(b) $y = \sqrt{z}\sin^{-1}(z)$

Solution

(a)

$$\dot{f}(t) = -\frac{4}{\sqrt{1-t^2}} - \frac{10}{1+t^2}$$

(b)

$$\dot{y} = \frac{1}{2}z^{-\frac{1}{2}}\sin^{-1}(z) + \frac{\sqrt{z}}{\sqrt{1-z^2}}$$

3.6 Derivatives of Hyperbolic Functions

In many physical situations combinations of e^x and e^x arise fairly often. Because of this these combinations are given names. There are six hyperbolic functions and they are defined as follows.

$$\begin{aligned} \sinh x =& \frac{\mathbf{e}^x - \mathbf{e}^{-x}}{2} & \cosh x &= \frac{\mathbf{e}^x + \mathbf{e}^{-x}}{2} \\ \tanh x =& \frac{\sinh^x}{\cosh x} & \coth x &= \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} \\ \mathrm{sech} x =& \frac{1}{\cosh x} & \mathrm{csch} x &= \frac{1}{\sinh x} \end{aligned}$$

Here are the graphs of the three main hyperbolic functions. We also have the following facts about the hyperbolic functions.

$$\begin{aligned} \sinh(-x) &= -\sinh(x) & \cosh(-x) &= \cosh(x) \\ \cosh^2(x) - \sinh^2(x) &= 1 & 1 - \tanh^2(x) &= \mathrm{sec}h^2(x) \end{aligned}$$

Because the hyperbolic functions are defined in terms of exponential functions, the derivatives of it is

$$\frac{d}{dx}\left(\mathbf{e}^{-x}\right) = -\mathbf{e}^{-x}$$

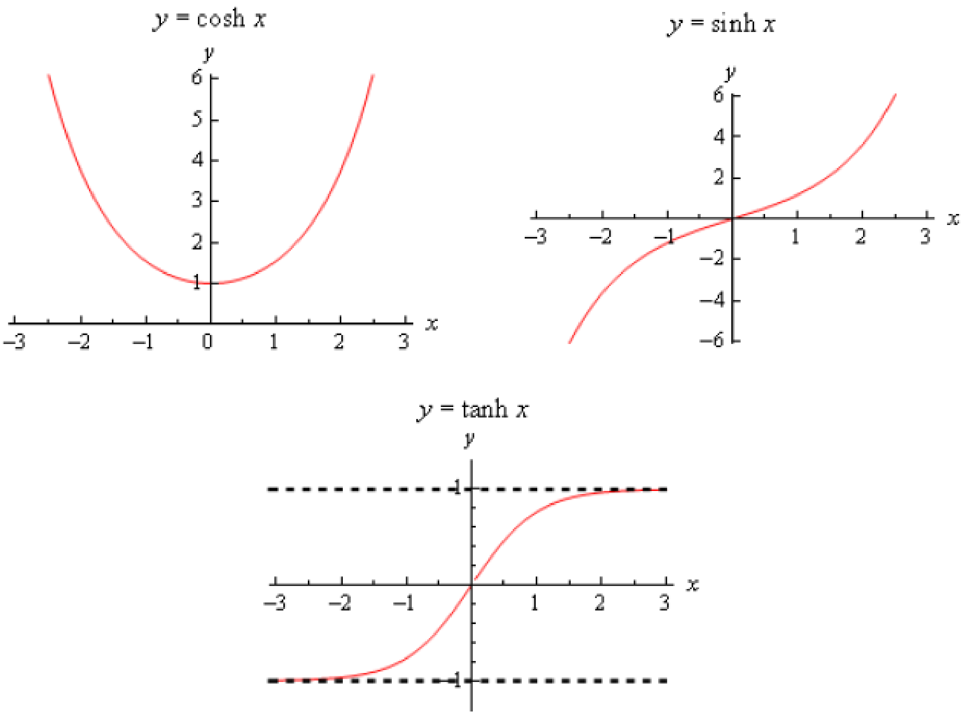

FIGURE 3.3

then,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\frac{\mathbf{e}^x - \mathbf{e}^{-x}}{2} = \frac{\mathbf{e}^x - (-\mathbf{e}^{-x})}{2} = \frac{\mathbf{e}^x + \mathbf{e}^{-x}}{2} = \cosh x$$

Then the other derivatives are

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \mathrm{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\mathrm{csch}^2 x$$
$$\frac{d}{dx}(\mathrm{sech} x) = -\,\mathrm{sech} x \tanh x \qquad \frac{d}{dx}(\mathrm{csch} x) = -\mathrm{csch} x \coth x$$

Example Differentiate each of the following functions.

