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Logic and Propositional Calculus

4.1 INTRODUCTION

Many proofs in mathematics and many algorithms in computer science use logical expressions such as

"IF p THEN q" or "IF p_1 AND p_2, THEN q_1 OR q_2"

It is therefore necessary to know the cases in which these expressions are either TRUE or FALSE: what we refer to as the truth values of such expressions. We discuss these issues in this section.

We also investigate the truth value of quantified statements, which are statements which use the logical quantifiers "for every" and "there exists".

4.2 PROPOSITIONS AND COMPOUND PROPOSITIONS

A *proposition* (or *statement*) is a declarative sentence which is true or false, but not both. Consider, for example, the following eight sentences:

(i) Paris is in France.
(ii) $1 + 1 = 2$.
(iii) $2 + 2 = 3$.
(iv) London is in Denmark.
(v) $9 < 6$.
(vi) $x = 2$ is a solution of $x^2 = 4$.
(vii) Where are you going?
(viii) Do your homework.

All of them are propositions except (vii) and (viii). Moreover, (i), (ii), and (vi) are true, whereas (iii), (iv), and (v) are false.

Compound Propositions

Many propositions are *composite*, that is, composed of *subpropositions* and various connectives discussed subsequently. Such composite propositions are called *compound propositions*. A proposition is said to be *primitive* if it cannot be broken down into simpler propositions, that is, if it is not composite.

EXAMPLE 4.1

(*a*) "Roses are red and violets are blue" is a compound proposition with subpropositions "Roses are red" and "Violets are blue".

(*b*) "John is intelligent or studies every night" is a compound proposition with subpropositions "John is intelligent" and "John studies every night".

(*c*) The above propositions (i) through (vi) are all primitive propositions; they cannot be broken down into simpler propositions.

> The fundamental property of a compound proposition is that its truth value is completely determined by the truth values of its subpropositions together with the way in which they are connected to form the compound propositions. The next section studies some of these connectives.

4.3 BASIC LOGICAL OPERATIONS

This section discusses the three basic logical operations of conjunction, disjunction, and negation which correspond, respectively, to the English words "and", "or", and "not".

Conjunction, $p \wedge q$

Any two propositions can be combined by the word "and" to form a compound proposition called the *conjunction* of the original propositions. Symbolically,

$$p \wedge q$$

read "p and q", denotes the conjunction of p and q. Since $p \wedge q$ is a proposition it has a truth value, and this truth value depends only on the truth values of p and q. Specifically:

Definition 4.1: If p and q are true, then $p \wedge q$ is true; otherwise $p \wedge q$ is false.

The truth value of $p \wedge q$ may be defined equivalently by the table in Fig. 4-1(a). Here, the first line is a short way of saying that if p is true and q is true, then $p \wedge q$ is true. The second line says that if p is true and q is false, then $p \wedge q$ is false. And so on. Observe that there are four lines corresponding to the four possible combinations of T and F for the two subpropositions p and q. Note that $p \wedge q$ is true only when both p and q are true.

EXAMPLE 4.2 Consider the following four statements:

(i) Paris is in France and $2 + 2 = 4$.
(ii) Paris is in France and $2 + 2 = 5$.
(iii) Paris is in England and $2 + 2 = 4$.
(iv) Paris is in England and $2 + 2 = 5$.

Only the first statement is true. Each of the other statements is false, since at least one of its substatements is false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(a) "p and q"

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(b) "p or q"

p	$\neg p$
T	F
F	T

(c) "not p"

Fig. 4-1

Disjunction, $p \vee q$

Any two propositions can be combined by the word "or" to form a compound proposition called the *disjunction* of the original propositions. Symbolically,

$$p \vee q$$

read "p or q", denotes the disjunction of p and q. The truth value of $p \vee q$ depends only on the truth values of p and q as follows.

Definition 4.2: If p and q are false, then $p \vee q$ is false; otherwise $p \vee q$ is true.

The truth value of $p \vee q$ may be defined equivalently by the table in Fig. 4-1(b). Observe that $p \vee q$ is false only in the fourth case when both p and q are false.

EXAMPLE 4.3 Consider the following four statements:

(i) Paris is in France or $2 + 2 = 4$.

(ii) Paris is in France or $2 + 2 = 5$.

(iii) Paris is in England or $2 + 2 = 4$.

(iv) Paris is in England or $2 + 2 = 5$.

Only the last statement (iv) is false. Each of the other statements is true since at least one of its substatements is true.

Remark: The English word "or" is commonly used in two distinct ways. Sometimes it is used in the sense of "p or q or both", i.e., at least one of the two alternatives occurs, as above, and sometimes it is used in the sense of "p or q but not both", i.e., exactly one of the two alternatives occurs. For example, the sentence "He will go to Harvard or to Yale" uses "or" in the latter sense, called the *exclusive disjunction*. Unless otherwise stated, "or" shall be used in the former sense. This discussion points out the precision we gain from our symbolic language: $p \vee q$ is defined by its truth table and *always* means "p and/or q".

Negation, $\neg p$

Given any proposition p, another proposition, called the *negation* of p, can be formed by writing "It is not the case that ..." or "It is false that ..." before p or, if possible, by inserting in p the word "not". Symbolically,

$$\neg p$$

read "not p", denotes the negation of p. The truth value of $\neg p$ depends on the truth value of p as follows.

Definition 4.3: If p is true, then $\neg p$ is false; and if p is false, then $\neg p$ is true.

The truth value of $\neg p$ may be defined equivalently by the table in Fig. 4-1(c). Thus the truth value of the negation of p is always the opposite of the truth value of p.

EXAMPLE 4.4 Consider the following six statements:

(a_1) Paris is in France.

(a_2) It is not the case that Paris is in France.

(a_3) Paris is not in France.

(b_1) $2 + 2 = 5$.

(b_2) It is not the case that $2 + 2 = 5$.

(b_3) $2 + 2 \neq 5$.

Then (a_2) and (a_3) are each the negation of (a_1); and (b_2) and (b_3) are each the negation of (b_1). Since (a_1) is true, (a_2) and (a_3) are false; and since (b_1) is false, (b_2) and (b_3) are true.

Remark: The logical notation for the connectives "and", "or", and "not" is not completely standardized. For example, some texts use:

$p \;\&\; q$, $p \cdot q$ or pq	for $p \wedge q$
$p + q$	for $p \vee q$
p', $\bar{p}$ or $\sim p$	for $\neg p$

4.4 PROPOSITIONS AND TRUTH TABLES

Let $P(p, q, \ldots)$ denote an expression constructed from logical variables p, q, ..., which take on the value TRUE (T) or FALSE (F), and the logical connectives $\wedge$, $\vee$, and $\neg$ (and others discussed subsequently). Such an expression $P(p, q, \ldots)$ will be called a *proposition*.

The main property of a proposition $P(p, q, \ldots)$ is that its truth value depends exclusively upon the truth values of its variables, that is, the truth value of a proposition is known once the truth value of each of its variables is known. A simple concise way to show this relationship is through a *truth table*. We describe a way to obtain such a truth table below.

Consider, for example, the proposition $\neg (p \wedge \neg q)$. Figure 4-2(*a*) indicates how the truth table of $\neg (p \wedge \neg q)$ is constructed. Observe that the first columns of the table are for the variables $p, q, \ldots$ and that there are enough rows in the table to allow for all possible combinations of T and F for these *variables*. (For 2 variables, as above, 4 rows are necessary; for 3 variables, 8 rows are necessary; and, in general, for n variables, 2^n rows are required.) There is then a column for each "elementary" stage of the construction of the proposition, the truth value at each step being determined from the previous stages by the definitions of the connectives $\wedge, \vee, \neg$. Finally we obtain the truth value of the proposition, which appears in the last column.

The actual truth table of the proposition $\neg (p \wedge \neg q)$ is shown in Fig. 4-2(*b*). It consists precisely of the columns in Fig. 4-2(*a*) which appear under the variables and under the proposition; the other columns were merely used in the construction of the truth table.

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(*a*)

p	q	$\neg(p \wedge \neg q)$
T	T	T
T	F	F
F	T	T
F	F	T

(*b*)

Fig. 4-2

Remark: In order to avoid an excessive number of parentheses, we sometimes adopt an order of precedence for the logical connectives. Specifically,

$\neg$ has precedence over $\wedge$ which has precedence over $\vee$

For example, $\neg p \wedge q$ means $(\neg p) \wedge q$ and not $\neg (p \wedge q)$.

Alternative Method for Constructing a Truth Table

Another way to construct the truth table for $\neg (p \wedge \neg q)$ follows:

(*a*) First we construct the truth table shown in Fig. 4-3. That is, first we list all the variables and the combinations of their truth values. Then the proposition is written on the top row to the right of its variables with sufficient space so that there is a column under each variable and each connective in the proposition. Also there is a final row labeled "Step".

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T					
T	F					
F	T					
F	F					
Step						

Fig. 4-3

(*b*) Next, additional truth values are entered into the truth table in various steps as shown in Fig. 4-4. That is, first the truth values of the variables are entered under the variables in the proposition, and then there is a column of truth values entered under each logical operation. We also indicate the step in which each column of truth values is entered in the table.

The truth table of the proposition then consists of the original columns under the variables and the last step, that is, the last column entered into the table.

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T		T			T
T	F		T			F
F	T		F			T
F	F		F			F
Step			1			1

(*a*)

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T		T		F	T
T	F		T		T	F
F	T		F		F	T
F	F		F		T	F
Step			1		2	1

(*b*)

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T		T	F	F	T
T	F		T	T	T	F
F	T		F	F	F	T
F	F		F	F	T	F
Step			1	3	2	1

(*c*)

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T	T	T	F	F	T
F	T	F	T	T	T	F
F	F	T	F	F	F	T
F	F	T	F	F	T	F
Step		4	1	3	2	1

(*d*)

Fig. 4-4

4.5 TAUTOLOGIES AND CONTRADICTIONS

Some propositions $P(p, q, \ldots)$ contain only T in the last column of their truth tables or, in other words, they are true for any truth values of their variables. Such propositions are called *tautologies*. Analogously, a proposition $P(p, q, \ldots)$ is called a *contradiction* if it contains only F in the last column of its truth table or, in other words, if it is false for any truth values of its variables. For example, the proposition "p or not p", that is, $p \vee \neg p$, is a tautology, and the proposition "p and not p", that is, $p \wedge \neg p$, is a contradiction. This is verified by looking at their truth tables in Fig. 4-5. (The truth tables have only two rows since each proposition has only the one variable p.)

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

(*a*) $p \vee \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

(*b*) $p \wedge \neg p$

Fig. 4-5

Note that the negation of a tautology is a contradiction since it is always false, and the negation of a contradiction is a tautology since it is always true.

Now let $P(p, q, \ldots)$ be a tautology, and let $P_1(p, q, \ldots), P_2(p, q, \ldots), \ldots$ be any propositions. Since $P(p, q, \ldots)$ does not depend upon the particular truth values of its variables $p, q, \ldots$, we can substitute P_1 for p, P_2 for $q, \ldots$ in the tautology $P(p, q, \ldots)$ and still have a tautology. In other words:

Theorem 4.1 **(Principle of Substitution):** If $P(p, q, \ldots)$ is a tautology, then $P(P_1, P_2, \ldots)$ is a tautology for any propositions $P_1, P_2, \ldots$.

4.6 LOGICAL EQUIVALENCE

Two propositions $P(p, q, \ldots)$ and $Q(p, q, \ldots)$ are said to be *logically equivalent*, or simply *equivalent* or *equal*, denoted by

$$P(p, q, \ldots) \equiv Q(p, q, \ldots)$$

if they have identical truth tables. Consider, for example, the truth tables of $\neg (p \wedge q)$ and $\neg p \vee \neg q$ appearing in Fig. 4-6. Observe that both truth tables are the same, that is, both propositions are false in the first case and true in the other three cases. Accordingly, we can write

$$\neg (p \wedge q) \equiv \neg p \vee \neg q$$

In other words, the propositions are logically equivalent.

Remark: Consider the statement

"It is not the case that roses are red and violets are blue"

This statement can be written in the form $\neg (p \vee q)$ where:

p is "roses are red" and q is "violets are blue"

However, as noted above, $\neg (p \wedge q) \equiv \neg p \vee \neg q$. Thus the statement

"Roses are not red, or violets are not Blue."

has the same meaning as the given statement.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(*a*) $\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(*b*) $\neg p \vee \neg q$

Fig. 4-6

4.7 ALGEBRA OF PROPOSITIONS

Propositions satisfy various laws which are listed in Table 4-1. (In this table, T and F are restricted to the truth values "true" and "false", respectively.) We state this result formally.

