Chapter 1

Functions

1.1 Review: Numbers

1.1.1 Different Kinds of Numbers

The simplest numbers are the ***positive integers***

$$1; 2; 3; 4;$$

the number zero

$$0;$$

and the ***negative integers***

$$...; -4; -3; -2; -1 :$$

Together these form the integers or "whole numbers."

Next, there are the numbers you get by dividing one whole number by another (nonzero) whole number. These are the so called fractions or ***rational numbers*** such as

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{4}{3}, \cdots$$

and

$$-\tfrac{1}{2}, -\tfrac{1}{3}, -\tfrac{2}{3}, -\tfrac{1}{4}, -\tfrac{2}{4}, -\tfrac{3}{4}, -\tfrac{4}{3}, \cdots$$

By definition, any whole number is a rational number (in particular zero is a rational number).

You can add, subtract, multiply and divide any pair of rational numbers and the result will again be a rational number (provided you don't try to divide by zero).

One can represent certain fractions as decimal fractions, e.g.

$$\frac{279}{25} = \frac{1116}{100} = 11.16$$

Not all fractions can be represented as decimal fractions. For instance, expanding $\frac{1}{3}$ into a decimal fraction leads to an unending decimal fraction

$$\frac{1}{3} = 0.333333333333333333...$$

It is impossible to write the complete decimal expansion of $\frac{1}{3}$ because it contains infinitely many digits. But we can describe the expansion: each digit is a three. An electronic calculator, which always represents numbers as ***finite*** decimal numbers, can never hold the number $\frac{1}{3}$ exactly.

Every fraction can be written as a decimal fraction which may or may not be finite. If the decimal expansion doesn't end, then it must repeat. For instance,

$$\frac{1}{7} = 0.142857142857142857$$

Conversely, any infinite repeating decimal expansion represents a rational number.

A ***real number*** is specified by a possibly unending decimal expansion. For instance,

$$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769...$$

Of course you can never write all the digits in the decimal expansion, so you only write the first few digits and hide the others behind dots.

Why are Real Numbers Called Real?

At some point it becomes useful to assume that there is a number whose square is -1. No real number has this property since the square of any real number is positive, so it was decided to call this new imagined number "***imaginary***" and to refer to the numbers we already have (rationales, $\sqrt{2}$-like things) as "***real***".

1.1.2 The Real Number Line and Intervals

It is customary to visualize the real numbers as points on a straight line. We imagine a line, and choose one point on this line, which we call the origin. We also decide which direction we call "***left***" and hence which we call "***right***". Some draw the number line vertically and use the words "***up***" and "***down***".

To plot any real number x one marks off a distance x from the origin, to the right (up) if $x > 0$, to the left (down) if $x < 0$.

The ***distance along the number line*** between two numbers x and y is $|x - y|$. In particular, the distance is never a negative number. See figure 1.1.

FIGURE 1.1

The collection of all real numbers between two given real numbers form an interval. The following notation is used:

1. $(a; b)$ is the set of all real numbers x which satisfy $a < x < b$.
2. $[a; b)$ is the set of all real numbers x which satisfy $a \leq x < b$.
3. $(a; b]$ is the set of all real numbers x which satisfy $a < x \leq b$.
4. $[a; b]$ is the set of all real numbers x which satisfy $a \leq x \leq b$.

If the endpoint is not included then it may be ∞ or $-\infty$. E.g. $(-\infty; 2]$ is the interval of all real numbers (both positive and negative) which are ≤ 2.

1.1.3 Set Notations

A common way of describing a set is to say it is the collection of all real numbers which satisfy a certain condition. One uses this notation

$$\mathcal{A} = \{x \mid x \text{ satisfies this or that condition}\}$$

For instance, the interval $(a; b)$ can be described as

$$(a; b) = \{x \mid a < x < b\}$$

The set

$$\mathcal{B} = \{x \mid x^2 - 1 > 0\}$$

consists of all real numbers x for which $x^2 - 1 > 0$, i.e. it consists of all real numbers x for which either $x > 1$ or $x < -1$ holds. This set consists of two parts: the interval $(-\infty; -1)$ and the interval $(1; \infty)$.

You can try to draw a set of real numbers by drawing the number line and colouring the points belonging to that set red, or by marking them in some other way.

