ADVANCE MATHMATICS

Rearrange By

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Refence

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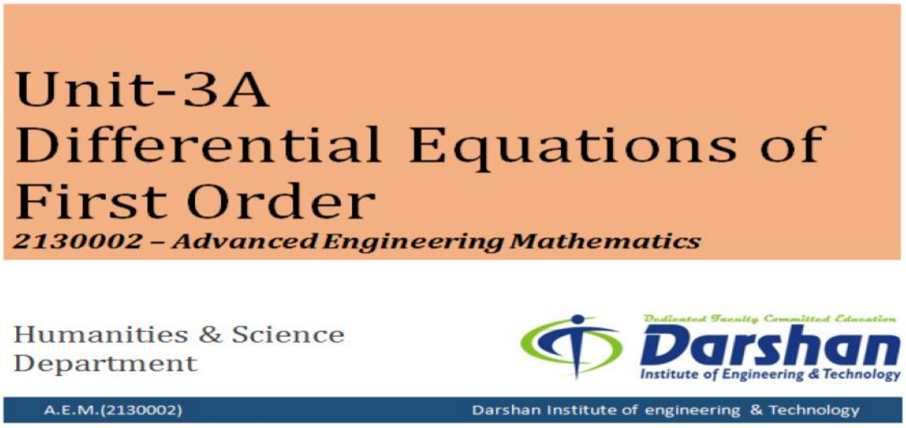
Unit-3A
Differential Equations of First Order
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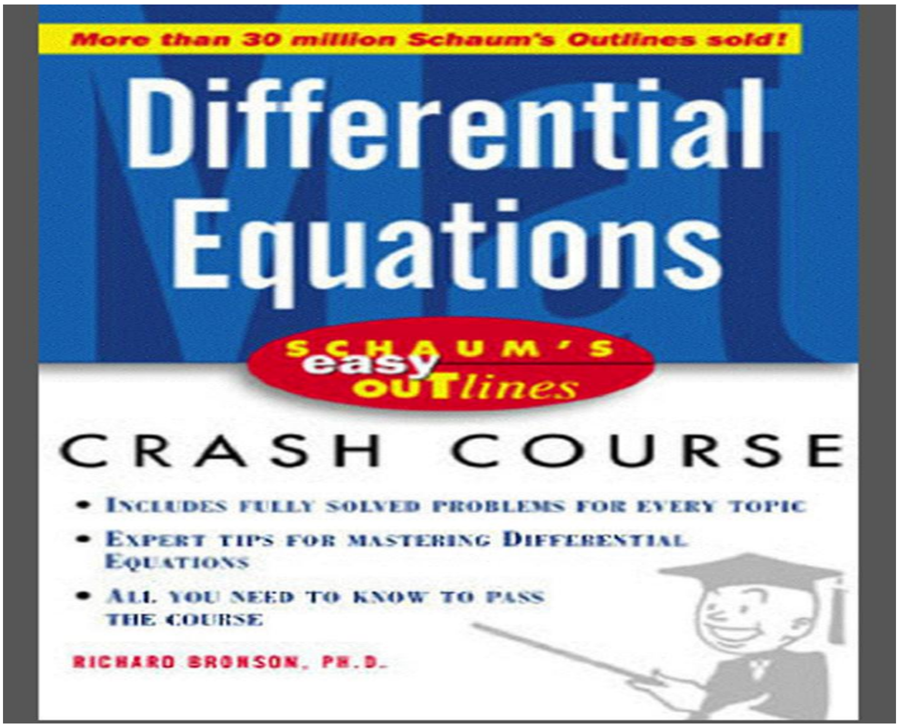
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CONTENTS

Introduction

- Let $y = f(x)$

Differentiate both the sides w.r.to x

$$\Rightarrow \frac{dy}{dx} = y' = f'(x)$$

Differentiate again both the sides w.r.to x

$$\Rightarrow \frac{d^2y}{dx^2} = y'' = f''(x)$$

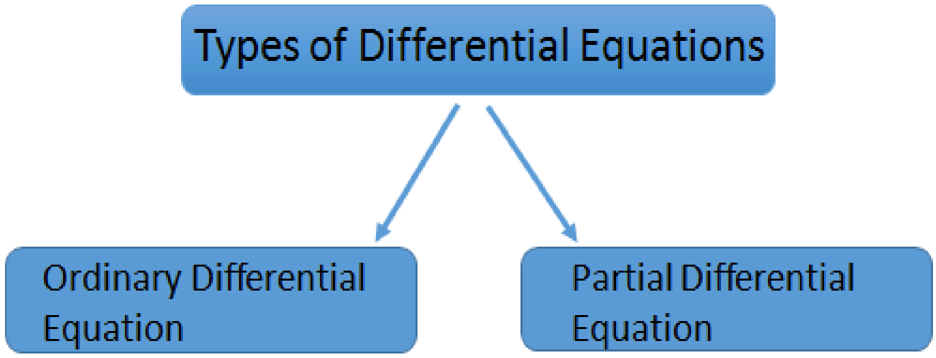
Continuing this process we can find n^{th} order derivative of the given function.

$$\frac{d^ny}{dx^n} = y^{(n)} = f^{(n)}(x)$$

- **Differential Equation**

An eqn. which involves differential co-efficient is called a Differential Equation.

$e.g.\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 0$



as:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 5x$$

$$xdy + (x - y)dx = 0$$

$$y''' - y = 2$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial v}{\partial t} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u$$

Types of Differential Equations

- **Ordinary differential Equation**

An eqn. which involves function of single variable and ordinary derivatives of that function then it is called an Ordinary Differential Equation.

e.g. $\frac{dy}{dx} + y = 0$

- **Partial Differential Equation**

An eqn. which involves function of two or more variables and partial derivatives of that function then it is called a Partial Differential Equation.

$e.g. \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 0$

Order of DE

- **Order of DE**

The order of highest derivative which appeared in a differential equation is "Order of D.E".

$e.g. \left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + 5y = 0$ has order 1.

Degree of DE

➢ Degree of DE

When a D.E. is in a polynomial form of derivatives, the highest power of highest order derivative occurring in D.E. is called a "Degree of D.E.".

$e.g.\ \left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + 5y = 0$ has degree 2.

M-1 Examples on Order and Degree

Ex.1 Find order and degree of $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{4}}$.

Solution:

Here, $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{4}}$

Taking Fourth power both the sides,

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^4 = y + \left(\frac{dy}{dx}\right)^2$

In above differential equation the order of highest derivative is 2

Now, DE is in polynomial form and highest power of the highest derivative is 4

Therefore, Order is 2 and Degree is 4.

Ex 3 Find order and degree of $\left(\frac{d^2y}{dx^2}\right)^3 = \left[x + \sin\left(\frac{dy}{dx}\right)\right]^2$.

Solution:

Here, $\left(\frac{d^2y}{dx^2}\right)^3 = \left[x + \sin\left(\frac{dy}{dx}\right)\right]^2$

In above differential equation the order of highest derivat 2 but the DE is not in polynomial form. So, degree of the given undefined.

Therefore, Order is 2 and degree is undefined.

Types of solution

- General solution

A solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation, is called the General solution or complete integral or complete primitive.

- Particular solution

The solution obtained from the general solution by giving a particular value to the arbitrary constants is called a particular solution.

Exercises: Find order and degree of the following D.E

$$\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^5 - 8y^3 = x^4$$

$$y^{(5)} - y'' = 2$$

Linear and Nonlinear DE

A differential equation is called "LINEAR DIFFERENTIAL EQUATION" if the dependent variable and every derivatives in the equation occurs in the first degree only and they should not be multiplied together. Otherwise it is known as "NONLINEAR DIFFERENTIAL EQUATION"

e.g. $\frac{d^2y}{dx^2} + \frac{x^2dy}{dx} + y = 0$ is linear.

$\frac{d^2y}{dx^2} + \frac{ydy}{dx} + y = 0$ is non-linear.

Types of First Order and First Degree DE

- ✓ Variable Separable Equation
- ✓ Homogeneous Differential Equation
- ✓ Linear(Leibnitz's) Differential Equation
- ✓ Bernoulli's Equation
- ✓ Exact Differential Equation

Exercises: which of the following D.E is linear and non-linear.

$$3y^{(4)} - 2y''' + 5x^3 y' + 7y = x - 3e^{2x}$$

$$(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 2y = 3$$

$$(y-x)dx + x^2 dy = 0$$

$$3yy'' + 5y' + 7xy = 0$$

$$x\frac{d^3y}{dx^3} - 3\left(\frac{dy}{dx}\right)^2 + 2y = 4$$

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - y = \sin y$$

$$\frac{dy}{dx} = \sqrt{3x+2y}$$

$$\frac{dy}{dx} = \frac{1}{2x+y}$$

Examples:

$$y = A\cos x + B\sin x \qquad (1)$$

Show that the above equation is a solution of the following DE

$$y'' + y = 0 \qquad (2)$$

Solutions:

$$y' = -A\sin x + B\cos x \qquad (3)$$

$$y'' = -A\cos x - B\sin x \qquad (4)$$

Insert (1) and (4) into (2)

$$= -A\cos x - B\sin x + A\cos x + B\sin x$$

$$= 0$$

Form a suitable DE using $y = A\cos x + B\sin x$

Solutions:

$$y' = -A\sin x + B\cos x$$

$$y'' = -A\cos x - B\sin x$$

$$= -(A\cos x + B\sin x)$$

$$\therefore \quad y'' = -y \quad => \frac{d^2y}{dx^2} + y = 0$$

EXERCISE:

Show that $y = A\cos(\ln x) + B\sin(\ln x)$ is the solution of the following DE $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Variable Separable Method

If a differential equation of type $\frac{dy}{dx} = f(x, y)$ can be converted into $M(x)dx = N(y)dy$, then it is known as a Variable Separable Equation.

The general solution of a Variable Separable Equation is

$$\int M(x)dx = \int N(y)dy + c$$

Where, c is an arbitrary constant.

➤ Note:

For convenience, the arbitrary constant can be chosen in any suitable form. e. g. $log c, \tan^{-1} c, e^c, \sin c$,etc.

M-2 Examples on Variable Separable DE

Ex. 1 Solve $9\,y\,y' + 4\,x = 0$

Solution:

Here, $9\,y\,y' + 4\,x = 0$

$$\Rightarrow 9y\frac{dy}{dx} = -4x$$

$$\Rightarrow 9y\,dy = -4x\,dx$$

Integrate both the sides,

$$\Rightarrow 9\int y\,dy = -4\int x\,dx$$

$$\Rightarrow \frac{9y^2}{2} = -\frac{4x^2}{2} + c'$$

$$\Rightarrow 9y^2 = -4x^2 + 2c'$$

$$\Rightarrow 9y^2 + 4x^2 = c \qquad (\because c = 2c')$$

Ex. 3 Solve $xy' + y = 0\, ; y(2) = -2$

Solution: Here, $xy' + y = 0$

$$\Rightarrow x\frac{dy}{dx} + y = 0$$

$$\Rightarrow x\, dy = -y\, dx$$

$$\Rightarrow \frac{1}{y}\, dy = -\frac{1}{x}\, dx$$

Integrate both the sides,

$\Rightarrow \log y = -\log x + \log c \qquad \Rightarrow \log y + \log x = \log c$

$\Rightarrow \log xy = \log c \qquad \Rightarrow xy = c$

Now, $y(2) = -2$

Here,

$xy = c$

$\Rightarrow (2)(-2) = c$

$\Rightarrow -4 = c$

Therefore, the solution is

$xy = -4$

Ex. 8 Solve $xy\dfrac{dy}{dx} = 1 + x + y + xy$.

Solution:

Here, $xy\frac{dy}{dx} = 1 + x + y + xy$

$$\Rightarrow xy\frac{dy}{dx} = 1 + x + y(1 + x)$$

$$\Rightarrow xy\frac{dy}{dx} = (1 + x)(1 + y)$$

$$\Rightarrow \frac{y}{1 + y}\,dy = \frac{1 + x}{x}\,dx$$

$$\Rightarrow \frac{(1 + y) - 1}{1 + y}\,dy = \left[\frac{1}{x} + 1\right]\,dx$$

$$\Rightarrow \left[\frac{1 + y}{1 + y} - \frac{1}{1 + y}\right]\,dy = \left[\frac{1}{x} + 1\right]dx$$

$$\Rightarrow \left[1 - \frac{1}{1 + y}\right]dy = \left[\frac{1}{x} + 1\right]dx$$

Integrate both the sides,

$$\Rightarrow \int\left[1 - \frac{1}{1 + y}\right]dy = \int\left[\frac{1}{x} + 1\right]dx$$

$$\Rightarrow y - \log(1 + y) = \log x + x + \log c$$

Exercise

1) Solve the following equations

a. $x\frac{dy}{dx} = \cot y$

b. $\frac{dy}{dx} = \frac{y}{x(x+1)}$

c. $\frac{dy}{dx} + (1 + y^2) = 0, \quad y(0) = 0$

d. $\sqrt{xy}\frac{dy}{dx} = \sqrt{4 - x}$

2)

Show that the DE $\frac{dy}{dx} = (x + y)^2$ can be reduced to a separable equation by using substitution $z = x + y$. Then obtain the solution for the original DE.

