Mathematics

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1^{st} Class

Chapter One
Functions

1.1 The Set of Numbers

1) N = set of natural numbers.
N = {1, 2, 3, 4……}

2) Z = set of integers.
Z= {……., -3, -2, -1, 0, 1, 2, 3…}

3) Quotient, Q = set of rational numbers.
$Q = [x{:}\, x = \frac{p}{q}\ p \text{ and } q \text{ are integers } q \neq 0\ ]$

<u>Ex:</u> $\frac{3}{2}$, - $\frac{4}{5}$, $\frac{3}{1}$, $\frac{-7}{1}$

4) Irr = set of irrational numbers
Irr = {*X* : *X is not a rational number*}

<u>Ex:</u> $\sqrt{2}$, $\sqrt{3}$, $-\sqrt{7}$

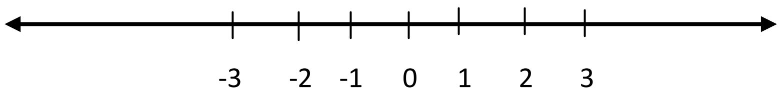
5) R: set of real numbers
R = set of all rational and irrational numbers

<u>Note:</u>

R = Q U Irr

N ⊂ Z ⊂ Q

<u>Note:</u> The set of real numbers is represented by a line called a line of numbers:



1.2 Finite Intervals

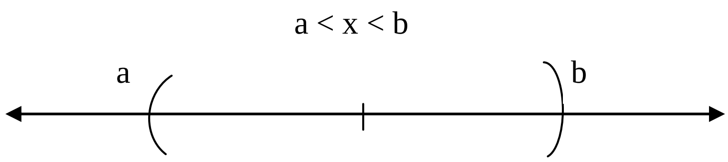
The set of values that a variable χ may take on is called the domain of χ . The domains of the variables in many applications of calculus are intervals like those shows below.

1. Open intervals

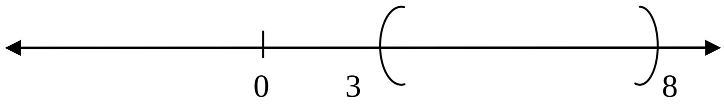
Is the set of all real numbers that lie strictly between two fixed numbers a and b:

In symbols

$(\boldsymbol{a}, \boldsymbol{b})$ ***or*** $\boldsymbol{a < x < b}$



Ex: The open interval : (3, 8) represented as { 3 < x < 8 }

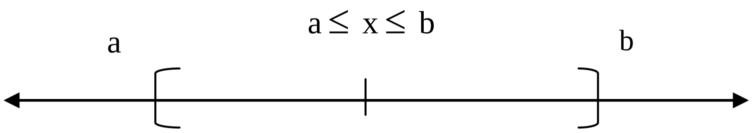


2. Close Intervals

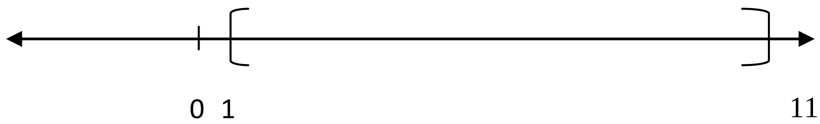
The closed intervals contain both endpoints:

In symbols

$[\boldsymbol{a}, \boldsymbol{b}]$ **or** $\boldsymbol{a \le \chi \le \mathrm{b}}$



Ex: The closed interval : [1, 11] represented as { 1 ≤ x ≤ 11 }



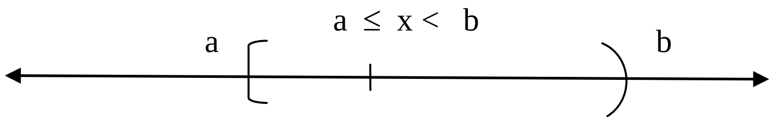
3. Half – open intervals

Contain one opened intervals but not both end points. two types from this intervals.

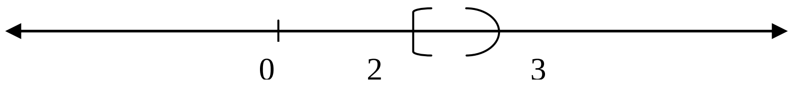
- **Left- closed , right -open.**

In symbols:

$[\boldsymbol{a}, \boldsymbol{b})$ **or** $\boldsymbol{a \le x < \mathrm{b}}$



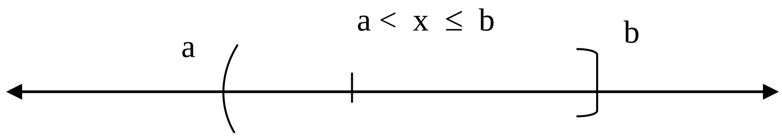
Ex: The interval :[2,3) represented as {2 ≤ x < 3}



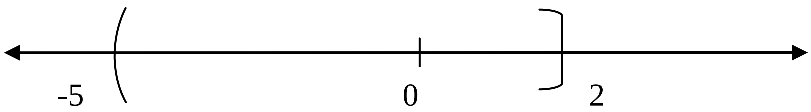
- **Left- open , right -closed.**

In symbols:

$(\boldsymbol{a}, \boldsymbol{b}]$ **or** $\boldsymbol{a} < \mathbf{x} \leq \mathbf{b}$



Ex: The interval (-5,2] represented as {-5 < x <= 2}



1.3 Infinite Intervals

There are four possible "infinite intervals":

$(\boldsymbol{a}, +\infty)$ ***where*** $\boldsymbol{x} > a$

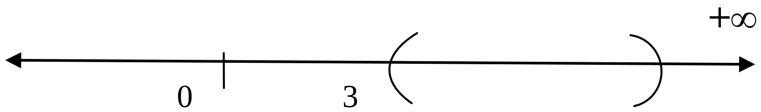
$[\boldsymbol{a}, +\infty)$ ***where*** $\boldsymbol{x} \geq \boldsymbol{a}$

$(-\infty, \boldsymbol{a})$ ***where*** $\boldsymbol{x} < a$

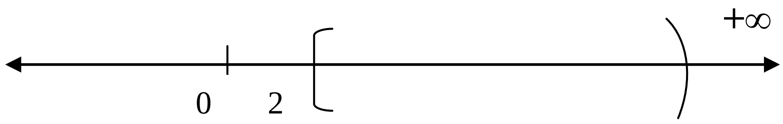
$(-\infty, \boldsymbol{a}]$ ***where*** $\boldsymbol{x} \leq \boldsymbol{a}$

We show no limited intervals by ends using this notation: **(-**∞**,** +∞**)**.

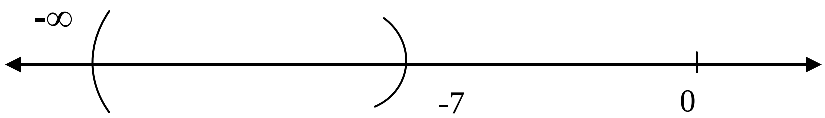
Ex: The infinite interval $(3, +\infty)$ where $x > 3$ represented as:



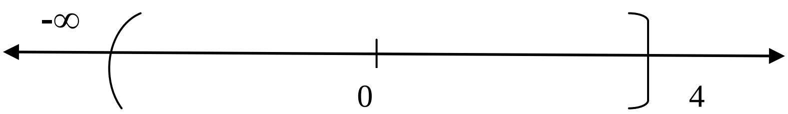
Ex: The infinite interval $[2, +\infty)$ where $x \geq 2$ represented as:



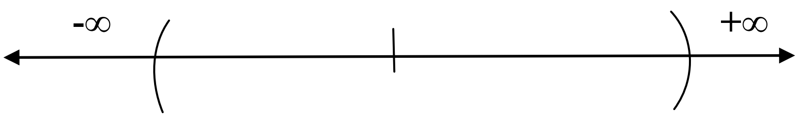
Ex: The infinite interval $(-\infty, \mathbf{-7})$ where $x< 7$ represented as:



Ex: The infinite interval $(-\infty, 4]$ where $x\leq 4$ represented as:



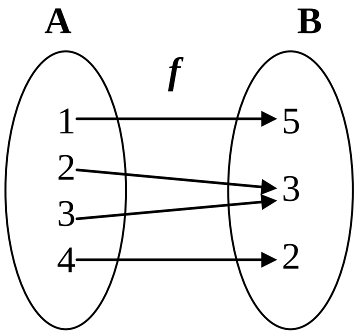
Ex: The infinite interval $\mathbf{(-\infty, +\infty)}$ represented as:



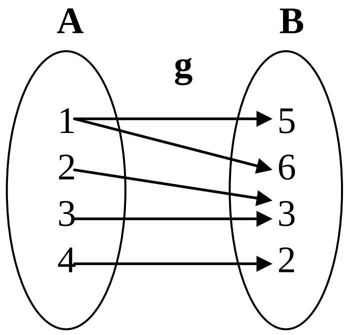
1.4 Function

A function, say F is a relation between the elements of two sets say A and B such that for every $\chi \in A$ there exists one and only one $Y \in B$ with Y = F(X).

The set A which contain the values of χ is called the domain of function F. The set B which contains the values of Y corresponding to the values of χ is called the range of the function F. χ is called the independer variable of the function F, while Y is called the dependant variable of F.



$f : A \rightarrow B$ is a function. Every element in A has associated with it exactly one element of B .



$g : A \rightarrow B$ is not a function. The element 1 in set X is assigned two elements, 5 and 6 in set Y.

Note:

1. Sometimes the domain is denoted by DF and the range by RF.
2. Y is called the image of χ .

Ex: Let the domain of χ be the set {0,1,2,3,4}. Assign to each value of χ the number $Y = x^2$. The function so defined is the set of pairs, { (0,0), (1,1), (2,4), (3,9), (4,16) }.

Ex: Let the domain of χ be the closed interval $-2 \le \chi \le 2$. Assign to each value of χ the number $y = \chi^2$ .The set of order pairs (χ , y) such that $-2 \le \chi \le 2$ And $y = \chi^2$ is a function.

Ex: The function that pairs with each value of χ different from 2 the number

$$f(x) = \frac{x}{x-2}$$

$$y = f(x) = \frac{x}{x-2} \quad x \neq 2$$

Ex: Let the function of, $\mathbf{f(x) = x^2}$: the value of x=4, find the f(x).
In fact we can write f(4) = 16.

1.5 Find Domain of Function

1) Domain of Polynomial Function

A polynomial of degree n with independent variable, written **Fn(x)** or simply $f(\chi)$ is an expression of the form:

$$Fn(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n \text{ , } a_0 \neq 0$$

Where a_0 , $a1$,......., a_n are constant (numbers).
The domain of polynomial function includes all real numbers R.

Ex:

(i) $f(x) = 5X$ polynomial of degree one.

(ii) $f(x) = 3x^5 - 2x + 7$ polynomial of degree five.

(iii) $f(x) = 8$ polynomial of degree Zero.

2) Domain of Rational Function

The domain of rational function include all real numbers R excepted the numbers that make the denominator equal to zero.

Ex:

$$f(x) = \frac{x}{x^2-4}$$

$$f(x) = \frac{x}{x^2-4} \quad x \neq \mp 2$$

3) Domain of Root Function

The domain of root function include all real numbers R that make the root greater than or equal to zero.

Ex:

$$f(x) = \sqrt{x-1}$$
$$x - 1 \geq 0 \rightarrow x \geq 1$$

1.6 Algebraic of Functions

Let $f(\chi)$ and $g(\chi)$ be two function.

1) $(f \mp g)(x) = f(x) \mp g(x)$
2) $(f.g)(x) = f(x).g(x)$
3) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad if\ g(x) \neq 0$

Ex: Let $f(x) = x + 2$, $g(x) = \sqrt{x - 3}$ evaluate the $f \mp g$, $f.g$ and $\frac{f}{g}$

Sol:

1) $(f \mp g)(x) = f(x) \mp g(x) = (x + 2) \mp (\sqrt{x - 3})$
2) $(f.g)(x) = f(x).g(x) = (x + 2)(\sqrt{x - 3})$
3) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{(x+2)}{(\sqrt{x-3})}$ $\{x: x \rangle 3\}$

1.7 Functions Types

1) Composite Function

Let $f(\chi)$ and $g(\chi)$ be two functions

We define: $(fog)(x) = f(g(x))$

Ex: Let $f(x) = x^2$, $g(x) = x - 7$ evaluate $(fog)(x)$and $(gof)(x)$

Sol: $(fog)(x) = f[g(x)] = f(x - 7) = (x - 7)^2$

$(gof)(x) = g[f(x)] = g(x^2) = x^2 - 7$

$\therefore (fog) \neq (gof)$

Ex: Let $f(x) = 5x - 4$, $g(x) = 2x + 1$ evaluate fog

Sol: $(fog)(x) = f[g(x)] = 5(2x + 1) - 4 = 10x + 5 - 4 = 10x + 1$

2) Inverse Function

Given a function F with domain A and the range B.
The inverse function of f written f^{-1}, with x= $f^{-1}(y)$.
Ex: Let the function, $f(x) = 2x + 3$, find the inverse function.

Sol: $f(x) = 2x + 3$

$Y = 2x + 3$

$Y - 3 = 2x$

$(Y - 3)/2 = x$

$x = (Y - 3)/2$

$\therefore x = f^{-1}(y)$

3) Even Function

f (x) is even if f (-x) = f (x)

Ex: f (x) =(x^2) is even

Sol: since $f(-x) = (-x^2) = (x^2) = f(x)$

4) Odd Function

f (x) is odd if $f(-x) = -f(x)$.

Ex: $f(x) = x^3$ is odd

Sol: since $f(-x) = -x^3 = -f(x)$

5) Absolute Function

We define the absolute value function $f(x) = |x|$. The absolute values of X:

$$|X| = \begin{cases} +x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

Ex: find the absolute function for, $f(x) = |x - 2|$

Sol: For $f(x) = |x - 2|$ we have

$$f(x) = \begin{cases} +(x-2) & if \quad x - 2 \geq 0 \;\; or \;\; x \geq 2 \\ -(x-2) & if \quad x - 2 < 0 \;\; or \;\; x < 2 \end{cases}$$

That is,

$$f(x) = \begin{cases} x - 2 & for \quad x \geq 2 \\ -x + 2 & for \quad x < 2 \end{cases}$$

6) Signum Function

The signum function is represented as sgn(x).

$$f(x) = sgn(x) = \begin{cases} 1 & when \quad x > 0 \\ 0 & when \quad x = 0 \\ -1 & when \quad x < 0 \end{cases}$$

Ex: sgn(-122) = -1.

sgn(0) = 0.

sgn(412) = 1.