Some sets can be very difficult to draw. For instance,

$$\mathcal{A} = \{x \mid x \text{ is a rational number}\}$$

If $\mathcal{A}$ and $\mathcal{B}$ are two sets then the union of $\mathcal{A}$ and $\mathcal{B}$ is the set which contains all numbers that belong either to $\mathcal{A}$ or to $\mathcal{B}$. The following notation is used

$$\mathcal{A} \bigcup \mathcal{B} = \{x \mid x \text{ belongs to } \mathcal{A} \text{ or to } \mathcal{B} \text{ or both } \}$$

Similarly, the intersection of two sets $\mathcal{A}$ and $\mathcal{B}$ is the set of numbers which belong to both sets. This notation is used:

$$\mathcal{A} \bigcap \mathcal{B} = \{x \mid x \text{ belongs to both } \mathcal{A} \text{ and } \mathcal{B} \text{ }\}$$

1.2 Functions

In this section we're going to make sure that you're familiar with functions and function notation. Both will appear in almost every section in a Calculus class and so you will need to be able to deal with them.

what exactly is a function?

An equation will be a function if for any x in the domain of the equation (the domain is all the $x's$ that can be plugged into the equation) the equation will yield exactly one value of y.

This is usually easier to understand with an example.

Example 1 Determine if each of the following are functions. (a) $y = x^2 + 1$

(b) $y^2 = x + 1$

Solution

(a) This first one is a function. Given an x there is only one way to square it and then add 1 to the result and so no matter what value of x you put into the equation there is only one possible value of y.

(b) The only difference between this equation and the first is that we moved the exponent off the x and onto the y. This small change is all that is required, in this case, to change the equation from a function to something that isn't a function.

To see that this isn't a function is fairly simple. Choose a value of x, say $x = 3$ and plug this into the equation.

$$y^2 = 3 + 1 = 4$$

Now, there are two possible values of y that we could use here. We could use $y = 2$ or $y = -2$.

Since there are two possible values of y that we get from a single x this equation isn't a function.

Note that this only needs to be the case for a single value of x to make an equation not be a function. For instance we could have used $x = -1$ and in this case we would get a single y ($y = 0$).

However, because of what happens at $x = 3$ this equation will not be a function.

Function Notation

Consider the following function

$$y = 2x^2 - 5x + 3$$

we can write this as any of the following.

$$\begin{aligned} f(x) &= 2x^2 - 5x + 3 \\ g(x) &= 2x^2 - 5x + 3 \\ h(x) &= 2x^2 - 5x + 3 \\ y(x) &= 2x^2 - 5x + 3 \end{aligned}$$

Recall that this is NOT a letter times x, this is just a fancy way of writing y.

Now, ***how do we actually evaluate the function?*** That's really simple. Everywhere we see an x on the right side we will substitute whatever is in the parenthesis on the left side. For our function if $x = -3$ this gives,

$$f(-3) = 2(-3)^2 - 5(-3) + 3 = 36$$

Example 2 Given $f(x) = -x^2 + 6x - 11$ find each of the following.

(a) $f(2)$

(b) $f(-10)$

(c) $f(t)$

(d) $f(t-3)$

(e) $f(x-3)$

(f) $f(4x-1)$

Solution

(a) $f(2) = -(2)^2 + 6(2) - 11 = -3$

(f) $f(4x-1) = -(4x-1)^2 + 6(4x-1) - 11 = -16x^2 + 32x - 18$

Finding Roots of Functions

A root of a function is nothing more than a number for which the function is zero. In other words, finding the roots of a function, $g(x)$, is equivalent to solving

$$g(x) = 0$$

Example 3 Determine all the roots of $f(t) = 9t^3 - 18t^2 + 6t$

Solution

So we will need to solve,

$$9t^3 - 18t^2 + 6t = 0$$

First, we should factor the equation as much as possible. Doing this gives,

$$3t(3t^2 - 6t + 2) = 0$$

Next recall that if a product of two things are zero then one (or both) of them had to be zero. This means that,

$$3t = 0$$
$$\text{OR,}$$
$$3t^2 - 6t + 2$$

From the first it's clear that one of the roots must then be $t = 0$. To get the remaining roots we will need to use the quadratic formula on the second equation. Doing this gives,