3)

Solve the initial value problem

$$\frac{dy}{dx} = \frac{y\cos x}{1 + 2y^2}, \qquad y(0) = 1$$

Reducible to variable separable Eq.

- If a differential equation of type $\frac{dy}{dx} = f(x, y)$ can be converted into $\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right)$ then it can be converted into variable separable equation by taking $y = vx$ & $\frac{dy}{dx} = x\frac{dv}{dx} + v$.

Definition (Homogeneous function of degree n)

A function $F(x, y)$ is called *homogeneous of degree n* if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$.

For a polynomial, homogeneous says that all of the terms have the same degree.

If F is homogeneous of degree 0, then F is a function of y/x.

Ex. 13 Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$.

Solution:

Here, $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$ (1)

Let, $\frac{y}{x} = v$

Differentiate above equation both the sides, w.r.to x

$$\Rightarrow \frac{x\frac{dy}{dx} - y}{x^2} = \frac{dv}{dx}$$

$$\Rightarrow x\frac{dy}{dx} - y = x^2\frac{dv}{dx}$$

Dividing above equation both the sides by x

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = x\frac{dv}{dx}$$

Therefore, by eq. (1)

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right) \quad \Rightarrow \frac{dy}{dx} - \frac{y}{x} = \tan\left(\frac{y}{x}\right)$$

$$\Rightarrow x\frac{dv}{dx} = \tan v$$

$$\Rightarrow \frac{1}{\tan v} dv = \frac{1}{x} dx$$

Integrate both the sides,

$$\Rightarrow \int \frac{1}{\tan v} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \int \cot v \, dv = \int \frac{1}{x} dx$$

$$\Rightarrow \log \sin v = \log x + \log c$$

$$\Rightarrow \sin v = xc \qquad \Rightarrow \sin \frac{y}{x} = xc \qquad \left(\because v = \frac{y}{x}\right)$$

Leibnitz's (linear) Equation

	Form - 1	Form -2
Form of DE	$\frac{dy}{dx} + P(x)y = Q(x)$	$\frac{dx}{dy} + P(y)x = Q(y)$
Integrating factor	$I.F. = e^{\int p(x)\,dx}$	$I.F. = e^{\int p(y)\,dy}$
Solution	$y\,(I.F.) = \int Q(x)\,(I.F.)dx + c$	$x\,(I.F.) = \int Q(y)\,(I.F.)dy + c$

M-3 Examples on Liebnitz's DE

Ex. 1 Solve $y' + y\sin x = e^{\cos x}$

Solution:

Here, $y' + y\sin x = e^{\cos x}$

Comparing it with $y' + P(x)y = Q(x)$

We have,

$P(x) = \sin x \qquad and \qquad Q(x) = e^{\cos x}$

Now, $\int p(x)\,dx = \int \sin x\,dx$

$$= -\cos x$$

$$\text{I.F.} = e^{\int p(x)\,dx}$$

$$= e^{-\cos x}$$

The general solution is,

$$y(I.F.) = \int Q(x)\,(I.F.)dx + c$$

$$\Rightarrow ye^{-\cos x} = \int e^{\cos x} \cdot e^{-\cos x} dx + c$$

$$\Rightarrow ye^{-\cos x} = \int 1\,dx + c$$

$$\Rightarrow ye^{-\cos x} = x + c$$

Ex. 4 Solve $(x+1)\frac{dy}{dx} - y = (x+1)^2 e^{3x}$

Solution:

Here, $(x+1)\frac{dy}{dx} - y = (x+1)^2 e^{3x}$

Dividing it by $(x+1)$

$$\frac{dy}{dx} - \frac{1}{x+1}y = (x+1)e^{3x}$$

Comparing it with $\frac{dy}{dx} + P(x)y = Q(x)$

We have,

$$P(x) = -\frac{1}{x+1} \quad and \quad Q(x) = (x+1)e^{3x}$$

Now, $\int p(x)\,dx = \int -\frac{1}{x+1}\,dx$

$$= -\log(x+1) = \log(x+1)^{-1}$$

$$I.F. = e^{\int p(x)\,dx}$$

$$= e^{\log(x+1)^{(-1)}}$$

$$= \frac{1}{x+1}$$

The general solution is,

$$y(I.F.) = \int Q(x)\,(I.F.)dx + c$$

$$\Rightarrow y\left(\frac{1}{x+1}\right) = \int (x+1)e^{3x} \cdot \left(\frac{1}{x+1}\right) dx + c$$

$$\Rightarrow y\left(\frac{1}{x+1}\right) = \int e^{3x} dx + c$$

$$\Rightarrow y\left(\frac{1}{x+1}\right) = \frac{e^{3x}}{3} + c$$

Ex. 7 Solve $\frac{dy}{dx} + (\cot x)y = 2\cos x$

Solution:

$$\frac{dy}{dx} + (\cot x)y = 2\cos x$$

Comparing it with $y' + P(x)y = Q(x)$

We have,

$P(x) = \cot x$ *and* $Q(x) = 2\cos x$

Now, $\int p(x)\,dx = \int \cot x$

$$= \log \sin x$$

$I.F. = e^{\int p(x)\,dx}$

$= e^{\log \sin x}$

$= \sin x$

The general solution is,

$$y(I.F.) = \int Q(x)\,(I.F.)dx \ + c$$

$$\Rightarrow y \sin x = \int 2\cos x \cdot \sin x \, dx \ + c$$

$$\Rightarrow \ y \sin x = \int \sin 2x \, dx + c$$

$$\Rightarrow y \sin x = -\frac{\cos 2x}{2} + c$$

Bernoulli's Differential Equation

- A differential equation of the form

  $$\frac{dy}{dx} + P(x)y = Q(x)\boxed{y^n} \quad \text{OR} \quad \frac{dx}{dy} + P(y)x = Q(y)\boxed{x^n}$$

  is known as Bernoulli's Differential Equation.

- Here, n is real number $(n \neq 0,1)$
- This type of differential equation can be converted into linear differential equation.

Process to reduce the Bernoulli's DE into Linear DE

➢ Case-1

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad (1)$$

Dividing both sides of equation (1) by y^n,

$$\Rightarrow y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \qquad (2)$$

Let $y^{1-n} = v$

Let $y^{1-n} = v \Rightarrow (1-n)y^{(-n)}\frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow y^{(-n)}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx}$$

Then from equation (2) $\left(\text{i.e. } y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)\right)$

$$\Rightarrow \frac{1}{(1-n)}\frac{dv}{dx} + P(x)v = Q(x)$$

$\Rightarrow \frac{dv}{dx} + P(x)(1-n)v = Q(x)(1-n)$ which is a linear DE

➢Case-2

A differential of the form

$$\frac{dy}{dx} + P(x)f(y) = Q(x)\,g(y)$$

Dividing both sides of equation by $g(y)$

$$\Rightarrow \frac{1}{g(y)}\frac{dy}{dx} + P(x)\frac{f(y)}{g(y)} = Q(x) \qquad (3)$$

Let $\frac{f(y)}{g(y)} = v$

Differentiate with respect to x both the sides,

Equation (3) becomes Linear Differential equation.

M-4 Examples on Bernoulli's DE

Ex.3 Solve $\frac{dy}{dx}+\frac{y}{x}=x^2y^6$

Solution:

$$\frac{dy}{dx}+\frac{y}{x}=x^2y^6 \qquad (1)$$

Divide the above equation both the sides by y^6

$$\Rightarrow \frac{1}{y^6}\frac{dy}{dx}+\frac{y}{xy^6}=x^2$$

$$\Rightarrow y^{-6}\frac{dy}{dx}+\boxed{y^{-5}}\,\frac{1}{x}=x^2 \qquad (2)$$

Let $y^{-5}=v$

$$\Rightarrow -5y^{-6}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow y^{-6}\frac{dy}{dx}=-\frac{1}{5}\frac{dv}{dx}$$

Now, from equation (2) $\left(i.e.\, y^{-6}\frac{dy}{dx}+\frac{y^{-5}}{x}=x^2\right)$

$$-\frac{1}{5}\frac{dv}{dx}+\frac{v}{x}=x^2$$

$$\Rightarrow \frac{dv}{dx} - \frac{5v}{x} = -5x^2$$

Which is a linear differential equation.

Now, compare the above equation with $\frac{dv}{dx} + p(x)v = Q(x)$

$$\Rightarrow P(x) = -\frac{5}{x} \quad and \quad Q(x) = -5x^2$$

$$\int P(x)dx = -5\int \frac{1}{x}\, dx = -5\log x = \log x^{-5}$$

$$I.F. = e^{\int P dx}$$

$$= e^{\log x^{-5}}$$

$$= x^{-5}$$

The general solution is,

$$v(I.F.) = \int Q(x)\,(I.F.)dx + c$$

$$\Rightarrow vx^{-5} = \int -5x^2\,(x^{-5})dx + c$$

$$\Rightarrow vx^{-5} = -5\int x^{-3}\,dx + c$$

$$\Rightarrow vx^{-5} = -5\int x^{-3}\,dx + c$$

$$\Rightarrow vx^{-5} = \frac{5x^{-2}}{2} + c$$

$$\Rightarrow y^{-5}x^{-5} = \frac{5x^{-2}}{2} + c$$

Which is required solution.

Ex.5 Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^{x}\sec y$

Solution:

$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)\,e^{x}\sec y \qquad (1)$$

Divide both the sides by $\sec y$

$$\Rightarrow \cos y\frac{dy}{dx} - \cos y\,\frac{\tan y}{1+x} = (1+x)e^{x}$$

$$\Rightarrow \cos y\frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^{x} \qquad (2)$$

Let $\sin y = v$

$$\Rightarrow \cos y \frac{dy}{dx} = \frac{dv}{dx}$$

Now, from equation (2) $\left(i.e. \cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^{x}\right)$

$$\frac{dv}{dx} - \frac{v}{1+x} = (1+x)e^{x}$$

Which is a linear differential equation.

Now, compare this equation with $\frac{dv}{dx} + p(x)v = Q(x)$

$$\Rightarrow P(x) = -\frac{1}{1+x} \qquad and \qquad Q(x) = (1+x)e^{x}$$

$$\int P(x)\, dx = -\int \frac{1}{1+x} dx$$

$$= -\log(1+x)$$

$$= \log(1+x)^{-1}$$

I.F.$= e^{\int P\, dx} = e^{\log(1+x)^{-1}} = \frac{1}{1+x}$

The general solution is,

$$v(I.F.) = \int Q(x)\,(I.F.)dx + c$$

$$\Rightarrow \frac{v}{1+x} = \int (1+x)e^{x}\frac{1}{1+x}dx + c$$

$$\Rightarrow \frac{\sin y}{1+x} = e^{x} + c$$

Which is required solution.

Ex.7 Solve $\frac{dy}{dx} - 2\,\mathrm{y}\tan x = y^2\tan^2 x$

Solution:

$$\frac{dy}{dx} - 2\,\mathrm{y}\tan x = y^2\tan^2 x \qquad (1)$$

Divide both the sides by y^2

$$\Rightarrow y^{-2}\frac{dy}{dx} - 2\,\mathrm{y}^{-1}\tan x = \tan^2 x \qquad (2)$$

Let $y^{-1} = v$

$$\Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dv}{dx} \quad \Rightarrow y^{-2}\frac{dy}{dx} = -\frac{dv}{dx}$$

Now, from equation (2) $\left(i.e.\, y^{-2}\frac{dy}{dx} - 2\,y^{-1}\tan x = \tan^2 x\right)$

$$-\frac{dv}{dx} - 2v\tan x = \tan^2 x$$

$$\Rightarrow \frac{dv}{dx} + 2v\tan x = -\tan^2 x$$

Which is a linear differential equation.

Now, compare this equation with $\frac{dv}{dx} + p(x)v = Q(x)$

$$\Rightarrow P(x) = 2\tan x \qquad and \qquad Q(x) = -\tan^2 x$$

Now, $\int P(x)\,dx = 2\int \tan x\,dx$

$$= 2\log\sec x$$

$$= \log\sec^2 x$$

$\text{I.F.} = e^{\int P\,dx}$

$$= e^{\log\sec^2 x}$$

$$= \sec^2 x$$

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The general solution is,

$$v(I.F.) = \int Q(x)\,(I.F.)dx + c$$

$$\left(\because \int f^n(x) f'(x)\,dx = \frac{f^{n+1}(x)}{n+1} \right)$$

$$v \sec^2 x = -\int \tan^2 x \sec^2 x \;dx + c$$

$$v \sec^2 x = -\frac{\tan^3 x}{3} + c$$

$$\frac{\sec^2 x}{y} = -\frac{\tan^3 x}{3} + c$$

Exact Differential Equation

- A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be Exact Differential Equation if it can be derived from its primitive by direct differential without any further transformation such as elimination etc.
- Necessary and Sufficient Condition:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

where 1^{st} order partial derivative of M & N must exist at all points of f(x, y)

- The general solution of Exact Differential Equation is

$$\int_{y=constant} M(x,y)dx + \int (terms\ of\ N\ free\ from\ x)dy = c$$

M-5 Examples on Exact DE

Ex. 1 Solve $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$

Solution:

Compare the given equation with $M(x, y)dx + N(x, y)dy = 0$

Therefore, we get that

$M = x^3 + 3xy^2$ and $N = y^3 + 3x^2y$

Now, differentiate M and N w.r.to y and x respectively.