1.8 Graphs of Functions

The set of points in the plane whose coordinate pairs are also the ordered pairs of function is called the graph of function. Graph a function includes three steps as:

1. Make a table of pairs from the function.
2. Plot enough of the corresponding points to learn the shape of the graph. Add more pairs to the table if necessary.
3. Complete the sketch by connecting the points.

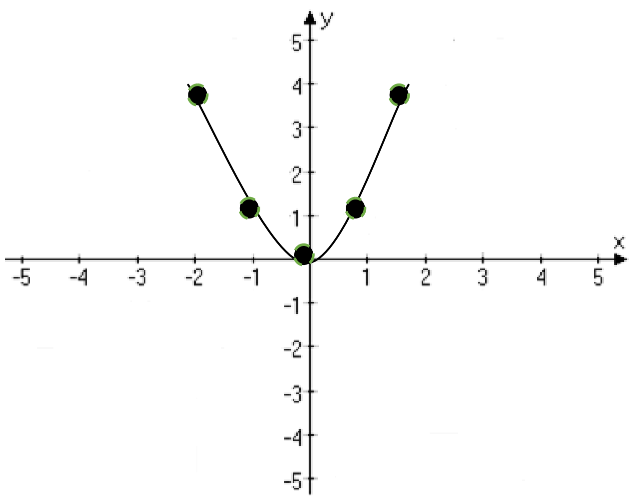
Ex: Sketch the graphs of the following function. $y = x^2$

Sol:

1. Make a table of pairs from the function.

x	$y = x^2$	(x , y)
-2	4	(-2,4)
-1	1	(-1,1)
0	0	(0,0)
1	1	(1,1)
2	4	(2,4)

2. Plot enough of the corresponding points and Complete the sketch by connecting the points.

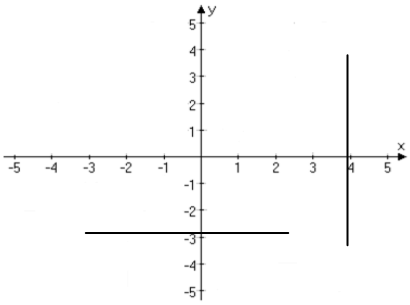


Ex: Sketch the graphs of the linear functions:

1) $x = 4$

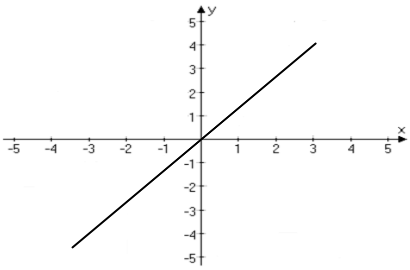
2) $y = -3$

Sol:



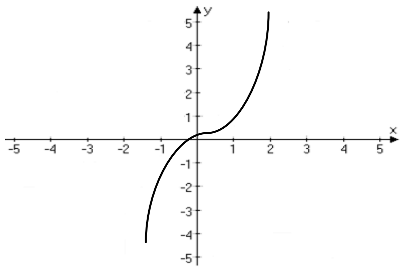
Ex: Sketch the graphs of the equal function: $y = x$

Sol:



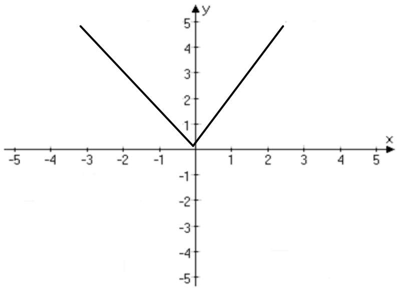
Ex: Sketch the graphs of the power function: $y = x^3$

Sol:



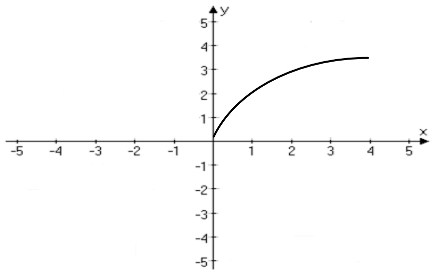
Ex: Sketch the graphs of the absolute value function: $y = |x|$

Sol:



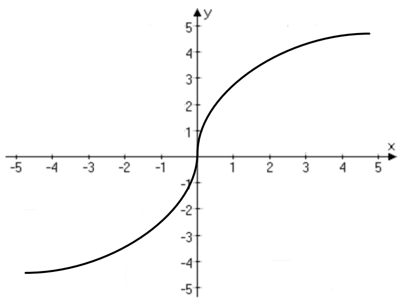
Ex: Sketch the graphs of the root function: $y = \sqrt{x}$

Sol:



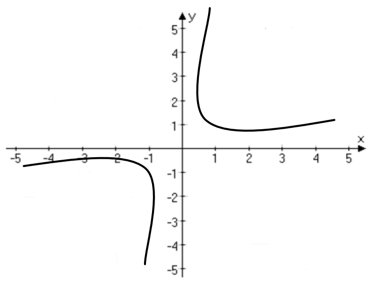
Ex: Sketch the graphs of the root function: $y = \sqrt[3]{x}$

Sol:



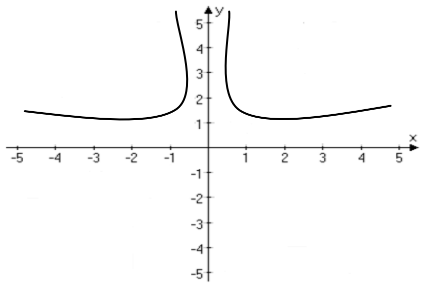
Ex: Sketch the graphs of the rational function: $y = \frac{1}{x}$

Sol:



Ex: Sketch the graphs of the rational function: $y = \frac{1}{x^2}$

Sol:



1.9 Limits

We say that L is a right hand limit for $f(\chi)$ when X approaches C for the right, written

$\lim_{x\to c^+} f(x) = L^+$

Similarly, L is the left – hand limit for $f(\chi)$ when X approaches C for the left, written

$\lim_{x\to c^-} f(x) = L^-$

Ex: Find the $\lim_{x\to 1} \frac{(x^2-1)}{(x-1)}$

Sol: $\lim_{x\to 1} \frac{x\mp 1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{(x-1)}$

$\lim_{x\to 1} x + 1 = 1 + 1 = 2$

Theorem 1:

If $\lim_{x\to c} f(x) = L_1$, $\lim_{x\to c} g(x) = L_2$

Then

1) $\lim_{x\to c}[f(x) \mp g(x)] = L1 + L2$

Ex: Find the $\lim_{x\to 2}\left[f(3x^2+1) \mp g(\sqrt{2x^2+1})\right]$

Sol: $\lim_{x\to 2}\left[f(3x^2+1) + g(\sqrt{2x^2+1})\right] = (3(2^2)+1+(\sqrt{2(2^2)+1})$

$(13) + (3) = 16$

$\lim_{x\to 2}\left[f(3x^2+1) - g(\sqrt{2x^2+1})\right] = (3(2^2)+1-(\sqrt{2(2^2)+1})$

$(13) - (3) = 10$

2) $\lim_{x\to c}[f(x).g(x)] = L1.L2$

Ex: Find the $\lim_{x\to 2}\left[f(3x^2+1).g(\sqrt{2x^2+1})\right]$

Sol: $\lim_{x\to 2}\left[f(3x^2+1).g(\sqrt{2x^2+1})\right] = (3(2^2)+1).(\sqrt{2(2^2)+1})$

$(13).(3) = 39$

3) $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{L1}{L2} \quad if L2 \neq 0$

Ex: Find the $\lim_{x\to 2}\frac{(x^3-8)}{(x^2-4)}$

Sol: $\lim_{x\to 2}\frac{(x^3-8)}{(x^2-4)} = \lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \lim_{x\to 2}\frac{x^2+2x+4}{x+2} = \frac{12}{4} = 3$

4) $\lim_{x\to c}[K\, f(x)] = KL_1$ Where K is a constant

Ex: Find the $\lim_{x\to 2}(4x^2)$

Sol: $\lim_{x\to 2}(4x^2) = 4\lim_{x\to 2} x^2 = 4(2^2) = 16$

5) $\lim_{x\to c}[a_0 + a_1x_1 + a_2x^2 + \cdots + a_nx^n]$

$= a_0 + a_1c + a_2c^2 \cdots + a_nc^n$

Ex: Find the $\lim_{x\to 2}(x^2-9)$

Sol: $\lim_{x\to 2}(x^2-9) = 4 - 9 = -5$

6) *Lim K* = *K* , K is constant

Ex: Find the $\lim_{x\to -1} 3$

Sol: $\lim_{x\to -1} 3 = 3$

Ex: Find the $\lim_{x\to 0}\frac{(x^2+3)}{(x^2+2x+1)}$

Sol: $\lim_{x\to 0}\frac{(x^2+3)}{(x^2+2x+1)} = \frac{\lim_{x\to 0}(x^2+3)}{\lim_{x\to 0}(x^2+2x+1)} = \frac{3}{1} = 3$

Note:

❖ The limit of the absolute value function includes applied limit from left hand and right hand, L^+ ,L^- .

Ex: Find the $\lim_{x\to 0} |x|$

Sol: $\lim_{x\to 0}\begin{cases} x \ if & x \geq 0 \\ -x \ if & x < 0 \end{cases}$

L$^+$ = $\lim_{x\to 0^+}|x| = 0$

L$^-$ = $\lim_{x\to 0^-}|x| = 0$

$\therefore\ L^+ = L^-$

Ex: Find the $f(X) = \frac{|x+1|}{x^2-1}$, at $x = -1$

Sol: $x^2 - 1 \neq 0 \to (x+1)(x-1) \neq 0 \to x \neq -1, x \neq +1$

$$|x+1| = \begin{cases} +(x+1)\ if \quad x+1 \geq 0, X \geq -1 \\ -(x+1) if\ \ x+1 < 0, X < -1 \end{cases}$$

L$^+$= $\lim_{x\to -1^+} \frac{(x+1)}{(x^2-1)} = \lim_{x\to -1^+} \frac{(x+1)}{(x-1)(x+1)} = \lim_{x\to -1^+} \frac{1}{x-1} = -\frac{1}{2}$

L$^-$= $\lim_{x\to -1^-} \frac{-(x+1)}{(x^2-1)} = \lim_{x\to -1^-} \frac{-(x+1)}{(x-1)(x+1)} = \lim_{x\to -1^-} \frac{-1}{x-1} = \frac{1}{2}$

$\therefore L^+ \neq L^-$

Ex: Find the $f(x) = \begin{cases} x^2+1, & x < -1 \\ 3x+1, & x \geq -1 \end{cases}$, at $x = -1$

Sol: $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} 3x+1 = 3(-1)+1 = -2$

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^2+1 = (-1)^2+1 = 2$

$\therefore \lim_{x\to -1^+} f(x) \neq \lim_{x\to -1^-} f(x)$

Ex: Find the $\lim_{x\to 3} \sqrt{2x-6}$, at $x = 3$

Sol: $2x - 6 \geq 0 \to 2x \geq 6 \to x \geq 3$

L$^+$ = $\lim_{x\to 3} \sqrt{2x-6} = \sqrt{6-6} = 0$

L$^-$ = not exit

$\therefore$ L$^+$ $\neq$ L$^-$ The limit is not exit

Ex: Find the $f(x) = \frac{|x|}{x}$, at $\chi = 0$

Sol: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{-x}{x} = -1$

$\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$

$lim\ \frac{|x|}{x}$ does not exist

1.10 Infinity Limits

This type of limits include division by the highest exponent of x from numerator and denominator.

1) $\lim_{x\to 0^+} \frac{1}{x} = \infty$
2) $\lim_{x\to 0^-} \frac{1}{x} = -\infty$
3) $\lim_{x\to \infty} \frac{1}{x} = 0$

Ex: Find the $\lim_{x\to\infty} \frac{(x^3+x^2+2)}{(x^2+1)}$

Sol: $\lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{x^2}{x^3}+\frac{2}{x^3}}{\frac{x^2}{x^3}+\frac{1}{x^3}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{2}{x^3}}{\frac{1}{x}+\frac{1}{x^3}} = \frac{1+\frac{1}{\infty}+\frac{1}{\infty}}{\frac{1}{\infty}+\frac{1}{\infty}} = \frac{1}{0} = \infty$

Ex: Find the $\lim_{x\to\infty} \frac{(x^2+2x+1)}{(5x^2+2)}$

Sol: $\lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{2x}{x^2}+\frac{1}{x^2}}{5\frac{x^2}{x^2}+\frac{2}{x^2}} = \lim_{x\to\infty} \frac{1+\frac{2}{x}+\frac{1}{x^2}}{5+\frac{2}{x^2}} = \frac{1+\frac{2}{\infty}+\frac{1}{\infty}}{5+\frac{2}{\infty}} = \frac{1}{5}$

Ex: Find the $\lim_{x\to-\infty} \frac{2}{x+1}$

Sol: $\lim_{x\to-\infty} \frac{\frac{2}{x}}{\frac{x}{x}+\frac{1}{x}} = \lim_{x\to-\infty} \frac{\frac{2}{x}}{1+\frac{1}{x}} = \frac{0}{1+0} = 0$

1.11 Continuity

A function f is said to be continuous at $\chi = C$ provided the following conditions are satisfied:

1) $f(C)$ is defined
2) $\lim_{x\to c} f(x)$ exists
3) $\lim_{x\to c} f(x) = f(x)$

Ex: Check the continuity of the function at $x = 0$.