Theorem 4.2: Propositions satisfy the laws of Table 4-1.

Table 4-1 Laws of the algebra of propositions

Idempotent laws	
(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws	
(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws	
(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws	
(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws	
(5a) $p \vee \mathrm{F} \equiv p$	(5b) $p \wedge \mathrm{T} \equiv p$
(6a) $p \vee \mathrm{T} \equiv \mathrm{T}$	(6b) $p \wedge \mathrm{F} \equiv \mathrm{F}$
Complement laws	
(7a) $p \vee \neg p \equiv \mathrm{T}$	(7b) $p \wedge \neg p \equiv \mathrm{F}$
(8a) $\neg \mathrm{T} \equiv \mathrm{F}$	(8b) $\neg \mathrm{F} \equiv \mathrm{T}$
Involution law	
(9) $\neg\neg p \equiv p$	
DeMorgan's laws	
(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

4.8 CONDITIONAL AND BICONDITIONAL STATEMENTS

Many statements, particularly in mathematics, are of the form "If p then q". Such statements are called *conditional* statements and are denoted by

$$p \rightarrow q$$

The conditional $p \rightarrow q$ is frequently read "p implies q" or "p only if q".

Another common statement is of the form "p if and only if q". Such statements are called *biconditional* statements and are denoted by

$$p \leftrightarrow q$$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(a) $p \rightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(b) $p \leftrightarrow q$

Fig. 4-7

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

$\neg p \vee q$

Fig. 4-8

The truth values of $p \to q$ and $p \leftrightarrow q$ are defined by the tables in Fig. 4-7. Observe that:

(*a*) The conditional $p \to q$ is false only when the first part p is true and the second part q is false. Accordingly, when p is false, the conditional $p \to q$ is true regardless of the truth value of q.

(*b*) The biconditional $p \leftrightarrow q$ is true whenever p and q have the same truth values and false otherwise.

The truth table of the proposition $\neg p \vee q$ appears in Fig. 4-8. Observe that the truth tables of $\neg p \vee q$ and $p \to q$ are identical, that is, they are both false only in the second case. Accordingly, $p \to q$ is logically equivalent to $\neg p \vee q$; that is,

$$p \to q \equiv \neg p \vee q$$

In other words, the conditional statement "If p then q" is logically equivalent to the statement "Not p or q" which only involves the connectives $\vee$ and $\neg$ and thus was already a part of our language. We may regard $p \to q$ as an abbreviation for an oft-recurring statement.

4.9 ARGUMENTS

An *argument* is an assertion that a given set of propositions $P_1, P_2, \ldots, P_n$, called *premises*, yields (has a consequence) another proposition Q, called the *conclusion*. Such an argument is denoted by

$$P_1, P_2, \ldots, P_n \vdash Q$$

The notion of a "logical argument" or "valid argument" is formalized as follows:

Definition 4.4: An argument $P_1, P_2, \ldots, P_n \vdash Q$ is said to be *valid* if Q is true whenever all the premises $P_1, P_2, \ldots, P_n$ are true.

An argument which is not valid is called a *fallacy*.

EXAMPLE 4.5

(*a*) The following argument is valid:

$$p, p \to q \vdash q \qquad \textit{(Law of Detachment)}$$

The proof of this rule follows from the truth table in Fig. 4-9. Specifically, p and $p \to q$ are true simultaneously only in Case (row) 1, and in this case q is true.

(*b*) The following argument is a fallacy:

$$p \to q, q \vdash p$$

For $p \to q$ and q are both true in Case (row) 3 in the truth table in Fig. 4-9, but in this case p is false.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Fig. 4-9

Now the propositions $P_1, P_2, \ldots, P_n$ are true simultaneously if and only if the proposition $P_1 \wedge P_2 \wedge \ldots \wedge P_n$ is true. Thus the argument $P_1, P_2, \ldots, P_n \vdash Q$ is valid if and only if Q is true whenever $P_1 \wedge P_2 \wedge \ldots \wedge P_n$ is true or, equivalently, if the proposition $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \to Q$ is a tautology. We state this result formally.

Theorem 4.3: The argument $P_1, P_2, \ldots, P_n \vdash Q$ is valid if and only if the proposition $(P_1 \wedge P_2 \ldots \wedge P_n) \rightarrow Q$ is a tautology.

We apply this theorem in the next example.

EXAMPLE 4.6 A fundamental principle of logical reasoning states:

"If p implies q and q implies r, then p implies r."

That is, the following argument is valid:

$$p \rightarrow q,\ q \rightarrow r \vdash p \rightarrow r \text{ (Law of Syllogism)}$$

This fact is verified by the truth table in Fig. 4-10 which shows that the following proposition is a tautology:

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

Equivalently, the argument is valid since the premises $p \rightarrow q$ and $q \rightarrow r$ are true simultaneously only in Cases (rows) 1, 5, 7 and 8, and in these cases the conclusion $p \rightarrow r$ is also true. (Observe that the truth table required $2^3 = 8$ lines since there are three variables p, q and r.)

p	q	r	$[(p$	$\rightarrow$	$q)$	$\wedge$	$(q$	$\rightarrow$	$r)]$	$\rightarrow$	$(p$	$\rightarrow$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	F	F	T	T	F	F
T	F	T	T	F	F	F	F	T	T	T	T	T	T
T	F	F	T	F	F	F	F	T	F	T	T	F	F
F	T	T	F	T	T	T	T	T	T	T	F	T	T
F	T	F	F	T	T	F	T	F	F	T	F	T	F
F	F	T	F	T	F	T	F	T	T	T	F	T	T
F	F	F	F	T	F	T	F	T	F	T	F	T	F
Step			1	2	1	3	1	2	1	4	1	2	1

Fig. 4-10

We now apply the above theory to arguments involving specific statements. We emphasize that the validity of an argument does not depend upon the truth values nor the content of the statements appearing in the argument, but upon the particular form of the argument. This is illustrated in the following example.

EXAMPLE 4.7 Consider the following argument:

S_1: If a man is a bachelor, he is unhappy.
S_2: If a man is unhappy, he dies young.

....

S: Bachelors die young.

Here the statement S below the line denotes the conclusion of the argument, and the statements S_1 and S_2 above the line denote the premises. We claim that the argument $S_1, S_2 \vdash S$ is valid. For the argument is of the form

$$p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$$

where p is "He is a bachelor", q is "He is unhappy" and r is "He dies young"; and by Example 4.6 this argument (Law of Syllogism) is valid.

4.10 LOGICAL IMPLICATION

A proposition $P(p,q,\ldots)$ is said to *logically imply* a proposition $Q(p,q,\ldots)$, written

$$P(p,q,\ldots) \Rightarrow Q(p,q,\ldots)$$

if $Q(p,q,\ldots)$ is true whenever $P(p,q,\ldots)$ is true.

EXAMPLE 4.8 We claim that p logically implies $p \vee q$. For consider the truth table in Fig. 4-11. Observe that p is true in Cases (rows) 1 and 2, and in these cases $p \vee q$ is also true. Thus $p \Rightarrow p \vee q$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Fig. 4-11

Now if $Q(p,q,\ldots)$ is true whenever $P(p,q,\ldots)$ is true, then the argument

$$P(p,q,\ldots) \vdash Q(p,q,\ldots)$$

is valid; and conversely. Furthermore, the argument $P \vdash Q$ is valid if and only if the conditional statement $P \rightarrow Q$ is always true, i.e., a tautology. We state this result formally.

Theorem 4.4: For any propositions $P(p,q,\ldots)$ and $Q(p,q,\ldots)$, the following three statements are equivalent:

(i) $P(p,q,\ldots)$ logically implies $Q(p,q,\ldots)$.

(ii) The argument $P(p,q,\ldots) \vdash Q(p,q,\ldots)$ is valid.

(iii) The proposition $P(p,q,\ldots) \rightarrow Q(p,q,\ldots)$ is a tautology.

We note that some logicians and many texts use the word "implies" in the same sense as we use "logically implies", and so they distinguish between "implies" and "if... then". These two distinct concepts are, of course, intimately related as seen in the above theorem.

4.11 PROPOSITIONAL FUNCTIONS, QUANTIFIERS

Let A be a given set. A *propositional function* (or: an *open sentence* or *condition*) defined on A is an expression

$$p(x)$$

which has the property that $p(a)$ is true or false for each $a \in A$. That is, $p(x)$ becomes a statement (with a truth value) whenever any element $a \in A$ is substituted for the variable x. The set A is called the *domain* of $p(x)$, and the set T_p of all elements of A for which $p(a)$ is true is called the *truth set* of $p(x)$. In other words,

$$T_p = \{x\colon x \in A,\ p(x) \text{ is true}\} \quad \text{or} \quad T_p = \{x\colon p(x)\}$$

Frequently, when A is some set of numbers, the condition $p(x)$ has the form of an equation or inequality involving the variable x.

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الفصل الثاني

Set Theory

2.1 INTRODUCTION

The concept of a *set* appears in all mathematics. This chapter introduces the notation and terminology of set theory which is basic and used throughout the text.

Though logic is formally treated in Chapter 4, we introduce Venn diagram representation of sets here, and we show how it can be applied to logical arguments. The relation between set theory and logic will be further explored when we discuss Boolean algebra in Chapter 15.

This chapter closes with the formal definition of mathematical induction, with examples.

2.2 SETS AND ELEMENTS

A *set* may be viewed as a collection of objects, the *elements* or *members* of the set. We ordinarily use capital letters, $A, B, X, Y, \ldots$, to denote sets, and lowercase letters, $a, b, x, y, \ldots$, to denote elements of sets. The statement "p is an element of A", or, equivalently, "p belongs to A", is written

$$p \in A$$

The statement that p is not an element of A, that is, the negation of $p \in A$, is written

$$p \notin A$$

The fact that a set is completely determined when its members are specified is formally stated as the principle of extension.

Principle of Extension: Two sets A and B are equal if and only if they have the same members.

As usual, we write $A = B$ if the sets A and B are equal, and we write $A \neq B$ if the sets are not equal.

Specifying Sets

There are essentially two ways to specify a particular set. One way, if possible, is to list its members. For example,

$$A = \{a, e, i, o, u\}$$

denotes the set A whose elements are the letters a, e, i, o, u. Note that the elements are separated by commas and enclosed in braces { }. The second way is to state those properties which characterized the elements in the set. For example,

$$B = \{x : x \text{ is an even integer}, x > 0\}$$

which reads "B is the set of x such that x is an even integer and x is greater than 0", denotes the set B whose elements are the positive integers. A letter, usually x, is used to denote a typical member of the set; the colon is read as "such that" and the comma as "and".

EXAMPLE 2.1

(*a*) The set A above can also be written as

$$A = \{x : x \text{ is a letter in the English alphabet}, x \text{ is a vowel}\}$$

Observe that $b \notin A$, $e \in A$, and $p \notin A$.

(*b*) We could not list all the elements of the above set B although frequently we specify the set by writing

$$B = \{2, 4, 6, \ldots\}$$

where we assume that everyone knows what we mean. Observe that $8 \in B$ but $7 \notin B$.

(*c*) Let $E = \{x: x^2 - 3x + 2 = 0\}$. In other words, E consists of those numbers which are solutions of the equation $x^2 - 3x + 2 = 0$, sometimes called the *solution set* of the given equation. Since the solutions of the equation are 1 and 2, we could also write $E = \{1, 2\}$.

(*d*) Let $E = \{x: x^2 - 3x + 2 = 0\}$, $F = \{2, 1\}$ and $G = \{1, 2, 2, 1, \frac{6}{3}\}$. Then $E = F = G$. Observe that a set does not depend on the way in which its elements are displayed. A set remains the same if its elements are repeated or rearranged.

Some sets will occur very often in the text and so we use special symbols for them. Unless otherwise specified, we will let

$$\begin{aligned}
\mathbf{N} &= \text{the set of positive integers: } 1, 2, 3, \ldots \\
\mathbf{Z} &= \text{the set of integers: } \ldots, -2, -1, 0, 1, 2, \ldots \\
\mathbf{Q} &= \text{the set of rational numbers} \\
\mathbf{R} &= \text{the set of real numbers} \\
\mathbf{C} &= \text{the set of complex numbers}
\end{aligned}$$

Even if we can list the elements of a set, it may not be practical to do so. For example, we would not list the members of the set of people born in the world during the year 1976 although theoretically it is possible to compile such a list. That is, we describe a set by listing its elements only if the set contains a few elements; otherwise we describe a set by the property which characterizes its elements.

