## Course Weekly Outline

## Course Name: Data Structures

| Course Instructor | Dr. Mohammed Salah Ibrahim |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
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| Title | Teacher |  |  |  |  |
| Course Coordinator | Dr. Mohammed Salah Ibrahim |  |  |  |  |
| Course Objective | 1- Learning different data structures <br> 2- Understand why this data structure is better than the other one. <br> 3- Learning how to choose the best data structure for your algorithm. <br> 4- learn how to deal with your problem, building its algorithm and fitting the best data structures to it. |  |  |  |  |
| Course Description | This course covers all data structure types. It starts with defining algorithms and their complexity from the time and space prospection. Then, a list of data structure and their description is presented. The course describes every data structure in detail. In addition to that, it gives the reason to why we need this data structure and where to use it. This course includes many projects that give more understanding to the data structure studied. These projects talks about real life problems that we ask student to use one of the data structure that has been presented in the course to solve it. |  |  |  |  |
| Textbook | Introduction to Algorithm, third Edition, Thomas H. Cormen Algorithms, fourth edition, Robert Sedgewick and Kevin Wayne |  |  |  |  |
| References | Introduction to Algorithm, third Edition, Thomas H. Cormen Algorithms, fourth edition, Robert Sedgewick and Kevin Wayne |  |  |  |  |
| Course Assessments | Term Tests | Laboratory | Quizzes | Project | Final Exam |
|  | \%20 | \%10 | \%5 | \%15 | \%50 |
| General Notes |  |  |  |  |  |

Republic of Iraq
The Ministry of Higher Education \& Scientific Research


## Course Weekly Outline

| Z d d | Date | Topics Covered | Lab. Experiment Assignments | Notes |
| :---: | :---: | :---: | :---: | :---: |
| 1 |  | Introduction to Data Structures |  |  |
| 2 |  | Algorithms and Complexity |  |  |
| 3 |  | Arrays and Pointers | Accountant application using arrays |  |
| 4 |  | Linked List 1 |  |  |
| 5 |  | Linked List 2 | Student information system using linked list |  |
| 6 |  | First exam |  |  |
| 7 |  | Stack | Color cubes games using Stack |  |
| 8 |  | Queue | A snake game using queu |  |
| 9 |  | Tree 1 |  |  |


| 10 | Tree 2 |  |  |  |
| :--- | :--- | :--- | :--- | :--- |
| 11 | Graph 1 |  |  |  |
| 12 | Graph 2 | Social Media <br> connections using <br> Graph data structure |  |  |
| 13 |  | Hashing 1 |  |  |
| 14 | Hashing 2 | Simple search engine <br> application using <br> hashtable data structure |  |  |
| 15 |  | Second try exam |  |  |



Instructor Signature:

# Data Structure <br> Lecture 1: Introduction 

Prepared by
Dr. Mohammed Salah Al-Obiadi


- Variable is any entity that can take on different values.
- Consider the below equation:


## $x^{2}+2 y-2=1$

## Variables and Data Types

- This equation has variables $x$ and $y$, which hold values (data).
- Data Type is a set of data with predefined values.
- The variables $x$ and $y$ in the above equation can take any values such as
- Integer numbers (10, 20), Real numbers $(0.23,5.5)$, or Boolean (0 or 1).
- There are two types of data types:
- System-defined data types.
- User-defined data types.
- These are the data types that are defined by system are called primitive data types.
- Examples of such data types are int, float, char, double, bool, etc.
- Each data type has some bytes to store data.


## System-defined data types:

- For example:
- int may take 2 bytes or 4 bytes of memory.
- float may take 3 bytes or 4 bytes of memory.
- char may take 1 byte of memory.
- Symbols in ASCII codes like,,+- , /, @, \#, etc... take 1 byte of memory. (See Appendix A for the list of ASCII codes)
- If we have $x+y$, and $x$ is int and $y$ is float. Assume int takes 2 bytes, and float takes 3 bytes, then the equation $x+y$ take $2+1+3=6$ bytes of memory.
- Note that the symbol ' + ' has 1 byte in the memory.


## User defined data types

- If the system-defined data types are not enough, then most programming languages allow the users to define their own data types.
- Good examples of user defined data types are: structures in C/C + + and classes in Java.
- Here is an example of new defined data type called "newType":

```
struct newType {
    int datal;
    float data 2;
    ."
    char data;
;
```


## Data Structures

3
Data structure is a particular way of storing and organizing data in a computer so that it can be used efficiently.

而
General data structure types include arrays, files, linked lists, stacks, queues, trees, graphs and so on.

## 囲 Data structures are classified into two types:

$\square$
1- Linear data structures: elements can be accessed in a sequential order (e.g. Arrays, Linked Lists, Stacks and Queues).

2- Non - linear data structures: elements can be accessed in a random order (e.g. Trees, tables, sets, graphs).