$f(x) = 5$

Sol: 1) $f(5) = 5$

2) $\lim_{x\to o^+} 5 = 5$

$\lim_{x\to 0^-} 5 = 5$

3) $f(x) = \lim_{x\to 0} f(x) = 5$

The function continuous at constant

Ex: Check the continuity of the function at $\chi = 3$.

$$f(x) = \begin{cases} x-2, & x \neq 3 \\ 1, & x = 3 \end{cases}$$

Sol: 1) f (3) =1

2) $\lim_{x\to 3}(x-2) = 3-2 = 1$

3) $f(3) = \lim_{x\to 3} f(x)$

The function continuous at χ =3

Ex: Is the function $f(x) = \frac{x^3+2x^2+5x-1}{x^2-3x+2}$ continuous at $x = 1$.

Sol: $x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$(x-2) = 0 \rightarrow x = 2$

$(x-1) = 0 \rightarrow x = 1$

$\therefore f(x)\ where\ f(1) does\ not\ exist\ at\ x = 1$

Ex: Check the continuity of the function at $x = 0$.

$$f(x) = \begin{cases} \frac{x^3+3x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Sol: 1) f (0) =0

2) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^3+3x}{x}$

$= \lim_{x\to 0} \frac{x(x^2+3)}{x} = \lim_{x\to 0}(x^2+3) = 0+3 = 3$

3) $\lim_{x\to 3} f(x) \neq f(0)$

The function is not continuous at $x = 0$

Exercises:

Q1: Find the domain of the function.

1) $f(x) = \frac{x^2+x-2}{x^2-x-2}$
2) $f(x) = -\sqrt{-2x+3}$
3) $f(x) = |x+3|$

Q2: Evaluate the: $f \mp g \, , f.g \ and \ \frac{f}{g}$

1) $f(x) = (2x+3), g(x) = x^2$, find $f+g$.
2) $f(x) = (5x+1), g(x) = 3x-2$, find $f-g$.
3) $f(x) = (2x+3), g(x) = x^2$, find $f.g$.
4) Let $f(x) = (2x+3) \ and \ g(x) = -x^2+5, find \ (fog)(x), (gof)(x)$

Q3: Evaluate each of the following limits:

1) $\lim_{x\to -3} \frac{(x^2+4x+3)}{(x+3)}$
2) $\lim_{x\to 1} \frac{(x^2-1)}{(x-1)}$
3) $\lim_{x\to 2} \frac{x-2}{|x-2|}$
4) $\lim_{x\to 3} |x-3|$
5) $\lim_{x\to 3} \frac{3x+9}{x^2-9}$
6) $\lim_{x\to \infty} \frac{t+1}{t^2+1}$
7) $f(x) = \frac{1}{2}$, Check the continuity of the function at $x = 0$.
8) $f(x) = \sqrt{x-x^2}$,Check the continuity of the function $at \ x = \frac{1}{2}$
9) $f(x) = \begin{cases} 2x, & 0 < x < 3 \\ 9-x, & x \geq 0 \end{cases}$, Check the continuity of the function at $x = 3$.
10) $f(x) = \begin{cases} \sqrt[3]{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$,Check the continuity of the function at $x = 0$.

Chapter Two
Transcendental Function

2.1 Trigonometric Function

1) $\sin\theta = \frac{y}{r}$, $y = r\sin\theta$

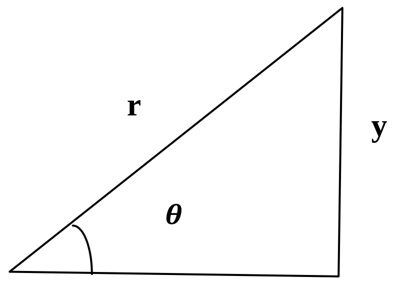
2) $\cos\theta = \frac{x}{r}$, $x = r\cos\theta$

3) $\tan\theta = \frac{y}{x}$

4) $\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$

5) $\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$

6) $\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$



7) $\cos^2\theta + \sin^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$\cos^2\theta = 1 - \sin^2\theta$

8) $\sec^2\theta - \tan^2\theta = 1$

$\sec^2\theta = \tan^2\theta + 1$

$\tan^2\theta = \sec^2\theta - 1$

9) $\csc^2\theta - \cot^2\theta = 1$

$\csc^2\theta = \cot^2\theta + 1$

$\cot^2\theta = \csc^2\theta - 1$

❖ **Addition and subtraction of trigonometric functions:**

1) $Cos(A+B) = CosA\, CosB + SinA\, SinB$

2) $Cos(A-B) = CosA\, CosB - SinA\, SinB$

3) $Sin(A+B) = SinA\, CosB + CosA\, SinB$

4) $Sin(A-B) = SinA\, CosB - CosA\, SinB$

5) $Tan(A+B) = \frac{tanA + tanB}{1 - tanA \cdot tanB}$

6) $Tan(A-B) = \frac{tanA - tanB}{1 + tanA \cdot tanB}$

1) $sinA + sinB = 2sin\frac{A+B}{2}cos\frac{A-B}{2}$

2) $sinA - sinB = 2sin\frac{A-B}{2}cos\frac{A+B}{2}$

3) $cosA + cosB = 2cos\frac{A+B}{2}cos\frac{A+B}{2}$

4) $cosA - cosB = -2sin\frac{A+B}{2}sin\frac{A-B}{2}$

- **Complications of trigonometric functions:**

1) $sin\,2\theta = 2sin\,\theta \cdot cos\theta$
2) $cos2\theta = cos^2\theta - sin^2\theta = 1 - 2sin^2\theta = 2sin^2\theta - 1$
3) $sin^2\theta = \frac{1}{2}[1 - cos2\theta]$
4) $cos^2\theta = \frac{1}{2}[1 + cos2\theta]$
5) $tan^2\theta = \frac{1 - cos\,2\theta}{1 + cos\,2\theta}$

A table for measuring the special angles:

Degree	0° 0	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$	90° or $\frac{\pi}{2}$	180° or π	270° or $\frac{3\pi}{2}$	360° or 2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	$-\infty$	0

2.2 Graphs of Trigonometric Functions

1) $sin(-\theta) = -sin(\theta)$

2) $cos(-\theta) = cos(\theta)$

3) $tan(-\theta) = -tan(\theta)$

4) $cot(-\theta) = -cot(\theta)$

5) $sec(-\theta) = sec(\theta)$

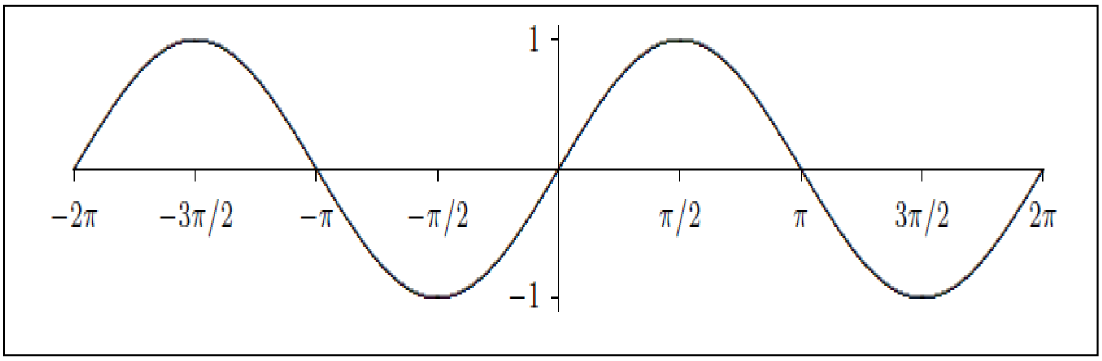
6) $csc(-\theta) = -csc(\theta)$

❖ **Y=Sin x**

Domain: $-\infty < \boldsymbol{x} < \infty$

Range: $-\mathbf{1} \leq \mathbf{y} \leq \mathbf{1}$

x	0	$\pi/2$	π	$3\pi/2$	2π
y	0	1	0	-1	0

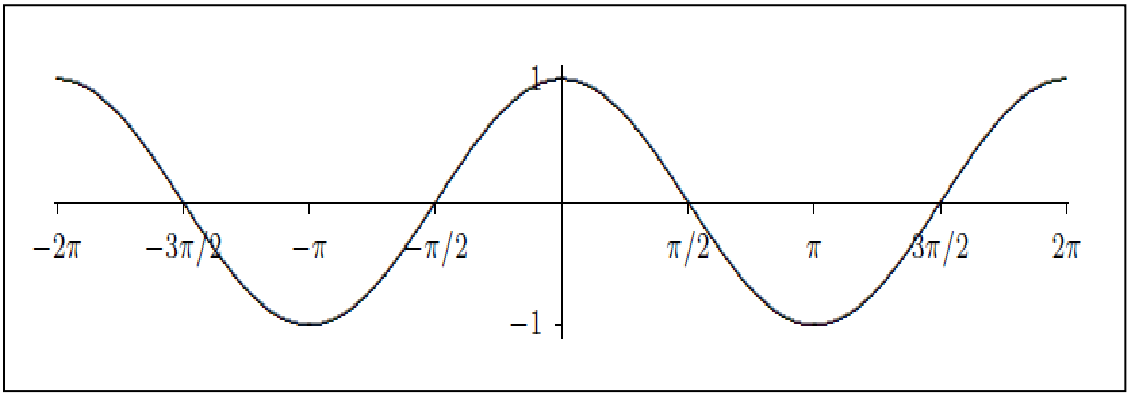


❖ **Y=Cos x**

Domain: $-\infty < \boldsymbol{x} < \infty$

Range: $-\mathbf{1} \leq \boldsymbol{y} \leq \mathbf{1}$

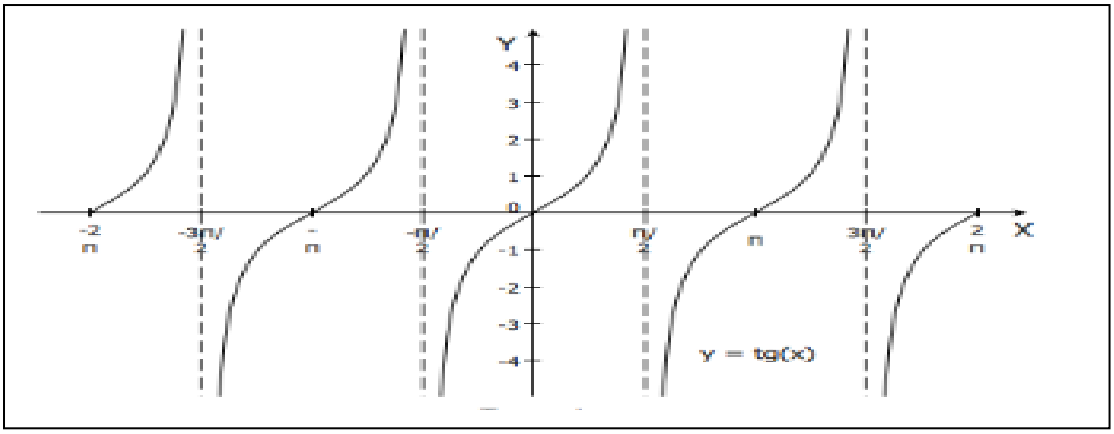
x	0	$\pi/2$	π	$3\pi/2$	2π
y	1	0	-1	0	1



❖ Y=Tan x

Domain: $\mathbf{x} \neq \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \ldots$

