Lec.1/ Matrices, Inverse matrices by elementary row

- **Matrices**
- When a system of equations has more than two equations, it is more convenient to use matrices and vectors in solution.
- The size of the matrix is described by the number of its rows and columns. A matrix of n rows and m columns is represented by (n x m) matrix.
- $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n\,x\,m} \quad , \quad i = 1, 2, \dots, n\,; \quad j = 1, 2, \dots, m$

- Types of matrices:
- **Square matrix**: it is a matrix that includes number of rows equals to number of columns (n=m).
- $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}_{2x2}, B = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 2 & 1 \\ 1 & 8 & 0 \end{bmatrix}_{3x3}$
- Diagonal matrix: it is a square matrix which all of its elements are zeros except the elements on the main diagonal.
- $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

- **Identity matrix:** it is a diagonal matrix but the elements on the main diagonal are equal to 1 and it is denoted by $I_{n.}$
- $I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- **Transpose matrix: Transpose of A is denoted by A^T which means writing the rows of A as columns in A^T.**
- A= $\begin{bmatrix} 9 & 7 & 5 \\ 1 & 2 & 4 \end{bmatrix}_{2x3}$, $A^T = \begin{bmatrix} 9 & 1 \\ 7 & 2 \\ 5 & 4 \end{bmatrix}_{3x2}$

- Matrix addition: if $A = [a_{ij}]$, $B = [b_{ij}]$ and both A&B are m x n matrices, then
- $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$
- Ex:
- $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$

Note: for any scalar (number) c , it can multiply the matrix A by c as follows:

cA = c$[a_{ij}] = [ca_{ij}]$

Ex:

$3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix}$

- Note:
- The matrix with only one column, m x 1 in size is called a column vector, while with only one row, 1 x n in size is called a row vector.
- Matrix multiplication: let A be m x k matrix and B be k x n matrix then C=AB is an m x n matrix, where
- $C_{ij} = \sum_{t=1}^{k} a_{it}\, b_{tj}$
- i = 1, 2, …., m , j = 1, 2, …., n

- Ex:
- A= $\begin{bmatrix} 3 & 7 & 1 \\ -2 & 1 & -3 \end{bmatrix}_{2x3}$, $B = \begin{bmatrix} 5 & -2 \\ 0 & 3 \\ 1 & -1 \end{bmatrix}_{3x2}$
- $AB = \begin{bmatrix} 16 & 14 \\ -13 & 10 \end{bmatrix}_{2x2}$
- $BA = \begin{bmatrix} 19 & 33 & 11 \\ -6 & 3 & -9 \\ 5 & 6 & 4 \end{bmatrix}_{3x3}$

Solution of system of linear equations using

Gauss Jordan elimination method

- In the Gaussian elimination method, we write simpler equivalent augmented matrices, where each row of an augmented matrix represents an equation that can perform the row operations on the augmented matrix.
- Steps:
- 1) Construct the augmented matrix (A:B).
- 2) Applying row operations including (adding or subtracting two rows, interchange two rows, multiplying any row by any constant except zero.

- Let A be a matrix, X a column vector, B a column vector then the system of linear equations is denoted by $\boldsymbol{AX = B}$
- The solution to a system of linear equations starts by the augmented matrix as shown for the following system:
- $x - 2y = -5$
- $3x + y = 6$

- ***Note: Number of variables equals to the number of equations***

- Depends on the coefficients of x, y and the constants on the right-hand side of the equation. The matrix of coefficients for this system is 2 x 2 matrix

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

- If we insert the constants from the right-hand side of the system into matrix of coefficients, we get the 2x3 matrix.

$$\left[\begin{array}{cc|c} 1 & -2 & -5 \\ 3 & 1 & 6 \end{array}\right]$$

- We use a vertical line between the coefficients and the constants to represent the equal signs. This matrix is augmented matrix of the system also it can be written as:

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

- **Application of Gauss Jordan elimination method to solve the system (AX = B)**
- Ex1: use Gaussian elimination method to solve the following system of equations
- $x - 3y = 11$
- $2x + y = 1$
- Sol: (note: no. of variables = no. of equations =2)
- The augmented matrix:
- $\begin{bmatrix} 1 & -3 & | & 11 \\ 2 & 1 & | & 1 \end{bmatrix}$