The fact that we can describe a set in terms of a property is formally stated as the *principle of abstraction*.

Principle of Abstraction: Given any set U and any property P, there is a set A such that the elements of A are exactly those members of U which have the property P.

1.3 UNIVERSAL SET AND EMPTY SET

In any application of the theory of sets, the members of all sets under investigation usually belong to some fixed large set called the *universal set*. For example, in plane geometry, the universal set consists of all the points in the plane, and in human population studies the universal set consists of all the people in the world. We will let the symbol

$$U$$

denote the universal set unless otherwise stated or implied.

For a given set U and a property P, there may not be any elements of U which have property P. For example, the set

$$S = \{x: x \text{ is a positive integer, } x^2 = 3\}$$

has no elements since no positive integer has the required property.

The set with no elements is called the *empty set* or *null set* and is denoted by

$$\varnothing$$

There is only one empty set. That is, if S and T are both empty, then $S = T$ since they have exactly the same elements, namely, none.

1.4 SUBSETS

If every element in a set A is also an element of a set B, then A is called a *subset* of B. We also say that A is *contained* in B or that B *contains* A. This relationship is written

$$A \subseteq B \qquad \text{or} \qquad B \supseteq A$$

If A is not a subset of B, i.e., if at least one element of A does not belong to B, we write $A \not\subseteq B$ or $B \not\supseteq A$.

EXAMPLE 1.2

(*a*) Consider the sets

$$A = \{1,3,4,5,8,9\} \qquad B = \{1,2,3,5,7\} \qquad C = \{1,5\}$$

Then $C \subseteq A$ and $C \subseteq B$ since 1 and 5, the elements of C, are also members of A and B. But $B \not\subseteq A$ since some of its elements, e.g., 2 and 7, do not belong to A. Furthermore, since the elements of A, B, and C must also belong to the universal set U, we have that U must at least contain the set $\{1,2,3,4,5,6,7,8,9\}$.

(*b*) Let **N**, **Z**, **Q**, and **R** be defined as in Section 1.2. Then

$$\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$$

(*c*) The set $E = \{2,4,6\}$ is a subset of the set $F = \{6,2,4\}$, since each number 2, 4, and 6 belonging to E also belongs to F. In fact, $E = F$. In a similar manner it can be shown that every set is a subset of itself.

The following properties of sets should be noted:

(i) Every set A is a subset of the universal set U since, by definition, all the elements of A belong to U. Also the empty set $\varnothing$ is a subset of A.

(ii) Every set A is a subset of itself since, trivially, the elements of A belong to A.

(iii) If every element of A belongs to a set B, and every element of B belongs to a set C, then clearly every element of A belongs to C. In other words, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(iv) If $A \subseteq B$ and $B \subseteq A$, then A and B have the same elements, i.e., $A = B$. Conversely, if $A = B$ then $A \subseteq B$ and $B \subseteq A$ since every set is a subset of itself.

We state these results formally.

Theorem 1.1: (i) For any set A, we have $\varnothing \subseteq A \subseteq U$.
(ii) For any set A, we have $A \subseteq A$.
(iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
(iv) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

If $A \subseteq B$, then it is still possible that $A = B$. When $A \subseteq B$ but $A \neq B$, we say A is a *proper subset* of B. We will write $A \subset B$ when A is a proper subset of B. For example, suppose

$$A = \{1,3\} \qquad B = \{1,2,3\}, \qquad C = \{1,3,2\}$$

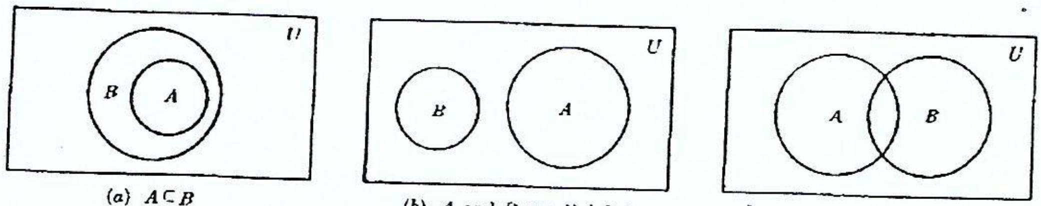
Then A and B are both subsets of C; but A is a proper subset of C, whereas B is not a proper subset of C since $B = C$.

1.5 VENN DIAGRAMS

A Venn diagram is a pictoral representation of sets in which sets are represented by enclosed areas in the plane.

The universal set U is represented by the interior of a rectangle, and the other sets are represented by disks lying within the rectangle. If $A \subseteq B$, then the disk representing A will be entirely within the disk representing B as in Fig. 1-1(*a*). If A and B are disjoint, i.e., if they have no elements in common, then the disk representing A will be separated from the disk representing B as in Fig. 1-1(*b*).

However, if A and B are two arbitrary sets, it is possible that some objects are in A but not in B, some are in B but not in A, some are in both A and B, and some are in neither A nor B; hence in general we represent A and B as in Fig. 1-1(*c*).



(a) $A \subset B$ (b) A and B are disjoint (c)

Fig. 1-1

Arguments and Venn Diagrams

Many verbal statements are essentially statements about sets and can therefore be described by Venn diagrams.

Hence Venn diagrams can sometimes be used to determine whether or not an argument is valid. Consider the following example.

EXAMPLE 1.3 Show that the following argument (adapted from a book on logic by Lewis Carroll, the author of *Alice in Wonderland*) is valid:

S_1: My saucepans are the only things I have that are made of tin.
S_2: I find all your presents very useful.
S_3: None of my saucepans is of the slightest use.

S: Your presents to me are not made of tin.

(The statements S_1, S_2, and S_3 above the horizontal line denote the assumptions, and the statement S below the line denotes the conclusion. The argument is valid if the conclusion S follows logically from the assumptions S_1, S_2, and S_3.)

By S_1 the tin objects are contained in the set of saucepans and by S_3 the set of saucepans and the set of useful things are disjoint: hence draw the Venn diagram of Fig. 1-2.

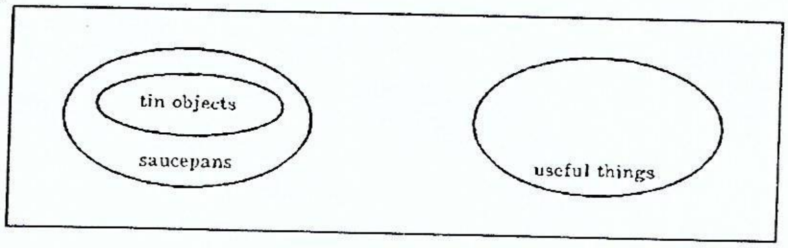


Fig. 1-2

By S_2 the set of "your presents" is a subset of the set of useful things; hence draw Fig. 1-3.

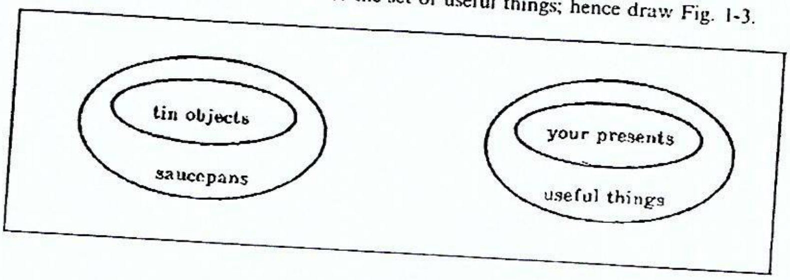


Fig. 1-3

The conclusion is clearly valid by the above Venn diagram because the set of "your presents" is disjoint from the set of tin objects.

1.6 SET OPERATIONS

This section introduces a number of important operations on sets.

Union and Intersection

The *union* of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to A or to B; that is,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

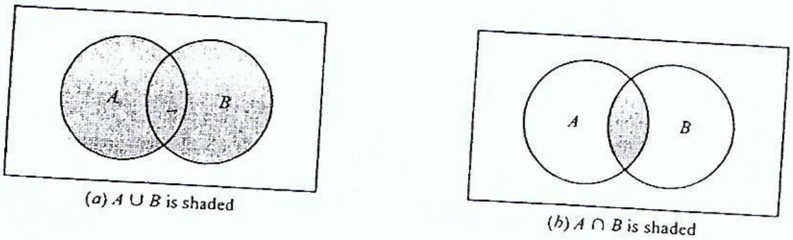
Here "or" is used in the sense of and/or. Figure 1-4(a) is a Venn diagram in which $A \cup B$ is shaded.

The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B; that is,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Figure 1-4(b) is a Venn diagram in which $A \cap B$ is shaded.

If $A \cap B = \varnothing$, that is, if A and B do not have any elements in common, then A and B are said to be *disjoint* or *nonintersecting*.



(a) $A \cup B$ is shaded

(b) $A \cap B$ is shaded

Fig. 1-4

EXAMPLE 1.4

(a) Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{2, 3, 5, 7\}$. Then

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\} \qquad A \cap B = \{3, 4\}$$
$$A \cup C = \{1, 2, 3, 4, 5, 7\} \qquad A \cap C = \{2, 3\}$$

(*b*) Let M denote the set of male students in a university C, and let F denote the set of female students in C. Then

$$M \cup F = C$$

since each student in C belongs to either M or F. On the other hand,

$$M \cap F = \emptyset$$

since no student belongs to both M and F.

The operation of set inclusion is closely related to the operations of union and intersection, as shown by the following theorem.

Theorem 1.2: The following are equivalent: $A \subseteq B$, $A \cap B = A$, and $A \cup B = B$.

Note: This theorem is proved in Problem 1.27. Other conditions equivalent to $A \subseteq B$ are given in Problem 1.37.

Complements

Recall that all sets under consideration at a particular time are subsets of a fixed universal set U. The *absolute complement* or, simply, *complement* of a set A, denoted by A^c, is the set of elements which belong to U but which do not belong to A; that is,

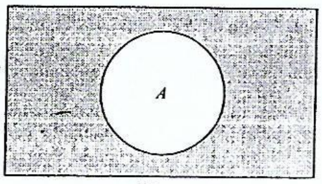
$$A^c = \{x: x \in U, x \notin A\}$$

Some texts denote the complement of A by A' or $\bar{A}$. Figure 1-5(*a*) is a Venn diagram in which A^c is shaded.

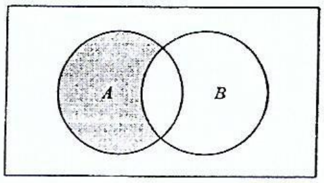
The *relative complement* of a set B with respect to a set A or, simply, the *difference* of A and B, denoted by $A \backslash B$, is the set of elements which belong to A but which do not belong to B; that is

$$A \backslash B = \{x: x \in A, x \notin B\}$$

The set $A \backslash B$ is read "A minus B". Many texts denote $A \backslash B$ by $A - B$ or $A \sim B$. Figure 1-5(*b*) is a Venn diagram in which $A \backslash B$ is shaded.



(*a*) A^c is shaded



(*b*) $A \backslash B$ is shaded

Fig. 1-5

EXAMPLE 1.5 Suppose $U = \mathbf{N} = \{1, 2, 3, \ldots\}$, the positive integers, is the universal set. Let

$$A = \{1, 2, 3, 4, \}, \quad B = \{3, 4, 5, 6, 7\}, \quad C = \{6, 7, 8, 9\}$$

and let $E = \{2, 4, 6, 8, \ldots\}$, the even integers. Then

$$A^c = \{5, 6, 7, 8, \ldots\}, \quad B^c = \{1, 2, 8, 9, 10, \ldots\}, \quad C^c = \{1, 2, 3, 4, 5, 10, 11, \ldots\}$$

and

$$A \backslash B = \{1, 2\}, \quad B \backslash C = \{3, 4, 5\}, \quad B \backslash A = \{5, 6, 7\}, \quad C \backslash E = \{7, 9\}$$

Also, $E^c = \{1, 3, 5, \ldots\}$, the odd integers.