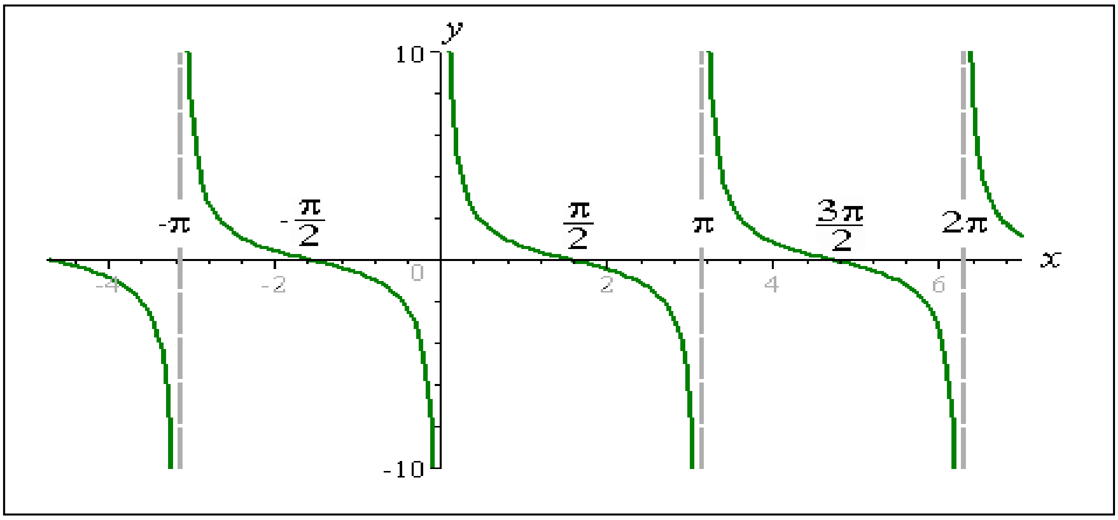
Range: $-\infty < y < \infty$



❖ Y=Cot x

Domain: $x \neq \mp\pi, \mp2\pi, \ldots$

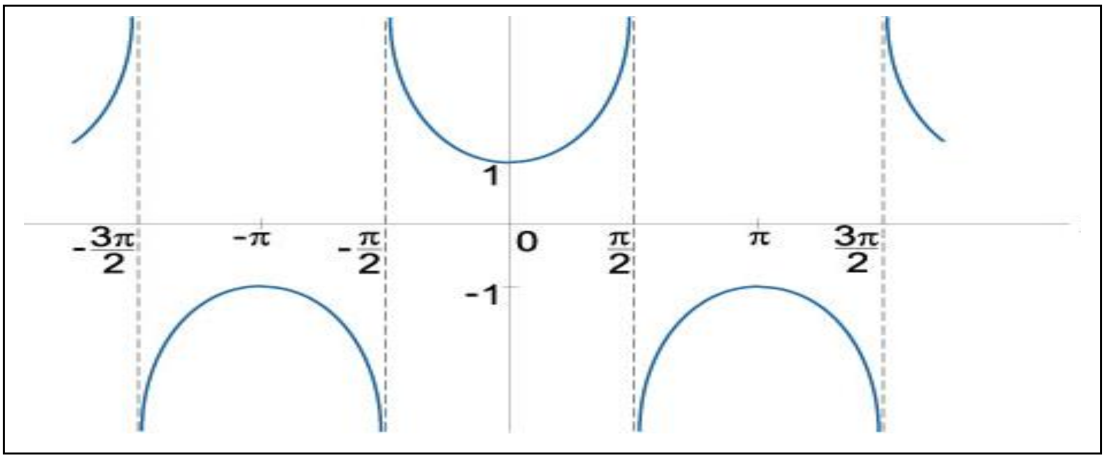
Range: $-\infty < y < \infty$



- **Y=Sec x**

  Domain: $x \neq \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \ldots$

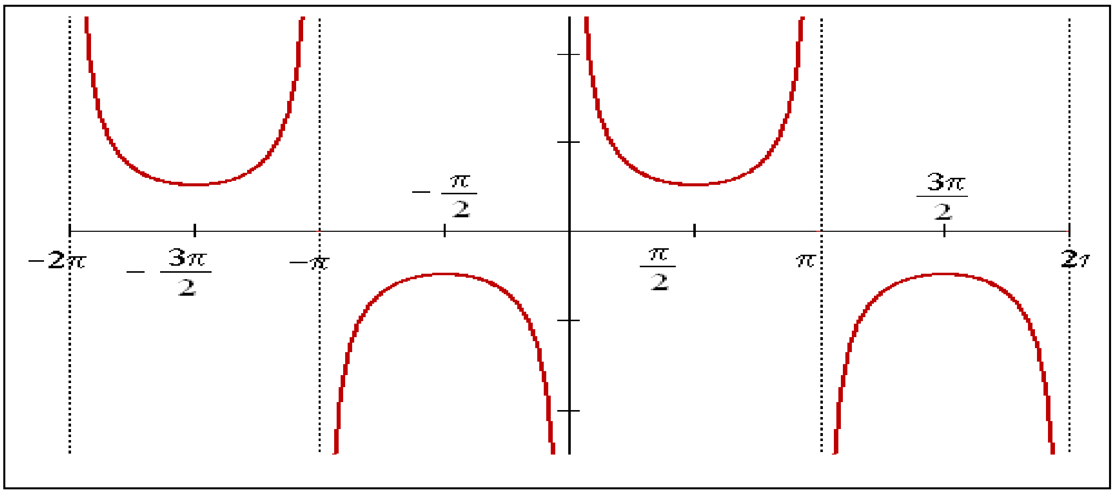
  Range: $y \leq -1$ and $y \geq 1$



- **Y=Csc x**

  Domain: $x \neq \mp\pi, \mp2\pi, \ldots$

  Range: $y \leq -1$ and $y \geq 1$



2.3 Limits of Trigonometric Functions

1) $lim_{\theta\to0}\, Sin\theta = 0$

2) $lim_{\theta\to0} \frac{Sin\theta}{\theta} = 1$

3) $lim_{\theta\to0}\, Cos\theta = 1$

4) $lim_{\theta\to0} \frac{Cos\theta - 1}{\theta} = 0$

Ex: Find the following limits.

1) $\lim_{x\to 0} \frac{\text{Sin}3x}{x}$

Sol: $\lim_{x\to 0} \frac{Sin3x}{x} \cdot \frac{3}{3} = \lim_{x\to 0} \frac{3Sin3x}{3x}$

$3\lim_{x\to 0} \frac{Sin3x}{3x} = 3 \cdot (1) = 3$

2) $\lim_{x\to 0} \frac{1-\cos x}{\sin x}$

Sol: $\lim_{x\to 0} \frac{\frac{1-\cos x}{x}}{\frac{\sin x}{x}} = \frac{0}{1} = 0$

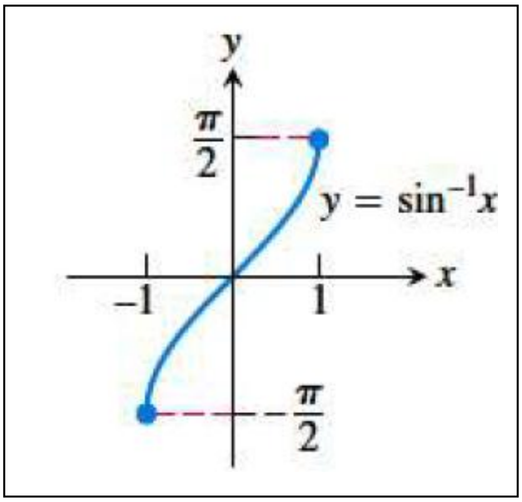
3) $\lim_{x\to 0} \frac{Sinx}{\cos x+1}$

Sol: $\lim_{x\to 0} \frac{Sinx}{\cos x+1} = \frac{Sin\,0}{\cos 0+1} = \frac{0}{2} = 0$

2.4 Inverse of Trigonometric Functions

1) **Inverse Sine (Sin^{-1})**

$y = \sin^{-1} x \to x = \sin y$



Ex: $\sin x : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$

Sol: $\therefore \sin^{-1} x : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

Domain: $\sin^{-1} = [-1, 1]$

Range: $\sin^{-1} = [-\frac{\pi}{2}, \frac{\pi}{2}]$

Domain: $-1 \le x \le 1$

Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Ex: Find $\sin^{-1} \frac{\sqrt{2}}{2}$

Sol: $\frac{\sqrt{2}}{2} \to \frac{\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} \to \frac{1}{\sqrt{2}}$

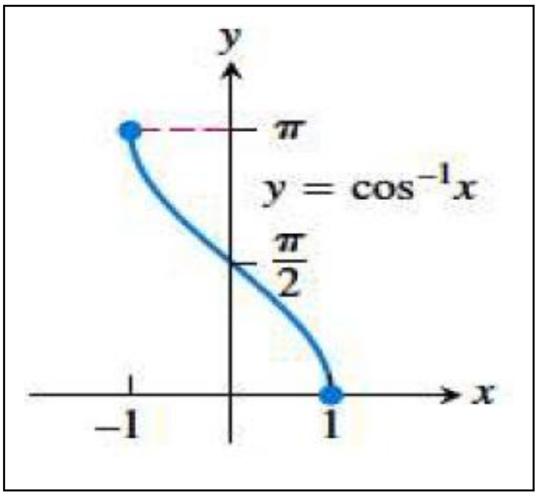
$y = \sin^{-1} \frac{1}{\sqrt{2}}$

$x = \sin y = \frac{1}{\sqrt{2}} \to y = \frac{\pi}{4}$

$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

2) Inverse Cosine (Cos^{-1})

$\mathbf{y = \cos^{-1} x \rightarrow x = \cos y}$



Ex: $\cos x: [0, \pi] \rightarrow [1, -1]$

Sol: $\therefore \cos^{-1} x : [1, -1] \rightarrow [0, \pi]$

Domain: $\mathbf{\cos^{-1} = [1, -1]}$

Range: $cos^{-1} = [0, \pi]$

Domain: $-1 \leq x \leq 1$

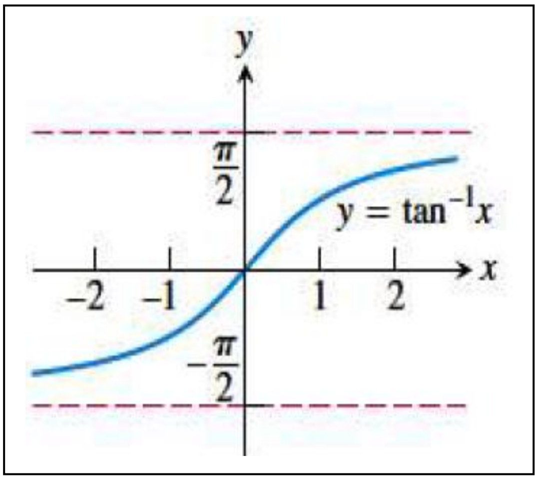
Range: $0 \leq y \leq \pi$

Ex: Find $\cos^{-1}(-1)$

Sol: $x = \cos y = -1 \rightarrow y = \pi$

3) Inverse of tangent (tan^{-1})

$\mathbf{y = \tan^{-1} x \rightarrow x = \tan y}$



Ex: $\tan x: [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow [-\infty, \infty]$

Sol: $\therefore \tan^{-1} x : [-\infty, \infty] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

Domain: $tan^{-1} = R$

Range: $\cos^{-1} = [-\frac{\pi}{2}, \frac{\pi}{2}]$

Domain: $-\infty < x < \infty$

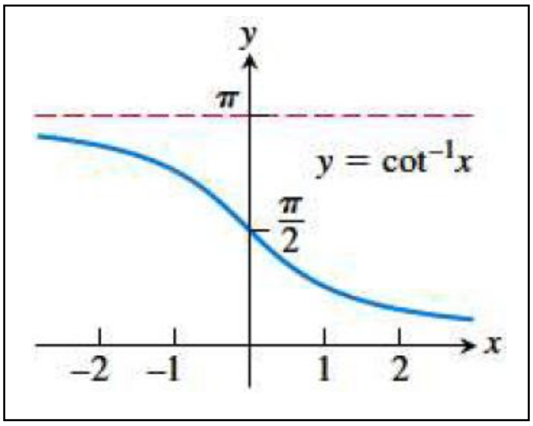
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

4) Inverse of cotangent (Cot^{-1})

$\mathbf{y = cot^{-1} x \rightarrow x = coty}$

Domain: R
Range: (0 , π)

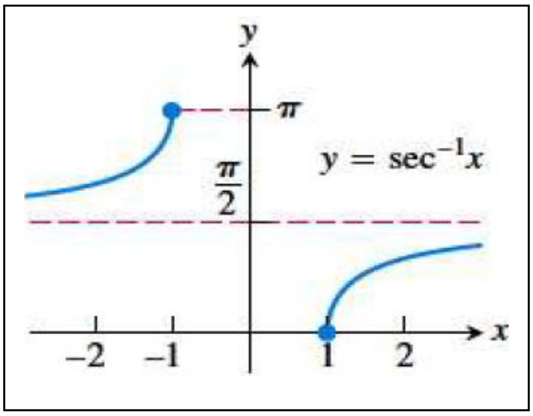
Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$



5) Inverse secant (Sec^{-1})

$\mathbf{y = sec^{-1} x \rightarrow x = secy}$

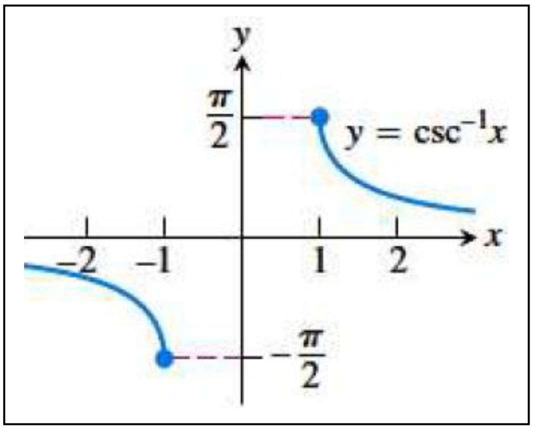
Domain: $x \leq -1$ or $x \geq 1$
Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$



6) Inverse Csc (Csc^{-1})

$\mathbf{y = csc^{-1} x \rightarrow x = cscy}$

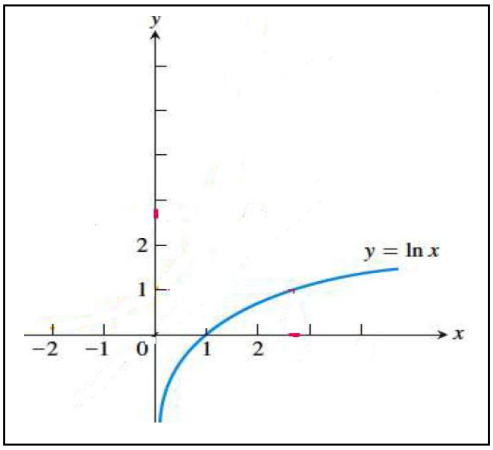
Domain: $x \leq -1$ or $x \geq 1$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$



2.5 Natural Logarithmic Function

$\mathbf{y = \ln x} \ \rightarrow \mathbf{x} > 0$

Range: R
Domain: $(0,\infty)$



Theorem:

1) $ln\,1 = 0$, $ln\,2 = 0.69$, $ln\,10 = 2.3$
2) $ln(x \cdot y) = ln\,x + ln\,y$
3) $ln(\frac{x}{y}) = ln\,x - ln\,y$
4) $ln(\frac{1}{x}) = ln\,1 - ln\,y = -\,ln\,y$
5) $ln\,x^a = a\,ln\,x$
6) $ln\,e = 1$

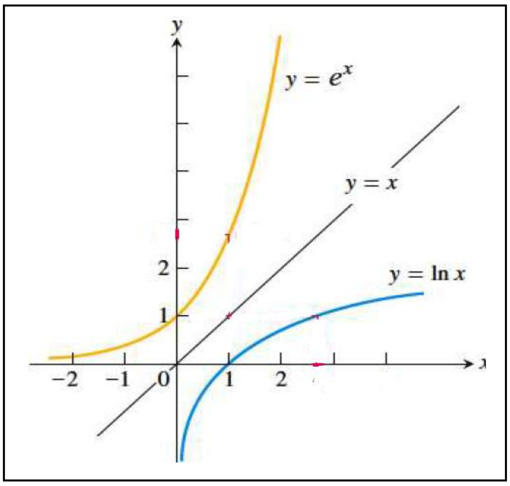
Ex: Find the following if you know $(\ln 2 = 0.69)$

1) $\ln 16$
   Sol: $\ln 16 = \ln 2^4 = 4\ln 2 = 4 \cdot (0.69) = 2.76$
2) $\ln\sqrt{2}$
   Sol: $\ln\sqrt{2} = \ln 2^{\frac{1}{2}} = \frac{1}{2}\ln 2 = \frac{1}{2} \cdot (0.69) = 0.345$
3) $2\ln 3 - 3\ln 2$
   Sol: $= \ln 3^2 - \ln 2^3$