$\Rightarrow \frac{\partial M}{\partial y} = 6xy$ and $\frac{\partial N}{\partial x} = 6xy$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Therefore, the given differential equation is Exact.

The general solution is

$$\int_{y=constant} M(x, y)dx + \int (terms\ of\ N\ free\ from\ x)dy = c$$

$$\Rightarrow \int_{y=constant} (x^3 + 3xy^2)dx + \int y^3 dy = c$$

$$\Rightarrow \frac{x^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} = c$$

Ex.8 Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

Solution:

Here,

$$\frac{dy}{dx} = -\frac{y\cos x + \sin y + y}{\sin x + x\cos y + x}$$

$$\Rightarrow (\sin x + x\cos y + x)dy = -(y\cos x + \sin y + y)dx$$

$$\Rightarrow (y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$$

Compare the given equation with $M(x, y)dx + N(x, y)dy = 0$

$M = y\cos x + \sin y + y$ and $N = \sin x + x\cos y + x$

$M = y\cos x + \sin y + y$ and $N = \sin x + x\cos y$

Now, differentiate M and N w.r.to y and x respectively.

$\Rightarrow \frac{\partial M}{\partial y} = \cos x + \cos y + 1$ and $\frac{\partial N}{\partial x} = \cos x + co$

$\Rightarrow \frac{\partial M}{\partial} = \frac{\partial N}{\partial}$

The general solution is

$$\Rightarrow \int_{y=constant} M(x, y)dx + \int (terms\ of\ N\ free\ from\ x)dy = c$$

$$\Rightarrow \int_{y=constant} (y\cos x + \sin y + y)\, dx = c$$

$$\Rightarrow y\sin x + x\sin y + yx = c$$

Ex.10 Solve $(y^2e^{xy^2}+4x^3)dx+\left(2xye^{xy^2}-3y^2\right)dy=0$

Solution:

Compare the given equation with $M(x,y)dx+N(x,y)dy=0$

Therefore, we get that

$M=y^2e^{xy^2}+4x^3$ and $N=2xye^{xy^2}-3y^2$

Now, differentiate M and N w.r.to y and x respectively.

$$\Rightarrow\frac{\partial M}{\partial y}=2ye^{xy^2}+2xy^3e^{xy^2} \quad \text{and} \quad \frac{\partial N}{\partial x}=2ye^{xy^2}+2xy^3e^{xy^2}$$

$$\Rightarrow\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Therefore, the given differential equation is Exact.

The general solution is

$$\Rightarrow\int_{y=constant} M(x,y)dx+\int(terms\ of\ N\ free\ from\ x)dy=c$$

$$\Rightarrow\int_{y=constant}(y^2e^{xy^2}+4x^3)\,dx-3\int(y^2)dy=c$$

$$\Rightarrow\frac{y^2e^{xy^2}}{y^2}+x^4-y^3=c \quad \Rightarrow e^{xy^2}+x^4-y^3=c$$

Non Exact Differential Equation

- A differential equation which is not exact differential equation is known as Non-Exact Differential Equation.
- i.e. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$
- We reduce the non-exact differential equation into exact differential equation by multiplying it with I.F.

Standard rules for finding I.F.

Condition	Type of equation	I.F.
$Mx + Ny \neq 0$	Homogeneous	$\frac{1}{Mx + Ny}$
$Mx - Ny \neq 0$	Non Homogeneous	$\frac{1}{Mx - Ny}$
$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$	-	$I.F. = e^{\int f(x)\,dx}$
$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$	-	$I.F. = e^{\int g(y)\,dy}$

Non Exact Differential Equation

- Now , multiply I.F. with the given differential equation to get new M and N.
- Therefore, the general solution is

$$\int_{y=constant} M\,dx + \int (terms\ of\ \ N\ free\ from\ x)dy = c$$

Where, c is any constant.

Homogeneous Differential Equation

- Homogeneous DE

  A differential equation is called homogeneous differential equation if each term has same degree.

- E.g. $(xy - 2y^2)dx - (x^2 - 3xy)\,dy = 0$

  Here, each term is of degree 2

  Therefore, this diff. equation is homogeneous.

Non-Homogeneous Differential Equation

- When the function is not homogeneous, it is called non homogeneous function.
- E.g. $(x^2y - 2y^2)dx - (x^2 - 3xy^2)\,dy = 0$

  Here, each term doesn't have same degree

  Therefore, this diff. equation is Non-homogeneous.

M-6 Examples on Non-Exact DE

Ex.2 Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$.

Solution:

Compare the given equation with $M(x,y)dx + N(x,y)dy = 0$

Therefore, we get that

$M = x^2y - 2xy^2$ and $N = -x^3 + 3x^2y$

Now, differentiate M and N w.r.to y and x respectively.

$\Rightarrow \dfrac{\partial M}{\partial y} = x^2 - 4xy$ and $\dfrac{\partial N}{\partial x} = -3x^2 + 6xy$

$\Rightarrow \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

- Here, the given differential equation is homogenous as each term is of order 3.

- Therefore, $I.F. = \frac{1}{Mx+Ny}$

$$= \frac{1}{(x^2y-2xy^2)x+(-x^3+3x^2y)y}$$

$$= \frac{1}{x^3y-2x^2y^2-x^3y+3x^2y^2}$$

$$= \frac{1}{x^2y^2}$$

- Now, multiply I.F. with the given differential equation to reduce it into exact form.

$$\frac{1}{x^2y^2}(x^2y - 2xy^2)dx - \frac{1}{x^2y^2}(x^3 - 3x^2y)dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

Therefore, $M = \frac{1}{y} - \frac{2}{x}$ *and* $N = -\frac{x}{y^2} + \frac{3}{y}$

Therefore, the given differential equation is Exact.

The general solution is

$$\Rightarrow \int_{y=constant} M(x,y)dx + \int (terms\ of\ N\ free\ from\ x)dy = c$$

$$\Rightarrow \int_{y=constant} \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y} dy = c$$

$$\Rightarrow \frac{1}{y}x - 2\log x + 3\log y = c$$

Ex.5 Solve $(x^2y^2+2)ydx+(2-x^2y^2)xdy=0$.

Solution:

Compare the given equation with $M(x,y)dx+N(x,y)dy=0$

Therefore, we get that

$M=x^2y^3+2y$ and $N=2x-x^3y^2$

Now, differentiate M and N w.r.to y and x respectively.

$\Rightarrow \frac{\partial M}{\partial y}=3y^2x^2+2$ and $\frac{\partial N}{\partial x}=2-3x^2y^2$

$\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

- Here, the given differential equation is non-homogenous and it is of the form $f(x,y)y\,dx+g(x,y)x\,dy=0$
- Therefore, $I.F.=\frac{1}{Mx-Ny}$

$$=\frac{1}{(x^2y^3+2y)x-(2x-x^3y^2)y}$$

$$=\frac{1}{x^3y^3+2xy-2xy+x^3y^3}$$

$$=\frac{1}{2x^3y^3}$$

➢ Now, multiply I.F. with the given differential equation to reduce it into exact form.

$$\frac{1}{2x^3y^3}(x^2y^3+2y)dx+\frac{1}{2x^3y^3}(2x-x^3y^2)dy=0$$

$$\left(\frac{1}{2x}+\frac{1}{x^3y^2}\right)dx+\left(\frac{1}{x^2y^3}-\frac{1}{2y}\right)dy=0$$

Therefore, $M=\frac{1}{2x}+\frac{1}{x^3y^2}$ $\quad and \quad$ $N=\frac{1}{x^2y^3}-\frac{1}{2y}$

Therefore, the given differential equation is Exact.

The general solution is

$$\Rightarrow\int_{y=constant} M(x,y)dx+\int(terms\ of\ N\ free\ from\ x)dy=c$$

$$\Rightarrow\int_{y=constant}\left(\frac{1}{2x}+\frac{1}{x^3y^2}\right)dx-\frac{1}{2}\int\frac{1}{y}dy=c$$

$$\Rightarrow\frac{1}{2}\log x-\frac{1}{2x^2y^2}-\frac{1}{2}\log y=c\Rightarrow\log\frac{x}{y}-\frac{1}{x^2y^2}=2c$$

Ex.9 Solve $(x^2 + y^2 + 3)dx - 2xydy = 0$.

Solution:

Compare the given equation with $M(x, y)dx + N(x, y)dy = 0$

Therefore, we get that

$M = x^2 + y^2 + 3$ and $N = -2xy$

Now, differentiate M and N w.r.to y and x respectively.

$\Rightarrow \frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = -2y$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

- Here, the given differential equation is non-homogenous.
- Here, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{1}{2xy}(2y + 2y)$

$$= -\frac{4y}{2xy}$$

$$= -\frac{2}{x} = f(x)$$

Therefore, $I.F. = e^{\int f(x)dx}$

$$= e^{-\int \frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2}$$

- Now, multiply I.F. with the given differential equation to reduce it into exact form.

$x^{-2}(x^2 + y^2 + 3)\, dx + x^{-2}(-2xy)\, dy = 0$

$(1 + x^{-2}y^2 + 3x^{-2})\, dx - 2x^{-1}y\, dy = 0$

Therefore, $M = 1 + x^{-2}y^2 + 3x^{-2} and$ $N = -2x^{-1}y$

Therefore, the given differential equation is Exact.

The general solution is

$$\Rightarrow \int_{y=constant} M(x,y)dx + \int (terms\ of\ N\ free\ from\ x)dy = c$$

$$\Rightarrow \int_{y=constant} (1 + x^{-2}y^2 + 3x^{-2})\ dx = c$$

$$\Rightarrow x - \frac{y^2}{x} - \frac{3}{x} = c \quad \Rightarrow x^2 - y^2 - 3 = cx$$

Ex.10 Solve $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$.

Solution:

Compare the given equation with $M(x,y)dx + N(x,y)dy = 0$

Therefore, we get that

$M = 3x^2y^4 + 2xy$ and $N = 2x^3y^3 - x^2$

Now, differentiate M and N w.r.to y and x respectively.

$$\Rightarrow \frac{\partial M}{\partial y} = 12x^2y^3 + 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

- Here, the given differential equation is non-homogenous.
- Here, $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{(3x^2y^4+2xy)}(6x^2y^3 - 2x - 12x^2y^3 - 2x)$

$$= \frac{-6x^2y^3-4x}{(3x^2y^4+2xy)}$$

$$== \frac{-2x(3xy^3+2)}{xy(3xy^3+2)} = -\frac{2}{y} = g(y)$$

Therefore, $I.F. = e^{\int g(y)dy}$

$$= e^{-\int \frac{2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2}$$

- Now, multiply I.F. with the given differential equation to reduce it into exact form.

$y^{-2}(3x^2y^4 + 2xy)\,dx + y^{-2}(2x^3y^3 - x^2)\,dy = 0$

$(3x^2y^2 + 2xy^{-1})\,dx + (2x^3y - x^2y^{-2})\,dy = 0$

Therefore, $M = 3x^2y^2 + 2xy^{-1}$ *and* $N = 2x^3y - x^2y^{-2}$

Now, the given differential equation is Exact.

The general solution is

$$\Rightarrow \int_{y=constant} M(x,y)dx + \int (terms\ of\ N\ free\ from\ x)dy = c$$

$$\Rightarrow \int_{y=constant} (3x^2y^2 + 2xy^{-1})\,dx = c$$

$$\Rightarrow x^3y^2 + \frac{x^2}{y} = c$$

Orthogonal Trajectory

- **Trajectory**

  A Curve which cuts every member of a given family of curves according to some definite rule is called trajectory.

- **Orthogonal Trajectory**

  A curve which cuts every member of a given family at right angles is a called an Orthogonal Trajectory.

Method for finding Orthogonal trajectory of $f(x, y, c) = 0$

1. Differentiate $f(x, y, c) = 0$ ... (1) w.r.t. x.
2. Eliminate c by using eq^n ... (1) and its derivative.
3. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$. This will give you differential equation of the orthogonal trajectories.
4. Solve the differential equation to get the equation of the orthogonal trajectories.

M-7 Examples on Orthogonal Trajectory

Ex.1 Find orthogonal trajectories of $y = x^2 + c$.