Fundamental Products

Consider n distinct sets $A_1, A_2, \ldots, A_n$. A *fundamental product* of the sets is a set of the form

$$A_1^* \cap A_2^* \cap \cdots \cap A_n^*$$

where A_i^* is either A_i or A_i^c. We note that (1) there are 2^n such fundamental products, (2) any two such fundamental products are disjoint, and (3) the universal set U is the union of all the fundamental products (Problem 1.64). There is a geometrical description of these sets which is illustrated below.

EXAMPLE 1.6 Consider three sets A, B, and C. The following lists the eight fundamental products of the three sets:

$$P_1 = A \cap B \cap C, \quad P_3 = A \cap B^c \cap C, \quad P_5 = A^c \cap B \cap C, \quad P_7 = A^c \cap B^c \cap C$$
$$P_2 = A \cap B \cap C^c \quad P_4 = A \cap B^c \cap C^c. \quad P_6 = A^c \cap B \cap C^c \quad P_8 = A^c \cap B^c \cap C^c$$

These eight products correspond precisely to the eight disjoint regions in the Venn diagram of sets A, B, C in Fig. 1-6 as indicated by the labeling of the regions.

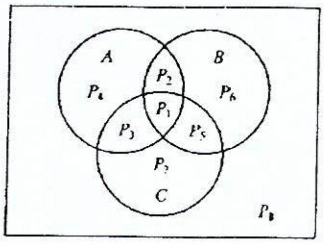
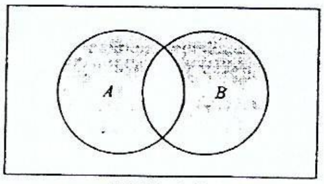


Fig. 1-6



$A \oplus B$ is shaded

Fig. 1-7

Symmetric Difference

The *symmetric difference* of sets A and B, denoted by $A \oplus B$, consists of those elements which belong to A or B but not to both; that is,

$$A \ominus B = (A \cup B) \backslash (A \cap B)$$

One can also show (Problem 1.18) that

$$A \oplus B = (A \backslash B) \cup (B \backslash A)$$

For example, suppose $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{4, 5, 6, 7, 8, 9\}$. Then

$$A \backslash B = \{1, 2, 3\}, \qquad B \backslash A = \{7, 8, 9\} \qquad \text{and so} \qquad A \oplus B = \{1, 2, 3, 7, 8, 9\}$$

Figure 1-7 is a Venn diagram in which $A \oplus B$ is shaded.

1.7 ALGEBRA OF SETS AND DUALITY

Sets under the operations of union, intersection, and complement satisfy various laws or identities which are listed in Table 1-1. In fact, we formally state this:

Theorem 1.3: Sets satisfy the laws in Table 1-1.

There are two methods of proving equations involving set operations. One way is to use what it means for an object x to be an element of each side, and the other way is to use Venn diagrams. For example, consider the first of DeMorgan's laws.

$$(A \cup B)^c = A^c \cap B^c$$

Table 1-1 Laws of the algebra of sets

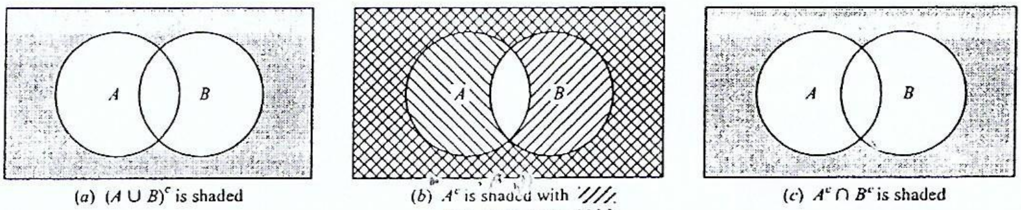
Idempotent laws	
(1a) $A \cup A = A$	(1b) $A \cap A = A$
Associative laws	
(2a) $(A \cup B) \cup C = A \cup (B \cup C)$	(2b) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws	
(3a) $A \cup B = B \cup A$	(3b) $A \cap B = B \cap A$
Distributive laws	
(4a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(4b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws	
(5a) $A \cup \varnothing = A$	(5b) $A \cap U = A$
(6a) $A \cup U = U$	(6b) $A \cap \varnothing = \varnothing$
Involution laws	
(7) $(A^c)^c = A$	
Complement laws	
(8a) $A \cup A^c = U$	(8b) $A \cap A^c = \varnothing$
(9a) $U^c = \varnothing$	(9b) $\varnothing^c = U$
DeMorgan's laws	
(10a) $(A \cup B)^c = A^c \cap B^c$	(10b) $(A \cap B)^c = A^c \cup B^c$

Method 1: We first show that $(A \cup B)^c \subseteq A^c \cap B^c$. If $x \in (A \cup B)^c$, then $x \notin A \cup B$. Thus $x \notin A$ and $x \notin B$, and so $x \in A^c$ and $x \in B^c$. Hence $x \in A^c \cap B^c$.

Next we show that $A^c \cap B^c \subseteq (A \cup B)^c$. Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$, so $x \notin A$ and $x \notin B$. Hence $x \notin A \cup B$, so $x \in (A \cup B)^c$.

We have proven that every element of $(A \cup B)^c$ belongs to $A^c \cap B^c$ and that every element of $A^c \cap B^c$ belongs to $(A \cup B)^c$. Together, these inclusions prove that the sets have the same elements, i.e., that $(A \cup B)^c = A^c \cap B^c$.

Method 2: From the Venn diagram for $A \cup B$ in Fig. 1-4, we see that $(A \cup B)^c$ is represented by the shaded area in Fig. 1-8(*a*). To find $A^c \cap B^c$, the area in both A^c and B^c, we shaded A^c with strokes in one direction and B^c with strokes in another direction as in Fig. 1-8(*b*). Then $A^c \cap B^c$ is represented by the crosshatched area, which is shaded in Fig. 1-8(*c*). Since $(A \cup B)^c$ and $A^c \cap B^c$ are represented by the same area, they are equal.



(*a*) $(A \cup B)^c$ is shaded   (*b*) A^c is shaded with ////, B^c is shaded with \\\\\\\\   (*c*) $A^c \cap B^c$ is shaded

Fig. 1-8

Duality

Note that the identities in Table 1-1 are arranged in pairs, as, for example, (2*a*) and (2*b*). We now consider the principle behind this arrangement. Suppose E is an equation of set algebra. The *dual* E^* of E is the equation obtained by replacing each occurrence of $\cup$, $\cap$, U, and $\varnothing$ in E by $\cap$, $\cup$, $\varnothing$ and U, respectively. For example, the dual of

$$(U \cap A) \cup (B \cap A) = A \quad \text{is} \quad (\varnothing \cup A) \cap (B \cup A) = A$$

Observe that the pairs of laws in Table 1-1 are duals of each other. It is a fact of set algebra, called the *principle of duality*, that, if any equation E is an identity, then its dual E^* is also an identity.

1.8 FINITE SETS, COUNTING PRINCIPLE

A set is said to be finite if it contains exactly m distinct elements where m denotes some nonnegative integer. Otherwise, a set is said to be infinite. For example, the empty set $\varnothing$ and the set of letters of the English alphabet are finite sets, whereas the set of even positive integers, $\{2, 4, 6, \ldots\}$, is infinite.

The notation $n(A)$ will denote the number of elements in a finite set A. Some texts use $\#(A)$, $|A|$ or card(A) instead of $n(A)$.

Lemma 1.4: If A and B are disjoint finite sets, then $A \cup B$ is finite and

$$n(A \cup B) = n(A) + n(B)$$

Proof. In counting the elements of $A \cup B$, first count those that are in A. There are $n(A)$ of these. The only other elements of $A \cup B$ are those that are in B but not in A. But since A and B are disjoint, no element of B is in A, so there are $n(B)$ elements that are in B but not in A. Therefore, $n(A \cup B) = n(A) + n(B)$.

We also have a formula for $n(A \cup B)$ even when they are not disjoint. This is proved in Problem 1.28.

Theorem 1.5: If A and B are finite sets, then $A \cup B$ and $A \cap B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

We can apply this result to obtain a similar formula for three sets:

Corollary 1.6: If A, B, and C are finite sets, then so is $A \cup B \cup C$, and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Mathematical induction (Section 1.10) may be used to further generalize this result to any finite number of sets.

EXAMPLE 1.7 Consider the following data for 120 mathematics students at a college concerning the languages French, German, and Russian:

65 study French
45 study German
42 study Russian
20 study French and German
25 study French and Russian
15 study German and Russian
8 study all three languages.

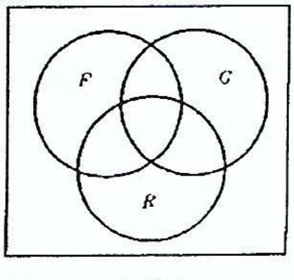


Fig. 1-9

Let F, G, and R denote the sets of students studying French, German and Russian, respectively. We wish to find the number of students who study at least one of the three languages, and to fill in the correct number of students in each of the eight regions of the Venn diagram shown in Fig. 1-9.

By Corollary 1.6,

$$(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(F \cap G \cap R)$$
$$= 65 + 45 + 42 - 20 - 25 - 15 + 8 - 100$$

That is, $n(F \cup G \cup R) = 100$ students study at least one of the three languages.

We now use this result to fill in the Venn diagram. We have:

8 study all three languages,
$20 - 8 = 12$ study French and German but not Russian
$25 - 8 = 17$ study French and Russian but not German
$15 - 8 = 7$ study German and Russian but not French
$65 - 12 - 8 - 17 = 28$ study only French
$45 - 12 - 8 - 7 = 18$ study only German
$42 - 17 - 8 - 7 = 10$ study only Russian
$120 - 100 = 20$ do not study any of the languages

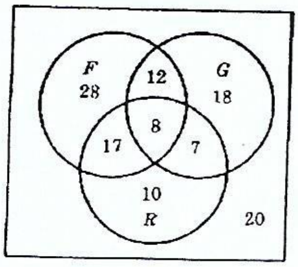


Fig. 1-10

Accordingly, the completed diagram appears in Fig. 1-10. Observe that $28 + 18 + 10 = 56$ students study only one of the languages.

1.9 CLASSES OF SETS, POWER SETS, PARTITIONS

Given a set S, we might wish to talk about some of its subsets. Thus we would be considering a set of sets. Whenever such a situation occurs, to avoid confusion we will speak of a *class* of sets or *collection* of sets rather than a set of sets. If we wish to consider some of the sets in a given class of sets, then we speak of a *subclass* or *subcollection*.

EXAMPLE 1.8 Suppose $S = \{1, 2, 3, 4\}$. Let A be the class of subsets of S which contain exactly three elements of S. Then

$$A = [\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}]$$

The elements of A are the sets $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, and $\{2,3,4\}$.

Let B be the class of subsets of S which contain 2 and two other elements of S. Then

$$B = [\{1,2,3\}, \{1,2,4\}, \{2,3,4\}]$$

The elements of B are the sets $\{1,2,3\}$, $\{1,2,4\}$, and $\{2,3,4\}$. Thus B is a subclass of A, since every element of B is also an element of A. (To avoid confusion, we will sometimes enclose the sets of a class in brackets instead of braces.)

Power Sets

For a given set S, we may speak of the class of all subsets of S. This class is called the *power set* of S, and will be denoted by Power(S). If S is finite, then so is Power(S). In fact, the number of elements in Power(S) is 2 raised to the power of S; that is,

$$n(\text{Power}(S)) = 2^{n(S)}$$

(For this reason, the power set of S is sometimes denoted by 2^S.)

EXAMPLE 1.9 Suppose $S = \{1, 2, 3\}$. Then

$$\text{Power}(S) = [\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, S]$$

Note that the empty set $\varnothing$ belongs to Power(S) since $\varnothing$ is a subset of S. Similarly, S belongs to Power(S). As expected from the above remark, Power(S) has $2^3 = 8$ elements.

Partitions

Let S be a nonempty set. A partition of S is a subdivision of S into nonoverlapping, nonempty subsets. Precisely, a *partition* of S is a collection $\{A_i\}$ of nonempty subsets of S such that:

(i) Each a in S belongs to one of the A_i.

(ii) The sets of $\{A_i\}$ are mutually disjoint; that is, if

$$A_i \neq A_j \quad \text{then} \quad A_i \cap A_j = \varnothing$$

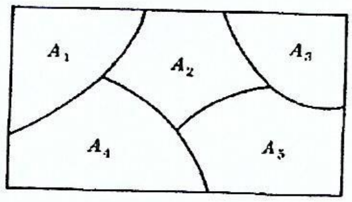


Fig. 1-11

The subsets in a partition are called *cells*. Figure 1-11 is a Venn diagram of a partition of the rectangular set S of points into five cells, A_1, A_2, A_3, A_4, and A_5.