   $= \ln 9 - \ln 8$

   $= \ln\frac{9}{8} = \ln 1.125 = 0.118$

2.6 Exponential function e^x



1) $\mathbf{y = e^x, e = 2.7182}$

Domain: R

Range: R (0,∞)

$e = 2.7182$

$e^0 = (2.7182)^0 = 1 > 0$

$e^2 = (2.7182)^2 = 7.29 > 0$

2) $\mathbf{y = e^{-x}}$

$e^{-1} = (2.7182)^{-1} = \frac{1}{2.7182} > 0$

<u>Theorem:</u>

1) $e^0 = 1$
2) $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
3) $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4) $(e^x)^r = e^{rx}$, $\forall r \in R$
5) $e^{-x} = \frac{1}{e^x}$
6) $e^{lnx} = ln\, e^x = x$

<u>Ex:</u> Evaluate the following:

1) $\mathbf{\ln e^{(-x)^2}}$

<u>Sol:</u> $\mathbf{\ln e^{(-x)^2} = -x^2}$

2) $\mathbf{e^{2\ln x}}$

<u>Sol:</u> $\mathbf{e^{2\ln x} = e^{\ln x^2} = x^2}$

3) $e^{(\ln x - 2\ln y)}$

<u>Sol:</u> $= e^{(\ln x - \ln y^2)}$

$= e^{\left(\ln \frac{x}{y^2}\right)} = \frac{x}{y^2}$

2.7 Exponential function a^x

$y = a^x, \qquad a > 0$

$a = e^{\ln a}, \qquad a > 0$

$a^u = (e^{\ln a}) \rightarrow a^u = e^{u \ln a}$

Theorem:

1) $a^0 = 1$
2) $a^1 = a$
3) $a^x \cdot a^y = a^{x+y}$
4) $(a^m)^n = a^{m \cdot n}$
5) $(a^{1/n})^m = (a^{m/n}) = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$
6) $\frac{a^x}{a^y} = a^{x-y}$
7) $a^{-x} = \frac{1}{a^x}$

Ex: Evaluate the following:

1) $4^{\frac{1}{2}}$

   Sol: $= \sqrt[2]{4} = 2$

2) $7^{-2} \cdot 7^6 \cdot 7^{-4}$

   Sol: $= 7^{-2+6-4} = 7^0 = 1$

3) 10^{-3}

   Sol: $= \frac{1}{10^3} = \frac{1}{1000}$

4) $\frac{15^6}{15^5}$

   Sol: $= 15^{6-5} = 15^1 = 15$

5) $(2^5)^2$

   Sol: $= 2^{10} = 1024$

6) $(3^{20})^{\frac{1}{10}}$

   Sol: $= 3^{\frac{20}{10}} = 3^2 = 9$

7) $\mathbf{8^{-\frac{2}{3}}}$

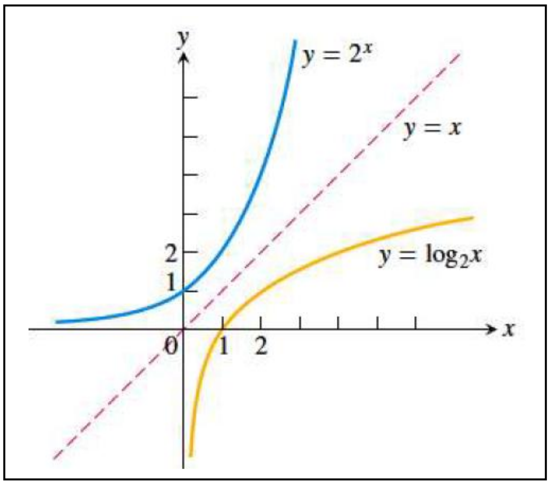
Sol: $\frac{1}{(8)^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4}$

2.8 Normal Logarithmic

$y = \log_a x \quad , \rightarrow x = a^y$

$a > 0$

Domain: $(0,\infty)$
Range: R



Ex: Evaluate the following:

1) $log_2(8) = 3$
2) $log_5(625) = 4$
3) $log_2(64) = 6$
4) $log_3(81) = 4$
5) $log_2(32) = 5$

Theorem:

1) $log_a 1 = 0$
2) $log_a a = 1$
3) $log_a(x \cdot y) = log_a x + log_a y$
4) $log_a(\frac{x}{y}) = log_a x - log_a y$
5) $log_a x^y = y \cdot log_a x$
6) $log_a x = \frac{ln\, x}{ln\, a}$
7) $log_e y = ln\, y = x \rightarrow y = e^x$

Ex: Evaluate the: $y = \log_{10} x$

Sol: $\log_{10} x \rightarrow x = 10^y$

Ex: Evaluate the: $3 = \log_{10} 2x$

Sol: $2x = 10^3 \rightarrow x = \frac{1000}{2} = 500$

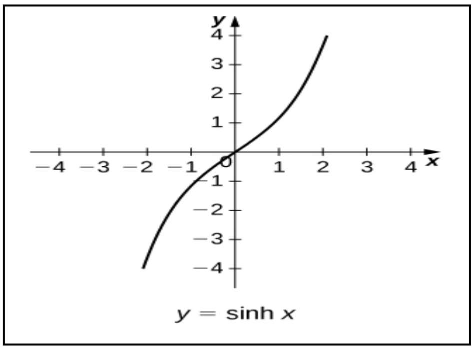
2.9 Hyperbolic Functions and Graphs

1) Hyperbolic Sine (Sinh)

y =SinhX , $\mathbf{SinhX} = \frac{\mathbf{e^{x}-e^{-x}}}{\mathbf{2}}$

Domain: R

Range: R

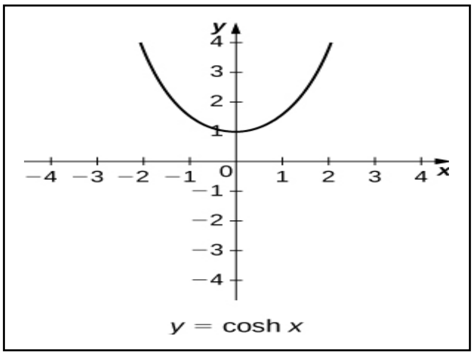


2) Hyperbolic Cosin (cosh)

y = cosh X , $\text{coshX} = \frac{e^{x}+e^{-x}}{2}$

Domain: R

Range: (1,∞)

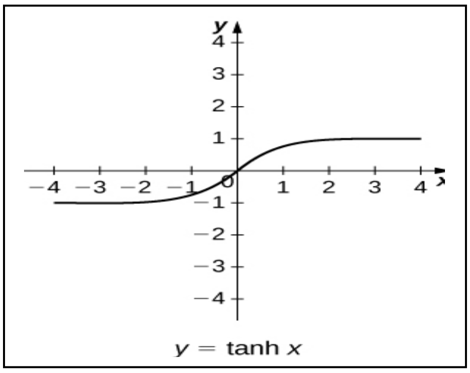


3) Hyperbolic tangent (tanh)

y = tanh X , $\mathbf{tanhX} = \frac{\mathbf{e^{x}-e^{-x}}}{\mathbf{e^{x}+e^{-x}}}$

Domain: R

Range: (-1,1)

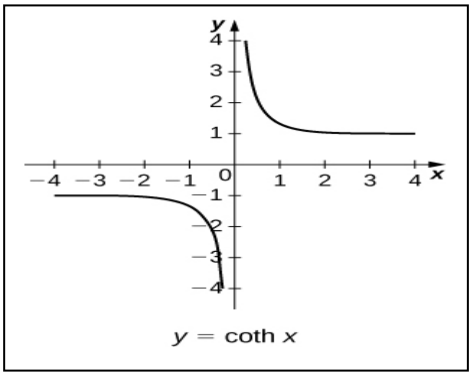


4) Hyperbolic cotangent (coth)

y = coth X , $\coth X = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

Domain: R - {0}

Range: $\{y: y < -1 \ or \ \ y > 1\}$

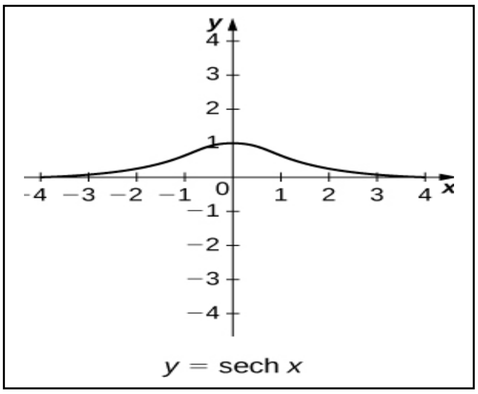


5) Hyperbolic Secant (Sech)

y = Sech X , $\mathbf{SechX} = \frac{\mathbf{2}}{\mathbf{e^{x}+e^{-x}}}$

Domain: R

Range: (0,1)

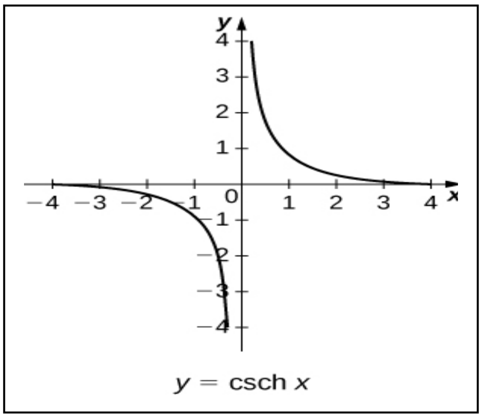


6) Hyperbolic cosecant (Csch)

y = Csch X , $\text{Csch}X = \frac{2}{e^{x}-e^{-x}}$

Domain: R - {0}

Range: R - {0}



Exercises:

Q1: Find the limit of the following:

1) $lim_{x\to 0}\frac{tan\ x}{x}$
2) $lim_{\theta\to 0}\frac{sin\ 2\theta}{\theta}$

Q2: Find the inverse of the following:

1) $sin^{-1}(\frac{\sqrt{3}}{2})$
2) $cos^{-1}(-1)$
3) $tan^{-1}(-1)$

Q3: Find the following if you know ($\ln 2 = 0.69$, $\ln 3 = 1.09$)

1) $ln\frac{1}{2}$
2) $ln(\sqrt[3]{\frac{2}{3}})$
3) $e^{ln\ \frac{1}{x}}$
4) e^{x+lnx}
5) $log_{10}(3x-5)=2$

Chapter Three
Derivative

3.1 Derivative by Definition

Let $y = f(x)$ and let P(x, y) be fixed point on the curve, and Q $(x + \Delta x, y + \Delta y)$ is another point on the curve as see in the figure.

$y = f(x)$, and

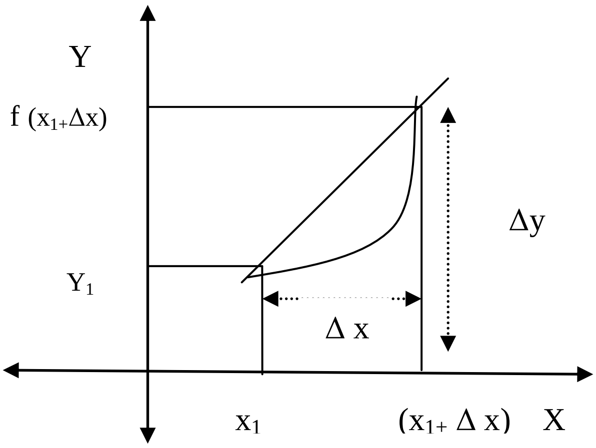
$y + \Delta y = f(x + \Delta x)$

$\Delta y = f(x + \Delta x) - y$

Divided by Δx

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\therefore M = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



Slope of the curve y= f(x) at a point (x,f(x)) is the slope of tangent for this curve at point (x,f(x)).

The derivative of a function f at a point x, written $f'(x)$, is given by:

$$f'(x) = M = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If this limit exists.

We define the limit may exist for some value of x . At each point x where limit does exist, then f is said to have a derivative or to be differentiable. Notations for first derivative as:

$$y' = f'(x) = \frac{dy}{dx} = \frac{df}{dx} = Df(x)$$

Notes:

1) $\boldsymbol{M = f'(x)}$

2) Tangent line equation معادلة المستقيم المماس

$y - y_1 = M(x - x_1)$

Ex: If $f(x) = x^2 + 1$, find f ′ (x) by definition.

Sol:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 1 - (x^2+1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 1 - x^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

Ex: If $f(x) = |x|$, find f ′ (x) at x=0 by definition.

Sol:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad |x| = \begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{cases}$$

$$= \lim_{\Delta x \to 0} \frac{|x+\Delta x| - |x|}{\Delta x} \qquad |\Delta x| = \begin{cases} \Delta x, & if\ x \geq 0 \\ -\Delta x, & if\ x < 0 \end{cases}$$

Where x = 0

$$= \lim_{\Delta x \to 0} \frac{|0+\Delta x| - |0|}{\Delta x} = \frac{|\Delta x|}{\Delta x}$$

$$L^+ = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$$L^- = \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x} = -1$$

$$\therefore L^+ \neq L^-$$

The limit does not exist due the function is not differentiable at x=0.

Ex: Find the slope of the tangent line to the function $f(x)$ = 3x+5 at the point (4,2), then write the equation of the tangent line.