Solution:

Here, $y = x^2 + c$

Now, differentiate both the sides w.r.to x

$\Rightarrow \frac{dy}{dx} = 2x$

$\Rightarrow -\frac{dx}{dy} = 2x$ $\left(\because \text{Replacing } \frac{dy}{dx} \text{ by } -\frac{dx}{dy}\right)$

$\Rightarrow \frac{1}{x}\, dx = -2dy$

$\Rightarrow \frac{1}{x}\, dx = -2dy$

Integrate both the sides,

$\Rightarrow \log x = -2y + c$

$\Rightarrow \log x + 2y = c$

Method for finding Orthogonal trajectory of $f(r,\theta,c)=0$

1. Differentiate $f(r,\theta,c)=0 \ldots (1)$ w.r.t. θ.
2. Eliminate c by using eqn ...(1) and its derivative
3. Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$. This will give you differential eqn of the orthogonal trajectories.
4. Solve the differential equation to get the equation of the orthogonal trajectories.

Ex.4 Find Orthogonal trajectories of $r^n = a^n \cos n\theta$.

Solution:

Here, $r^n = a^n \cos n\theta \qquad (1)$

Differentiate both the sides w.r.to θ

$$\Rightarrow n r^{n-1}\frac{dr}{d\theta} = -n a^n \sin n\theta$$

$$\Rightarrow r^{n-1}\frac{dr}{d\theta} = -\frac{r^n}{\cos n\theta}\sin n\theta \qquad \left(\because By\ eq^n(1), a^n = \frac{r^n}{\cos n\theta}\right)$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

$$\Rightarrow -r\frac{d\theta}{dr} = -\tan n\theta \qquad \left(\text{Replacing } \frac{dr}{d\theta} \text{ by } -r^2\frac{d\theta}{dr}\right)$$

$$\Rightarrow \cot n\theta\, d\theta = \frac{1}{r}\, dr$$

$$\Rightarrow \frac{\log\sin n\theta}{n} = \log r + \log c$$

$$\Rightarrow \log\sin n\theta = n\log rc \qquad \Rightarrow \sin n\theta = (rc)^n \qquad \Rightarrow r^n = c^n \sin n\theta$$

Second Order Differential Equations

general linear second order differential equation is in the form.

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = g(t) \qquad (1)$$

In fact, we will rarely look at non-constant coefficient linear second order differential equations. In the case where we assume constant coefficients we will use the following differential equation.

$$ay'' + by' + cy = g(t) \qquad (2)$$

Initially we will make our life easier by looking at differential equations with $g(t) = 0$. When $g(t) = 0$ we call the differential equation **homogeneous** and when $g(t) \neq 0$ we call the differential equation **nonhomogeneous**.

: Real & Distinct Roots

We start with the differential equation.

$$ay'' + by' + cy = 0$$

Write down the characteristic equation.

$$ar^2 + br + c = 0$$

Solve the characteristic equation for the two roots, r_1 and r_2. This gives the two solutions

$$y_1(t) = \mathbf{e}^{r_1 t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{r_2 t}$$

Now, if the two roots are real and distinct (*i.e.* $r_1 \neq r_2$) it will turn out that these two solutions are "nice enough" to form the general solution

$$y(t) = c_1 \mathbf{e}^{r_1 t} + c_2 \mathbf{e}^{r_2 t}$$

Example 1 Solve the following I

$$y'' + 11y' + 24y = 0 \qquad y(0) = 0 \qquad y'(0) = -7$$

Solution

The characteristic equation is

$$r^2 + 11r + 24 = 0$$

$$(r+8)(r+3) = 0$$

Its roots are $r_1 = -8$ and $r_2 = -3$ and so the general solution and its derivative is.

$$y(t) = c_1 \mathbf{e}^{-8t} + c_2 \mathbf{e}^{-3t}$$

$$y'(t) = -8c_1 \mathbf{e}^{-8t} - 3c_2 \mathbf{e}^{-3t}$$

Now, plug in the initial conditions to get the following system of equations.

$$0 = y(0) = c_1 + c_2$$

$$-7 = y'(0) = -8c_1 - 3c_2$$

Solving this system gives $c_1 = \frac{7}{5}$ and $c_2 = -\frac{7}{5}$. The actual solution to the differential equation is then

$$y(t) = \frac{7}{5}\mathbf{e}^{-8t} - \frac{7}{5}\mathbf{e}^{-3t}$$

Example 2 Solve the following

$$y'' + 3y' - 10y = 0 \qquad y(0) = 4 \qquad y'(0) = -2$$

Solution

The characteristic equation is

$$r^2 + 3r - 10 = 0$$

$$(r+5)(r-2) = 0$$

Its roots are r_1 = - 5 and r_2 = 2 and so the general solution and its derivative is.

$$y(t) = c_1\mathbf{e}^{-5t} + c_2\mathbf{e}^{2t}$$

$$y'(t) = -5c_1\mathbf{e}^{-5t} + 2c_2\mathbf{e}^{2t}$$

Now, plug in the initial conditions to get the following system of equations.

$$4 = y(0) = c_1 + c_2$$

$$-2 = y'(0) = -5c_1 + 2c_2$$

Solving this system gives $c_1 = \frac{10}{7}$ and $c_2 = \frac{18}{7}$. The actual solution to the differential equation is then

$$y(t) = \frac{10}{7}\mathbf{e}^{-5t} + \frac{18}{7}\mathbf{e}^{2t}$$

Find the general solution to the following differential equation.

$$y'' - 6y' - 2y = 0$$

Solution

The characteristic equation is.

$$r^2 - 6r - 2 = 0$$

The roots of this equation are.

$$r_{1,2} = 3 \pm \sqrt{11}$$

Now, do NOT get excited about these roots they are just two real numbers.

$$r_1 = 3 + \sqrt{11} \qquad \text{and} \qquad r_1 = 3 - \sqrt{11}$$

Admittedly they are not as nice looking as we may be used to, but they are just real numbers. Therefore, the general solution is

$$y(t) = c_1 \mathbf{e}^{(3+\sqrt{11})t} + c_2 \mathbf{e}^{(3-\sqrt{11})t}$$

Complex Roots

In this section we will be looking at solutions to the differential equation

$$ay'' + by' + cy = 0$$

in which roots of the characteristic equation,

$$ar^2 + br + c = 0$$

are complex roots in the form $r_{1,2} = \lambda \pm \mu i$.

the general form of the solution gives the following solutions to the

differential equation.

$$y_1(t) = \mathbf{e}^{(\lambda+\mu i)t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{(\lambda-\mu i)t}$$

To do this we'll need Euler's Formula.

$$\mathbf{e}^{i\theta} = \cos\theta + i\sin\theta$$

$$y_1(t) = \mathbf{e}^{\lambda t}\mathbf{e}^{i\mu t} = \mathbf{e}^{\lambda t}\left(\cos(\mu t) + i\sin(\mu t)\right)$$
$$y_2(t) = \mathbf{e}^{\lambda t}\mathbf{e}^{-i\mu t} = \mathbf{e}^{\lambda t}\left(\cos(\mu t) - i\sin(\mu t)\right)$$

$$y(t) = c_1 \mathbf{e}^{\lambda t} \cos(\mu t) + c_2 \mathbf{e}^{\lambda t} \sin(\mu t)$$

Example 1)

$$y'' - 4y' + 9y = 0 \qquad y(0) = 0 \qquad y'(0) = -8$$

Solution

The characteristic equation for this differential equation is.

$$r^2 - 4r + 9 = 0$$

The roots of this equation are $r_{1,2} = 2 \pm \sqrt{5}\,i$. The general solution to the differential equation is then.

$$y(t) = c_1 \mathbf{e}^{2t} \cos\left(\sqrt{5}t\right) + c_2 \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right)$$

the following.

$$0 = y(0) = c_1$$

In other words, the first term will drop out in order to meet the first condition. This makes the solution, along with its derivative

$$y(t) = c_2 \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right)$$
$$y'(t) = 2c_2 \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right) + \sqrt{5}c_2 \mathbf{e}^{2t} \cos\left(\sqrt{5}t\right)$$

A much nicer derivative than if we'd done the original solution. Now, apply the second initial condition to the derivative to get.

$$-8 = y'(0) = \sqrt{5}c_2 \qquad \Rightarrow \qquad c_2 = -\frac{8}{\sqrt{5}}$$

The actual solution is then.

$$y(t) = -\frac{8}{\sqrt{5}} \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right)$$

Example 2)

$$y'' - 8y' + 17y = 0 \qquad y(0) = -4 \quad y'(0) = -1$$

Solution

The characteristic equation this time is.

$$r^2 - 8r + 17 = 0$$

The roots of this are $r_{1,2} = 4 \pm i$. The general solution as well as its derivative is

$$y(t) = c_1 \mathbf{e}^{4t}\cos(t) + c_2 \mathbf{e}^{4t}\sin(t)$$
$$y'(t) = 4c_1 \mathbf{e}^{4t}\cos(t) - c_1 \mathbf{e}^{4t}\sin(t) + 4c_2 \mathbf{e}^{4t}\sin(t) + c_2 \mathbf{e}^{4t}\cos(t)$$

Notice that this time we will need the derivative from the start as we won't be having one of the terms drop out. Applying the initial conditions gives the following system.

$$-4 = y(0) = c_1$$
$$-1 = y'(0) = 4c_1 + c_2$$

Solving this system gives $c_1 = -4$ and $c_2 = 15$. The actual solution to the IVP is then.

$$y(t) = -4\mathbf{e}^{4t}\cos(t) + 15\mathbf{e}^{4t}\sin(t)$$

Repeated Roots

In this case we want solutions to

$$ay'' + by' + cy = 0$$

where solutions to the characteristic equation

$$ar^2 + br + c = 0$$

are double roots $r_1 = r_2 = r$.

This leads to a problem however. Recall that the solutions are

$$y_1(t) = \mathbf{e}^{r_1 t} = \mathbf{e}^{rt} \qquad y_2(t) = \mathbf{e}^{r_2 t} = \mathbf{e}^{rt}$$

Example 1)

$$y'' - 4y' + 4y = 0 \qquad y(0) = 12 \qquad y'(0) = -3$$

Solution

The characteristic equation and its roots are.

$$r^2 - 4r + 4 = (r-2)^2 = 0 \qquad r_{1,2} = 2$$

The general solution and its derivative are

$$y(t) = c_1 \mathbf{e}^{2t} + c_2 t\mathbf{e}^{2t}$$
$$y'(t) = 2c_1 \mathbf{e}^{2t} + c_2 \mathbf{e}^{2t} + 2c_2 t\mathbf{e}^{2t}$$

Don't forget to product rule the second term! Plugging in the initial conditions gives the following system.

$$12 = y(0) = c_1$$
$$-3 = y'(0) = 2c_1 + c_2$$

This system is easily solve to get $c_1 = 12$ and $c_2 = -27$. The actual solution to the IVP is then.

$$y(t) = 12\mathbf{e}^{2t} - 27t\mathbf{e}^{2t}$$

Example2)

$$16y'' - 40y' + 25y = 0 \qquad y(0) = 3 \qquad y'(0) = -\frac{9}{4}$$

Solution

The characteristic equation and its roots are.

$$16r^2 - 40r + 25 = (4r-5)^2 = 0 \qquad r_{1,2} = \frac{5}{4}$$

The general solution and its derivative are

$$y(t) = c_1 \mathbf{e}^{\frac{5t}{4}} + c_2 t\mathbf{e}^{\frac{5t}{4}}$$
$$y'(t) = \frac{5}{4} c_1 \mathbf{e}^{\frac{5t}{4}} + c_2 \mathbf{e}^{\frac{5t}{4}} + \frac{5}{4} c_2 t\mathbf{e}^{\frac{5t}{4}}$$

$$3 = y(0) = c_1$$
$$-\frac{9}{4} = y'(0) = \frac{5}{4} c_1 + c_2$$

This system is easily solve to get $c_1 = 3$ and $c_2 = -6$. The actual solution to the IVP is then.

$$y(t) = 3\mathbf{e}^{\frac{5t}{4}} - 6t\mathbf{e}^{\frac{5t}{4}}$$

Nonhomogeneous Differential Equations

linear nonhomogeneous differential equation is

$$y'' + p(t)y' + q(t)y = g(t)$$

First, we will call

$$y'' + p(t)y' + q(t)y = 0$$

the associated homogeneous differential equation to (1).

Undetermined Coefficients

the complementary solution

the particular solution

Example 1 Determine a particular solution to

$$y'' - 4y' - 12y = 3e^{5t}$$

Solution

The point here is to find a particular solution, however the first thing that we're going to do is find the complementary solution to this differential equation. Recall that the complementary solution comes from solving,

$$y'' - 4y' - 12y = 0$$

The characteristic equation for this differential equation and its roots are.

$$r^2 - 4r - 12 = (r-6)(r+2) = 0 \qquad \Rightarrow \qquad r_1 = -2, \quad r_2 = 6$$

The complementary solution is then,

$$y_c(t) = c_1 e^{-2t} + c_2 e^{6t}$$

$$Y_P(t) = A\mathbf{e}^{5t}$$

Now, all that we need to do is do a couple of derivatives, plug this into the differential equation and see if we can determine what *A* needs to be.