EXAMPLE 1.10 Consider the following collections of subsets of $S = \{1, 2, \ldots, 8, 9\}$:

(i) $[\{1,3,5\}, \{2,6\}, \{4,8,9\}]$

(ii) $[\{1,3,5\}, \{2,4,6,8\}, \{5,7,9\}]$

(iii) $[\{1,3,5\}, \{2,4,6,8\}, \{7,9\}]$

Then (i) is not a partition of S since 7 in S does not belong to any of the subsets. Furthermore, (ii) is not a partition of S since $\{1,3,5\}$ and $\{5,7,9\}$ are not disjoint. On the other hand, (iii) is a partition of S.

Generalized Set Operations

The set operations of union and intersection were defined above for two sets. These operations can be extended to any number of sets, finite or infinite, as follows.

Consider first a finite number of sets, say, $A_1, A_2, \ldots, A_m$. The union and intersection of these sets are denoted and defined, respectively, by

$$A_1 \cup A_2 \cup \cdots \cup A_m = \cup_{i=1}^{m} A_i = \{x: x \in A_i \text{ for some } A_i\}$$
$$A_1 \cap A_2 \cap \cdots \cap A_m = \cap_{i=1}^{m} A_i = \{x: x \in A_i \text{ for every } A_i\}$$

That is, the union consists of those elements which belong to at least one of the sets, and the intersection consists of those elements which belong to all the sets.

Now let $\mathcal{A}$ be any collection of sets. The union and the intersection of the sets in the collection $\mathcal{A}$ is denoted and defined, respectively, by

$$\cup(A: A \in \mathcal{A}) = \{x: x \in A \text{ for some } A \in \mathcal{A}\}$$
$$\cap(A: A \in \mathcal{A}) = \{x: x \in A \text{ for every } A \in \mathcal{A}\}$$

That is, the union consists of those elements which belong to at least one of the sets in the collection $\mathcal{A}$, and the intersection consists of those elements which belong to every set in the collection $\mathcal{A}$.

EXAMPLE 1.11 Consider the sets

$$A_1 = \{1, 2, 3, \ldots\} = \mathbf{N}, \quad A_2 = \{2, 3, 4, \ldots\}, \quad A_3 = \{3, 4, 5, \ldots\}, \quad A_n = \{n, n+1, n+2, \ldots\}$$

Then the union and intersection of the sets are as follows:

$$\cup(A_n: n \in \mathbf{N}) = \mathbf{N} \quad \text{and} \quad \cap(A_n: n \in \mathbf{N}) = \varnothing$$

DeMorgan's laws also hold for the above generalized operations. That is:

Theorem 1.7: Let $\mathcal{A}$ be a collection of sets. Then

(i) $(\cup(A: A \in \mathcal{A}))^c = \cap(A^c: A \in \mathcal{A})$

(ii) $(\cap(A: A \in \mathcal{A}))^c = \cup(A^c: A \in \mathcal{A})$

1.10 MATHEMATICAL INDUCTION

An essential property of the set

$$N = \{1, 2, 3, \ldots\}$$

which is used in many proofs, follows:

Principle of Mathematical Induction I: Let P be a proposition defined on the positive integers **N**, i.e., $P(n)$ is either true or false for each n in N. Suppose P has the following two properties:

(i) $P(1)$ is true.

(ii) $P(n+1)$ is true whenever $P(n)$ is true.

Then P is true for every positive integer.

We shall not prove this principle. In fact, this principle is usually given as one of the axioms when N is developed axiomatically.

EXAMPLE 1.12 Let P be the proposition that the sum of the first n odd numbers is n^2; that is,

$$P(n):\ 1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

(The nth odd number is $2n - 1$, and the next odd number is $2n + 1$). Observe that $P(n)$ is true for $n = 1$, that is,

$$P(1):\ 1 = 1^2$$

Assuming $P(n)$ is true, we add $2n + 1$ to both sides of $P(n)$, obtaining

$$1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2$$

which is $P(n + 1)$. That is, $P(n + 1)$ is true whenever $P(n)$ is true. By the principle of mathematical induction, P is true for all n.

There is a form of the principle of mathematical induction which is sometimes more convenient to use. Although it appears different, it is really equivalent to the principle of induction.

Principle of Mathematical Induction II: Let P be a proposition defined on the positive integers N such that:

(i) $P(1)$ is true.

(ii) $P(n)$ is true whenever $P(k)$ is true for all $1 \leq k < n$.

Then P is true for every positive integer.

Remark: Sometimes one wants to prove that a proposition P is true for the set of integers

$$\{a, a + 1, a + 2, \ldots\}$$

where a is any integer, possibly zero. This can be done by simply replacing 1 by a in either of the above Principles of Mathematical Induction.

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الفصل الثالث

Relations

3.1 INTRODUCTION

The reader is familiar with many relations which are used in mathematics and computer science, e.g., "less than", "is parallel to", "is a subset of", and so on. In a certain sense, these relations consider the existence or nonexistence of a certain connection between pairs of objects taken in a definite order. Formally, we define a relation in terms of these "ordered pairs".

There are three kinds of relations which play a major role in our theory: (i) equivalence relations, (ii) order relations, (iii) functions. Equivalence relations are mainly covered in this chapter. Order relations are introduced here, but will also be discussed in Chapter 14. Functions are covered in the next chapter.

Relations, as noted above, will be defined in terms of ordered pairs (a, b) of elements, where a is designated as the first element and b as the second element. In particular,

$$(a,b) = (c,d)$$

if and only if $a = c$ and $b = d$. Thus $(a,b) \neq (b,a)$ unless $a = b$. This contrasts with sets studied in Chapter 1, where the order of elements is irrelevant; for example, $\{3,5\} = \{5,3\}$.

Although matrices will be covered in Chapter 5, we have included their connection with relations here for completeness. These sections, however, can be ignored at a first reading by those with no previous knowledge of matrix theory.

3.2 PRODUCT SETS

Consider two arbitrary sets A and B. The set of all ordered pairs (a,b) where $a \in A$ and $b \in B$ is called the *product*, or *Cartesian product*, of A and B. A short designation of this product is $A \times B$, which is read "A cross B". By definition,

$$A \times B = \{(a,b): a \in A \text{ and } b \in B\}$$

One frequently writes A^2 instead of $A \times A$.

EXAMPLE 3.1 R denotes the set of real numbers and so $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the set of ordered pairs of real numbers. The reader is familiar with the geometrical representation of $\mathbf{R}^2$ as points in the plane as in Fig. 2-1. Here each point P represents an ordered pair (a,b) of real numbers and vice versa; the vertical line through P meets the x axis at a, and the horizontal line through P meets the y axis at b. $\mathbf{R}^2$ is frequently called the *Cartesian plane*.

EXAMPLE 3.2 Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$$
$$B \times A = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$$

Also $$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$$

There are two things worth noting in the above example. First of all $A \times B \neq B \times A$. The Cartesian product deals with ordered pairs, so naturally the order in which the sets are considered is important. Secondly, using $n(S)$ for the number of elements in a set S, we have

$$n(A \times B) = 6 = 2 \cdot 3 = n(A) \cdot n(B)$$

In fact, $n(A \times B) = n(A) \cdot n(B)$ for any finite sets A and B. This follows from the observation that, for an ordered pair (a,b) in $A \times B$, there are $n(A)$ possibilities for a, and for each of these there are $n(B)$ possibilities for b.

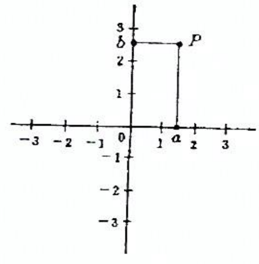


Fig. 3-1

The idea of a product of sets can be extended to any finite number of sets. For any sets $A_1, A_2, \dots, A_n$, the set of all ordered n-tuples $(a_1, a_2, \dots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n$ is called the *product* of the sets $A_1, \dots, A_n$ and is denoted by

$$A_1 \times A_2 \times \cdots \times A_n \qquad \text{or} \qquad \prod_{i=1}^{n} A_i$$

Just as we write A^2 instead of $A \times A$, so we write A^n instead of $A \times A \times \cdots \times A$, where there are n factors all equal to A. For example, $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ denotes the usual three-dimensional space.

3.3 RELATIONS

We begin with a definition.

Definition. Let A and B be sets. A *binary relation* or, simply, *relation* from A to B is a subset of $A \times B$.

Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair $a \in A$ and $b \in B$, exactly one of the following is true:

(i) $(a, b) \in R$; we than say "a is *R-related* to b", written aRb.

(ii) $(a, b) \notin R$; we then say "a is not R-related to b", written $a\not{R}b$.

If R is a relation from a set A to itself, that is, if R is a subset of $A^2 = A \times A$, then we say that R is a relation on A.

The *domain* of a relation R is the set of all first elements of the ordered pairs which belong to R, and the *range* of R is the set of second elements.

Although n-ary relations, which involve ordered n-tuples, are introduced in Section 2.12, the term relation shall mean binary relation unless otherwise stated or implied.

EXAMPLE 3.3

(*a*) Let $A = (1, 2, 3)$ and $B = \{x, y, z\}$, and let $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from A to B since R is a subset of $A \times B$. With respect to this relation,

$$1Ry, \quad 1Rz, \quad 3Ry, \qquad \text{but} \qquad 1\not{R}x, \quad 2\not{R}x, \quad 2\not{R}y, \quad 2\not{R}z, \quad 3\not{R}x, \quad 3\not{R}z$$

The domain of R is $\{1, 3\}$ and the range is $\{y, z\}$.

(*b*) Let A = {eggs, milk, corn} and B = {cows, goats, hens}. We can define a relation R from A to B by $(a, b) \in R$ if a is produced by b. In other words,

R = {(eggs, hens), (milk, cows), (milk, goats)}

With respect to this relation,

eggs R hens, milk R cows, etc.

(*c*) Suppose we say that two countries are *adjacent* if they have some part of their boundaries in common. Then "is adjacent to" is a relation R on the countries of the earth. Thus

(Italy, Switzerland) $\in R$ but (Canada, Mexico) $\notin R$

(*d*) Set inclusion $\subseteq$ is a relation on any collection of sets. For, given any pair of sets A and B, either $A \subseteq B$ or $A \not\subseteq B$.

(*e*) A familiar relation on the set $\mathbf{Z}$ of integers is "m divides n". A common notation for this relation is to write $m|n$ when m divides n. Thus $6|30$ but $7\nmid 25$.

(*f*) Consider the set L of lines in the plane. Perpendicularity, written $\perp$, is a relation on L. That is, given any pair of lines a and b, either $a \perp b$ or $a \not\perp b$. Similarly, "is parallel to", written $\|$, is a relation on L since either $a \| b$ or $a \nparallel b$.

(g) Let A be any set. An important relation on A is that of *equality*,

$$\{(a,a): a \in A\}$$

which is usually denoted by "=". This relation is also called the *identity* or *diagonal relation* on A and it will also be denoted by Δ_A or simply Δ.

(h) Let A be any set. Then $A \times A$ and $\varnothing$ are subsets of $A \times A$ and hence are relations on A called the *universal relation* and *empty relation*, respectively.

Inverse Relation

Let R be any relation from a set A to a set B. The *inverse* of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is,

$$R^{-1} = \{(b,a): (a,b) \in R\}$$

For example, the inverse of the relation $R = \{(1,y), (1,z), (3,y)\}$ from $A = \{1,2,3\}$ to $B = \{x,y,z\}$ follows:

$$R^{-1} = \{(y,1), (z,1), (y,3)\}$$

Clearly, if R is any relation, then $(R^{-1})^{-1} = R$. Also, the domain and range of R^{-1} are equal, respectively, to the range and domain of R. Moreover, if R is a relation on A, then R^{-1} is also a relation on A.

3.4 PICTORIAL REPRESENTATIONS OF RELATIONS

First we consider a relation S on the set $\mathbf{R}$ of real numbers; that is, S is a subset of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. Since $\mathbf{R}^2$ can be represented by the set of points in the plane, we can picture S by emphasizing those points in the plane which belong to S. The pictorial representation of the relation is sometimes called the *graph* of the relation.