Now we want to get this matrix $\begin{bmatrix} 1 & 0 & | & x \\ 0 & 1 & | & y \end{bmatrix}$ by applying row operations as follows:

- $\begin{bmatrix} 1 & -3 & | & 11 \\ 2 & 1 & | & 1 \end{bmatrix} \rightarrow R_2' = -2R_1 + R_2 \rightarrow \begin{bmatrix} 1 & -3 & | & 11 \\ 0 & 7 & | & -21 \end{bmatrix} \rightarrow R_2' = \frac{1}{7}R_2$

- $\begin{bmatrix} 1 & -3 & | & 11 \\ 0 & 1 & | & -3 \end{bmatrix} \rightarrow R_1' = 3R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -3 \end{bmatrix}$
- The solution to the system is (x = 2 and y = -3)
- To check the result, substitute the values of x & y in any equation, such as in ($x - 3y = 11$)

$2 - 3(-3) = 11$

$2 + 9 = 11$

$11 = 11 \rightarrow$ L.H.S = R.H.S

- Ex2: use Gaussian elimination method to solve the following system of equations
- $2x - y + z = -3$
- $x + y - z = 6$
- $3x - y - z = 4$

- $\begin{bmatrix} 2 & -1 & 1 & | & -3 \\ 1 & 1 & -1 & | & 6 \\ 3 & -1 & -1 & | & 4 \end{bmatrix} \rightarrow$ we want to get this matrix $\begin{bmatrix} 1 & 0 & 0 & | & x \\ 0 & 1 & 0 & | & y \\ 0 & 0 & 1 & | & z \end{bmatrix}$

- $R_1 \leftrightarrow R_2 \rightarrow \begin{bmatrix} 1 & 1 & -1 & | & 6 \\ 2 & -1 & 1 & | & -3 \\ 3 & -1 & -1 & | & 4 \end{bmatrix}$,
- $R_3' = -3R_1 + R_3$ & $R_2' = -2R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 1 & -1 & | & 6 \\ 0 & -3 & 3 & | & -15 \\ 0 & -4 & 2 & | & -14 \end{bmatrix}$

- $R_2' = -\frac{1}{3}R_2 \rightarrow \begin{bmatrix} 1 & 1 & -1 & | & 6 \\ 0 & 1 & -1 & | & 5 \\ 0 & -4 & 2 & & -14 \end{bmatrix}$
- $R_1' = -R_2 + R_1$ & $R_3' = 4R_2 + R_3 \rightarrow \begin{bmatrix} 1 & 1 & 0 & | & 1 \\ 0 & 1 & -1 & | & 5 \\ 0 & 0 & -2 & & 6 \end{bmatrix}$
- $R_3' = -\frac{1}{2}R_3 \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & -1 & | & 5 \\ 0 & 0 & 1 & & -3 \end{bmatrix}$
- $R_2' = R_3 + R_2 \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & & -3 \end{bmatrix}$
- The solution to the system is (x = 1, y = 2 and z = -3)

- To check the result, substitute the values of x=1 , y=2 & z=-3 in any equation, such as in equation ($2x - y + z = -3$)

$$2(1) - 1(2) + (-3) = -3$$
$$2 - 2 - 3 = -3$$
$-3 = -3 \rightarrow$ L.H.S = R.H.S

- **Homework 2:** Solve the following equations by Gauss-Jordan Elimination Method, and <u>check the results.</u>

1) $2x - y = 18$
   $3x + y = 2$

2) $3x - 2y + 8z = 9$
   $-2x + 2y + z = 3$
   $x + 2y - 3z = 8$

Lec.1/ Matrices, Inverse matrices by elementary row

- Determinant
- Determinant is a value that can be calculated from the elements of a ***square matrix***. The determinant of a matrix A is denoted det(A), or the symbol for determinant is two vertical lines either side, $|A|$ means the determinant of A.
- It used to find the inverse of a matrix and useful in calculus for several applications.
- It used to check whether or not a matrix can be inverted, where if det (A)=0 then there is no inverse.

- The calculation of determinant is as follows:

For 2x2 matrix,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Ex1: find the determinant of A

A=$\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ = $\begin{vmatrix} 1 & 3 \\ -2 & 5 \end{vmatrix}$= 1(5)-3(-2)= 5+6=11

For 3x3 matrix,

To each element of a 3x3 matrix there corresponds a 2x2 matrix that is obtained by deleting the row and column of that element. The determinate of the 2x2 matrix is called the minor of that element.