Plugging into the differential equation gives

$$25A\mathbf{e}^{5t} - 4\left(5A\mathbf{e}^{5t}\right) - 12\left(A\mathbf{e}^{5t}\right) = 3\mathbf{e}^{5t}$$
$$-7A\mathbf{e}^{5t} = 3\mathbf{e}^{5t}$$

So, in order for our guess to be a solution we will need to choose *A* so that the coefficients of the exponentials on either side of the equal sign are the same. In other words we need to choose *A* so that,

$$-7A = 3 \qquad \Rightarrow \qquad A = -\frac{3}{7}$$

Okay, we found a value for the coefficient. This means that we guessed correctly. A particular solution to the differential equation is then,

$$Y_P(t) = -\frac{3}{7}\mathbf{e}^{5t}$$

Example 2 Solve the following IVP

$$y'' - 4y' - 12y = 3\mathbf{e}^{5t} \qquad y(0) = \frac{18}{7} \qquad y'(0) = -\frac{1}{7}$$

Solution

We know that the general solution will be of the form,

$$y(t) = y_c(t) + Y_P(t)$$

$$y(t) = c_1\mathbf{e}^{-2t} + c_2\mathbf{e}^{6t} - \frac{3}{7}\mathbf{e}^{5t}$$
$$y'(t) = -2c_1\mathbf{e}^{-2t} + 6c_2\mathbf{e}^{6t} - \frac{15}{7}\mathbf{e}^{5t}$$

Now, apply the initial conditions to these.

$$\frac{18}{7} = y(0) = c_1 + c_2 - \frac{3}{7}$$
$$-\frac{1}{7} = y'(0) = -2c_1 + 6c_2 - \frac{15}{7}$$

Solving this system gives $c_1 = 2$ and $c_2 = 1$. The actual solution is then.

$$y(t) = 2\mathbf{e}^{-2t} + \mathbf{e}^{6t} - \frac{3}{7}\mathbf{e}^{5t}$$

Example 3 Find a particular solution for the following differential equation.

$$y'' - 4y' - 12y = \sin(2t)$$

Solution

$$Y_P(t) = A\sin(2t)$$

Differentiating and plugging into the differential equation gives,

$$-4A\sin(2t) - 4(2A\cos(2t)) - 12(A\sin(2t)) = \sin(2t)$$

Collecting like terms yields

$$-16A\sin(2t) - 8A\cos(2t) = \sin(2t)$$

We need to pick *A* so that we get the same function on both sides of the equal sign. This means that the coefficients of the sines and cosines must be equal. Or,

$$\cos(2t): \quad -8A = 0 \quad \Rightarrow \quad A = 0$$

$$\sin(2t): \quad -16A = 1 \quad \Rightarrow \quad A = -\frac{1}{16}$$

$$Y_P(t) = A\cos(2t) + B\sin(2t)$$

$$-4A\cos(2t) - 4B\sin(2t) - 4(-2A\sin(2t) + 2B\cos(2t)) - 12(A\cos(2t) + B\sin(2t)) = \sin(2t)$$

$$(-4A - 8B - 12A)\cos(2t) + (-4B + 8A - 12B)\sin(2t) = \sin(2t)$$

$$(-16A - 8B)\cos(2t) + (8A - 16B)\sin(2t) = \sin(2t)$$

Now, set the coefficients equal

$$\cos(2t): \quad -16A - 8B = 0$$

$$\sin(2t): \quad 8A - 16B = 1$$

Solving this system gives us

$$A = \frac{1}{40} \qquad B = -\frac{1}{20}$$

We found constants and this time we guessed correctly. A particular solution to the differential equation is then,

$$Y_P(t) = \frac{1}{40}\cos(2t) - \frac{1}{20}\sin(2t)$$

Now that we've gone over the three basic kinds of functions that we can use undetermined coefficients on let's summarize.

g(t)	$Y_P(t)$ guess
$a\mathbf{e}^{\beta t}$	$A\mathbf{e}^{\beta t}$
$a\cos(\beta t)$	$A\cos(\beta t)+B\sin(\beta t)$
$b\sin(\beta t)$	$A\cos(\beta t)+B\sin(\beta t)$
$a\cos(\beta t)+b\sin(\beta t)$	$A\cos(\beta t)+B\sin(\beta t)$
n^{th} degree polynomial	$A_n t^n + A_{n-1}t^{n-1}+\cdots A_1 t + A_0$

Fact

If $Y_{P1}(t)$ is a particular solution for

$$y''+p(t)y'+q(t)y=g_1(t)$$

and if $Y_{P2}(t)$ is a particular solution for

$$y''+p(t)y'+q(t)y=g_2(t)$$

then $Y_{P1}(t)+Y_{P2}(t)$ is a particular solution for

$$y''+p(t)y'+q(t)y=g_1(t)+g_2(t)$$

Find a particular solution for the following differential equation.

$$y''-4y'-12y=\mathbf{e}^{6t}$$

$$y_c(t)=c_1\mathbf{e}^{-2t}+c_2\mathbf{e}^{6t}$$

$$Y_P(t)=A\mathbf{e}^{6t}$$

Plugging this into the differential equation gives,

$$36A\mathbf{e}^{6t}-24A\mathbf{e}^{6t}-12A\mathbf{e}^{6t}=\mathbf{e}^{6t}$$

$$0=\mathbf{e}^{6t}$$

$$y'' - 4y' - 12y = 0$$

$$Y_P(t) = Ate^{6t}$$

Plugging this into our differential equation gives,

$$\left(12Ae^{6t} + 36Ate^{6t}\right) - 4\left(Ae^{6t} + 6Ate^{6t}\right) - 12Ate^{6t} = e^{6t}$$

$$(36A - 24A - 12A)te^{6t} + (12A - 4A)e^{6t} = e^{6t}$$

$$8Ae^{6t} = e^{6t}$$

Now, we can set coefficients equal.

$$8A = 1 \qquad \Rightarrow \qquad A = \frac{1}{8}$$

So, the particular solution in this case is,

$$Y_P(t) = \frac{t}{8}e^{6t}$$

Exercises

Solve $y'' - y' - 2y = 0.$

Solve $y'' - 8y' + 16y = 0.$

Solve $y'' - y' - 2y = 4x^2.$

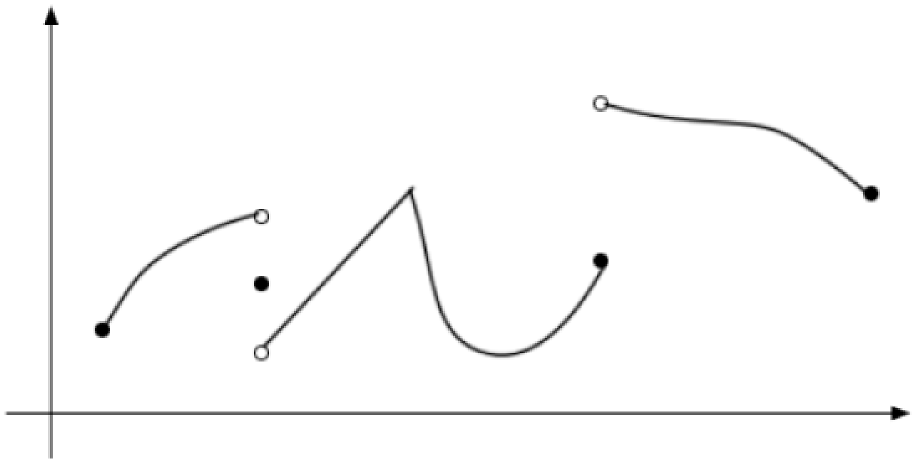
Solve $y'' - y' - 2y = \sin 2x.$

Laplace Transforms

we will be looking at how to use Laplace transforms to solve differential equations.

Before we start with the definition of the Laplace transform we need to get another definition out of the way.

A function is called **piecewise continuous** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (*i.e.* the subinterval without its endpoints) and has a finite limit at the endpoints of each subinterval. Below is a sketch of a piecewise continuous function.



In other words, a piecewise continuous function is a function that has a finite number of breaks in it and doesn't blow up to infinity anywhere.

Now, let's take a look at the definition of the Laplace transform.

Definition

Suppose that *f(t)* is a piecewise continuous function. The Laplace transform of *f(t)* is denoted $\mathcal{L}\{f(t)\}$ and defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty \mathbf{e}^{-st} f(t)\,dt \qquad \textbf{(1)}$$

There is an alternate notation for Laplace transforms. For the sake of convenience we will often denote Laplace transforms as,

$$\mathcal{L}\{f(t)\} = F(s)$$

the integral in the definition of the transform is called an improper integral

Example 1 If $c \neq 0$, evaluate the following integral.

$$\int_0^\infty \mathbf{e}^{ct}\, dt$$

Solution

Remember that you need to convert improper integrals to limits as follows,

$$\int_0^\infty \mathbf{e}^{ct}\, dt = \lim_{n\to\infty} \int_0^n \mathbf{e}^{ct}\, dt$$

Now, do the integral, then evaluate the limit.

$$\begin{aligned}\int_0^\infty \mathbf{e}^{ct}\, dt &= \lim_{n\to\infty} \int_0^n \mathbf{e}^{ct}\, dt \\ &= \lim_{n\to\infty} \left(\frac{1}{c}\mathbf{e}^{ct} \right)\Bigg|_0^n \\ &= \lim_{n\to\infty} \left(\frac{1}{c}\mathbf{e}^{cn} - \frac{1}{c} \right)\end{aligned}$$

Now, at this point, we've got to be careful. The value of *c* will affect our answer. We've already assumed that *c* was non-zero, now we need to worry about the sign of *c*. If *c* is positive the exponential will go to infinity. On the other hand, if *c* is negative the exponential will go to zero.

So, the integral is only convergent (*i.e.* the limit exists and is finite) provided *c<0*. In this case we get,

$$\int_0^\infty \mathbf{e}^{ct}\, dt = -\frac{1}{c} \qquad \text{provided } c < 0 \qquad \textbf{(2)}$$

Example 2 Compute $\mathcal{L}\{1\}$.

Solution

There's not really a whole lot do here other than plug the function *f(t)* = 1 into (1)

$$\mathcal{L}\{1\} = \int_0^\infty \mathbf{e}^{-st}\, dt$$

Now, at this point notice that this is nothing more than the integral in the previous example with $c = -s$. Therefore, all we need to do is reuse (2) with the appropriate substitution. Doing this gives,

$$\mathcal{L}\{1\} = \int_0^\infty \mathbf{e}^{-st}\, dt = -\frac{1}{-s} \qquad \text{provided } -s < 0$$

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \text{provided } s > 0$$

Example 3 Compute $\mathcal{L}\left\{\mathbf{e}^{at}\right\}$

Solution
Plug the function into the definition of the transform and do a little simplification.

$$\mathcal{L}\left\{\mathbf{e}^{at}\right\} = \int_0^\infty \mathbf{e}^{-st}\mathbf{e}^{at}\,dt = \int_0^\infty \mathbf{e}^{(a-s)t}\,dt$$

Once again, notice that we can use (2) provided $c = a - s$. So, let's do this.

$$\begin{aligned}\mathcal{L}\left\{\mathbf{e}^{at}\right\} &= \int_0^\infty \mathbf{e}^{(a-s)t}\,dt \\ &= -\frac{1}{a-s} \qquad \text{provided } a - s < 0 \\ &= \frac{1}{s-a} \qquad \text{provided } s > a\end{aligned}$$

Example 4 Compute $\mathcal{L}\left\{\sin(at)\right\}$.