Sol:

Slope of the tangent line

$$M = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$M = \lim_{\Delta x \to 0} \frac{f(4+\Delta x) - f(4)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(4+\Delta x) + 5 - (3.4+5)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{12+3\Delta x+5-12-5}{\Delta x} = \lim_{\Delta x\to 0}\frac{3\Delta x}{\Delta x} = \lim_{\Delta x\to 0} 3 \ = 3$$

equation of the tangent line:

$$y - y_1 = M(x - x_1)$$

$$\text{y} - 2 = 3(x - 4)$$

$$\text{y} - 2 = 3x - 12 \rightarrow y = 3x - 10$$

3.2 Rules of Derivations

1) The constant function

Let $f(x) = k$, where k is some real constant. Then

$$f'(x) = (k)' = 0$$

Ex:

$$f'(8) = (8)' = 0$$

$$f'(-5) = (-5)' = 0$$

$$f'(0{,}2321) = (0{,}2321)' = 0$$

2) The identity function

$f(x) = x$

Let $f(x) = x$, the identity function of x. Then

$$f'(x) = (x)' = 1$$

3) A function of the form x^n

Let $f(x) = x^n$, a function of x, and x a real constant. We have

$$f'(x) = (x^n)' = n\, x^{n-1}$$

Ex: $f(x) = x^4$

Sol: $f'(\text{x}^4) = 4x^{4-1} = 4x^3$

Ex: $f(x) = x^{\frac{1}{2}}$

Sol: $f'(\text{x}^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{\frac{-1}{2}}$

Ex: $f(x) = x^{-2}$

Sol: $f'(x^{-2}) = -2x^{-2-1} = -2x^{-3}$

Ex: $f(x) = x^{\frac{-1}{3}}$

Sol: $f'(x^{\frac{-1}{3}}) = \frac{-1}{3}x^{\frac{-1}{3}-1} = \frac{-1}{3}x^{\frac{-4}{3}}$

Ex: $f(x) = \sqrt{x}$

Sol: $f(x) = \sqrt{x} \rightarrow f(x) = x^{\frac{1}{2}}$

$f'(x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{\frac{-1}{2}}$

4) Constant multiples

Let k be a real constant and $f(x)$ any given function. Then
$(k\,f(x))' = kf'(x)$

Ex: $f(x) = 4x^2$
Sol: $f'(4\,x^2) = 4(x^2)' = 4(2x) = 8x$

5) Addition and subtraction of functions

Let $f(x)$ and $g(x)$be two functions. Then
$(f(x) \mp g(x))' = f'(x) \mp g'(x)$
Ex: $f(x) = 3\sqrt{x} + 2x - \frac{8}{x}$

Sol: $f'(3\sqrt{x} + 2x - \frac{8}{x}) = (3x^{\frac{1}{2}})' + (2x)' - (8x^{-1})'$

$= 3(x^{\frac{1}{2}})' + 2(x)' - 8(x^{-1})'$

$= 3(\frac{1}{2}x^{\frac{-1}{2}}) + 2(1) - 8(x^{-2})$

$= \frac{3}{2}x^{\frac{-1}{2}} + 2 - 8x^{-2}$

6) Product rule

Let $f(x)$ and $g(x)$ be two functions. Then the derivate of the product

$$(f(x).\,g(x))' = f'(x)g(x) + f(x)g'(x)$$

Ex: $f(x) = x^2(x^3 + 6)$

Sol: $f'(x^2(x^3 + 6)) = 2.\,x^{2-1}.\,(x^3 + 6) + x^2.\,(3x^{3-1} + 0)$

$$= 2x.(x^3 + 6) + x^2.(3x^2)$$

$$= 2x^4 + 12x + 3x^4$$

$$= 5x^4 + 12x$$

7) Quotient rule

Let $f(x)$ and $g(x)$ be two functions. Then the derivative of the quotient.

$$\left(\frac{f(x)}{g(x}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

<u>Ex:</u> $f(x) = \frac{x^2+1}{x^2}$

<u>Sol:</u> $f'\left(\frac{x^2+1}{x^2}\right) = \frac{(2x)\cdot(x^2)-(x^2+1).(2x)}{[x^2]^2}$

$$= \frac{2x^3-2x^3-2x}{x^4}$$

$$= \frac{-2x}{x^4} = \frac{-2}{x^3}$$

8) Exponential function

Let $f(x) = [f(x)]^n$, function of x, and n is integer number and f(x) differentiable function. We have

$$([f(x)]^n)' = n.[f(x)]^{n-1}.f'(x)$$

<u>Ex:</u> $f(x) = [3x^2 + 1]^2$

<u>Sol:</u> $f'([3x^2 + 1]^2) = 2.[3x^2 + 1]^{2-1}.(3.2x^{2-1} + 0)$

$$= 2.[3x^2 + 1].(6x)$$

$$= 12x[3x^2 + 1] = 36x^3 + 12x$$

<u>Ex:</u> Find the slope of the tangent line to the function $f(x) = x^2 + 2x$ at the point (1,3), then write the equation of the tangent line.

Sol: $f(x) = x^2 + 2x$

$M = f'(x) = 2x + 2 = 2(x+1)$ at point (1, 3)

$M = 2(1+1) \rightarrow M = 4$

$y - y_1 = M(x - x_1)$

$y - 3 = 4(x-1) \rightarrow y = 4x - 4 + 3 = 4x - 1$

3.3 Derivatives of Higher Order

If f is a differentiable function, then f′ is also a function. So, f′ may have a derivative, denoted by (f′)′ = f″. This new function f″ is called the second derivative of f.

Other Notations: $y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

The third derivative f ‴ is the derivative of the second derivative as:

f ‴ = (f ″)′.

Other Notations: $y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$

Similarly, **the fourth derivative** f⁗ is the derivative of the third derivative as: f⁗ = (f‴)′. And so on.

Ex: Find the y', y'', y''', y'''' $or\ y^4$ for the function $y = 2x^3 + x^2 - 1$

Sol:

$y' = 6x^2 + 2x$

$y'' = 12x + 2$

$y''' = 12$

$y'''' = y^4 = 0$

Ex: Find the y', y'', y''', y'''' $or\ y^4$ for the function $y = t^4 + 4t$

Sol:

$y' = 4t^3 + 4$

$y'' = 12t^2$

$y''' = 24t$

$y'''' = y^4 = 24$

3.4 Derivatives of Composite Functions

Ex: Let $(x) = 5x - 3, \quad g(x) = 2x^2 + 7$, find

1. $(fog)(x)$ and $(fog)'(x)$

2. $(gof)(x)$ and $(gof)'(x)$ → **H.W**

Sol 1:

$(fog)(x) = f(g(x)) = f(2x^2 + 7) = 5(2x^2 + 7) - 3 = 10x^2 + 32$

$(fog)'(x) = 10x^2 + 32 = 20x$

3.5 Implicit Derivative

Ex: find dy/dx if

$x^5 + 4xy^3 - 3y^5 = 2$

Sol: $5x^4 + 4x3y^2(dy/dx) + 4y^3 - 15y^4(dy/dx) = 0$

$(12xy^2 - 15y^4)\, dy/dx = -5x^4 - 4y^3$

$dy/dx = \frac{(-5x^4 - 4y^3)}{(12xy^2 - 15y^4)}$

3.6 Chain Rule

1) If y = f (x), and x = g (t), then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Ex: Let $y = 3x - 1, \quad x = 2t$, find $\frac{dy}{dt}$ using chain rule.

Sol: $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$= (3) \cdot (2) = 6$

2) If y = f (t), and t = g (x), then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Ex: Let $y = 5u^2 + 3,\ \ u = 3x + 1$, find $\frac{dy}{dx}$ using chain rule.

Sol:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 10u \cdot 3 = 10(3x + 1) \cdot 3 = (30x + 10) \cdot 3 = 90x + 30$$

3) If y = f (t), and x = g (t), then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Ex: Let $y = t^2,\ \ x = 3t + 1$, find $\frac{dy}{dx}$ using chain rule.

Sol:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{2t}{3}$$

3.7 Applications of Derivative

If s = s(t) is the position function of an object that moves in a straight line, we know that its first derivative represents the **velocity v(t)** of the object as a function of time:

$$v(t) = s^{'}(t) = \frac{ds}{dt}$$

Thus the **acceleration a(t)** function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v^{'}(t) = s^{''}(t)$$

Ex: Find velocity and acceleration at time t to a moving an object as:
$s = 2t^3 - 5t^2 + 4t - 3.$

Sol:
$v = \frac{ds}{dt} = 6t^2 - 10t + 4$

$a = \frac{dv}{dt} = 12t - 10$

3.8 Derivative of trigonometric functions

$F(x) = y, \quad \frac{d}{dx} = y', \; x = u$

1) $y = \sin(x) \rightarrow y' = \cos(x)$
 $or\ y = \sin(u) \rightarrow y' = \cos)(u) \cdot \frac{du}{dx}$
2) $y = \cos(x) \rightarrow y' = -\sin(x)$
 $or\ y = \cos(u) \rightarrow y' = -\sin(u) \cdot \frac{du}{dx}$
3) $y = \tan(x) \rightarrow y' = sec^2(x)$
 $or\ y = \tan(u) \rightarrow y' = sec^2(u) \cdot \frac{du}{dx}$
4) $y = \cot(x) \rightarrow y' = -csc^2(x)$
 $or\ y = \cot(u) \rightarrow y' = -csc^2(u) \cdot \frac{du}{dx}$
5) $y = \sec(x) \rightarrow y' = \sec(x) \cdot \tan(x)$
 $or\ y = \sec(u) \rightarrow y' = \sec(u) \cdot \tan(u) \cdot \frac{du}{dx}$
6) $y = \csc(x) \rightarrow y' = -\csc(x) \cdot \cot(x)$
 $or\ y = \csc(u) \rightarrow y' = -\csc(u) \cdot \cot(u) \cdot \frac{du}{dx}$

Ex: Find the y' for the $y = \sin 5x$
Sol: $y' = \cos 5x \cdot 5 = 5\cos(5x)$

Ex: Find the y' for the $y = 4\sec(x^2) - 3\cot x$
Sol: $y' = 4 \cdot \sec(x^2) \cdot \tan(x^2) \cdot 2x - 3(-csc^2(x) \cdot 1)$
$y' = 8x\ \sec(x^2) \cdot \tan(x^2) + 3\ csc^2(x)$

Ex: Find the y' for the $y = \sqrt{sin^2(x) + 2}$
Sol: $y = \sqrt{sin^2(x) + 2} = [sin^2(x) + 2]^{1/2}$
$y' = \frac{1}{2}[sin^2(x) + 2]^{-1/2} \cdot (2\sin(x) \cdot \cos(x) + 0)$

$$y' = \frac{2\sin(x)\cdot\cos(x)}{2\sqrt{sin^2(x)+2}} = \frac{\sin(x)\cdot\cos(x)}{\sqrt{sin^2(x)+2}}$$

3.9 Derivative of the inverse trigonometric functions

$F(x) = u$

1) $\frac{d(\sin^{-1}u)}{dx} = \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$

2) $\frac{d(\cos^{-1}u)}{dx} = \frac{-1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$

3) $\frac{d(\tan^{-1}u)}{dx} = \frac{1}{1+u^2}\cdot\frac{du}{dx}$

4) $\frac{d(\cot^{-1}u)}{dx} = \frac{-1}{1+u^2}\cdot\frac{du}{dx}$

5) $\frac{d(\sec^{-1}u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}}\cdot\frac{du}{dx}$

6) $\frac{d(\csc^{-1}u)}{dx} = \frac{-1}{|u|\sqrt{u^2-1}}\cdot\frac{du}{dx}$

Ex: Find y' for the $y = \sin^{-1}\sqrt{x}$

Sol: $y' = \frac{1}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}$

Ex: Find y' for the $y = \sqrt{\csc^{-1}3x}$

Sol: $y = (\csc^{-1}3x)^{1/2}$

$$y' = \frac{1}{2}(\csc^{-1}3x)^{-1/2}\cdot\frac{-1}{|3x|\sqrt{(3x)^2-1}}\cdot(3)$$

$$y' = \frac{1}{2\sqrt{\csc^{-1}3x}}\cdot\frac{-3}{|3x|\sqrt{9x^2-1}} = \frac{-1}{2|x|\sqrt{9x^2-1}\cdot\csc^{-1}3x}$$

3.10 Derivative Natural Logarithmic Function

$y = \ln x \to y' = \frac{1}{x}$

$y = \ln u$

$y' = \frac{dy}{dx} = \frac{1}{u}\cdot\frac{dy}{dx}$

Ex: Find y' for the $y = x^3\cdot\ln(x)$

Sol: $y' = x^3\cdot\frac{1}{x} + \ln(x)\cdot 3x^2$

$= x^2 + 3x^2\ln(x)$

3.11 Derivative Exponential function e^x, a^x

1) $y = e^x \rightarrow y' = e^x . 1$

$y = e^u \rightarrow y' = e^u \cdot \frac{du}{dx}$, $u = x$

Ex: Find y' for the $y = e^{x^2+x+1}$

Sol: $y' = e^{x^2+x+1} \cdot (2x + 2)$

2) $y = a^x \rightarrow y' = a^x \cdot lna$

$y = a^x \rightarrow y' = a^u \cdot lna \cdot \frac{du}{dx}$, $u = x$,

Ex: Find y' for the $y = 5^{2x}$

Sol: $y = 5^{2x} \rightarrow y' = 5^{2x} \cdot ln5 \cdot (2)$

$y' = 2 \cdot 5^{2x} \cdot ln5$

Ex: Find y' for the $y = 6^{sinx + lnx + 3}$

Sol: $y' = 6^{sinx + lnx + 3} \cdot \ln 6 \cdot (\cos x + \frac{1}{x})$

3.12 Derivative Normal Logarithmic

$y = \log_a x$, $y = \frac{\ln x}{\ln a}$, $a \neq 1, a > 0$

$y = \frac{\ln x}{\ln a} \rightarrow y' = \frac{1}{x \cdot \ln a}$

$y = \log_a u \rightarrow y' = \frac{1}{u \cdot \ln a} \cdot \frac{du}{dx}$

Ex: Find y' for the $y = \log_3 x^2 + 2x$

Sol: $y' = \frac{1}{(x^2+2x) \cdot \ln 3} \cdot (2x + 2) = \frac{(2x+2)}{(x^2+2x) \cdot \ln 3}$

Ex: Find y' for the $y = \log_5 e^{x+1}$

Sol: $y' = \frac{1}{e^{x+1} \cdot \ln 5} \cdot e^{x+1} = \frac{e^{x+1}}{e^{x+1} \cdot \ln 5} = \frac{1}{\ln 5}$

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Exercises:

Q 1:

1) Find $f'(x) if\ f(x) = x^2 - x$ by definition.

2) Find the slope of the tangent line to the function $g(x) = 3x^2 - x + 1$ at the point (1,3). Write the equation of the tangent line.