(a) $f(x) = 2x^5 \cosh x$

(b) $h(t) = \frac{\sinh t}{t+1}$

Solution

(a)

$$\dot{f}(x) = 10x^4 \cosh x + 2x^5 \sinh x$$

(b)

$$\dot{h}(t) = \frac{(t+1)\cosh t - \sinh t}{(t+1)^2}$$

3.7 Chain Rule

There are two forms of the chain rule.

Chain Rule

Suppose that we have two functions $f(x)$ and $g(x)$ and they are both differentiable.

1- If we define $F(x) = (fog)(x)$ then the derivative of $F(x)$ is,

$$\dot{F}(x) = \dot{f}\,(g(x))\,\dot{g}(x)$$

2- If we have $y = f(u)$ and $u = g(x)$ then the derivative of y is,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example Use the Chain Rule to differentiate $R(z) = \sqrt{5z - 8}$.

Solution

The two functions that we needed for the composition are,

$$f(z) = \sqrt{z} \qquad\qquad g(z) = 5z - 8$$

and their derivatives,

$$\dot{f}(z) = \frac{1}{2\sqrt{z}} \qquad\qquad \dot{g}(z) = 5$$

So, using the chain rule we get,

$$\begin{aligned}\dot{R}(z) &= \dot{f}\,(g(z))\,\dot{g}(z)\\ &= \frac{1}{2\sqrt{5z-8}}\,(5)\\ &= \frac{5}{2\sqrt{5z-8}}\end{aligned}$$

Let's take the function from the previous example and rewrite it slightly. The

$$R(z) = \underbrace{(5z-8)}_{\text{inside function}}{}^{\underbrace{\frac{1}{2}}_{\text{outside function}}}$$

derivative is then, In general this is how we think of the chain rule. We identify

$$R'(z) = \overbrace{\frac{1}{2}\underbrace{(5z-8)}_{\substack{\text{inside function}\\ \text{left alone}}}{}^{-\frac{1}{2}}}^{\substack{\text{derivative of}\\ \text{outside function}}}\ \underbrace{(5)}_{\substack{\text{derivative of}\\ \text{inside function}}}$$

the "inside function" and the "outside function". We then differentiate the outside function leaving the inside function alone and multiply all of this by the derivative of the inside function. In its general form this is,

$$F'(x) = \underbrace{f'}_{\substack{\text{derivative of}\\ \text{outside function}}}\ \underbrace{(g(x))}_{\substack{\text{inside function}\\ \text{left alone}}}\ \underbrace{g'(x)}_{\substack{\text{times derivative}\\ \text{of inside function}}}$$

Example Differentiate each of the following.

(a) $f(x) = \sin\left(3x^2+2\right)$

(b) $f(t) = \left(2t^3+\cos(t)\right)^{50}$

(c) $h(w) = \mathbf{e}^{w^4-3w^2+9}$

(d) $g(x) = \ln\left(x^{-4}+x^4\right)$

(e) $y = \sec\left(1-5x\right)$

(f) $P(t) = \cos^4(t) + \cos(t^4)$

Solution

(a) It looks like the outside function is the sine and the inside function is $3x^2+x$. The derivative is then.

$$\dot{f}(x) = (6x+1)\cos(3x^2+x)$$

(b) In this case the outside function is the exponent of 50 and the inside function is all the stuff on the inside of the parenthesis. The derivative is then.

$$\dot{f}(t) = 50\left(6t^2 - \sin(t)\right)\left(2t^3 + \cos(t)\right)^{49}$$

(c)

$$\dot{h}(w) = \left(4w^3 - 6w\right)\mathbf{e}^{w^4-3w^2+9}$$

(d)

$$\dot{g}(x) = \frac{1}{x^{-4}+x^4}\left(-4x^{-5}+4x^3\right) = \frac{-4x^{-5}+4x^3}{x^{-4}+x^4}$$

(e)

$$\dot{y} = -5\sec(1-5x)\tan(1-5x)$$

(f) There are two points to this problem. First, there are two terms and each will require a different application of the chain rule. That will often be the case so don't expect just a single chain rule when doing these problems. Second, we need to be very careful in choosing the outside and inside function for each term.
The first term can actually be written as,

$$\cos^4(t) = (\cos(t))^4$$

So, in the first term the outside function is the exponent of 4 and the inside function is the cosine. In the second term it's exactly the opposite. In the second term the outside function is the cosine and the inside function is t^4. Here's the