Solution
Note that we're going to leave it to you to check most of the integration here. Plug the function into the definition. This time let's also use the alternate notation.

$$\begin{aligned}\mathcal{L}\left\{\sin(at)\right\} &= F(s) \\ &= \int_0^\infty \mathbf{e}^{-st}\sin(at)\,dt \\ &= \lim_{n\to\infty}\int_0^n \mathbf{e}^{-st}\sin(at)\,dt\end{aligned}$$

Now, if we integrate by parts we will arrive at,

$$F(s) = \lim_{n\to\infty}\left(-\left(\frac{1}{a}\mathbf{e}^{-st}\cos(at)\right)\Bigg|_0^n - \frac{s}{a}\int_0^n \mathbf{e}^{-st}\cos(at)\,dt\right)$$

$$F(s) = \lim_{n\to\infty}\left(\frac{1}{a}\left(1-\mathbf{e}^{-sn}\cos(an)\right) - \frac{s}{a}\left(\left(\frac{1}{a}\mathbf{e}^{-st}\sin(at)\right)\Bigg|_0^n + \frac{s}{a}\int_0^n \mathbf{e}^{-st}\sin(at)\,dt\right)\right)$$

Now, evaluate the second term, take the limit and simplify.

$$\begin{aligned}F(s) &= \lim_{n\to\infty}\left(\frac{1}{a}\left(1-\mathbf{e}^{-sn}\cos(an)\right) - \frac{s}{a}\left(\frac{1}{a}\mathbf{e}^{-sn}\sin(an) + \frac{s}{a}\int_0^n \mathbf{e}^{-st}\sin(at)\,dt\right)\right) \\ &= \frac{1}{a} - \frac{s}{a}\left(\frac{s}{a}\int_0^\infty \mathbf{e}^{-st}\sin(at)\,dt\right) \\ &= \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty \mathbf{e}^{-st}\sin(at)\,dt\end{aligned}$$

Now, notice that in the limits we had to assume that $s>0$ in order to do the following two limits.

$$\lim_{n\to\infty}\mathbf{e}^{-sn}\cos(an) = 0$$

$$\lim_{n\to\infty}\mathbf{e}^{-sn}\sin(an) = 0$$

$$F(s)=\frac{1}{a}-\frac{s^2}{a^2}F(s)$$

Now, simply solve for *F(s)* to get,

$$\mathcal{L}\{\sin(at)\}=F(s)=\frac{a}{s^2+a^2} \qquad \text{provided } s>0$$

Fact

Given *f(t)* and *g(t)* then,

$$\mathcal{L}\{af(t)+bg(t)\}=aF(s)+bG(s)$$

for any constants *a* and *b*.

Table Of Laplace Transforms

	$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$
2.	e^{at}	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$
4.	$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$
6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$
8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$

10.	$t\cos(at)$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\dfrac{2a^3}{(s^2+a^2)^2}$
12.	$\sin(at)+at\cos(at)$	$\dfrac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\dfrac{s(s^2-a^2)}{(s^2+a^2)^2}$
14.	$\cos(at)+at\sin(at)$	$\dfrac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\dfrac{s\sin(b)+a\cos(b)}{s^2+a^2}$
16.	$\cos(at+b)$	$\dfrac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$
18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$
22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$
24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$

25.	$u_c(t)=u(t-c)$ Heaviside Function	$\dfrac{e^{-cs}}{s}$
26.	$\delta(t-c)$ Dirac Delta Function	e^{-cs}
27.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
28.	$u_c(t)g(t)$	$e^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$e^{ct}f(t)$	$F(s-c)$
30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$
32.	$\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$
36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

Table Notes

Recall the definition of hyperbolic functions.

$$\cosh(t)=\frac{e^t+e^{-t}}{2} \qquad \sinh(t)=\frac{e^t-e^{-t}}{2}$$

Formula #4 uses the Gamma function which is defined as

$$\Gamma(t) = \int_0^\infty \mathbf{e}^{-x} x^{t-1}\, dx$$

If n is a positive integer then,

$$\Gamma(n+1) = n!$$

The Gamma function is an extension of the normal factorial function. Here are a couple of quick facts for the Gamma function

$$\Gamma(p+1) = p\Gamma(p)$$

$$p(p+1)(p+2)\cdots(p+n-1) = \frac{\Gamma(p+n)}{\Gamma(p)}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Example 1 Find the Laplace transforms of the given functions.

(a) $f(t) = 6\mathbf{e}^{-5t} + \mathbf{e}^{3t} + 5t^3 - 9$

(b) $g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

(c) $h(t) = 3\sinh(2t) + 3\sin(2t)$

(d) $g(t) = \mathbf{e}^{3t} + \cos(6t) - \mathbf{e}^{3t}\cos(6t)$

Solution

Okay, there's not really a whole lot to do here other than go to the table, transform the individual functions up, put any constants back in and then add or subtract the results.

We'll do these examples in a little more detail than is typically used since this is the first time we're using the tables.

(a) $f(t) = 6\mathbf{e}^{-5t} + \mathbf{e}^{3t} + 5t^3 - 9$

$$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

(b) $g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

$$G(s) = 4\frac{s}{s^2+(4)^2} - 9\frac{4}{s^2+(4)^2} + 2\frac{s}{s^2+(10)^2}$$
$$= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$

(c) $h(t) = 3\sinh(2t) + 3\sin(2t)$

$$H(s) = 3\frac{2}{s^2-(2)^2} + 3\frac{2}{s^2+(2)^2}$$
$$= \frac{6}{s^2-4} + \frac{6}{s^2+4}$$

(d) $g(t) = \mathbf{e}^{3t} + \cos(6t) - \mathbf{e}^{3t}\cos(6t)$

$$G(s) = \frac{1}{s-3} + \frac{s}{s^2+(6)^2} - \frac{s-3}{(s-3)^2+(6)^2}$$
$$= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$

Example 2 Find the transform of each of the following functions.

(a) $f(t) = t\cosh(3t)$

(b) $h(t) = t^2\sin(2t)$

(c) $g(t) = t^{\frac{3}{2}}$

(d) $f(t) = (10t)^{\frac{3}{2}}$

(e) $f(t) = tg'(t)$

Solution

(a) $f(t) = t\cosh(3t)$

This function is not in the table of Laplace transforms. However, we can use #30 in the table to compute its transform. This will correspond to #30 if we take *n=1*.

$$F(s) = \mathcal{L}\{tg(t)\} = -G'(s), \quad \text{where } g(t) = \cosh(3t)$$

So, we then have,

$$G(s) = \frac{s}{s^2-9} \qquad G'(s) = -\frac{s^2+9}{(s^2-9)^2}$$

Using #30 we then have,

$$F(s) = \frac{s^2+9}{(s^2-9)^2}$$

(b) $h(t) = t^2 \sin(2t)$

This part will also use #30 in the table. In fact, we could use #30 in one of two ways. We could use it with $n = 1$.

$$H(s) = \mathcal{L}\{tf(t)\} = -F'(s), \qquad \text{where } f(t) = t\sin(2t)$$

Or we could use it with $n = 2$.

$$H(s) = \mathcal{L}\{t^2 f(t)\} = F''(s), \qquad \text{where } f(t) = \sin(2t)$$

Since it's less work to do one derivative, let's do it the first way. So, using #9 we have,

$$F(s) = \frac{4s}{\left(s^2+4\right)^2} \qquad F'(s) = -\frac{12s^2-16}{\left(s^2+4\right)^3}$$

The transform is then,

$$H(s) = \frac{12s^2-16}{\left(s^2+4\right)^3}$$

(c) $g(t) = t^{\frac{3}{2}}$

This part can be done using either #6 (with $n = 2$) or #32 (along with #5). We will use #32 so we can see an example of this. In order to use #32 we'll need to notice that

$$\int_0^t \sqrt{v}\, dv = \frac{2}{3} t^{\frac{3}{2}} \qquad \Rightarrow \qquad t^{\frac{3}{2}} = \frac{3}{2}\int_0^t \sqrt{v}\, dv$$

Now, using #5,

$$f(t) = \sqrt{t} \qquad F(s) = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$$

we get the following.

$$G(s) = \frac{3}{2}\left(\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\right)\left(\frac{1}{s}\right) = \frac{3\sqrt{\pi}}{4s^{\frac{5}{2}}}$$

This is what we would have gotten had we used #6.

(d) $f(t) = (10t)^{\frac{3}{2}}$

For this part we will use #24 along with the answer from the previous part. To see this note that if

$$g(t) = t^{\frac{3}{2}}$$

then

$$f(t) = g(10t)$$

Therefore, the transform is.

$$\begin{aligned} F(s) &= \frac{1}{10} G\left(\frac{s}{10}\right) \\ &= \frac{1}{10}\left(\frac{3\sqrt{\pi}}{4\left(\frac{s}{10}\right)^{\frac{5}{2}}}\right) \\ &= 10^{\frac{3}{2}} \frac{3\sqrt{\pi}}{4s^{\frac{5}{2}}} \end{aligned}$$

(e) $f(t) = tg'(t)$

This final part will again use #30 from the table as well as #35.

$$\begin{aligned} \mathcal{L}\{tg'(t)\} &= -\frac{d}{ds}\mathcal{L}\{g'\} \\ &= -\frac{d}{ds}\{sG(s) - g(0)\} \\ &= -(G(s) + sG'(s) - 0) \\ &= -G(s) - sG'(s) \end{aligned}$$

Remember that *g(0)* is just a constant so when we differentiate it we will get zero!

Inverse Laplace Transforms

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Fact

Given the two Laplace transforms *F(s)* and *G(s)* then

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}$$

for any constants *a* and *b*.

Example 1 Find the inverse transform of each of the following.

(a) $F(s)=\frac{6}{s}-\frac{1}{s-8}+\frac{4}{s-3}$

(b) $H(s)=\frac{19}{s+2}-\frac{1}{3s-5}+\frac{7}{s^5}$

(c) $F(s)=\frac{6s}{s^2+25}+\frac{3}{s^2+25}$

(d) $G(s)=\frac{8}{3s^2+12}+\frac{3}{s^2-49}$

Solution

(a) $F(s)=\frac{6}{s}-\frac{1}{s-8}+\frac{4}{s-3}$

$$F(s)=6\frac{1}{s}-\frac{1}{s-8}+4\frac{1}{s-3}$$
$$f(t)=6(1)-\mathbf{e}^{8t}+4\left(\mathbf{e}^{3t}\right)$$
$$=6-\mathbf{e}^{8t}+4\mathbf{e}^{3t}$$

(b) $H(s)=\frac{19}{s+2}-\frac{1}{3s-5}+\frac{7}{s^5}$

$$H(s)=\frac{19}{s-(-2)}-\frac{1}{3\left(s-\frac{5}{3}\right)}+\frac{7\frac{4!}{4!}}{s^{4+1}}$$
$$=19\frac{1}{s-(-2)}-\frac{1}{3}\frac{1}{s-\frac{5}{3}}+\frac{7}{4!}\frac{4!}{s^{4+1}}$$
$$h(t)=19\mathbf{e}^{-2t}-\frac{1}{3}\mathbf{e}^{\frac{5t}{3}}+\frac{7}{24}t^4$$

(d) $G(s)=\frac{8}{3s^2+12}+\frac{3}{s^2-49}$

$$G(s)=\frac{1}{3}\frac{8}{s^2+4}+\frac{3}{s^2-49}$$
$$=\frac{1}{3}\frac{(4)(2)}{s^2+(2)^2}+\frac{3\frac{7}{7}}{s^2-(7)^2}$$

$$g(t)=\frac{4}{3}\sin(2t)+\frac{3}{7}\sinh(7t)$$

Exercises

Example 2 Find the inverse transform of each of the following.

(a) $F(s) = \dfrac{6s-5}{s^2+7}$

(b) $F(s) = \dfrac{1-3s}{s^2+8s+21}$

(c) $G(s) = \dfrac{3s-2}{2s^2-6s-2}$

(d) $H(s) = \dfrac{s+7}{s^2-3s-10}$

Example 3 Find the inverse transform of each of the following.

(a) $G(s) = \dfrac{86s-78}{(s+3)(s-4)(5s-1)}$

(b) $F(s) = \dfrac{2-5s}{(s-6)(s^2+11)}$

(c) $G(s) = \dfrac{25}{s^3(s^2+4s+5)}$

Table

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

Solutions of Linear Differential Equations with Constant

Fact

Suppose that $f, f', f'', \ldots f^{(n-1)}$ are all continuous functions and $f^{(n)}$ is a piecewise continuous function. Then,

$$\mathcal{L}\left\{f^{(n)}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$
$$\mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0)$$

Example 1 Solve the following

$$y'' - 10y' + 9y = 5t, \qquad y(0) = -1 \quad y'(0) = 2$$

$$\mathcal{L}\{y''\} - 10\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{5t\}$$

Using the appropriate formulas from our table of Laplace transforms gives us the following.