Sequential access


Random access


## Operations Performed in Data Structure

1- Traversing
2- Insertion
3- Deletion

4- Merging
5- Sorting
6- Searching

## Appendix A: <br> ASCII TABLE

|  | Hex | Char |  | Hex | Char |  | Hex | Char | Dec | Hex | Char |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | [NULL] | 32 | 20 | [SPACE] | 64 | 40 | ( | 96 | 60 | - |
| 1 | 1 | [START OF HEADING] | 33 | 21 | ! | 65 | 41 | A | 97 | 61 | a |
| 2 | 2 | [START OF TEXT] | 34 | 22 | " | 66 | 42 | B | 98 | 62 | b |
| 3 | 3 | [END OF TEXT] | 35 | 23 | \# | 67 | 43 | C | 99 | 63 | c |
| 4 | 4 | [END OF TRANSMISSION] | 36 | 24 | \$ | 68 | 44 | D | 100 | 64 | d |
| 5 | 5 | [ENQUIRY] | 37 | 25 | \% | 69 | 45 | E | 101 | 65 | e |
| 6 | 6 | [ACKNOWLEDGE] | 38 | 26 | \& | 70 | 46 | F | 102 | 66 | $f$ |
| 7 | 7 | [BELL] | 39 | 27 | 1 | 71 | 47 | G | 103 | 67 | g |
| 8 | 8 | [BACKSPACE] | 40 | 28 | 1 | 72 | 48 | H | 104 | 68 | h |
| 9 | 9 | [HORIZONTAL TAB] | 41 | 29 | ) | 73 | 49 | I | 105 | 69 | i |
| 10 | A | [LINE FEED] | 42 | 2A | * | 74 | 4A | J | 106 | 6A | j |
| 11 | B | [VERTICAL TAB] | 43 | 2B | + | 75 | 4B | K | 107 | 6B | k |
| 12 | C | [FORM FEED] | 44 | 2C | , | 76 | 4C | L | 108 | 6C | I |
| 13 | D | [CARRIAGE RETURN] | 45 | 2D | * | 77 | 4D | M | 109 | 6D | m |
| 14 | E | [SHIFT OUT] | 46 | 2E | , | 78 | 4E | N | 110 | 6E | n |
| 15 | F | [SHIFT IN] | 47 | 2F | 1 | 79 | 4F | 0 | 111 | 6 F | 0 |
| 16 | 10 | [DATA LINK ESCAPE] | 48 | 30 | 0 | 80 | 50 | P | 112 | 70 | p |
| 17 | 11 | [DEVICE CONTROL 1] | 49 | 31 | 1 | 81 | 51 | Q | 113 | 71 | q |
| 18 | 12 | [DEVICE CONTROL 2] | 50 | 32 | 2 | 82 | 52 | R | 114 | 72 | r |
| 19 | 13 | [DEVICE CONTROL 3] | 51 | 33 | 3 | 83 | 53 | S | 115 | 73 | 5 |
| 20 | 14 | [DEVICE CONTROL 4] | 52 | 34 | 4 | 84 | 54 | T | 116 | 74 | t |
| 21 | 15 | [NEGATIVE ACKNOWLEDGE] | 53 | 35 | 5 | 85 | 55 | U | 117 | 75 | u |
| 22 | 16 | [SYNCHRONOUS IDLE] | 54 | 36 | 6 | 86 | 56 | V | 118 | 76 | $v$ |
| 23 | 17 | [ENG OF TRANS. BLOCK] | 55 | 37 | 7 | 87 | 57 | W | 119 | 77 | w |
| 24 | 18 | [CANCEL] | 56 | 38 | 8 | 88 | 58 | X | 120 | 78 | x |
| 25 | 19 | [END OF MEDIUM] | 57 | 39 | 9 | 89 | 59 | Y | 121 | 79 | y |
| 26 | 1 A | [SUBSTITUTE] | 58 | 3A | : | 90 | 5A | Z | 122 | 7A | $z$ |
| 27 | 1 B | [ESCAPE] | 59 | 3B | ; | 91 | 5B | [ | 123 | 7B | \{ |
| 28 | 1 C | [FILE SEPARATOR] | 60 | 3C | $<$ | 92 | 5 C | 1 | 124 | 7C | 1 |
| 29 | 1D | [GROUP SEPARATOR] | 61 | 3D | = | 93 | 5D | ] | 125 | 7D | \} |
| 30 | 1 E | [RECORD SEPARATOR] | 62 | 3E | $>$ | 94 | 5E | ヘ | 126 | 7E | $\sim$ |
| 31 | 1 F | [UNIT SEPARATOR] | 63 | 3F | ? | 95 | 5F | - | 127 | 7F | [DEL] |

## Data Structure Lecture 2: <br> Algorithms and Complexity

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## What is an Algorithm?

- Steps of preparing a frying egg.

1. Get the frying pan.
2. Get the oil.
a. Do we have oil?
i. If yes, put it in the pan.
ii. If no, do we want to buy oil?
a. If yes, then go out and buy.

b. If no, no egg today.
3. Turn on the stove, etc...

An algorithm is the step-by-step clear instructions to solve a given problem.

## Criteria for judging Algorithms

- There are two main criteria for judging Algorithms:

1. Correctness: does the algorithm give solution to the problem in a finite number of steps?
2. Efficiency: how much resources (in terms of memory and time) does it take to execute the program.


## Complexity of an Algorithm

1. Space Complexity of a program is the amount of memory it needs to run to completion.
2. Time complexity of a program is the amount of computer time it needs to run to completion. The time complexity is of two types such as
a) Compilation time
b) Runtime

- Big-O Notation [Upper Bounding Function]: The $\mathrm{O}(\mathrm{g}(\mathrm{n}))$ represents the upper bound computation a program can cause to the computer. $\mathrm{f}(\mathrm{n})=\mathrm{O}(\mathrm{g}(\mathrm{n}))($ read as $f$ of $n$ is big oh of $g$ of $n$ )
- Example-1 Find upper bound for:
- $f(n)=3 n+8$
- solution $f(n)=O(n)$
- $f(n)=\mathrm{n}^{2}+1$
- Solution $f(n)=O\left(n^{2}\right)$
- $f(n)=16 \mathrm{n}^{3}+45 \mathrm{n}^{2}+12 \mathrm{n}$
- Solution $f(n)=O\left(\max \left(n^{3}, n^{2}, n\right)\right)=O\left(n^{3}\right)$
- $f(n)=n^{4}+100 n^{2}+50$
- Solution $f(n)=O\left(\max \left(n^{4}, n^{2}\right)\right)=O\left(n^{4}\right)$
- $f(n)=410$
- Solution $f(n)=O(1)$


## Example a program of $O(1)$ :

- Problem: To find out the greater between two numbers

```
bool max_value (int a, int b) // function that accept two
numbers
{
            if (a> b) // Compare the two numbers
                        return true; // if first is greater return true
            else
                return false. // otherwise return false
}
```

- This function does not have any loop and will not cost the computer a lot of computations, so it's $f(n)=O(1)$ means a constant computations.


## Example a program of $O(n)$ :

- Problem: Program to search a number from a list of numbers

```
bool search (int arr [], int number, int n)
{
bool found=false;
for (int i=0; i<n; i++)
{
            if (arr[i] ==number)
                {
                found=true;
                break;
                } // end of if
    } // end of for
    return found;
    } // end of function
```

- This function has a for loop that require $n$ time implementations from the computer, so it's $f(n)=O(n)$.