- $det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- Ex2: find the determinant of A

- $\begin{vmatrix} 3 & 8 & 1 \\ 6 & 2 & -1 \\ -1 & -4 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & -1 \\ -4 & 1 \end{vmatrix} - 8\begin{vmatrix} 6 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ -1 & -4 \end{vmatrix}$

- = 3(2-4) -8(6-1) +(-24+2) = -68

- Notice the **+ - +** pattern for the numbers of the first row.

- Finding the inverse of matrices by elementary row method: Also called the Gauss Jordan elimination method.
- Construct the augmented matrix $(\boldsymbol{A} : \boldsymbol{I})$
- Using row operations: Change the rows using (1) adding or subtracting the row by another row, 2) multiplying the row by a constant and 3) swapping rows) until convert matrix **A** into the Identity Matrix **I**, $(\boldsymbol{I} : \boldsymbol{A}^{-1})$

Note:

1) *Augmented matrices appear in Linear algebra as two appended matrices and are useful for solving systems of linear equations.*
2) *It can check the result through multiplying the original matrix by the inverse matrix to get the identity matrix* $(\boldsymbol{A}\,\boldsymbol{A}^{-1} = \boldsymbol{I})$

- Ex1: Find A^{-1} using elementary row method (Gaussian elimination)
- $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \rightarrow \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \rightarrow R_1 = \frac{1}{2}R_1 \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$
- $R_2' = R_2 - R_1 \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{7}{2} & \frac{-1}{2} & 1 \end{array}\right]$, $R_2' = \frac{2}{7}R_2 \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{-1}{7} & \frac{2}{7} \end{array}\right]$
- $R_1' = R_1 - \frac{1}{2}R_2 \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{7} & \frac{-1}{7} \\ 0 & 1 & \frac{-1}{7} & \frac{2}{7} \end{array}\right]$, $\rightarrow A^{-1} = \begin{bmatrix} \frac{4}{7} & \frac{-1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix}$
- $\boldsymbol{A}\,\boldsymbol{A}^{-1} = \boldsymbol{I} \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \frac{4}{7} & \frac{-1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- Ex2: find A^{-1}
- $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 0 & 2 \end{bmatrix}$
- $\begin{bmatrix} 2 & -1 & 3 & | & 1 & 0 & 0 \\ 1 & 0 & -2 & | & 0 & 1 & 0 \\ 4 & 0 & 2 & | & 0 & 0 & 1 \end{bmatrix} \rightarrow R_1 = \frac{1}{2}R_1 \rightarrow \begin{bmatrix} 1 & \frac{-1}{2} & \frac{3}{2} & | & \frac{1}{2} & 0 & 0 \\ 1 & 0 & -2 & | & 0 & 1 & 0 \\ 4 & 0 & 2 & | & 0 & 0 & 1 \end{bmatrix}$
- $\rightarrow R_2 = R_2 - R_1 \rightarrow \begin{bmatrix} 1 & \frac{-1}{2} & \frac{3}{2} & | & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{-7}{2} & | & \frac{-1}{2} & 1 & 0 \\ 4 & 0 & 2 & | & 0 & 0 & 1 \end{bmatrix}$

- $\rightarrow R_3 = R_3 - 4R_1 \begin{bmatrix} 1 & \frac{-1}{2} & \frac{3}{2} & | & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{-7}{2} & | & \frac{-1}{2} & 1 & 0 \\ 0 & 2 & -4 & | & -2 & 0 & 1 \end{bmatrix}$
- $\rightarrow R_2 = 2R_2 \begin{bmatrix} 1 & \frac{-1}{2} & \frac{3}{2} & | & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -7 & | & -1 & 2 & 0 \\ 0 & 2 & -4 & | & -2 & 0 & 1 \end{bmatrix}$
- $\rightarrow R_1 = \frac{1}{2}R_2 + R_1 ,\ R_3 = -2R_2 + R_3 \begin{bmatrix} 1 & 0 & \frac{-4}{2} & | & 0 & 1 & 0 \\ 0 & 1 & -7 & | & -1 & 2 & 0 \\ 0 & 0 & 10 & | & 0 & -4 & 1 \end{bmatrix}$
- $\rightarrow R_3 = \frac{1}{10}R_3 \begin{bmatrix} 1 & 0 & -2 & | & 0 & 1 & 0 \\ 0 & 1 & -7 & | & -1 & 2 & 0 \\ 0 & 0 & 1 & | & 0 & \frac{-2}{5} & \frac{1}{10} \end{bmatrix}$