$$s^2 Y(s) - sy(0) - y'(0) - 10\left(sY(s) - y(0)\right) + 9Y(s) = \frac{5}{s^2}$$

$$\left(s^2 - 10s + 9\right)Y(s) + s - 12 = \frac{5}{s^2}$$
$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

$$5 + 12s^2 - s^3 = As(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$$

$$s=0 \qquad 5=9B \qquad \Rightarrow \qquad B=\frac{5}{9}$$
$$s=1 \qquad 16=-8D \qquad \Rightarrow \qquad D=-2$$
$$s=9 \qquad 248=648C \qquad \Rightarrow \qquad C=\frac{31}{81}$$
$$s=2 \qquad 45=-14A+\frac{4345}{81} \qquad \Rightarrow \qquad A=\frac{50}{81}$$

$$Y(s)=\frac{\frac{50}{81}}{s}+\frac{\frac{5}{9}}{s^2}+\frac{\frac{31}{81}}{s-9}-\frac{2}{s-1}$$

$$y(t)=\frac{50}{81}+\frac{5}{9}t+\frac{31}{81}e^{9t}-2e^{t}$$

Example 2 Solve the following

$$2y''+3y'-2y=te^{-2t}, \qquad y(0)=0 \quad y'(0)=-2$$

$$2\left(s^2Y(s)-sy(0)-y'(0)\right)+3\left(sY(s)-y(0)\right)-2Y(s)=\frac{1}{(s+2)^2}$$
$$\left(2s^2+3s-2\right)Y(s)+4=\frac{1}{(s+2)^2}$$

$$Y(s)=\frac{1}{(2s-1)(s+2)^3}-\frac{4}{(2s-1)(s+2)}$$

$$Y(s)=\frac{1-4(s+2)^2}{(2s-1)(s+2)^3}$$
$$=\frac{-4s^2-16s-15}{(2s-1)(s+2)^3}$$

$$Y(s)=\frac{A}{2s-1}+\frac{B}{s+2}+\frac{C}{(s+2)^2}+\frac{D}{(s+2)^3}$$

$$\begin{aligned}-4s^2-16s-15 &= A(s+2)^3+B(2s-1)(s+2)^2+C(2s-1)(s+2)+D(2s-1)\\ &=(A+2B)s^3+(6A+7B+2C)s^2+(12A+4B+3C+2D)s\\ &\qquad +8A-4B-2C-D\end{aligned}$$

$$\left.\begin{aligned}A+2B&=0\\ 6A+7B+2C&=-4\\ 12A+4B+3C+2D&=-16\\ 8A-4B-2C-D&=-15\end{aligned}\right\} \Rightarrow \quad \begin{aligned}A&=-\frac{192}{125} & B&=\frac{96}{125}\\ C&=-\frac{2}{25} & D&=-\frac{1}{5}\end{aligned}$$

$$Y(s)=\frac{1}{125}\left(\frac{-192}{2\left(s-\frac{1}{2}\right)}+\frac{96}{s+2}-\frac{10}{(s+2)^2}-\frac{25\frac{2!}{2!}}{(s+2)^3}\right)$$

$$y(t)=\frac{1}{125}\left(-96\mathrm{e}^{\frac{t}{2}}+96\mathrm{e}^{-2t}-10t\mathrm{e}^{-2t}-\frac{25}{2}t^2\mathrm{e}^{-2t}\right)$$

Exercises

Example 3 Solve the following

$$y''-6y'+15y=2\sin(3t), \qquad y(0)=-1 \quad y'(0)=-4$$

Example 4 Solve the following

$$y''+4y'=\cos(t-3)+4t, \qquad y(3)=0 \quad y'(3)=7$$

Power series

A power series about a point x_0 is an expression of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + \cdots$$

Example1

Find a power series expansion for $f'(x)$, with

$$f(x)=(1+x)^{-1}=\sum_{n=0}^{\infty}(-1)^n x^n.$$

Solution.

$$f'(x)=\sum_{n=0}^{\infty}[(-1)^n x^n]'=\sum_{n=1}^{\infty}(-1)^n n\,x^{n-1}=\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)\,x^n.$$

Example2

Find a power series expansion for $g(x)=\int_0^x f(t)\,\mathrm{d}t$ for

$$f(x)=(1+x)^{-1}=\sum_{n=0}^{\infty}(-1)^n x^n.$$

Solution. Compute

$$\int f(x)=\sum_{n=0}^{\infty}(-1)^n\int x^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}+C.$$

The constant C is determined through setting $x=0$:

$$C=g(0)=0.$$

Therefore

$$g(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1},$$

or if preferred,

$$g(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n.$$

Example3

Find at least the first four nonzero terms in a power series expansion about $x=0$ for a general solution to

$$z'' - x^2 z = 0.$$

Solution. We write

$$z(x) = a_0 + a_1 x + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

Substituting into the equation, we have

$$\begin{aligned}
0 &= z'' - x^2 z \\
&= \sum_{n=2}^{\infty} a_n n (n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} \\
&= \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n - \sum_{n=2}^{\infty} a_{n-2} x^n \\
&= 2 a_2 + 6 a_3 x + \sum_{n=2}^{\infty} [a_{n+2}(n+2)(n+1) - a_{n-2}] x^n.
\end{aligned}$$

Thus we have

$$\begin{aligned}
2 a_2 &= 0 \\
6 a_3 &= 0 \\
a_{n+2}(n+2)(n+1) &= a_{n-2} \Longrightarrow a_{n+2} = \frac{a_{n-2}}{(n+2)(n+1)}.
\end{aligned}$$

We conclude:

$$\begin{aligned}
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= \frac{a_0}{12} \\
a_5 &= \frac{a_1}{20}
\end{aligned}$$

As we only need 4 nonzero terms, we stop here. The solution is

$$z(x) = a_0 + a_1 x + \frac{a_0}{12} x^4 + \frac{a_1}{20} x^5 + \cdots.$$

Example 22. Find a power series expansion about $x = 0$ for a general solution to the given differential equation. Your answer should include a general formula for the coefficients.

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$$y'' - x\,y' + 4\,y = 0.$$

Solution. We write

$$y(x) = \sum_{n=0}^{\infty} a_n\,x^n.$$

Substituting into the equation, we have

$$\begin{aligned} 0 &= y'' - x\,y' + 4\,y \\ &= \sum_{n=2}^{\infty} a_n\,n\,(n-1)\,x^{n-2} - x\sum_{n=1}^{\infty} n\,a_n\,x^{n-1} + \sum_{n=0}^{\infty} 4\,a_n\,x^n \\ &= \sum_{n=0}^{\infty} a_{n+2}\,(n+2)\,(n+1)\,x^n - \sum_{n=1}^{\infty} n\,a_n\,x^n + \sum_{n=0}^{\infty} 4\,a_n\,x^n \\ &= (2\,a_2 + 4\,a_0) + \sum_{n=1}^{\infty} [(n+2)\,(n+1)\,a_{n+2} - (n-4)\,a_n]\,x^n. \end{aligned}$$

This gives

$$\begin{aligned} 2\,a_2 + 4\,a_0 &= 0 \\ (n+2)\,(n+1)\,a_{n+2} - (n-4)\,a_n &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} a_2 &= -2\,a_0, \\ a_{n+2} &= \frac{n-4}{(n+2)\,(n+1)}\,a_n. \end{aligned}$$

It is clear that we should discuss $n = 2\,k$ and $n = 2\,k - 1$ separately.

For even n, we have

$$a_{2k+1} = \frac{2\,k-5}{(2\,k+1)\,(2\,k)}\,a_{2k-1} = \frac{(2\,k-5)\,(2\,k-7)}{(2\,k+1)\cdots(2\,k-2)}\,a_{2k-3} = \cdots = \frac{(2\,k-5)\cdots(-3)}{(2\,k+1)!}\,a_1.$$

Summarizing, we have

$$y(x) = a_0\left[1 - 2\,x^2 + \frac{1}{3}\,x^4\right] + a_1\left[x + \sum_{k=1}^{\infty} \frac{(2\,k-5)\cdots(-3)}{(2\,k+1)!}\,x^{2k+1}\right].$$

Example 23. Find at least the first four nonzero terms in a power series expansion about $x = 0$ for the solution to the given initial value problem.

$$w'' + 3\,x\,w' - w = 0, \qquad w(0) = 2, \quad w'(0) = 0.$$

Solution. We write

$$w(x) = \sum_{n=0}^{\infty} a_n\,x^n.$$

Substituting into the equation, we obtain

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} a_n\,n\,(n-1)\,x^{n-2} + 3\,x \sum_{n=1}^{\infty} n\,a_n\,x^{n-1} - \sum_{n=0}^{\infty} a_n\,x^n \\ &= \sum_{n=0}^{\infty} a_{n+2}\,(n+2)\,(n+1)\,x^n + \sum_{n=1}^{\infty} 3\,n\,a_n\,x^n - \sum_{n=0}^{\infty} a_n\,x^n \\ &= 2\,a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)\,(n+1)\,a_{n+2} + 3\,n\,a_n - a_n\right] x^n. \end{aligned}$$

Therefore

$$\begin{aligned} 2\,a_2 - a_0 &= 0 \\ (n+2)\,(n+1)\,a_{n+2} + (3\,n - 1)\,a_n &= 0 \end{aligned}$$

which leads to

$$\begin{aligned} a_2 &= \frac{1}{2}\,a_0 \\ a_{n+2} &= \frac{1 - 3\,n}{(n+2)\,(n+1)}\,a_n. \end{aligned}$$

On the other hand, the initial values give

$$2 = w(0) = a_0, \quad 0 = w'(0) = a_1.$$

Therefore we can compute successively

$$\begin{aligned} a_2 &= \frac{1}{2}a_0 = 1, \\ a_3 &= \frac{-2}{6}a_1 = 0, \\ a_4 &= \frac{-5}{12}a_2 = -\frac{5}{12}, \\ a_5 &= \frac{-8}{20}a_3 = 0, \\ a_6 &= \frac{-11}{30}a_4 = \frac{11}{72}. \end{aligned}$$

We stop here as only four nonzero terms are required. Finally the answer is

$$w(x) = 2 + x^2 - \frac{5}{12}x^4 + \frac{11}{72}x^6 + \cdots.$$

Example 25. Consider

$$x^2 y'' + 3y' - xy = 0.$$

Solution. Write

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Substituting into the equation, we obtain

$$\begin{aligned} 0 &= x^2 \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + 3\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=2}^{\infty} a_n n(n-1) x^n + \sum_{n=0}^{\infty} 3(n+1) a_{n+1} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n \\ &= 3a_1 + (6a_2 - a_0)x + \sum_{n=2}^{\infty} [a_n n(n-1) + 3(n+1)a_{n+1} - a_{n-1}] x^n. \end{aligned}$$

This leads to

$$\begin{aligned} 3\,a_1 &= 0 \\ 6\,a_2 - a_0 &= 0 \\ 3\,(n+1)\,a_{n+1} + n\,(n-1)\,a_n - a_{n-1} &= 0, \qquad n \geqslant 2 \end{aligned}$$

which leads to

$$a_1 = 0, \quad a_2 = \frac{a_0}{6}, \quad a_{n+1} = \frac{a_{n-1} - n\,(n-1)\,a_n}{3\,(n+1)}.$$

Fourier Series

☆Fourier sine series

the Fourier sine series of an odd function $f(x)$ on $-L \le x \le L$ is given

Example 1 Find the Fourier sine series for $f(x) = x$ on $-L \le x \le L$.

Solution

First note that the function we're working with is in fact an odd function and so this is something we can do. There really isn't much to do here other than to compute the coefficients for $f(x) = x$.

Here is that work and note that we're going to leave the integration by parts details to you to verify. Don't forget that *n*, *L*, and π are constants!

$$B_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{L}{n^2\pi^2}\right)\left(L\sin\left(\frac{n\pi x}{L}\right) - n\pi x\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L$$

$$= \frac{2}{n^2\pi^2}\left(L\sin(n\pi) - n\pi L\cos(n\pi)\right)$$

These integrals can, on occasion, be somewhat messy especially when we use a general *L* for the endpoints of the interval instead of a specific number.

Now, taking advantage of the fact that *n* is an integer we know that $\sin(n\pi) = 0$ and that $\cos(n\pi) = (-1)^n$. We therefore have,

$$B_n = \frac{2}{n^2\pi^2}\left(-n\pi L(-1)^n\right) = \frac{(-1)^{n+1}\,2L}{n\pi} \qquad n = 1, 2, 3\ldots$$

The Fourier sine series is then,

$$x = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)$$

☆

Fourier cosine series.

$$f(x)=\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

this function on $-L \le x \le L$

the Fourier cosine series of an even function, $f(x)$ on $-L \le x \le L$ is

given by,

$$f(x)=\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \qquad A_n=\begin{cases}\dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx & n=0\\[2ex] \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n\neq 0\end{cases}$$

Finally, before we work an example, let's notice that because both $f(x)$ and the cosines are even the integrand in both of the integrals above is even and so we can write the formulas for the A_n's as follows,

$$A_n=\begin{cases}\dfrac{1}{L}\displaystyle\int_{0}^{L} f(x)\,dx & n=0\\[2ex] \dfrac{2}{L}\displaystyle\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n\neq 0\end{cases}$$

Now let's take a look at an example.

Example 1 Find the Fourier cosine series for $f(x) = x^2$ on $-L \le x \le L$.

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L x^2\,dx = \frac{1}{L}\left(\frac{L^3}{3}\right) = \frac{L^2}{3}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left(\frac{L}{n^3\pi^3}\right)\left(2Ln\pi x\cos\left(\frac{n\pi x}{L}\right) + \left(n^2\pi^2x^2 - 2L^2\right)\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L$$

$$= \frac{2}{n^3\pi^3}\left(2L^2n\pi\cos(n\pi) + \left(n^2\pi^2L^2 - 2L^2\right)\sin(n\pi)\right)$$

$$= \frac{4L^2(-1)^n}{n^2\pi^2} \qquad n = 1, 2, 3, \ldots$$

The coefficients are then,

$$A_0 = \frac{L^2}{3} \qquad A_n = \frac{4L^2(-1)^n}{n^2\pi^2}, \quad n = 1, 2, 3, \ldots$$

The Fourier cosine series is then,

$$x^2 = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) = \frac{L^2}{3} + \sum_{n=1}^{\infty}\frac{4L^2(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)$$

Note that we'll often strip out the $n = 0$ from the series as we've done here because it will almost always be different from the other coefficients and it allows us to actually plug the coefficients into the series.