3) $f(x) = \sqrt[3]{x}\,, find\ f'(x)$

4) $f(x) = 2\sqrt[3]{x} + 3, find\ f'(x)$

5) $f(x) = x^6 - \sqrt{x}\,, find\ f'(x)$

6) $f(x) = 4x^3 + 5x^2,\ find\ f'(x)$

7) $f(x) = 4 - 4x + \frac{2}{x},\ find\ f'(x)$

8) $f(x) = \sqrt{(4x+2)^5},\ find\ f'(x)$

9) $If\ \ y = x^7 - 7x^3 + 10x^2,\ find\ y', y'', y''', y^4, y^5$

10) $If\ \ y = 4x^2 - 12 + \frac{4}{x^2},\ find\ y', y'', y''', y^4$

11) Let $y = u^3 + 1$, $u = 2x^2$, find $\frac{dy}{dx}$ using chain rule.

12) Let $y = 5t^3 + 3t$, $x = 3t^2 - 1$, find $\frac{dy}{dx}$ using chain rule.

13) Find $\frac{dy}{dx}$ $if\ x^2 + y^2 + yx.$

14) Find velocity and acceleration at time t to a moving an object as:

$s = 2t^4 - 3t^3 + 2t^2 - t$ at time =1 minute.

Q3 Find y' for the following:

1) $y = Sin(tan(x^2))$

2) $y = csc(x) \cdot sec(3x)$

3) $y = \sin^{-1} 2x \cdot \cos^{-1} 2x$

4) $y = \ln(3x+5)^2$

5) $\boldsymbol{y = e^{4x} - \ln(x)}$

6) $y = e^{(sinx + cosx)}$

Chapter Four
Sequences and Series

4.1 Sequences

A sequence is a set of numbers written in a given order. The numbers in the sequence are called terms as example:

$3,6,9,12,15 \rightarrow$ sequence

Definition:

- **A finite sequence** is a function whose domain is the set of integers as {3, 6, 9, 12, 15}.
- **An infinite sequence** is a function whose domain is the set of positive integers as {3, 6, 9, 12, 15,……..}.

Both sequences have the general rule $a_n = 3n$ where a_n represents the *n*th term of the sequence. The general rule can also be written using function notation: $f(n) = 3n$.

Ex: Write the sequence for the first six terms of the functions as:

$$a_n = 2n + 3$$

Sol: $a_1 = 2(1) + 3 = 5$

$a_2 = 2(2) + 3 = 7$

$a_3 = 2(3) + 3 = 9$

$a_4 = 2(4) + 3 = 11$

$a_5 = 2(5) + 3 = 13$

$a_6 = 2(6) + 3 = 15$

Sequence={5,7,9,11,13,15}

Ex: Write the sequence for the first six terms of the functions as:

$f(n) = (-2)^{n-1}$

Sol: $f(1) = (-2)^{1-1} = 1$

$f(2) = (-2)^{2-1} = -2$

$f(3) = (-2)^{3-1} = 4$

$f(4) = (-2)^{4-1} = -8$

$f(5) = (-2)^{5-1} = 16$

$f(6) = (-2)^{6-1} = -32$

Sequence={1,-2,4,-8,16,-32}

Ex: For each sequence, write the next term, and write the rule or function for the next term as:

a) $-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots\ldots$

Sol: You can write the terms as $\left(-\frac{1}{3}\right)^1, \left(-\frac{1}{3}\right)^2, \left(-\frac{1}{3}\right)^3, \left(-\frac{1}{3}\right)^4, \ldots.$

The next term is $a_5 = \left(-\frac{1}{3}\right)^5 = -\frac{1}{243}$. A rule for the *n*th term is $a_n = \left(-\frac{1}{3}\right)^n$.

b) 2, 6, 12, 20, . . .

Sol: You can write the terms as 1(2), 2(3), 3(4), 4(5), ...

The next term is $f(5) = 5(6) = 30$. A rule for the *n*th term is $f(n) = n(n + 1)$.

Ex: list the next three terms of the sequence :2,4,8,16……, and write a general expression for a_n.

Sol: each term of the sequence is a power 2: $2^1, 2^2, 2^3, 2^4$, and the next three terms should be : $2^5, 2^6, 2^7$ or 32,64,128.

$a_n = 2^n$

Ex: Write the first six terms of the sequence $a_n = n + 1$.

Sol: $a_1 = 1 + 1 = 2$

$a_2 = 2 + 1 = 3$

$a_3 = 3 + 1 = 4$

$a_4 = 4 + 1 = 5$

$a_5 = 5 + 1 = 6$

$a_6 = 6 + 1 = 7$

Sequence ={2,3,4,5,6,7}

Ex: Write the first three terms of the sequence $f(n) = \frac{1}{n+1}$.

Sol: $f(1) = \frac{1}{1+1} = \frac{1}{2}$

$f(2) = \frac{1}{2+1} = \frac{1}{3}$

$f(3) = \frac{1}{3+1} = \frac{1}{4}$

Sequence = $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$

Ex: Write the first four terms of the sequence $f(n) = \frac{n+2}{2n}$.

Sol: $f(1) = \frac{1+2}{2.1} = \frac{3}{2}$

$f(2) = \frac{2+2}{2.2} = \frac{4}{4} = 1$

$f(3) = \frac{3+2}{2.3} = \frac{5}{6}$

$f(4) = \frac{4+2}{2.4} = \frac{6}{8}$

Sequence = $\{\frac{3}{2}, 1, \frac{5}{6}, \frac{6}{8}\}$

Ex: Write the first four terms of the sequence $f(n) = \frac{3}{-n}$.

Sol: $f(1) = \frac{3}{-1} = -3$

$f(2) = \frac{3}{-2}$

$f(3) = \frac{3}{-3} = -1$

$f(4) = \frac{3}{-4}$

Sequence = $\{-3, \frac{3}{-2}, -1, \frac{3}{-4}\}$

Ex: Write the next term in the sequence. Then write a rule for the *n*th term. $\frac{1}{20}, \frac{2}{30}, \frac{3}{40}, \frac{4}{50}$

Sol: the next term equal = $\frac{5}{60}$

$f(n) = \frac{n}{(n+1)10}$

Ex: Write the next term in the sequence. Then write a rule for the *n*th term. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

Sol: the next term equal = $\frac{5}{6}$

$f(n) = \frac{n}{n+1}$

Ex: Write the first six terms of the sequence $f(n) = n^2$.

Sol: $f(1) = 1^2 = 1$

$f(2) = 2^2 = 4$

$f(3) = 3^2 = 9$

$f(4) = 4^2 = 16$

$f(5) = 5^2 = 25$

$f(6) = 6^2 = 36$

Sequence ={1,4,9,16,25,36}

4.2 Series

A series is the sum of the terms of a sequence. You can use summation notation to write a series as $\sum$.

Definition:

- **A finite series** is the sum of a finite number of terms of a sequence as {3+ 6+ 9+ 12+ 15}.
  For example shown above, the finite series, you can write as:
  $3 + 6 + 9 + 12 + 15 = \sum_{i=1}^{5} 3i$

- **An infinite series** is the sum of an infinite number of terms of a sequence as {3+ 6+ 9+ 12+ 15+ ⋯⋯⋯ }.
  For example shown above, the an infinite series, you can write as:
  $3 + 6 + 9 + 12 + 15 + \cdots = \sum_{i=1}^{\infty} 3i$

Ex: Write the summation notation for each series.

a) 5 + 10 + 15 + . . . + 100

Sol: The summation notation for the series is $\sum_{i=1}^{20} 5i$ where i = 1, 2, 3, . . . , 20

b) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} \cdots\cdots\cdots$

Sol: The summation notation for the series is $\sum_{i=1}^{\infty} \frac{i}{i+1}$ where i = 1, 2, 3, 4,……

Ex: Find the sum of the series $\sum_{i=1}^{6} 2i$.

Sol: $\sum_{i=1}^{6} 2i = 2(1) + 2(2) + 2(3) + 2(4) + 2(5) + 2(6)$

$= 2(1 + 2 + 3 + 4 + 5 + 6) = 2(21) = 42$

Ex: Find the sum of the series $\sum_{k=3}^{6}(2 + k^2)$.

Sol: $\sum_{k=3}^{6}(2+k^2) = (2\ +\ 3^2)\ +\ (2\ +\ 4^2)\ +\ (2\ +\ 5^2)\ +\ (2\ +\ 6^2)$

$= 11 + 18 + 27 + 38 = 94$

Ex: Write the summation notation for the series.

a) 4 + 8 + 12 + 16 + 20

Sol: The summation notation for the series is $\sum_{i=1}^{5} i \times 4$ where i = 1, 2, 3, 4,5.

b) 1 + 4 + 9 + 16 + 25 + 36

Sol: The summation notation for the series is $\sum_{i=1}^{6} i^2$ where i = 1, 2, 3, 4,5,6.

c) 12 + 20 + 30 + 42 + 56 + 72 + 90 + 110

Sol: The summation notation for the series is $\sum_{n=3}^{10}(n^2+n)$ where n = 3, 4,5,6,7,8,9,10.

Ex: Find the sum of the series $\sum_{k=1}^{7}(k+5)$.

Sol: $\sum_{k=1}^{7}(k+5) = (1+5)+(2+5)+(3+5)+(4+5)+(5+5)+(6+5)+(7+5) = 6+7+8+9+10+11+12 = 63$

Ex: Find the sum of the series $\sum_{n=1}^{3} 4n^3$.

Sol: $\sum_{n=1}^{3} 4n^3 = 4\times 1^3 + 4\times 2^3 + 4\times 3^3$

$= 4 + 32 + 108 = 144$

4.3 Power Series

If $\{a_n\}$ is a sequence of constants, the power series in x centered at 0 is represented as expression:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots + a_n x^n \cdots\cdots$$

Here are some power series centered at 0:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 \cdots\cdots,$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 \cdots\cdots,$$

$$\sum_{n=0}^{\infty} (n!) x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 \cdots\cdots,$$

And here is a power series centered at 1:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots\cdots,$$

Ex: Write the summation notation for the series at n=1.

$$x^3 + \frac{1}{2}x^7 + \frac{1}{3}x^{11} + \frac{1}{4}x^{15} + \frac{1}{5}x^{19} + \cdots\cdots,$$

Sol: $\sum_{n=1}^{\infty} \frac{1}{n} x^{4n-1}$

Ex: Find the terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

Sol: $= \frac{1}{8} - \frac{1}{11} + \frac{1}{16} - \frac{1}{23} + \cdots\cdots,$

4.4 Taylor's Series

If a function f can be represented by a power series in (x-a) called Taylor's series and has the form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!}$$

$$= f(a) + f'(x-a) + \frac{f''(a)(x-a)^2}{2!} \cdots + \frac{f^n(a)(x-a)^n}{n!} \cdots$$

Ex: Find the Taylor series for $f(x) = 1/x^2$ centered at $a = 1$

Sol: We have: $f(x) = 1/x^2$ $\quad so \quad f(1) = 1$

$f'(x) = -2x^{-3}$ $\quad so \quad f'(1) = -2$

$f''(x) = 6x^{-4}$ $\quad so\, f''(1) = 6$

$f^{(3)}(x) = -24x^{-5}$ $\quad so \quad f^{(3)}(1) = -24$

$f^{(4)}(x) = 120x^{-6}$ $\quad so \quad f^{(4)}(1) = 120$

Therefore,

$$\frac{1}{x^2} = 1 + -2(x-1) + \frac{6}{2!}(x-1)^2 + \frac{-24}{3!}(x-1)^3 + \frac{120}{4!}(x-1)^4 + \cdots$$

$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 - \cdots$$

4.5 Maclaurin Series

A Maclaurin series is simply a Taylor series with a = 0.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}$$

$$= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} \cdots + \frac{f^n(0)x^n}{n!} \cdots$$

Ex: Find the Maclaurin series for $f(x) = \frac{1}{(1+x)^2}$

Sol: $f(x) = \frac{1}{(1+x)^2}$ $\quad so \quad f(0) = 1$

$f'(x) = -2(1+x)^{-3}$ $\quad$ so $\quad f'(0) = -2$

$f''(x) = 6(1+x)^{-4}$ $\quad$ so $\quad f''(0) = 6$

$f^{(3)}(x) = -24(1+x)^{-5}$ $\quad$ so $\quad f^{(3)}(0) = -24$

$f^{(4)}(x) = 120(1+x)^{-6}$ $\quad$ so $\quad f^{(4)}(0) = 120$

Therefore,

$$\frac{1}{(1+x)^2} = 1 + \; -2x + \frac{6}{2!}x^2 + \frac{-24}{3!}x^3 + \frac{120}{4!}x^4 + \cdots.$$

$$= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - ..$$

Ex: Find the first three terms of the Maclaurin series for $f(x) = \sqrt{1+x}$

Sol: $f(x) = (1+x)^{1/2}$ $\quad$ so $\quad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\quad$ so $\quad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $\quad$ so $\quad f''(0) = -\frac{1}{4}$

Therefore, the first three terms of the Taylor series for $\sqrt{1+x}$ are:

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + \frac{1}{2}x + \frac{-1/4}{2}x^2$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

Exercises:

Q1: list the next three terms of the sequence :$1, \frac{1}{2}, \frac{1}{3}, \cdots$ and write a general expression for a_n.

Q2: Write the next term in the sequence. Then write a rule for the sequence. $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$

Q3: Find the sum of the series $\sum_{i=1}^{3} \frac{i(i+1)}{2}$.

Q4: Write the summation notation for each series.

$10 + 100 + 1000 + 10000$

Q5: Find the sum of the series $\sum_{n=0}^{3}(20 - 2n)$.

Q6: Find the Taylor series for $f(x) = \frac{1}{x}$ centered at $a = 1$

Q7: Find the first four terms of the Maclaurin series for $f(x) = \frac{1}{1-2x}$

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Computer Science Dept.
1st Class

Chapter Five
Integration

5.1 Integration

The process of finding the function whose derivative is given is called integration, it's the inverse of differentiation.