## Example a program of $O\left(n^{2}\right)$ :

Problem: Write a program to sort the series of numbers using Bubble sort

```
void array (int arr [], int n)
{
int i, j;
for (i=0; i<n; i++) // start of outer loop
{
        for (j=1; j<n-i; j++) // inner loop
        {
            if (arr [j+1] > arr[j]) // comparing the elements
                { // swapping if the adjacent is larger
                temp=arr [j+1];
                arr [j+1] =arr[j];
                arr[j] =temp;
            } // end of if
        } //end of inner for loop
} // end of outer for loop
```

- This function has two for loops that require $n \times n$ time implementations from the computer, so it's $f(n)=O(n \times n)=O\left(n^{2}\right)$.


## Big-O notation

Time
Compexity
Chart

# Data Structure Lecture 3: Arrays and Pointers 

Prepared by
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## Arrays data structures

- Arrays are widely used in any programming language.
- It is extremely useful in cases where we need to store the similar set of elements.
- It helps in reducing the program complexity.
- increases the programmer's productivity.
- Arrays can be categorized into the following:
- Single Dimensional array.
- Double Dimensional array.
- Multidimensional array.
- We will not study the Multidimensional array.


## Why we use Arrays

- Consider that we need to store grades of five students.
- In a normal way, we have to define five variables of the same type:

```
int main ()
    {
        int marks1, marks2, marks3, marks4, marks5;
        cout<<"enter marks1";
        cin>>marks1;
        cout<<"enter marks2";
        cin>>marks2;
        cout<<'"enter marks3";
        cin>>marks3;
        cout<<"enter marks4";
        cin>>marks4;
        cout<<"enter marks5";
        cin>>marks5;
        return 0;
    }
```

- Complexity of the above program will grow further upon increment of subjects.
- Consider we have 200 students, how the program will look like? What is the solution?
- Here the solutions lie with the usage of arrays.

A data
structure used to store set of similar data types.

Elements are
stored in continuous memory locations.

Size of the array should be constant.

## Declaration:

- Data type variable_name[bound] ;


## OneDimensional Array

## Examples:

- Int arr[10]; // an integer array with 10 elements.
- Char arr[20]; // a character array with 20 elements.
- float arr[15]; // a flaot array with 20 elements

The array elements are stored in a consecutive manner inside the memory.

For Example: int x[7];
Array
Element in
Memory
Let the $x[0]$ be at the memory address 568 , then the entire array can be represented in the memory as:

| $\mathrm{x}[0]$ | $\mathrm{X}[1]$ | $\mathrm{X}[2]$ | $\mathrm{X}[3]$ | $\mathrm{X}[4]$ | $\mathrm{X}[5]$ | $\mathrm{X}[6]$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 568 | 570 | 572 | 574 | 576 | 578 | 580 |

## Declaration:

- Data type variable_name[rows] [columns] ;


## Two-Dimensional

 Array
## Examples:

- Int arr[4][6]; // an integer 2-D array with 4 rows and 6 columns.
- Char arr[20][20]; // a character 2-D. array with 20 rows and 20 columns.
- float arr[5][10]; // a float 2-D array with 5 rows and 10 columns
int $x[3][4]=\{$
$\{1,2,3,4\}$,
$\{5,6,7,8\}$,
$\{2,4,6,3\}$,
\};
char $x[3][4]=\{$

$$
\text { \{'h', 'a', 'f', '7'\}, }
$$

# Examples of Twodimensional 

 arrays$$
\left\{\text { 'u' ' 'f', 'z', 'l' }^{\prime}\right. \text {, }
$$

$$
\text { \{'y', '8', 'j', 'm'\}, }
$$

$$
\}
$$

## Pointer is a variable that is capable to hold the address of another variable.

## POINTERS

Holding of addresses of another variable is needed in various instances that include:

| 1- To access | 2- To change <br> the value of <br> the array <br> element | variable from <br> function | 3-In dynamic <br> allocation of <br> memory. |
| :---: | :---: | :---: | :---: | | 4- In complex |
| :---: |
| programming, |
| such as link |
| list, tree, B |

How to know a variable is a pointer?

## Pointers are preceded with the symbol *.

## For instance:

- int *x, It means that this pointer can hold the address of integer type variable.
- char ${ }^{*} \mathrm{c}$, It means that this pointer can hold the address of char type variable.
- float *w, It means that this pointer can hold the address of float type variable.


## Example of Declaring pointer

int $x=8$;
int *p;// variable that is pointer of int type $\mathrm{p}=\alpha \mathrm{x}$; //p now holds the address of variable x cout<<p; // print the address of $x$; cout<<*p; // print the value pointed by p;

## Explaining Example of Declaring pointer

- Initially, the variable "x" is declared
- Assumes that it has been allocated the address location 1000.
- when int *p is declared, it is also allocated the address 925.
- When $\mathrm{p}=\& \mathrm{x}$, this means that p holds the address of variable x which is 1000 .
- Printing p will print address while printing * p will print x value.



## Pointer to pointer

- Sometimes, we need to store the address of a pointer.
- This can be accomplished with the help of pointer to pointer.
- Pointer to pointer is a variable that holds the address of another variable that is pointer type.
- Declaring pointer to pointer is different from the normal pointer type.
- In pointer to pointer notation two asterisk $\left({ }^{* *}\right)$ are preceded before the identifier.
- Example:-
- int **pp;
- int *p;
- $\mathrm{pp}=\& \mathrm{p}$;



## Data Structure Lecture 4: Linked List

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## What is a Linked List?

A linked list is a data structure used for storing collections of data.

## A linked list has the following properties:

1. Successive elements are connected by pointers.
2. The last element points to NULL.
3. Can grow or shrink in size during execution of a program.
4. Can be made just as long as required (until systems memory exhausts).
5. Does not waste memory space. It allocates memory as list grows.


## Linked List vs Arrays?

| Array | Linked list |
| :--- | :--- |
| Array elements store in a contiguous memory location. | Linked list elements can be stored anywhere in the memory |
| Array works with a static memory and cannot be changed at The Linked list works with dynamic memory means memory |  |
| the run time. | size can be changed at the run time. |
| Array elements are independent of each other. | Linked list elements are dependent on each other. As each <br> node contains the address of the next node. |

Array takes more time while performing any operation like Linked list takes less time while performing any operation insertion, deletion, etc. like insertion, deletion, etc.

Accessing any element in an array is faster as the element in Accessing an element in a linked list is slower as it starts an array can be directly accessed through the index. traversing from the first element of the linked list.