- $\rightarrow R_1 = 2R_3 + R_1, \quad R_2 = R_2 + 7R_3 \begin{bmatrix} 1 & 0 & 0 & | & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & 0 & | & -1 & \frac{-4}{5} & \frac{7}{10} \\ 0 & 0 & 1 & | & 0 & \frac{-2}{5} & \frac{1}{10} \end{bmatrix}$
- $A^{-1} = \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -1 & \frac{-4}{5} & \frac{7}{10} \\ 0 & \frac{-2}{5} & \frac{1}{10} \end{bmatrix}$

- **Homework:** find A^{-1} using Gaussian elimination and <u>check the result</u>

1) $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ 0 & 2 & 4 \end{bmatrix}$ **Ans**: $A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -2 & 2 & \frac{1}{2} \\ 1 & -1 & 0 \end{bmatrix}$

2) $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$ **Ans**: $A^{-1} = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$

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SOLUTION OF SYSTEM OF LINEAR EQUATIONS

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Lecture 3: **Crout's method or LU decomposition method.**

Crout's Method (LU Decomposition method)

It is a distinct method of solving a system of linear equations of the form $A\tilde{x}=b$, where the matrix A is decomposed into a product of a lower triangular matrix L and an upper triangular matrix U, that is $A=LU$

Explicitly, we can write it as

هذه المصفوفة الاصلية   هذه المصفوفة السفلى L   هذه المصفوفة العليا U

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Therefore, by LU-decomposition, the system of linear equations $A\underset{\sim}{x}=\underset{\sim}{b}$ can be solved in three steps:

I. Construct the lower triangular matrix L and upper triangular matrix U.
II. Using forward substitution, solve $L\tilde{y}=\underset{\sim}{b}$
III. Solve $U\tilde{x}=\tilde{y}$, backward substitution.

We further elaborate the process by considering a 3×3 matrix A. We consider solving the system of equation of the form $A\underset{\sim}{x}=\underset{\sim}{b}$, where,

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \tilde{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} and \quad b=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

The matrix A is factorized as a product of two matrices L (lower triangular matrix) and U (upper triangular matrix) as follows:

$$\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

L   U   A

$$\Rightarrow \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

This implies

$$l_{11} = a_{11},\ l_{21} = a_{21},\ l_{31} = a_{31};$$

$$l_{11}u_{12} = a_{12} \Rightarrow u_{12} = \frac{a_{12}}{l_{11}} = \frac{a_{12}}{a_{11}};$$

$$l_{11}u_{13} = a_{13} \Rightarrow u_{13} = \frac{a_{13}}{l_{11}} = \frac{a_{13}}{a_{11}};$$

$$l_{21}u_{12} + l_{22} = a_{22} \Rightarrow l_{22} = a_{22} - l_{21}u_{12};$$

$$l_{21}u_{13} + l_{22}u_{23} = a_{23} \Rightarrow u_{23} = \frac{1}{l_{22}}\left(a_{23} - l_{21}u_{13}\right);$$

$$l_{31}u_{12} + l_{32} = a_{32} \Rightarrow l_{32} = a_{32} - l_{31}u_{12};$$

$$l_{31}u_{13} + l_{32}u_{32} + l_{33} = a_{33} \Rightarrow l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

Once all the value of l_{ij}'s and u_{ij}'s are obtained, we can write

$$A\tilde{x} = \underset{\sim}{b} \text{ as } LU\,\tilde{x} = \underset{\sim}{b}$$

$$Let\ U\,\tilde{x} = \underset{\sim}{y}, \text{ then } L\,\underset{\sim}{y} = \underset{\sim}{b}$$

$$\Rightarrow \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} l_{11}y_1 \\ l_{21}y_1 + l_{22}y_2 \\ l_{31}y_1 + l_{32}y_2 + l_{33}y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\Rightarrow y_1 = \frac{b_1}{l_{11}},\ y_2 = \frac{1}{l_{22}}\left(b_2 - l_{21}y_1\right) \text{ and } y_3 = \frac{1}{l_{33}}\left(b_3 - l_{31}y_1 - l_{32}y_2\right)$$