In this case, before we actually proceed with this we'll need to define the even extension of a function, $f(x)$ on $-L \le x \le L$. So, given a function $f(x)$ we'll define the even extension of the function as,

$$g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le L \\ f(-x) & \text{if } -L \le x \le 0 \end{cases}$$

Showing that this is an even function is simple enough.

$$g(-x) = f(-(-x)) = f(x) = g(x) \qquad \text{for } 0 < x < L$$

Example 3 Find the Fourier cosine series for $f(x) = L - x$ on $0 \le x \le L$.

Solution
All we need to do is compute the coefficients so here is the work for that,

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L L - x\,dx = \frac{L}{2}$$

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L (L-x)\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\left(\frac{L}{n^2\pi^2}\right)\left(n\pi(L-x)\sin\left(\frac{n\pi x}{L}\right) - L\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L \\ &= \frac{2}{L}\left(\frac{L}{n^2\pi^2}\right)\left(-L\cos(n\pi) + L\right) = \frac{2L}{n^2\pi^2}\left(1+(-1)^{n+1}\right) \qquad n = 1,2,3,\ldots \end{aligned}$$

The Fourier cosine series is then,

$$f(x) = \frac{L}{2} + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}\left(1+(-1)^{n+1}\right)\cos\left(\frac{n\pi x}{L}\right)$$

Note that as we did with the first example in this section we stripped out the A_0 term before we plugged in the coefficients.

Example 4 Find the Fourier cosine series for $f(x) = x^3$ on $0 \le x \le L$.

Solution
The integral for A_0 is simple enough but the integral for the rest will be fairly messy as it will require three integration by parts. We'll leave most of the details of the actual integration to you to verify. Here's the work,

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L x^3\,dx = \frac{L^3}{4}$$

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x^3\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\left(\frac{L}{n^4\pi^4}\right)\left(n\pi x\left(n^2\pi^2x^2 - 6L^2\right)\sin\left(\frac{n\pi x}{L}\right) + \left(3Ln^2\pi^2x^2 - 6L^3\right)\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L \\ &= \frac{2}{L}\left(\frac{L}{n^4\pi^4}\right)\left(n\pi L\left(n^2\pi^2L^2 - 6L^2\right)\sin(n\pi) + \left(3L^3n^2\pi^2 - 6L^3\right)\cos(n\pi) + 6L^3\right) \\ &= \frac{2}{L}\left(\frac{3L^4}{n^4\pi^4}\right)\left(2+\left(n^2\pi^2-2\right)(-1)^n\right) = \frac{6L^3}{n^4\pi^4}\left(2+\left(n^2\pi^2-2\right)(-1)^n\right) \qquad n = 1,2,3,\ldots \end{aligned}$$

The Fourier cosine series for this function is then,

$$f(x) = \frac{L^3}{4} + \sum_{n=1}^{\infty}\frac{6L^3}{n^4\pi^4}\left(2+\left(n^2\pi^2-2\right)(-1)^n\right)\cos\left(\frac{n\pi x}{L}\right)$$

Fourier Series

Okay, in the previous two sections we've looked at Fourier sine and Fourier cosine series. It is now time to look at a Fourier series. With a Fourier series we are going to try to write a series representation for $f(x)$ on $-L \le x \le L$ in the form,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

following formulas that we derived when we proved the two sets were mutually orthogonal.

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \ne 0 \\ 0 & \text{if } n \ne m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \ne m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0$$

So, let's start off by multiplying both sides of the series above by $\cos\left(\frac{m\pi x}{L}\right)$ and integrating from –*L* to *L*. Doing this gives,

$$\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx = \int_{-L}^{L} \sum_{n=0}^{\infty} A_n \cos\left(\tfrac{n\pi x}{L}\right)\cos\left(\tfrac{m\pi x}{L}\right)dx + \int_{-L}^{L} \sum_{n=1}^{\infty} B_n \sin\left(\tfrac{n\pi x}{L}\right)\cos\left(\tfrac{m\pi x}{L}\right)dx$$

$$\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty} A_n \int_{-L}^{L} \cos\left(\tfrac{n\pi x}{L}\right)\cos\left(\tfrac{m\pi x}{L}\right)dx + \sum_{n=1}^{\infty} B_n \int_{-L}^{L} \sin\left(\tfrac{n\pi x}{L}\right)\cos\left(\tfrac{m\pi x}{L}\right)dx$$

$$\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx = \begin{cases} A_m(2L) & \text{if } n = m = 0 \\ A_m(L) & \text{if } n = m \ne 0 \end{cases}$$

Solving for A_m gives,

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$A_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx \qquad m = 1, 2, 3, \ldots$$

Solving for A_m gives,

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$A_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx \qquad m = 1, 2, 3, \ldots$$

Now, do it all over again only this time multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$, integrate both sides from –*L* to *L* and interchange the integral and summation to get,

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty} A_n \int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx + \sum_{n=1}^{\infty} B_n \int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

In this case the integral in the first series will always be zero and the second will be zero if $n \neq m$ and so we get,

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = B_m(L)$$

Finally, solving for B_m gives,

$$B_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad m = 1, 2, 3, \ldots$$

here in this section, as we've done in the previous two sections as well. Provided *n* in an integer then,

$$\cos(n\pi) = (-1)^n \qquad \sin(n\pi) = 0$$

Also, don't forget that sine is an odd function, *i.e.* $\sin(-x) = -\sin(x)$ and that cosine is an even function, *i.e.* $\cos(-x) = \cos(x)$. We'll also be making heavy use of these ideas without comment in many of the integral evaluations so be ready for these as well.

Now let's take a look at an example.

Example 1 Find the Fourier series for $f(x) = L - x$ on $-L \le x \le L$.

Solution

So, let's go ahead and just run through formulas for the coefficients.

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} L - x\,dx = L$$

$$\begin{aligned}
A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L}(L-x)\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{L}\left(\frac{L}{n^2\pi^2}\right)\left(n\pi(L-x)\sin\left(\frac{n\pi x}{L}\right) - L\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{L} \\
&= \frac{1}{L}\left(\frac{L}{n^2\pi^2}\right)\left(-2n\pi L\sin(-n\pi)\right) = 0 \qquad\qquad n = 1, 2, 3, \ldots
\end{aligned}$$

$$\begin{aligned}
B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{1}{L}\left(-\frac{L}{n^2\pi^2}\right)\left[L\sin\left(\frac{n\pi x}{L}\right) - n\pi(x-L)\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{-L}^{L} \\
&= \frac{1}{L}\left[\frac{L^2}{n^2\pi^2}\left(2n\pi\cos(n\pi) - 2\sin(n\pi)\right)\right] = \frac{2L(-1)^n}{n\pi} \qquad\qquad n = 1, 2, 3, \ldots
\end{aligned}$$

Note that in this case we had $A_0 \neq 0$ and $A_n = 0,\ n = 1, 2, 3, \ldots$ This will happen on occasion so don't get excited about this kind of thing when it happens.

The Fourier series is then,

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) \\
&= A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) = L + \sum_{n=1}^{\infty}\frac{2L(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{L}\right)
\end{aligned}$$

Example 2 Find the Fourier series for $f(x)=\begin{cases} L & \text{if } -L\le x\le 0 \\ 2x & \text{if } 0\le x\le L \end{cases}$ on $-L\le x\le L$.

Solution

Because of the piece-wise nature of the function the work for the coefficients is going to be a little unpleasant but let's get on with it.

$$A_0=\frac{1}{2L}\int_{-L}^{L}f(x)\,dx=\frac{1}{2L}\left[\int_{-L}^{0}f(x)\,dx+\int_{0}^{L}f(x)\,dx\right]$$
$$=\frac{1}{2L}\left[\int_{-L}^{0}L\,dx+\int_{0}^{L}2x\,dx\right]=\frac{1}{2L}\left[L^2+L^2\right]=L$$

$$A_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx=\frac{1}{L}\left[\int_{-L}^{0}f(x)\cos\left(\frac{n\pi x}{L}\right)dx+\int_{0}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right]$$
$$=\frac{1}{L}\left[\int_{-L}^{0}L\cos\left(\frac{n\pi x}{L}\right)dx+\int_{0}^{L}2x\cos\left(\frac{n\pi x}{L}\right)dx\right]$$

At this point it will probably be easier to do each of these individually.

$$\int_{-L}^{0}L\cos\left(\frac{n\pi x}{L}\right)dx=\left(\frac{L^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{0}=\frac{L^2}{n\pi}\sin(n\pi)=0$$

$$\int_{0}^{L}2x\cos\left(\frac{n\pi x}{L}\right)dx=\left(\frac{2L}{n^2\pi^2}\right)\left(L\cos\left(\frac{n\pi x}{L}\right)+n\pi x\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{0}^{L}$$

$$=\left(\frac{2L}{n^2\pi^2}\right)\left(L\cos(n\pi)+n\pi L\sin(n\pi)-L\cos(0)\right)$$
$$=\left(\frac{2L^2}{n^2\pi^2}\right)\left((-1)^n-1\right)$$

So, if we put all of this together we have,

$$A_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx=\frac{1}{L}\left[0+\left(\frac{2L^2}{n^2\pi^2}\right)\left((-1)^n-1\right)\right]$$
$$=\frac{2L}{n^2\pi^2}\left((-1)^n-1\right)\ ,\qquad n=1,2,3,\ldots$$

So, we've gotten the coefficients for the cosines taken care of and now we need to take care of the coefficients for the sines.

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[\int_{-L}^{0} f(x)\sin\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{1}{L}\left[\int_{-L}^{0} L\sin\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} 2x\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

As with the coefficients for the cosines will probably be easier to do each of these individually.

$$\int_{-L}^{0} L\sin\left(\frac{n\pi x}{L}\right)dx = \left(-\frac{L^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{0} = \frac{L^2}{n\pi}\left(-1+\cos(n\pi)\right) = \frac{L^2}{n\pi}\left((-1)^n - 1\right)$$

$$\int_{0}^{L} 2x\sin\left(\frac{n\pi x}{L}\right)dx = \left(\frac{2L}{n^2\pi^2}\right)\left(L\sin\left(\frac{n\pi x}{L}\right) - n\pi x\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{0}^{L}$$

$$= \left(\frac{2L}{n^2\pi^2}\right)\left(L\sin(n\pi) - n\pi L\cos(n\pi)\right)$$

$$= \left(\frac{2L^2}{n^2\pi^2}\right)\left(-n\pi(-1)^n\right) = -\frac{2L^2}{n\pi}(-1)^n$$

So, if we put all of this together we have,

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[\frac{L^2}{n\pi}\left((-1)^n - 1\right) - \frac{2L^2}{n\pi}(-1)^n\right]$$

$$= \frac{L}{n\pi}\left[-1-(-1)^n\right] = -\frac{L}{n\pi}\left(1+(-1)^n\right) \qquad n = 1, 2, 3, \ldots$$

So, after all that work the Fourier series is,

$$f(x) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right)$$

$$= A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right)$$

$$= L + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{L}\right) - \sum_{n=1}^{\infty}\frac{L}{n\pi}\left(1+(-1)^n\right)\sin\left(\frac{n\pi x}{L}\right)$$

Example 3 Find the Fourier series for $f(x) = x$ on $-L \le x \le L$.

Solution

Let's start with the integrals for A_n.

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} x\,dx = 0$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

In both cases note that we are integrating an odd function (*x* is odd and cosine is even so the product is odd) over the interval $[-L, L]$ and so we know that both of these integrals will be zero.

Next here is the integral for B_n

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$B_n = \frac{(-1)^{n+1}\,2L}{n\pi} \qquad\qquad n = 1, 2, 3, \ldots$$

In this case the Fourier series is,

$$f(x) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

Example 4 Find the Fourier series for $f(x) = x^2$ on $-L \le x \le L$.

Solution

Here are the integrals for the A_n and in this case because both the function and cosine are even we'll be integrating an even function and so can "simplify" the integral.

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} x^2\,dx = \frac{1}{L}\int_{0}^{L} x^2\,dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)dx$$

As with the previous example both of these integrals were done in Example 1 in the Fourier cosine series section and so we'll not bother redoing them here. The coefficients are,

$$A_0 = \frac{L^2}{3} \qquad\qquad A_n = \frac{4L^2(-1)^n}{n^2\pi^2}, \quad n = 1, 2, 3, \ldots$$

Next here is the integral for the B_n

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} x^2\sin\left(\frac{n\pi x}{L}\right)dx = 0$$

In this case the function is even and sine is odd so the product is odd and we're integrating over $-L \le x \le L$ and so the integral is zero.

The Fourier series is then,

$$f(x) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) = \frac{L^2}{3} + \sum_{n=1}^{\infty}\frac{4L^2(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)$$