In the case of an array, memory is allocated at compile-time. In the case of a linked list, memory is allocated at run time.
Memory utilization is inefficient in the array. For example, if Memory utilization is efficient as the memory can be the size of the array is 6 , and array consists of 3 elements allocated or deallocated at the run time.
then the rest of the space will be unused.
Arrays take $O(1)$ for access to an element.
Linked lists take $\mathrm{O}(\mathrm{n})$ for access to an element.

## Operation on Linked List

1- Traversal: To traverse all the nodes one after another.
2- Insertion: To add a node at the given position.
3- Deletion: To delete a node.
4- Searching: To search an element(s) by value.
5- Updating: To update a node.
6- Sorting: To arrange nodes in a linked list in a specific order.
7- Merging: To merge two linked lists into one.

## Types of Link List

1- Single Link List

2- Double Link List

3- Circular Link List

4- Doubly Circular linked list

## Single Link List



## STRUCTURE OF THE NODE OF A LINKED LIST

```
Struct tagname
{
    Data type member1;
    Data type member2;
    ......................
    ....................
    Data type membern;
    Struct tagname *var;
};
{
int info;
struct link *next;
};
```


## Example:

struct link

## LOGIC FOR CREATION

struct link start, *node;


We can't guarantee addresses will be in a continues form, so we need pointers to keep addresses.


## Algorithm For Creation Of Single Link List

Struct link start, *node
create(start, node) [start is the structure type of variable][node is the structure type of pointer] step-1 : node $=\&$ start
step-2 : node $\rightarrow$ next $=$ new $\operatorname{link}() / /$ allocate memory of size struct link for the node
node $=$ node $\rightarrow$ next
input : node $\rightarrow$ info
node $\rightarrow$ next $=$ null
step-3 : repeat step-2 to create more nodes
step-4 : return

## Algorithm For Traversing Of Single Link List

struct link start, *node;
traverse(start,node) [start is the structure type of variable] [node is the structure type of pointer]
step-1 : node = start.next
step-2 : repeat while (node!=null )
write : node $\rightarrow$ info
node $=$ node $\rightarrow$ next
end of loop
step-3 : return

## Insertion Into Linked List

## The insertion process with link list can be discussed in four different ways:

1. Insertion at Beginning.
2. Insertion at End.
3. Insertion when node number is known.
4. Insertion when information is known.

## Algorithm For Insertion At Beginning

struct start, *first, *node,* newnode
insbeg(start,first,node, newnode) [start is the structure variable] [node and first is the structure pointer]
step-1 : first = \&start //first saves start's address
node $=$ start.next
step-2 : newnode = new link () input : newnode $\rightarrow$ info
first $\rightarrow$ next $=$ newnode
newnode $\rightarrow$ next $:=$ node
step-3 : return

## Algorithm For Insertion At Last

struct start, *last, *node,* newnode
inslast(start,last,node,newnode)
step-1 : last = \&start //last's pointer saves start's address
node $=$ start.next
step-2 : repeat while(node != null)

$$
\begin{aligned}
& \text { node }=\text { node } \rightarrow \text { next } \\
& \text { last }=\text { last } \rightarrow \text { next }
\end{aligned}
$$

step-3 : newnode $\rightarrow$ next $=$ new $\operatorname{link}() / /$ allocate a memory to newnode
input : newnode $\rightarrow$ info
last $\rightarrow$ next $=$ newnode
newnode $\rightarrow$ next $=$ null
step-4 : return

## Data Structure Lecture 5: Linked List

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## Algorithm For Insertion Of Node When Node Number Is Known

(insert in previous position)

```
struct start, *previous, *node, *newnode
insnode(start, previous,node,no,newnode) [no is the node number]
step-1 : previous := \&start
    node := start.next
    count :=1
step-2 : repeat while(node != null)
if \((\) count \(=\) no \()\) then:
                newnode \(\rightarrow\) next \(=\) new \(\operatorname{link}()\) \allocate space in memory
                input : newnode \(\rightarrow\) info
                previous \(\rightarrow\) next \(=\) newnode
                newnode \(\rightarrow\) next \(=\) node
                return
else :
```

node $=$ node $\rightarrow$ next
previous $=$ previous $\rightarrow$ next
count:=count+1
step-3 : return

## Example: insert a new node at node number 3



## Algorithm For Insertion Of Node When Information Is Known

```
insertnode(start, previous,node,data,newnode) [data is information to insert]
step-1 : previous := &start
    node := start.next
step-2 : repeat while(node != null)
    if(node }->\mathrm{ info = data) then:
        newnode }->\mathrm{ next = new link()
        input: newnode }->\mathrm{ info /*insert data*/
        previous }->\mathrm{ next = newnode
        newnode }->\mathrm{ next = node
        return
    else:
        node = node }->\mathrm{ next
        previous = previous }->\mathrm{ next
step-3 : return
```


## Example: insert a new node at info $=7$



## Algorithm For Deletion From Beginning

```
dellbeg(start,first, node)
step-1 : first := &start
        node := start.next
step-2 : first }->\mathrm{ next = node }->\mathrm{ next
    free(node)
step-3 : return
```


## Example of Deleting a node in the beginning



## Algorithm For Deletion Of Node When Node Number Is Known

```
delnode(start,previous,node,no) [no is the node number]
step-1 : previous := &start
    node := start.next
    count :=1
step-2 : repeat while(node != null)
    if(count = no) then:
        previous }->\mathrm{ next =node }->\mathrm{ next
            free(node)
            return
        else :
        node = node }->\mathrm{ next
        previous:= previous }->\mathrm{ next
    count:=count+1
step-3 : return
```


## Example: delete a node at node number 3



```
dlelinfo(start,previous,node,data) [data is the information to insert]
step-1 : previous = &start
    node = start.next
step-2 : repeat while(node != null)
    if(node }->\mathrm{ info = data) then:
        previous }->\mathrm{ next =node }->\mathrm{ next
        free(node)
        return
    else :
        node = node }->\mathrm{ next
        previous = previous }->\mathrm{ next
step-3: return
```


## Example: delete a node at info =4



# Data Structure Lecture 6: Stack 

## Prepared by

Dr. Mohammed Salah Al-Obiadi

## STACK

- Stack is a linear data structure.
- Follows the principle of LIFO (Last in First Out).
- Any data structure use the LIFO principle, it can be called as STACK.