Frequently, the relation S consists of all ordered pairs of real numbers which satisfy some given equation

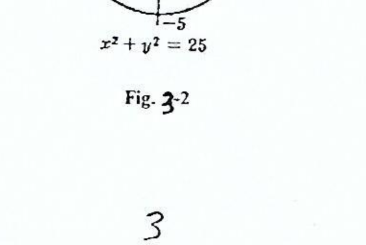
$$E(x,y) = 0$$

We usually identify the relation with the equation; that is, we speak of the relation $E(x,y) = 0$.

EXAMPLE 3.4 Consider the relation S defined by the equation

$$x^2 + y^2 = 25$$

That is, S consists of all ordered pairs (x,y) which satisfy the given equation. The graph of the equation is a circle having its center at the origin and radius 5. See Fig. 2-2.



$x^2 + y^2 = 25$

Fig. 3-2

Representations of Relations on Finite Sets

Suppose A and B are finite sets. The following are two ways of picturing a relation R from A to B.

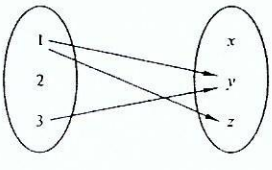
(i) Form a rectangular array whose rows are labeled by the elements of A and whose columns are labeled by the elements of B. Put a 1 or 0 in each position of the array according as $a \in A$ is or is not related to $b \in B$. This array is called the *matrix of the relation.*

(ii) Write down the elements of A and the elements of B in two disjoint disks, and then draw an arrow from $a \in A$ to $b \in B$ whenever a is related to b. This picture will be called the *arrow diagram* of the relation.

Figure 2-3 pictures the first relations in Example 2.3 by the above two ways.

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

(i)



(ii)

$R = \{(1, y),\ (1, z),\ (3, y)\}$

Fig. 3-3

Directed Graphs of Relations on Sets

There is another way of picturing a relation R when R is a relation from a finite set to itself. First we write down the elements of the set, and then we drawn an arrow from each element x to each element y whenever x is related to y. This diagram is called the *directed graph* of the relation. Figure 2-4, for example, shows the directed graph of the following relation R on the set $A = \{1, 2, 3, 4\}$:

$$R = \{(1,2),\ (2,2),\ (2,4),\ (3,2),\ (3,4),\ (4,1).\ (4,3)\}$$

Observe that there is an arrow from 2 to itself, since 2 is related to 2 under R.

These directed graphs will be studied in detail as a separate subject in Chapter 8. We mention it here mainly for completeness.

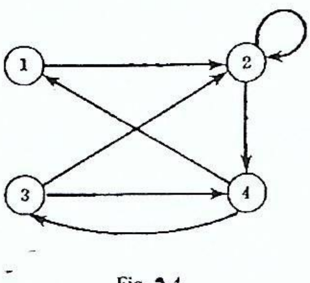


Fig. 3-4

3.5 COMPOSITION OF RELATIONS

Let A, B, and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of $A \times B$ and S is a subset of $B \times C$. Then R and S give rise to a relation from A to C denoted by $R \circ S$ and defined by

$$a(R \circ S)c \text{ if for some } b \in B \text{ we have } aRb \text{ and } bSc$$

That is,

$$R \circ S = \{(a, c): \text{there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}$$

The relation $R \circ S$ is called the *composition* of R and S; it is sometimes denoted simply by RS.

Suppose R is a relation on a set A, that is, R is a relation from a set A to itself. Then $R \circ R$, the composition of R with itself is always defined, and $R \circ R$ is sometimes denoted by R^2. Similarly, $R^3 = R^2 \circ R = R \circ R \circ R$, and so on. Thus R^n is defined for all positive n.

Warning: Many texts denote the composition of relations R and S by $S \circ R$ rather than $R \circ S$. This is done in order to conform with the usual use of $g \circ f$ to denote the composition of f and g where f and g are functions. Thus the reader may have to adjust his notation when using this text as a supplement with another text. However, when a relation R is composed with itself, then the meaning of $R \circ R$ is unambiguous.

The arrow diagrams of relations give us a geometrical interpretation of the composition $R \circ S$ as seen in the following example.

EXAMPLE 3.5 Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{x, y, z\}$ and let

$$R = \{(1, a), (2, d), (3, a)\ (3, b), (3, d)\} \quad \text{and} \quad S = \{(b, x), (b, z), (c, y), (d, z)\}$$

Consider the arrow diagrams of R and S as in Fig. 2-5. Observe that there is an arrow from 2 to d which is followed by an arrow from d to z. We can view these two arrows as a "path" which "connects" the element $2 \in A$ to the element $z \in C$. Thus

$$2(R \circ S)z \quad \text{since} \quad 2Rd \text{ and } dSz$$

Similarly there is a path from 3 to x and a path from 3 to z. Hence

$$3(R \circ S)x \quad \text{and} \quad 3(R \circ S)z$$

No other element of A is connected to an element of C. Accordingly,

$$R \circ S = \{(2, z), (3, x), (3, z)\}$$

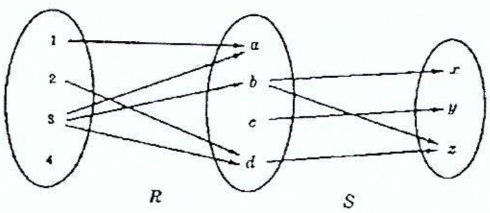


Fig. 3-5

Composition of Relations and Matrices

There is another way of finding $R \circ S$. Let M_R and M_S denote respectively the matrices of the relations R and S. Then

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad M_S = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Multiplying M_R and M_S we obtain the matrix

$$M = M_R M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

The nonzero entries in this matrix tell us which elements are related by $R \circ S$. Thus $M = M_R M_S$ and $M_{R \circ S}$ have the same nonzero entries.

Our first theorem tells us that the composition of relations is associative.

Theorem 3.1: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Then

$$(R \circ S) \circ T = R \circ (S \circ T)$$

We prove this theorem in Problem 3.11.

3.6 TYPES OF RELATIONS

Consider a given set A. This section discusses a number of important types of relations which are defined on A.

Reflexive Relations

A relation R on a set A is *reflexive* if aRa for every $a \in A$, that is, if $(a, a) \in R$ for every $a \in A$. Thus R is not reflexive if there exists an $a \in A$ such that $(a, a) \notin R$.

EXAMPLE 3.6 Consider the following five relations on the set $A = \{1, 2, 3, 4\}$:

$R_1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$
$R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
$R_3 = \{(1, 3), (2, 1)\}$
$R_4 = \varnothing$, the empty relation
$R_5 = A \times A$, the universal relation

Determine which of the relations are reflexive.

Since A contain the four elements 1, 2, 3, and 4, a relation R on A is reflexive if it contains the four pairs (1, 1), (2, 2), (3, 3), and (4, 4). Thus only R_2 and the universal relation $R_5 = A \times A$ are reflexive. Note that R_1, R_3, and R_4 are not reflexive since, for example, (2, 2) does not belong to any of them.

EXAMPLE 3.7 Consider the following five relations:

(1) Relation $\leq$ (less than or equal) on the set $\mathbf{Z}$ of integers
(2) Set inclusion $\subseteq$ on a collection C of sets
(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane
(4) Relation $\parallel$ (parallel) on the set L of lines in the plane
(5) Relation $|$ of divisibility on the set $\mathbf{N}$ of positive integers. (Recall $x|y$ if there exists z such that $xz = y$.)

Determine which of the relations are reflexive.

The relation (3) is not reflexive since no line is perpendicular to itself. Also (4) is not reflexive since no line is parallel to itself. The other relations are reflexive; that is, $x \leq x$ for every integer x in **Z**, $A \subseteq A$ for any set A in C, and $n|n$ for every positive integer n in **N**.

Symmetric and Antisymmetric Relations

A relation R on a set A is *symmetric* if whenever aRb then bRa, that is, if whenever $(a,b) \in R$ then $(b,a) \in R$. Thus R is not symmetric if there exists $a, b \in A$ such that $(a,b) \in R$ but $(b,a) \notin R$.

EXAMPLE 3.8

(*a*) Determine which of the relations in Example 2.6 are symmetric.

R_1 is not symmetric since $(1,2) \in R_1$ but $(2,1) \notin R_1$. R_3 is not symmetric since $(1,3) \in R_3$ but $(3,1) \notin R_3$. The other relations are symmetric.

(*b*) Determine which of the relations in Example 2.7 are symmetric.

The relation $\perp$ is symmetric since if line a is perpendicular to line b then b is perpendicular to a. Also, $\|$ is symmetric since if line a is parallel to line b then b is parallel to a. The other relations are not symmetric. For example, $3 \leq 4$ but $4 \not\leq 3$; $\{1,2\} \subseteq \{1,2,3\}$ but $\{1,2,3\} \not\subseteq \{1,2\}$, and $2|6$ but $6 \nmid 2$.

A relation R on a set A is *antisymmetric* if whenever aRb and bRa then $a = b$, that is, if whenever $(a,b), (b,a) \in R$ then $a = b$. Thus R is not antisymmetric if there exist $a, b \in A$ such that (a,b) and (b,a) belong to R, but $a \neq b$.

EXAMPLE 3.9

(*a*) Determine which of the relations in Example 2.6 are antisymmetric.

R_2 is not antisymmetric since $(1,2)$ and $(2,1)$ belong to R_2, but $1 \neq 2$. Similarly, the universal relation R_5 is not antisymmetric. All the other relations are antisymmetric.

(*b*) Determine which of the relations in Example 2.7 are antisymmetric.

The relation $\leq$ is antisymmetric since whenever $a \leq b$ and $b \leq a$ then $a = b$. Set inclusion $\subseteq$ is antisymmetric since whenever $A \subseteq B$ and $B \subseteq A$ then $A = B$. Also, divisibility on **N** is antisymmetric since whenever $m|n$ and $n|m$ then $m = n$. (Note that divisibility on **Z** is not antisymmetric since $3|-3$ and $-3|3$ but $3 \neq -3$.) The relation $\perp$ is not antisymmetric since we can have distinct lines a and b such that $a \perp b$ and $b \perp a$. Similarly, $\|$ is not antisymmetric.

Remark: The properties of being symmetric and being antisymmetric are not negatives of each other. For example, the relation $R = \{(1,3), (3,1), (2,3)\}$ is neither symmetric nor antisymmetric. On the other hand, the relation $R' = \{(1,1), (2,2)\}$ is both symmetric and antisymmetric.

Transitive Relations

A relation R on a set A is *transitive* if whenever aRb and bRc then aRc, that is, if whenever $(a,b), (b,c) \in R$ then $(a,c) \in R$. Thus R is not transitive if there exist $a, b, c \in A$ such that $(a,b), (b,c) \in R$ but $(a,c) \notin R$.

EXAMPLE 3.10

(*a*) Determine which of the relations in Example 2.6 are transitive.

The relation R_3 is not transitive since $(2,1), (1,3) \in R_3$ but $(2,3) \notin R_3$. All the other relations are transitive.

(*b*) Determine which of the relations in Example 2.7 are transitive.

The relations $\leq$, $\subseteq$, and $|$ are transitive. That is: (i) If $a \leq b$ and $b \leq c$, then $a \leq c$. (ii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. (iii) If $a|b$ and $b|c$, then $a|c$.

On the other hand the relation $\perp$ is not transitive. If $a \perp b$ and $b \perp c$, then it is not true that $a \perp c$. Since no line is parallel to itself, we can have $a \parallel b$ and $b \parallel a$, but $a \nparallel a$. Thus $\parallel$ is not transitive. (We note that the relation "is parallel or equal to" is a transitive relation on the set L of lines in the plane.)

The property of transitivity can also be expressed in terms of the composition of relations. For a relation R on A we define

$$R^2 = R \circ R \qquad \text{and, more generally,} \qquad R^n = R^{n-1} \circ R$$

Then we have the following result.

Theorem 3.2: A relation R is transitive if and only if $R^n \subseteq R$ for $n \geq 1$.

3.7 CLOSURE PROPERTIES

Consider a given set A and the collection of all relations on A. Let $\mathcal{P}$ be a property of such relations, such as being symmetric or being transitive. A relation with property $\mathcal{P}$ will be called a $\mathcal{P}$-relation. The $\mathcal{P}$-*closure* of an arbitrary relation R on A, written $\mathcal{P}(R)$, is a $\mathcal{P}$-relation such that

$$R \subseteq \mathcal{P}(R) \subseteq S$$

for every $\mathcal{P}$-relation S containing R. We will write

$$\text{reflexive}(R), \qquad \text{symmetric}(R), \qquad \text{and transitive}(R)$$

for the reflexive, symmetric, and transitive closures of R.

