 :

if you have two Latin square of order 2 , show that does not exist a pair of orthogonal 2×2 Latin square .

Exercise 6 :

Define the following functions: Hamming distance, Encoding, decoding and find E(0110) , D(111 111 000 111) and if H on $\sum^{5}$

find H(10101, 00011) .

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1- Lecture note in discrete mathematics , Marcel B. Finan , Arkansas Tech University , November 8 , 2002 .

2- Discrete and combinatorial mathematics , Ralph P. Grimaldi 4^{th} ed , 1998 .