## Operations Performed With STACK

1- PUSH: which adds an element to the collection.

2-POP: which removes the most recently added element.

## Overflow conditions

- During the PUSH (add) operation, we have to check the condition for overflow
- Condition for OVERFLOW
- Top = size -1 (for the STACK starts with 0)
- Example of stack of size 6.
- Now the stack has 6 items so we can't add any item.


## Item 6: We can't add Item 6 because Stack is full

```
Item 5 Top
Item 4
Item 3
Item 2
Item 1
Item 0
```


## Underflow conditions

- During the POP (delete) operation, we have to check the condition for underflow.
- Condition for
- Top =-1 (for the STACK starts with 0)
- Example of stack of size 6.
- Now the stack is empty and Top=-1, so we can't remove any item.

Stack is empty. There is nothing to delete


## EXAMPLES

STACK[5]

top $=-1($ CONDITION FOR EMPTY

top $=0$
PUSH(25)

top $=1$
PUSH(53)

top $=2$
PUSH(78)

| 5 | 25 | 53 | 78 |  |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | 2 | 3 | 4 |

top $=3$
PUSH(99)

| 5 | 25 | 53 | 78 | 99 |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | 2 | 3 | 4 |

top $=4$

## Can we do PUSH(76)??

No, because OVERFLOW (top $=$ size -1 Condition for OVERFLOW)
POP

| 5 | 25 | 53 | 78 |  |
| :--- | :--- | :--- | :--- | :--- |
| 0 | 1 | 2 | 3 | 4 | top $=3$