Generally speaking, $\mathcal{P}(R)$ need not exist. However, there is a general situation where $\mathcal{P}(R)$ will always exist. Suppose $\mathcal{P}$ is a property such that there is at least one $\mathcal{P}$-relation containing R and that the intersection of any $\mathcal{P}$-relations is again a $\mathcal{P}$-relation. Then one can prove (Problem 2.16) that

$$\mathcal{P}(R) = \cap(S\text{: } S \text{ is a } \mathcal{P}\text{-relation and } R \subseteq S)$$

Thus one can obtain $\mathcal{P}(R)$ from the "top-down", that is, as the intersection of relations. However, one usually wants to find $\mathcal{P}(R)$ from the "bottom-up", that is, by adjoining elements to R to obtain $\mathcal{P}(R)$. This we do below.

Reflexive and Symmetric Closures

The next theorem tells us how to easily obtain the reflexive and symmetric closures of a relation. Here $\Delta_A = \{(a,a)\text{: } a \in A\}$ is the *diagonal* or *equality* relation on A.

Theorem 3.3: Let R be a relation on a set A. Then:

(i) $R \cup \Delta_A$ is the reflexive closure of R.

(ii) $R \cup R^{-1}$ is the symmetric closure of R.

In other words, reflexive(R) is obtained by simply adding to R those elements (a,a) in the diagonal which do not already belong to R, and symmetric(R) is obtained by adding to R all pairs (b,a) whenever (a,b) belongs to R.

EXAMPLE 3.11

(*a*) Consider the following relation R on the set $A = \{1,2,3,4\}$:

$$R = \{(1,1), (1,3), (2,4), (3,1), (3,3), (4,3)\}$$

Then

$$\text{reflexive}(R) = R \cup \{(2,2), (4,4)\} \qquad \text{and} \qquad \text{symmetric}(R) = R \cup \{(4,2), (3,4)\}$$

(*b*) Consider the relation < (less than) on the set **N** of positive integers. Then

$$\text{reflexive}(<) = < \cup \Delta = \leq = \{(a, b) : a \leq b\}$$

$$\text{symmetric}(<) = < \cup > = \{(a, b) : a \neq b\}$$

Transitive Closure

Let R be a relation on a set A. Recall that $R^2 = R \circ R$ and $R^n = R^{n-1} \circ R$. We define

$$R^* = \bigcup_{i=1}^{\infty} R^i$$

The following theorem applies.

Theorem 3.4: R^* is the transitive closure of a relation R.

Suppose A is a finite set with n elements. Then we show in Chapter 8 on directed graphs that

$$R^* = R \cup R^2 \cup \cdots \cup R^n$$

This gives us the following result.

Theorem 3.5: Let R be a relation on a set A with n elements. Then

$$\text{transitive}(R) = R \cup R^2 \cup \cdots \cup R^n$$

Finding transitive(R) can take a lot of time when A has a large number of elements. An efficient way for doing this will be described in Chapter 8. Here we give a simple example where A has only three elements.

EXAMPLE 3.12 Consider the following relation R on $A = \{1, 2, 3\}$:

$$R = \{(1, 2), (2, 3), (3, 3)\}$$

Then

$$R^2 = R \circ R = \{(1, 3), (2, 3)(3, 3)\} \quad \text{and} \quad R^3 = R^2 \circ R = \{(1, 3), (2, 3), (3, 3)\}$$

Accordingly,

$$\text{transitive}(R) = R \cup R^2 \cup R^3 = \{(1, 2), (2, 3), (3, 3), (1, 3)\}$$

3.8 EQUIVALENCE RELATIONS

Consider a nonempty set S. A relation R on S is an *equivalence relation* if R is reflexive, symmetric, and transitive. That is, R is an equivalence relation on S if it has the following three properties:

(1) For every $a \in S$, aRa.
(2) If aRb, then bRa.
(3) If aRb and bRc, then aRc.

The general idea behind an equivalence relation is that it is a classification of objects which are in some way "alike". In fact, the relation "=" of equality on any set S is an equivalence relation; that is:

(1) $a = a$ for every $a \in S$.
(2) If $a = b$, then $b = a$.
(3) If $a = b$ and $b = c$, then $a = c$.

Other equivalence relations follow.

EXAMPLE 3.13

(*a*) Consider the set L of lines and the set T of triangles in the Euclidean plane. The relation "is parallel to or identical to" is an equivalence relation on L, and congruence and similarity are equivalence relations on T.

(*b*) The classification of animals by species, that is, the relation "is of the same species as", is an equivalence relation on the set of animals.

(*c*) The relation $\subseteq$ of set inclusion is not an equivalence relation. It is reflexive and transitive, but it is not symmetric since $A \subseteq B$ does not imply $B \subseteq A$.

(*d*) Let m be a fixed positive integer. Two integers a and b are said to be *congruent modulo m*, written

$$a \equiv b \pmod{m}$$

if m divides $a - b$. For example, for $m = 4$ we have $11 \equiv 3 \pmod 4$ since 4 divides $11 - 3$, and $22 \equiv 6 \pmod 4$ since 4 divides $22 - 6$. This relation of congruence modulo m is an equivalence relation.

Equivalence Relations and Partitions

This subsection explores the relationship between equivalence relations and partitions on a nonempty set S. Recall first that a partition P of S is a collection $\{A_i\}$ of nonempty subsets of S with the following two properties:

(1) Each $a \in S$ belongs to some A_i.
(2) If $A_i \neq A_j$, then $A_i \cap A_j = \emptyset$.

In other words, a partition P of S is a subdivision of S into disjoint nonempty sets. (See Section 1.9.)

Suppose R is an equivalence relation on a set S. For each a in S, let $[a]$ denote the set of elements of S to which a is related under R; that is,

$$[a] = \{x: (a, x) \in R\}$$

We call $[a]$ the *equivalence class* of a in S; any $b \in [a]$ is called a *representative* of the equivalence class.

The collection of all equivalence classes of elements of S under an equivalence relation R is denoted by S/R, that is,

$$S/R = \{[a]: a \in S\}$$

It is called the *quotient* set of S by R. The fundamental property of a quotient set is contained in the following theorem.

Theorem 3.6: Let R be an equivalence relation on a set S. Then the quotient set S/R is a partition of S. Specifically:

(i) For each a in S, we have $a \in [a]$.
(ii) $[a] = [b]$ if and only if $(a, b) \in R$.
(iii) If $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint.

Conversely, given a partition $\{A_i\}$ of the set S, there is an equivalence relation R on S such that the sets A_i are the equivalence classes.

This important theorem will be proved in Problem 2.21.

EXAMPLE 3.14

(*a*) Consider the following relation R on $S = \{1, 2, 3\}$:

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$$

One can show that R is reflexive, symmetric, and transitive, that is, that R is an equivalence relation. Under the relation R,

$$[1] = \{1, 2\} \qquad [2] = \{1, 2\} \qquad [3] = \{3\}$$

Observe that $[1] = [2]$ and that $S/R = \{[1], [3]\}$ is a partition of S. One can choose either $\{1, 3\}$ or $\{2, 3\}$ as a set of representatives of the equivalence classes.

(*b*) Let R_5 be the relation on the set $\mathbf{Z}$ of integers defined by

$$x \equiv y \pmod 5$$

which reads "x is congruent to y modulo 5" and which means that the difference $x - y$ is divisible by 5. Then R_5 is an equivalence relation on $\mathbf{Z}$. There are exactly five equivalence classes in the quotient set $\mathbf{Z}/R_5$ as follows:

$$A_0 = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$$
$$A_1 = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$$
$$A_2 = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$$
$$A_3 = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$$
$$A_4 = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$$

Observe that any integer x, which can be uniquely expressed in the form $x = 5q + r$ where $0 \le r < 5$, is a member of the equivalence class A_r where r is the remainder. As expected, the equivalence classes are disjoint and

$$\mathbf{Z} = A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$$

Usually one chooses $\{0, 1, 2, 3, 4\}$ or $\{-2, -1, 0, 1, 2\}$ as a set of representatives of the equivalence classes.

3.9 PARTIAL ORDERING RELATIONS

This section defines another important class of relations. A relation R on a set S is called a *partial ordering* or a *partial order* if R is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set* or *poset*. Partially ordered sets will be studied in more detail in Chapter 14, so here we simply give some examples.

EXAMPLE 3.15

(*a*) The relation $\subseteq$ of set inclusion is a partial ordering on any collection of sets since set inclusion has the three desired properties. That is,

(1) $A \subseteq A$ for any set A.
(2) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
(3) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(*b*) The relation $\le$ on the set $\mathbf{R}$ of real numbers is reflexive, antisymmetric, and transitive. Thus $\le$ is a partial ordering.

(*c*) The relation "a divides b" is a partial ordering on the set $\mathbf{N}$ of positive integers. However, "a divides b" is not a partial ordering on the set $\mathbf{Z}$ of integers since $a|b$ and $b|a$ does not imply $a = b$. For example, $3|-3$ and $-3|3$ but $3 \neq -3$.

3.10 *n*-ARY RELATIONS

All the relations discussed above were binary relations. By an *n-ary relation*, we mean a set of ordered n-tuples. For any set S, a subset of the product set S^n is called an n-ary relation on S. In particular, a subset of S^3 is called a *ternary relation* on S.

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Functions and Algorithms

4.1 INTRODUCTION

One of the most important concepts in mathematics is that of a function. The terms "map", "mapping", "transformation", and many others mean the same thing; the choice of which word to use in a given situation is usually determined by tradition and the mathematical background of the person using the term.

Related to the notion of a function is that of an algorithm. The notation for presenting an algorithm and a discussion of its complexity is also covered in this chapter.

4.2 FUNCTIONS

Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assignments is called a *function* from A into B. The set A is called the *domain* of the function, and the set B is called the *codomain*.

Functions are ordinarily denoted by symbols. For example, let f denote a function from A into B. Then we write

$$f: A \to B$$

which is read: "f is a function from A into B", or "f takes (or; maps) A into B". If $a \in A$, then $f(a)$ (read: "f of a") denotes the unique element of B which f assigns to a; it is called the *image* of a under f, or the *value* of f at a. The set of all image values is called the *range* or *image* of f. The image of $f: A \to B$ is denoted by Ran (f), Im (f) or $f(A)$.

Frequently, a function can be expressed by means of a mathematical formula. For example, consider the function which sends each real number into its square. We may describe this function by writing

$$f(x) = x^2 \qquad \text{or} \qquad x \mapsto x^2 \qquad \text{or} \qquad y = x^2$$

In the first notation, x is called a *variable* and the letter f denotes the function. In the second notation, the barred arrow $\mapsto$ is read "goes into". In the last notation, x is called the *independent variable* and y is called the *dependent variable* since the value of y will depend on the value of x.

Remark: Whenever a function is given by a formula in terms of a variable x, we assume, unless it is otherwise stated, that the domain of the function is **R** (or the largest subset of **R** for which the formula has meaning) and the codomain is **R**.

EXAMPLE 4.1

(*a*) Consider the function $f(x) = x^3$, i.e., f assigns to each real number its cube. Then the image of 2 is 8, and so we may write $f(2) = 8$.

(*b*) Let f assign to each country in the world its capital city. Here the domain of f is the set of countries in the world; the codomain is the list of cities of the world. The image of France is Paris; or, in other words, f(France) = Paris.

(*c*) Figure 3-1 defines a function f from $A = \{a, b, c, d\}$ into $B = \{r, s, t, u\}$ in the obvious way. Here

$$f(a) = s, \qquad f(b) = u, \qquad f(c) = r, \qquad f(d) = s$$

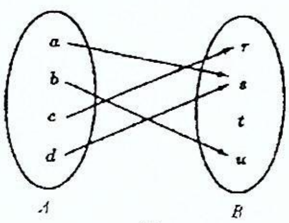


Fig. 4-1

The image of f is the set of image values, $\{r, s, u\}$. Note that t does not belong to the image of f because t is not the image of any element under f.

(*d*) Let A be any set. The function from A into A which assigns to each element that element itself is called the *identity function* on A and is usually denoted by 1_A or simply 1. In other words,

$$1_A(a) = a$$

for every element a in A.