5.2 Indefinite Integral

A function y=F(x) is called a solution of dy/dx=f(x) if dF(x)/dx=f(x). We say that F(x) is an integral of f(x) with respect to x and F(x) + c is also an integral of f(x) with a constant c as:

$\int f(x)\,dx = F(x) + c$

5.3 Rule Integration

1) $\int dx = x + c$
2) $\int a \cdot f(x)\,dx = a \int f(x)\,dx$
3) $\int (f(x) \pm g(x))dx = \int f(x)\,dx \pm \int g(x)\,dx$
4) $\int [a\,f(x) \pm b\,g(x)]dx = a\int f(x)\,dx \pm b \int g(x)\,dx$
5) $\int x^r\,dx = \frac{x^{r+1}}{r+1} + c, \qquad r \neq -1$
6) $\int [f(x)]^n \cdot f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c, \qquad n \neq -1$

Ex: Calculate integration of the: $\int 3dx$

Sol: $3\int dx = 3x + c$

Ex: Calculate integration of the: $\int 5x^2\,dx$

Sol: $5\int x^2\,dx = 5 \cdot \frac{x^3}{3} + c$

Ex: Calculate integration of the: $\int \frac{4}{x^3}\,dx$

Sol: $4\int x^{-3}\,dx = 4 \cdot \frac{x^{-2}}{-2} + c = \frac{-2}{x^2} + c$

Ex: Calculate integration of the: $\int \sqrt{x}dx$

Sol: $= \int x^{1/2}dx = \frac{x^{(\frac{1}{2}+1)}}{\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2\sqrt{x^3}}{3} + c$

Ex: Calculate integration of the: $\int (3 + 6x^2)dx$

Sol: $\int 3dx + 6\int x^2dx = 3x + 6 \cdot \frac{x^3}{3} + c = 3x + 2x^3 + c$

Ex: Calculate integration of the: $\int 6x^2\,(2x^3 - 6)^4dx$

Sol: $= \frac{(2x^3-6)^5}{5} + c$

Ex: Calculate integration of the: $\int \frac{(z+1)dz}{\sqrt[3]{z^2+2z+2}}$

Sol: $\int (z^2 + 2z + 2)^{-1/3}(z + 1)dz$

$$\frac{2}{2}\int (z^2 + 2z + 2)^{-1/3}(z + 1)dz = \frac{1}{2}\int (z^2 + 2z + 2)^{-1/3}(2z + 2)dz$$

$$\frac{1}{2} \cdot \frac{(z^2+2z+2)^{2/3}}{2/3} + c = \frac{3}{4}\sqrt[3]{(z^2 + 2z + 2)^2 + c}$$

5.4 Integral of Trigonometric Functions

1) $\int \sin(x)dx = -\cos(x) + c$

$or \int \sin(u) \cdot \frac{du}{dx} = -\cos(u) + c$

2) $\int \cos(x)dx = \sin(x) + c$

$or \int \cos(u) \cdot \frac{du}{dx} = \sin(u) + c$

3) $\int \sec^2(x)dx = \tan(x) + c$

$or \int \sec^2(u) \cdot \frac{du}{dx} = \tan(u) + c$

4) $\int \csc^2(x)dx = -\cot(x) + c$

$or \int \csc^2(u) \cdot \frac{du}{dx} = -\cot(u) + c$

5) $\int [\sec(x) \cdot \tan(x)]\,dx = \sec(x) + c$

$or \int [\sec(u) \cdot \tan(u)]\frac{du}{dx} = \sec(u) + c$

6) $\int \csc(x) \cdot \cot(x)\,dx = -\csc(x) + c$

$or \int \csc(u) \cdot \cot(u)\frac{du}{dx} = -\csc(u) + c$

Ex: Calculate integration of the: $\int \cos(2x)dx$

Sol: $\frac{2}{2}\int \cos(2x)\,dx = \frac{1}{2}\int \cos(2x)\cdot 2\,dx = \frac{1}{2}\sin(2x) + c$

Ex: Calculate integration of the: $\int x\sin(2x^2)dx$

Sol: $\frac{4}{4}\int \sin(2x^2)\,dx = \frac{1}{4}\int \sin(2x^2)\cdot 4x\,dx$

$= \frac{1}{4}(-\cos(2x^2) + c = -\frac{1}{4}\cos(2x^2) + c$

Ex: Calculate integration of the: $\int \csc^2(\sqrt{x})\cdot\frac{dx}{\sqrt{x}}$

Sol: $(\sqrt{x})' = (x^{1/2})' = \frac{1}{2}\cdot x^{-1/2} = \frac{1}{2\sqrt{x}}$

$\frac{2}{2}\int \csc^2(\sqrt{x})\cdot\frac{1}{\sqrt{x}}dx$

$= 2\int \csc^2(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}dx = -2\cot(\sqrt{x}) + c$

5.5 Integral of the Inverse Trigonometric Functions

1) $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + c$
2) $\int \frac{-du}{\sqrt{1-u^2}} = \cos^{-1}u + c$
3) $\int \frac{1}{1+u^2}du = \tan^{-1}u + c$
4) $\int \frac{-1}{1+u^2}du = \cot^{-1}u + c$
5) $\int \frac{du}{|u|\sqrt{u^2-1}} = \sec^{-1}u + c$
6) $\int \frac{-du}{|u|\sqrt{u^2-1}} = \csc^{-1}u + c$

Ex: Calculate integration of the: $\int \frac{dx}{\sqrt{1-4x^2}}$

Sol: $u^2 = 4x^2 = (2x)^2 \rightarrow\therefore u = 2x \rightarrow du = 2$

$\frac{2}{2}\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \sin^{-1}(2x) + c$

5.6 Integral Natural Logarithmic Function

$$\int \frac{1}{x} dx = \ln|x| + c\,, \qquad x \neq 0$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c\,, \qquad f(x) \neq 0$$

Ex: Calculate integration of the: $\int \frac{3x^2}{x^3} dx$
Sol: $f(x) = x^3 \rightarrow f'(x) = 3x^2$

$$\int \frac{3x^2}{x^3} dx = \ln|x^3| + c$$

Ex: Calculate integration of the: $\int \frac{sec^2(x)}{\tan(x)} dx$
Sol: $f(x) = \tan(x) \rightarrow f'(x) = sec^2(x)$

$$\int \frac{sec^2(x)}{\tan(x)} dx = \ln|\tan(x)| + c$$

5.7 Integral Exponential Function e^x, a^x

1) $\int e^x dx = e^x + c$

$$\int e^u \frac{du}{dx} = e^u + c$$

Ex: Calculate integration of the: $\int e^{x^4} \cdot (4x^3) dx$
Sol: $u = x^4 \rightarrow u' = 4x^3$

$$\int e^{x^4} \cdot (4x^3) dx = e^{x^4} + c$$

Ex: Calculate integration of the: $\int x\, e^{-x^2} dx$
Sol: $u = -x^2 \rightarrow u' = -2x$

$$\frac{-2}{-2}\int x\, e^{-x^2} dx = \frac{-1}{2}\int (-2x)\, e^{-x^2} dx = \frac{-1}{2} e^{-x^2} + \boldsymbol{c}$$

2) $\int a^x dx = \frac{a^x}{\ln a} + c$

$$\int a^u du = \frac{a^u}{\ln a} + c$$

Ex: Calculate integration of the: $\int 3^x dx$

Sol: $\int 3^x dx = \frac{3^x}{\ln 3} + c$

Ex: Calculate integration of the: $\int cos\theta\ 4^{-sin\theta}\, d\theta$

Sol: $u = -sin\theta,\ du = -cos\theta d\theta$

$$-\int 4^{-sin\theta}(-cos\theta)d\theta = \frac{4^{-sin\theta}}{\ln 4} + c$$

5.8 Integral Normal Logarithmic

$$\int \frac{du}{u \cdot \ln a} = \log_a u + c$$
$$or \int \frac{dx}{x \cdot \ln a} = \log_a x + c$$

Ex: Calculate integration of the: $\int \frac{xdx}{x^2 \cdot \ln 5}$

Sol: $u = x^2\ , du = 2x\ dx$

$$\frac{1}{2}\int \frac{2xdx}{x^2 \cdot \ln 5} = \frac{1}{2}\log_5 x^2 + c$$

5.9 Definite Integral

$$\int_a^b f = \int_a^b f(x)dx$$
$$= [f(x) + c]_a^b$$
$$= (f(b) + c) - (f(a) + c)$$
$$= f(b) + c - f(a) - c$$
$$= f(b) - f(a)$$

5.10 Rule Definite Integral

1) $\int_a^b (k_1 f(x) \pm k_2 g(x))dx = k_1 \int_a^b f(x)dx \pm k_2 \int_a^b g(x)dx$

2) $\int_a^a f(x)dx = 0$

3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\ ,\ at\ a \le c \le b, \quad c \in [a,b]$

Ex: Calculate integration of the: $\int_1^6 (3x^2 + 2x)dx$

Sol: $= \frac{3x^3}{3} + \frac{2x^2}{2} |_1^6 = x^3 + x^2|_1^6$

$= ((6)^3 + (6)^2) - ((1)^3 + (1)^2) = 216 + 36 - 2 = 250$

Ex: Calculate integration of the: $\int_0^2 \sqrt{4x+1}\, dx$

Sol: $\int_0^2 (4x+1)^{\frac{1}{2}}\, dx = \frac{1}{4}\int_0^2 (4x+1)^{\frac{1}{2}}.4\, dx$

$= \frac{1}{4}.\frac{(4x+1)^{\frac{3}{2}}}{\frac{3}{2}} |_0^2 = \frac{1}{6}.\sqrt{(4x+1)^3}|_0^2$

$= \frac{1}{6}(\sqrt{(8+1)^3} - \sqrt{(0+1)^3}$

$= \frac{1}{6}\left(\sqrt{9^3} - 1\right) = \frac{1}{6}\left(\sqrt{729} - 1\right) = \frac{1}{6}.(27-1) = \frac{26}{6} = \frac{13}{3}$

Ex: Calculate integration of the: $\int_0^{\pi} \sin x\, dx$

Sol: $= -\cos x\, |_0^{\pi}$

$= -\cos \pi + \cos 0 = -(-1) + 1 = 2$

Ex: Calculate integration of the: $\int_0^{\frac{\pi}{2}} \cos\theta\, d\theta$

Sol: $= sin\theta|_0^{\frac{\pi}{2}} = sin\frac{\pi}{2} - sin0 = 1$

Ex: Calculate integration of the: $\int_{-1}^2 |x|$

Sol: $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

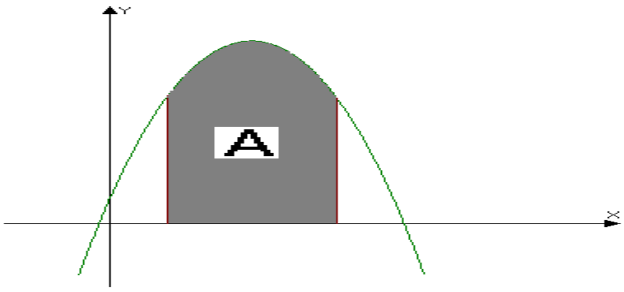
$\int_{-1}^2 |x| = \int_{-1}^0 -x + \int_0^2 x$

$= -\frac{x^2}{2}|_{-1}^0 + \frac{x^2}{2}|_0^2 = 0 + \frac{1}{2} + \frac{4}{2} - 0 = \frac{5}{2}$

5.11 Applications of the Definite Integral

❖ Area Calculation

Let, $y = f(x)$, connecting $to\ the\ interval$ $[a, b]$.
Therefore, the area can be calculated as:



$$\boldsymbol{A = \int_a^b f(x)\,dx}$$

Ex: Find the area of the region bounded by the curve function $y = x^2$ and the x-axis and lines x ,$x = 1$, $x = 3$

Sol: $f(x) \geq 0$

$$A = \int_1^3 x^2\,dx = \frac{x^3}{3}\Big|_1^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

Ex: Find the area of the region bounded by the curve function $y = -x^2$ and the x-axis and lines $x = -2$, $x = 2$

Sol: $f(x) = -x^2 \leq 0$

$$A = \int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} -x^2\,dx = -\frac{x^3}{3}$$

$$-\frac{2^3}{3} + \frac{(-2)^3}{3} = -\frac{8}{3} + \frac{-8}{3} = \left|-\frac{16}{3}\right| = \frac{16}{3}$$

Ex: Find the area of the region bounded by the x-axis and the curve function $f(x) = x^2 - 6x + 8$, connecting the points of intersection of the curve with the X-axis

Sol: $f(x) = 0$

$x^2 - 6x + 8 = 0 \leftrightarrow (x-2)(x-4) = 0 \leftrightarrow x = 2\ or\ x = 4$

$f(x) \leq 0,\ x\ \ in\ [2,4]$

$$A = \int_2^4 (x^2 - 6x + 8)dx = \left(\frac{x^3}{3} - \frac{6x^2}{2} + 8x\right)|_2^4$$

$$= \left(\frac{4^3}{3} - 3(4)^2 + 8(4)\right) - \left(\frac{2^3}{3} - 3(2)^2 + 8(2)\right)$$

$$= \left(\frac{64}{3} - 48 + 32\right) - \left(\frac{8}{3} - 12 + 16\right) = \frac{64-48}{3} - \frac{8+12}{3} = \frac{16}{3} - \frac{20}{3} = \left|-\frac{4}{3}\right| = \frac{4}{3}$$

Exercises:

Q1 Calculate the following integrals:

1) $\int x^5 dx$
2) $\int \frac{1}{x^3} dx$
3) $\int (5x^4 - \frac{10}{x^2})dx$
4) $\int 3x^5 dx$
5) $\int [5 + 6x]^2 \cdot 6dx$
6) $\int (sec^2(x) + 4x^8)dx$
7) $\int 3\csc(x) \cdot \cot(x) dx$
8) $\int \cot(5x) \cdot \csc(5x)\, dx$
9) $\int \frac{cosx}{\sqrt{1-sin^2x}} dx$
10) $\int \frac{2x}{x^2} dx$
11) $\int 3e^{4x}\, dx$
12) $\int x\, e^{-x^2}\, dx$
13) $\int 5^{2t-2}\, dt$
14) $\int \frac{\cos(t)dt}{\sin(t) \cdot \ln 4}$
15) $\int_1^2 x\, dx$
16) $\int_0^3 (x^3 - 4x + 1)\, dx$