| 5 | 25 | 53 |  |  |
| :--- | :--- | :--- | :--- | :--- |
| 0 | 1 | 2 | 3 |  |

$$
\text { top }=2
$$

```
POP
POP

\[
\text { top }=1
\]

top \(=-1 \quad\) Can we do POP??
No, because the stack is underflow (top \(=-1\) Condition for underflow)

\section*{Algorithm For Push Operation}

PUSH(stack[size], no, top) [no is the number to insert] [top is the position of the stack] step-1 : if (top = size - 1) then :
write: "overflow"
return
step-2 : top : = top +1
stack[top]:= no
step-3 : return

\section*{Algorithm For POP Operation}
\(\boldsymbol{p o p}(\) stack [size] , top) [stack[size] is the stack] [top is the position of the stack]
Step-1 : if (top =-1) then :
write : "underflow"
return
Step-2 : write : stack[top]
\[
\text { top }:=\text { top }-1
\]

Step-3 : return

\section*{Algorithm For Traverse Operation}

Traverse(stack[size], top)
Step-1 : if (top =-1) then :
write : "stack is empty"
return
Step-2 : set i : = 0
Step-3 : repeat for \(\mathrm{i}=\) top to 0 by -1 write : stack[i]
Step-4 : return

\section*{Algorithm For Update Operation}

Can you do it?

\section*{Applications of STACK}

1- Checking of the parenthesis of an expression

2- Reversing of a string


3- In Recursion


4- Evaluation of Expression


In a stack, insertion and removal of the item was permitted only from one end.

Item inserted at last removed first from the stack.

How to ensure that the items are removed in the order they have inserted?? Solution is the Queue.
Queue
Queue is a data structure that can be considered as open from both the ends.

Queue follows the principle of FIFO (First In First Out).

Insertion is accomplished from one end known as rear.

Removal of the item is taking place on the other end known as front.

\section*{Queue}

\section*{enqueue() operation \\  \\ REAR \\ }

\section*{1- Insert.}
- During the INSERT operation

\section*{commonly implemented operations} we have to check the condition for OVERFLOW

\section*{2- Delete.}
- During the DELETE operation we have to check the condition for UNDERFLOW.

\section*{Types of Queue}

\author{
1- Linear Queue
}

\section*{2- Circular Queue}

3- D - Queue (Double ended queue)

4- Priority Queue.

\section*{Linear Queue}
- A linear queue is a linear data structure that serves the request first, which has been arrived first.
- Overflow: insert an element with a filled QUEUE.
- Condition for OVERFLOW
- Rear = size -1
- UNDERFLOW: delete an element from an empty QUEUE.
- Condition for UNDERFLOW
- Front =-1 (for the QUEUE starts with 0 )
- CONDITION FOR EMPTY QUEUE
- Front = -1 and Rear = -1

\section*{Example: insert}

QUEUE[5]

front \(=-1\), rear \(=-1\)

front \(=0\), rear \(=0\)
 front \(=0\), rear \(=1\)
 front \(=0\), rear \(=2\)
 front \(=0\), rear \(=3\)

\[
\text { front }=0, \text { rear }=4
\]

INSERT(145)
"OVERFLOW"
(rear = size -1 Condition for OVERFLOW)

\section*{Example: Delete}

\section*{DELETE}
\begin{tabular}{|c|c|c|c|c|}
\hline & 25 & 53 & 78 & 99 \\
\hline 0 & 1 & 2 & 3 & 4 \\
\hline
\end{tabular}
front \(=1\), rear \(=4\)
DELETE
\begin{tabular}{|c|c|c|c|c|}
\hline & & 53 & 78 & 99 \\
\hline 0 & 1 & 2 & 3 & 4 \\
\hline
\end{tabular}
front \(=2\), rear \(=4\)

front \(=3\), rear \(=4\)
DELETE

front \(=4\), rear \(=4\)

front \(=-1\), rear \(=-1\)

DELETE
"UNDERFLOW"
( front \(=-1\) Condition for UNDERFLOW

\section*{Algorithm For Insert Operation}
- Insert(queue[size], front, rear, no)
- Step 1 : if (rear = size -1 ) then :
- write: "overflow"
- return
- Step 2 : if (rear \(=-1\) ) then :
- front := 0
- rear :=0
- else:
- rear :=rear+1
- Step 3: queue[rear] :=no
- Step 4: return

\section*{Algorithm For Delete Operation}
- Delet(queue[size], front, rear)
- Step 1 : if (front =-1) then :
- write : "underflow"
- return
- Step 2 : write: queue[front]
- Step 3 : if (front ==rear ) then :
- front :=-1
- rear :=-1
- else :
- front := front +1
- Step 4: return

\section*{Algorithm For Traverse Operation}
- Traverse(queue[size], front, rear)
- Step 1 : if (front =-1) then :
- write : "queue is empty"
- return
- Step 2 : set \(i=0\)
- Step 3 : repeat for \(\mathrm{i}=\) front to rear
- write : queue[i]
- Step 4: return

\section*{Algorithm For Update Operation}
- Update(queue[size], no, front, rear)
- Step-1 : if (rear = - 1) then :
- write : "stack is empty"
- return
- Step-2 : set i: =0
- Step-3 : repeat for \(i=\) front to rear
- if (no = queue[i]) then:
- queue[i] = new no
- return
- if \(\mathrm{i}=\) rear then:
- write : "update not completed"
- Step-4 : return

\section*{Applications of Queue}
\(\square\) 1- Operating systems schedule jobs (with equal priority) in the order of arrival (e.g., a print queue).


2- Simulation of real-world queues such as lines at a ticket counter or any other first come first-served scenarios.

3- Multiprogramming.


4- Asynchronous data transfer (file IO, pipes, sockets).


5- Waiting times of customers at call center.

6- Determining number of cashiers to have at a supermarket.


\section*{TREE}
- A TREE is a dynamic data structure that represents the hierarchical relationships between individual data items.
- It's a data structure in which the elements are arranged in the parent and child relationship manner.
- In a tree, nodes are organized in a hierarchical way in such a way that:
- Root is the beginning of the tree.
- Branches are Lines that connecting the nodes.
- Leaf nodes are nodes that have no children.

Figure 1 shows Example of a Tree


\section*{Tree Terminologies}

Node: Each element of a tree is called as node. In the previous figure there are 14 nodes.

Root is the beginning of the tree. In figure 1: A is the root node.

Parent: Parent of a node is the immediate predecessor of a node. In figure 1: B is the parent of E and F.
Child: Each immediate successor of a node is known as child. In figure 1: B, C, D are children of A.
Siblings: The child nodes of a given parent node are called siblings. In figure 1: H, I, J are siblings.
Degree of a Node: The number of sub-trees of a node in a given tree. In figure 1:
- The degree of node \(A\) is 3
- The degree of node B is 2
- The degree of node G is 1
- The degree of node F is 0

\section*{Tree Terminologies}

Degree of Tree: The maximum degree of nodes in a given tree. In the figure the maximum degree of nodes A and D is 3 . So the degree of Tree is 3.

Terminal Node: A node with degree zero is called terminal node or a leaf.

Level: The entire tree structured is leveled in such a way that the root is always at the level 0 , then its immediate children are at level 1, and their immediate children are at level \(\mathbf{2}\) and so on up

Path: Path is the sequence of consecutive edges from the source node to the destination node path between \(A\) and \(M\) is ( \(A, D\) ), \((D, I),(I, M)\).

Height: The height of node \(\mathbf{n}\) is the length of the longest path from n to leaf. The height of B is \(\mathbf{2}\) and F is \(\mathbf{0}\).

\section*{BINARY TREE}
- A binary tree is a special form of a tree in which every node of the tree can have at most two children.

OR
- In a binary tree the degree of each node is less than or equal to 2.


\section*{Types of Binary Tree}
1. Full Binary Tree
2. Perfect Binary Tree
3. Pathological Binary Tree

\section*{Full Binary Tree}

In a Full Binary Tree the out degree of every node is either 2 or Nil.


\section*{Perfect Binary Tree}

Perfect Binary Tree is a Binary Tree in which all nodes have 2 children and all the leaf nodes are at the same depth or same level.


\section*{Pathological Binary Tree}


\section*{Array Representation of a Tree}
- The ROOT node is always kept as the FIRST element of the array i.e/ in the 0Index the root node will be store. Then, in the successive memory locations the left child and right child are stored.
- Example:

ARRAY REPRESENTATION

\begin{tabular}{|c|}
\hline A \\
\hline B \\
\hline C \\
\hline D \\
\hline- \\
\hline- \\
\hline E \\
\hline
\end{tabular}

\section*{Linked List Representation of a Tree (Double Linked List)}


\section*{Operations Performed With the Binary Tree}
\(>\) Creation
\(>\) Insertion
\(>\) Deletion
\(>\) Searching
>Copying
\(>\) Merging
\(>\) Updating

\section*{Algorithm for Creation of Binary Tree}

Create (node, info) [node is the structure having both left and right pointer. info is data] Step-1 : if (node = null) then:

Node := new Node() allocate a memory to node
Node \(\rightarrow\) info := info
Node \(\rightarrow\) left := null
Node \(\rightarrow\) right := null
return
Step-2 : if node \(\rightarrow\) info>= info then:
create(node \(\rightarrow\) left, info)
else:
create(node \(\rightarrow\) right, info)
Step-3 : return(node)


\section*{Traversing With Tree}
- The tree traversing is the way to visit all the nodes of the tree on a specific order.
- The Tree traversal can be accomplished in four different ways:
- Inorder Traversal
- Pre Order Traversal.
- Post Order Traversal
- Level Order Traversal

\section*{Inorder traversal}
- Traverse the Left Subtree in INORDER(Left)
- Visit the Root node
- Traverse the Right Subtree in INORDER(Right)

\section*{Preorder Traversal}
- Visit the Root Node
- Traverse the Right Subtree in PREORDER(Left)
- Traverse the Right Subtree in PREORDER(Right)

\section*{Postorder Traversal}
- Traverse the Right Subtree in POSTORDER(Left)
- Traverse the Right Subtree in POSTORDER(Right)
- Visit the Root Node

\section*{Level Order Traversal}
- In this type of traversal the elements will be visited according to level wise but it is not so far used.

\section*{Examples}

1. Inorder: DBHEAFCIG
2. Preorder: ABDEHCFGI
3. Post Order: DHEBFIGCA
4. Level Order: ABCDEFGHI

\section*{Conversion Of A Tree From Inorder And Preorder}
- INORDER : D B HEAFCIG
- PREORDER : A B D E H C F GI
- Choose the ROOT from the preorder and from inorder find the nodes in left and right and this process will continue up to all the elements are chosen from the preorder/inorder.
- STEP1: From preorder A is the root and from inorder we will find that in the left of A \((\mathbf{D}, \mathbf{B}, \mathbf{H}, \mathbf{E})\) and in the right (F,C,I,G):
```