(*e*) Suppose S is a subset of A, that is, suppose $S \subseteq A$. The *inclusion map* or *embedding* of S into A, denoted by $i: S \hookrightarrow A$, is the function defined by

$$i(x) = x$$

for every $x \in S$; and the *restriction* to S of any function $f: A \to B$, denoted by $f|_S$, is the function from S to B defined by

$$f|_A(x) = f(x)$$

for every $x \in S$.

Functions as Relations

There is another point of view from which functions may be considered. First of all, every function $f: A \to B$ gives rise to a relation from A to B called the *graph of f* and defined by

$$\text{Graph of } f = \{(a, b): a \in A, b = f(a)\}$$

Two functions $f: A \to B$ and $g: A \to B$ are defined to be equal, written $f = g$, if $f(a) = g(a)$ for every $a \in A$; that is, if they have the same graph. Accordingly, we do not distinguish between a function and its graph. Now, such a graph relation has the property that each a in A belongs to a unique ordered pair (a, b) in the relation. On the other hand, any relation f from A to B that has this property gives rise to a function $f: A \to B$, where $f(a) = b$ for each (a, b) in f. Consequently, one may equivalently define a function as follows:

Definition: A function $f: A \to B$ is a relation from A to B (i.e., a subset of $A \times B$) such that each $a \in A$ belongs to a unique ordered pair (a, b) in f.

Although we do not distinguish between a function and its graph, we will still use the terminology "graph of f" when referring to f as a set of ordered pairs. Moreover, since the graph of f is a relation, we can draw its picture as was done for relations in general, and this pictorial representation is itself sometimes called the graph of f. Also, the defining condition of a function, that each $a \in A$ belongs to a unique pair (a, b) in f, is equivalent to the geometrical condition of each vertical line intersecting the graph in exactly one point.

EXAMPLE 4.2

(*a*) Let $f: A \to B$ be the function defined in Example 3.1(*c*). Then the graph of f is the following set of ordered pairs:

$$\{(a, s), (b, u), (c, r), (d, s)\}$$

(*b*) Consider the following relations on the set $A = \{1, 2, 3\}$:

$$f = \{(1, 3), (2, 3), (3, 1)\}$$
$$g = \{(1, 2), (3, 1)\}$$
$$h = \{(1, 3), (2, 1), (1, 2), (3, 1)\}$$

f is a function from A into A since each member of A appears as the first coordinate in exactly one ordered pair in f; here $f(1) = 3$, $f(2) = 3$ and $f(3) = 1$. g is not a function from A into A since $2 \in A$ is not the first coordinate of any pair in g and so g does not assign any image to 2. Also h is not a function from A into A

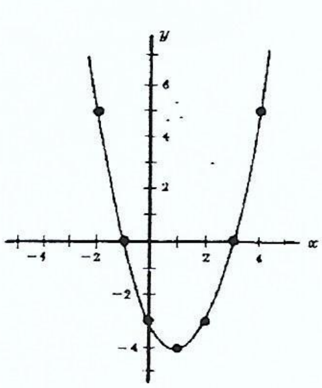
since $1 \in A$ appears as the first coordinate of two distinct ordered pairs in h, $(1,3)$ and $(1,2)$. If h is to be a function it cannot assign both 3 and 2 to the element $1 \in A$.

(c) By a real polynomial function, we mean a function $f: \mathbf{R} \to \mathbf{R}$ of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i are real numbers. Since $\mathbf{R}$ is an infinite set, it would be impossible to plot each point of the graph. However, the graph of such a function can be approximated by first plotting some of its points and then drawing a smooth curve through these points. The points are usually obtained from a table where various values are assigned to x and the corresponding values of $f(x)$ are computed. Figure 3-2 illustrates this technique using the function $f(x) = x^2 - 2x - 3$.

x	$f(x)$
−2	5
−1	0
0	−3
1	−4
2	−3
3	0
4	5



Graph of $f(x) = x^2 - 2x - 3$

Fig. 3-2

Composition Function

Consider functions $f: A \to B$ and $g: B \to C$; that is, where the codomain of f is the domain of g. Then we may define a new function from A to C, called the *composition* of f and g and written $g \circ f$, as follows:

$$(g \circ f)(a) \equiv g(f(a))$$

That is, we find the image of a under f and then find the image of $f(a)$ under g. This definition is not really new. If we view f and g as relations, then this function is the same as the composition of f and g as relations (see Section 2.6) except that here we use the functional notation $g \circ f$ for the composition of f and g instead of the notation $f \circ g$ which was used for relations.

Consider any function $f: A \to B$. Then

$$f \circ 1_A = f \quad \text{and} \quad 1_B \circ f = f$$

where 1_A and 1_B are the identity functions on A and B, respectively.

4.3 ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

A function $f: A \to B$ is said to be *one-to-one* (written 1-1) if different elements in the domain A have distinct images. Another way of saying the same thing is that f is *one-to-one* if $f(a) = f(a')$ implies $a = a'$.

A function $f: A \to B$ is said to be an *onto* function if each element of B is the image of some element of A. In other words, $f: A \to B$ is onto if the image of f is the entire codomain, i.e., if $f(A) = B$. In such a case we say that f is a function from A *onto* B or that f maps A *onto* B.

A function $f: A \to B$ is *invertible* if its inverse relation f^{-1} is a function from B to A. In general, the inverse relation f^{-1} may not be a function. The following theorem gives simple criteria which tells us when it is.

Theorem 4.1: A function $f: A \to B$ is invertible if and only if f is both one-to-one and onto.

If $f: A \to B$ is one-to-one and onto, then f is called a *one-to-one correspondence* between A and B. This terminology comes from the fact that each element of A will then correspond to a unique element of B and vice versa.

Some texts use the terms *injective* for a one-to-one function, *surjective* for an onto function, and *bijective* for a one-to-one correspondence.

EXAMPLE 4.3 Consider the functions $f_1: A \to B$, $f_2: B \to C$, $f_3: C \to D$ and $f_4: D \to E$ defined by the diagram of Fig. 3-3. Now f_1 is one-to-one since no element of B is the image of more than one element of A. Similarly, f_2 is one-to-one. However, neither f_3 nor f_4 is one-to-one since $f_3(r) = f_3(u)$ and $f_4(v) = f_4(w)$.

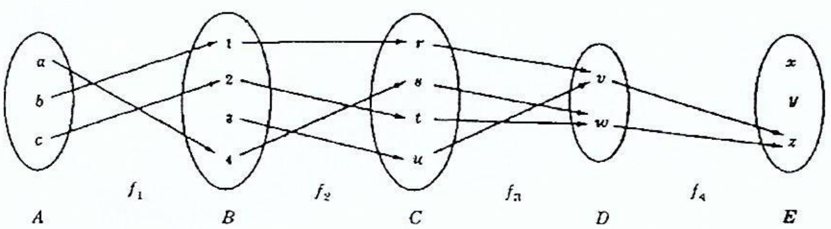


Fig 4-3

As far as being onto is concerned, f_2 and f_3 are both onto functions since every element of C is the image under f_2 of some element of B and every element of D is the image under f_3 of some element of C, i.e., $f_2(B) = C$ and $f_3(C) = D$. On the other hand, f_1 is not onto since $3 \in B$ is not the image under f_1 of any element of A, and f_4 is not onto since $x \in E$ is not the image under f_4 of any element of D.

Thus f_1 is one-to-one but not onto, f_3 is onto but not one-to-one and f_4 is neither one-to-one nor onto. However, f_2 is both one-to-one and onto, i.e., is a one-to-one correspondence between A and B. Hence f_2 is invertible and f_2^{-1} is a function from C to B.

Geometrical Characterization of One-to-One and Onto Functions

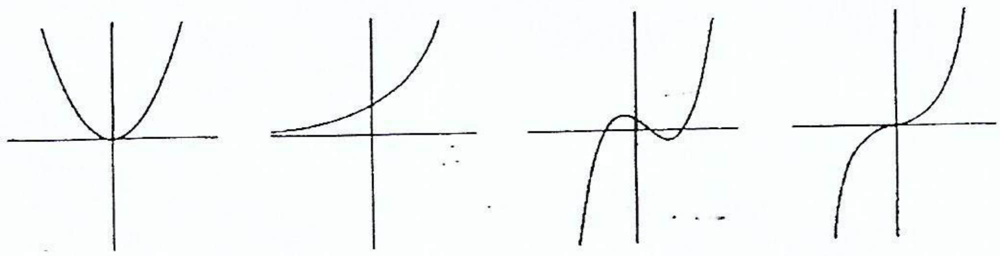
Since functions may be identified with their graphs, and since graphs may be plotted, we might wonder whether the concepts of being one-to-one and onto have geometrical meaning. We show that the answer is yes.

To say that a function $f: A \to B$ is one-to-one means that there are no two distinct pairs (a_1, b) and (a_2, b) in the graph of f; hence each horizontal line can intersect the graph of f in at most one point. On the other hand, to say that f is an onto function means that for every $b \in B$ there must be at least one $a \in A$ such that (a, b) belongs to the graph of f; hence each horizontal line must intersect the graph of f at least once. Accordingly, if f is both one-to-one and onto, i.e. invertible, then each horizontal line will intersect the graph of f in exactly one point.

EXAMPLE 4.4 Consider the following four functions from **R** into **R**:

$$f_1(x) = x^2, \qquad f_2(x) = 2^x, \qquad f_3(x) = x^3 - 2x^2 - 5x + 6, \qquad f_4(x) = x^3$$

The graphs of these functions appear in Fig. 3-4. Observe that there are horizontal lines which intersect the graph of f_1 twice and there are horizontal lines which do not intersect the graph of f_1 at all; hence f_1 is neither one-to-one nor onto. Similarly, f_2 is one-to-one but not onto, f_3 is onto but not one-to-one and f_4 is both one-to-one and onto. The inverse of f_4 is the cube root function, i.e., $f_4^{-1}(x) = \sqrt[3]{x}$.



$f_1(x) = x^2$ $\qquad$ $f_2(x) = 2^x$ $\qquad$ $f_3(x) = x^3 - 2x^2 - 5x + 6$ $\qquad$ $f_4(x) = x^3$

Fig. 3-4

4.4 MATHEMATICAL FUNCTIONS, EXPONENTIAL AND LOGARITHMIC FUNCTIONS

This section presents various mathematical functions which appear often in the analysis of algorithms, and in computer science in general, together with their notation. We also discuss the exponential and logarithmic functions, and their relationship.

Floor and Ceiling Functions

Let x be any real number. Then x lies between two integers called the floor and the ceiling of x. Specifically,

$\lfloor x \rfloor$, called the *floor* of x, denotes the greatest integer that does not exceed x.
$\lceil x \rceil$, called the *ceiling* of x, denotes the least integer that is not less than x.

If x is itself an integer, then $\lfloor x \rfloor = \lceil x \rceil$; otherwise $\lfloor x \rfloor + 1 = \lceil x \rceil$. For example,

$$\lfloor 3.14 \rfloor = 3, \quad \lfloor \sqrt{5} \rfloor = 2, \quad \lfloor -8.5 \rfloor = -9, \quad \lfloor 7 \rfloor = 7, \quad \lfloor -4 \rfloor = -4$$
$$\lceil 3.14 \rceil = 4, \quad \lceil \sqrt{5} \rceil = 3, \quad \lceil -8.5 \rceil = -8, \quad \lceil 7 \rceil = 7, \quad \lceil -4 \rceil = -4$$

Integer and Absolute Value Functions

Let x be any real number. The *integer value* of x, written INT(x), converts x into an integer by deleting (truncating) the fractional part of the number. Thus

$$\text{INT}(3.14) = 3, \quad \text{INT}(\sqrt{5}) = 2, \quad \text{INT}(-8.5) = -8, \quad \text{INT}(7) = 7$$

Observe that INT(x) $= \lfloor x \rfloor$ or INT(x) $= \lceil x \rceil$ according to whether x is positive or negative.

The *absolute value* of the real number x, written ABS(x) or $|x|$, is defined as the greater of x or $-x$. Hence ABS(0) = 0, and, for $x \neq 0$, ABS(x) $= x$ or ABS(x) $= -x$, depending on whether x is positive or negative. Thus

$$|-15| = 15, \quad |7| = 7, \quad |-3.33| = 3.33, \quad |4.44| = 4.44, \quad |-0.075| = 0.075$$

We note that $|x| = |-x|$ and, for $x \neq 0$, $|x|$ is positive.