A
(D,B,H,E) (F,C,I,G)

```
- STEP2: Again from Preorder 'B' will be chosen as PARENT and from Inorder in the left of B (D) and in the right (H,E):

- STEP3: From Preorder 'E' will chosen as the PARENT and from inorder on its left ' H ' is present:

- STEP4: From Preorder we will choose ' C ' as the PARENT and from inorder we observe that in the left of ' \(\mathbf{C}\) ' \((\mathbf{F})\) will placed and in the right (I,G)

(I,G)
- STEP5: From the PREORDR we observe that ' \(G\) ' is the parent and from the INORDER I will be used as the Left child of ' \(G\) '.


\section*{Conversion Of A Tree From Inorder And Postorder}
- INORDER: DBHEAFCIG
- POST ORDER : D HEBFIGCA
- Choose the ROOT from the postorder (from the right) and from inorder find the nodes in left and right and this process will continue up to all the elements are chosen from the postorder/inorder.
- STEP1 : From the right of POSTORDER 'A' will be chosen as the ROOT and from INORDER we observe that in the left of \(\mathrm{A}(\mathbf{D}, \mathbf{B}, \mathbf{H}, \mathbf{E}, \mathbf{A})\) and in the right ( \(\mathbf{F}, \mathbf{C}, \mathbf{I}, \mathbf{G}\) ) will be there.
- STEP2: From the POSTORDER 'C' will be chosen as the PARENT and from INORDER we observe that in the right of ' \(\mathrm{C}^{\prime}(\mathbf{I}, \mathbf{G})\) and to the left \((\mathbf{F})\) will be used.

(D,B,H,E,A) (F,C,I,G)

- STEP3: From the right of POSTORDER 'G' will be chosen as the PARENT and from inorder to the left of ' \(G\) ' (I) will be used.
(D,B,H,E)
- STEP4: From the right to postorder ' \(B\) ' will be chosen as the PARENT and from the INORDER to we observe that to the right of 'B' (H,E,A) and to the left (D) will be used.

- STEP5: From the right to postorder we will choose ' \(E\) ' as PARENT and from the Inorder to the left of ' \(E\) ' (H) will be used.


\section*{Applications of Binary Tree}
- Expression Tree
- Binary Search Tree
- Height Balanced Tree (AVL Tree)
- Threaded Binary Tree
- Heap Tree
- Huffman Tree
- Decision Tree
- Red Black Tree

\section*{Expression Tree}
- An expression tree is a Binary Tree which stores/represents the mathematical (arithmetic) expressions.
- The leaves of an expression tree are operands, such as constants or variable names and all the internal nodes are the operators.
- Formally we can define an expression Tree as a special kind of binary tree in which:
- Each leaf is an operand. Examples: a, b, c, 6, 100
- The root and internal nodes are operators. Examples: +, -, *, /, ^
- Subtrees are subexpressions with the root being an operator.

\section*{EXAMPLE}
- Represent an Expression Tree
- \(\mathbf{A}+(\mathbf{B} * \mathbf{C})-\left(\mathbf{D}^{\wedge} \mathbf{E}\right) / \mathbf{F}+\mathbf{G} * \mathbf{H}\)
- choose an operator in such a way that the terms in parenthesis will be in a side
- Choose a operator having higher precedence.

STEP1:


STEP2:


Step3:



\section*{Graph}
- GRAPH is a non-linear data structure in which the elements are arranged randomly inside the memory and are interconnected with each other
- A graph \(G\) is an ordered pair of sets \((\mathrm{V}, \mathrm{E})\) where
- V is the set of vertices and
- E is the edges which connect the vertices.


Vertices

\section*{Applications of Graph}
- Google maps uses graphs for building transportation systems.
- In Facebook, users are considered to be the vertices and if they are friends then there is an edge running between them.
- In World Wide Web, web pages are considered to be the vertices. There is an edge from a page \(u\) to other page \(v\) if there is a link of page \(v\) on page u.
- Path Optimization Algorithms, Path optimizations are primarily occupied with finding the best connection that fits some predefined criteria.
- Recommendation Search Engines: google uses graph to represent pages and their importance.

\section*{Graph Terminologies}
- Directed Graph: A graph in which every edge is directed is called undirected graph.

- Undirected Graph: A graph in which every edge is undirected


\section*{Graph Terminologies}

N
WEIGHTED GRAPH: A graph is said to be weighted if its edges have been assigned some non-negative value as weight.


Path is the sequence of consecutive edges from the source node to the destination node.

CYCLIC GRAPH A graph that has cycles is called as cyclic graph.

\(\square\) SOURCE A node which has no incoming edges, but has outgoing edges


\section*{Graph Terminologies}

A node, which has no outgoing edges but has incoming edges

\(B\) is a Sink node

DEGREE In an undirected graph the number of edges connected to a node is called the degree of that node. In graph-3 the degree of the node \(A\) is 3 and the degree of the node \(B\) is 2 .


REGULAR GRAPH A graph is regular if every node is adjacent to the same number of nodes


The major components of the graph are node and edges.

Like tree the graph can also be represented in two different ways such as
Representation of Graph
- ARray representation
- LINKED REPRESENTATION

Overall, there are four major approaches to represent the graph as
- Adjacency Matrix
- Adjacency Lists
- Adjacency Multilists
- Incedince Matrix

\section*{Adjacency Matrix}
- The nodes that are adjacent to one another are represented as matrix.
- The adjancy matrix of the graph G is a two-dimensional array of size n * n (Where n is the number of vertices in the graph) with the property that \(\mathrm{A}[I][\mathrm{J}]=1\), if the edge \((\mathrm{VI}, \mathrm{VJ})\) is in the set of edges and \(A[I][J]=0\) if there is no such edge

Example:

\begin{tabular}{|l|l|l|l|l|l|l|}
\hline & V1 & V2 & V3 & V4 & V5 & V6 \\
\hline V1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline V2 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline V3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline V4 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline V5 & 0 & 0 & 1 & 1 & 0 & 1 \\
\hline V6 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
\end{tabular}

If the graph was Undirected:

\begin{tabular}{|l|l|l|l|l|l|l|}
\hline & V1 & V2 & V3 & V4 & V5 & V6 \\
\hline V1 & 0 & 1 & 0 & 1 & 0 & 0 \\
\hline V2 & 1 & 0 & 0 & 1 & 0 & 0 \\
\hline V3 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline V4 & 1 & 1 & 0 & 0 & 1 & 0 \\
\hline V5 & 0 & 0 & 1 & 1 & 0 & 1 \\
\hline V6 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
\end{tabular}

\section*{Traversal of Graph}
- The Graph Traversal is of two types such as
- Breadth First Search (BFS).
- Depth First Search (DFS).


\section*{Hash Table}

- Hashing is a technique used for storing and retrieving information as quickly as possible.
- It is used to perform optimal searches and is useful in implementing symbol tables.
- Arrays, linked list, trees requires \(\mathrm{O}(\mathrm{N})\) or \(\log (\mathrm{N})\) to, search, traverse, add or delete, while with hashtable it requires only O(1) time to do these operations.
- A hash table is a array with mapping function.

\section*{Why hash table is important}
- If we have a set of names Ahmed, Yaser, Yasien, Mina, Mustafa, Lina.
- Let's suppose that you put these names in an array or a linked list.
- Let's suppose that you are looking for the name Lina.
- Then, you will need to match all the array or linked list elements to find that name because it's the last name in the list.
- This name costs loop of 6 to find it and if the list of names \(=n\), then it costs \(O(n)\) to find it.
- With hash table, you give the name Lina, and you get it immediately in \(\mathrm{O}(1)\). How is that? We will see in the next slides.

\section*{Time Complexity in Big O notation}

\section*{Algorithm}

Space
Search
Insert
Delete

Worst case
\(O(n)\)
\(O(n)\)
\(O(n)\)
\(O(n)\)

\section*{Components of Hashing}
- Hashing has four key components:
1. Hash Table
2. Hash Functions
3. Collisions
4. Collision Resolution Techniques

\section*{Hash Table}
- Hash table or hash map is a data structure that stores the keys and their associated values.
- Hash table is a kind of array.
- It uses a hash function to map keys to their associated values.
- The terms widely used with hash table are Key, value. See Figure down.


The hash function is used to transform the key into the index.

The hash function should map each possible key to a unique slot index.

\section*{Hash}

A hash function that maps each item into a unique slot is referred to as a perfect hash function.

A hash function can lead to collisions.

Collisions happen when a hash function for two keys produce same index.

Our goal is to create a hash function that minimizes the number of collisions

\section*{How to choose your hash function?}
- Example of phone number 436-555-4601.
- Consider the hash function \(\mathrm{h}(\mathrm{x})=\) sum of numbers \(\%\) hashtable size.
- The \% is the mode or division reminder and x is the phone number.
- Suppose hashtable size=10.

- So, \(\mathrm{h}(436-555-4601)=(4+3+6+5+5+5+4+6+0+1) \% 10=4\)
- Another phone (479-345-6537)

- \(\mathrm{h}(479-345-6537)=(4+7+9+3+4+5+6+5+3+7) \% 10=3\)
\begin{tabular}{llll|l|l|l|l|l|}
\hline & & & \(479-345-6537\) & \(436-555-4601\) & & & & \\
\hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{tabular}
- \(\mathrm{h}(564-767-6537)=(5+6+4+7+6+7+6+5+3+7) \% 10=6\)
\begin{tabular}{l|l|l|l|l|l|l|l|l|}
\hline & & & \(479-345-6537\) & \(436-555-4601\) & & \(564-767-6537\) & & \\
\hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{tabular}
- \(h(565-387-9865)=(5+6+5+3+8+7+9+8+6+5) \% 10=2\)
\begin{tabular}{ll|l|l|l|l|l|l|l|}
\hline & & \(565-387-9865\) & \(479-345-6537\) & \(436-555-4601\) & & \(564-767-6537\) & & \\
\hline
\end{tabular}
- \(h(343-387-9865)=(3+4+3+3+8+7+9+8+6+5) \% 10=6\), here we have collision because index 6 already full.
- Problems with hashtable:
1. Number of elements can be larger than hashtable size.
2. An input that can lead to same index which is called collision

\section*{Characteristics of Good Hash Functions}


\section*{Collisions}

Hash functions are used to map each key to a different address space

But it is not possible to create such a hash function and the problem is called collision.

Collision is the condition where two records are stored in the same location

\section*{Collision Resolution Techniques}
- Collision resolution is The process of finding an alternate location.
- There are a number of collision resolution techniques:
- Direct Chaining: An array of linked list application
- Separate chaining

- Open Addressing: Array-based implementation
- Linear probing (linear search)
- Quadratic probing (nonlinear search)
- Double hashing (use two hash functions)

The interval between probes is fixed at 1.

In linear probing, we search the hash table sequentially, starting from the original hash location.

\section*{Linear probing}

If a location is occupied, we check the next location.
\[
\text { rehash(key) }=(n+1) \% \text { tablesize }
\]

Where n is the current location found for key

\section*{Linear Probing Example}


Linear probing can lead to cluster of keys together.

\section*{Quadratic Probing}

The problem of Clustering can be eliminated if we use the quadratic probing method
In quadratic probing, we start from the original hash location n .

If a location is occupied, we check the next location.
\[
\text { rehash(key) = ( } \left.n+k^{2}\right) \% \text { tablesize }
\]

Where n is the current location found for key

If a location is occupied, we check the locations \(i+1^{2}\), \(i+2^{2}, i+3^{2}, i+4^{2}\)
- Let us assume that the table size is 11 (0..10)
- Insert keys
- \(31 \bmod 11=9\)
- \(19 \bmod 11=8\)
- \(2 \bmod 11=2\)
- \(13 \bmod 11=2 \rightarrow 2+1^{2} \bmod 11=3\)
- \(25 \bmod 11=3 \rightarrow 3+1^{2} \bmod 11=4\)
- \(24 \bmod 11=2 \rightarrow 2+1^{2} \bmod 11,2+2^{2}=6\)
- \(21 \bmod 11=10\)
- \(9 \bmod 11=9 \rightarrow 9+1^{2}, 9+2^{2} \bmod 11,9+3^{2}\) \(\bmod 11=7\)
\begin{tabular}{|c|}
\hline \\
\hline \\
\hline 2 \\
\hline 13 \\
\hline 25 \\
\hline 5 \\
\hline 24 \\
\hline 9 \\
\hline 19 \\
\hline 31 \\
\hline 21 \\
\hline
\end{tabular}```