By forward substitution we obtain, $U\underset{\sim}{x}=\underset{\sim}{y}$

$$\Rightarrow \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

By back substitution we get,

$$x_3 = y_3$$

$$x_2 + u_{23}x_3 = y_2 \Rightarrow x_2 = y_2 - u_{23}x_3$$

$$x_1 + u_{12}x_2 + u_{13}x_3 = y_1 \Rightarrow x_1 = y_1 - u_{12}x_2 - u_{13}x_3$$

Example 4. Solve the following system of linear equations, by Crout's method:

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$$10x_1 + 3x_2 + 4x_3 = +15$$

$$2x_1 - 10x_2 + 3x_3 = 37$$

$$3x_1 + 2x_2 - 10x_3 = -10$$

<u>**Solution:**</u> In matrix form, the given system of equation can be written as

$$\begin{pmatrix} 10 & 3 & 4 \\ 2 & -10 & 3 \\ 3 & 2 & -10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 15 \\ 37 \\ -10 \end{pmatrix}$$

which is of the form $A\underset{\sim}{x}=\underset{\sim}{b}$. Let $A = LU$, which implies

$$\begin{pmatrix} 10 & 3 & 4 \\ 2 & -10 & 3 \\ 3 & 2 & -10 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{pmatrix}$$

$$\Rightarrow l_{11} = 10,\ l_{21} = 2,\ l_{31} = 3;\ u_{12}\frac{3}{10},\ u_{13} = \frac{4}{10};$$

$$l_{21}u_{12} + l_{22} = -10 \Rightarrow l_{22} = -10 - 2\times\frac{3}{10} = -\frac{106}{10};$$

$$l_{21}u_{13} + l_{22}u_{23} = 3 \Rightarrow u_{23} = \frac{\left(3 - 2\times\frac{4}{10}\right)}{\left(-\frac{106}{10}\right)} = -\frac{11}{53};$$

$$l_{31}u_{12} + l_{32} = 2 \Rightarrow l_{32} = 2 - l_{31}u_{12} = 2 - 3\times\frac{3}{10} = \frac{11}{10};$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = -10 \Rightarrow l_{33} = -10 - l_{31}u_{13} - l_{32}u_{23}$$
$$= -10 - 3\times\frac{4}{10} + \frac{11}{10}\times\frac{11}{53} = -\frac{1163}{106}$$

Therefore, we get,

$$L = \begin{pmatrix} 10 & 0 & 0 \\ 2 & \frac{-106}{10} & 0 \\ 3 & \frac{11}{10} & \frac{-1163}{106} \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & \frac{3}{10} & \frac{4}{10} \\ 0 & 1 & \frac{-11}{53} \\ 0 & 0 & 1 \end{pmatrix}$$

Now, let $U\underset{\sim}{x} = \underset{\sim}{y}$, then $L\underset{\sim}{y} = \underset{\sim}{b}$ implies

$$\begin{pmatrix} 10 & 0 & 0 \\ 2 & \frac{-106}{10} & 0 \\ 3 & \frac{11}{10} & \frac{-1163}{106} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 15 \\ 37 \\ -10 \end{pmatrix}$$

This implies

$$10y_1 = 15 \Rightarrow y_1 = \frac{3}{2}$$
$$2y_1 - \frac{106}{10}y_2 = 37 \Rightarrow y_2 = \frac{-170}{53}$$
$$y_1 + \frac{11}{10}y_2 - \frac{1163}{106}y_3 = -10 \Rightarrow y_3 = 1$$

Thus, $y = \begin{pmatrix} 3/2 \\ \frac{-170}{53} \\ 1 \end{pmatrix}$ and $U\tilde{x} = \underset{\sim}{y}$ gives

$$\begin{pmatrix} 1 & \frac{3}{10} & \frac{4}{10} \\ 0 & 1 & \frac{-11}{53} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{-170}{53} \\ 1 \end{pmatrix}$$, which implies

$$x_1 + \frac{3}{10}x_2 - \frac{4}{10}x_3 = \frac{3}{2}$$
$$x_2 - \frac{11}{53}x_3 = \frac{-170}{53}$$
$$x_3 = 1$$

By back substitution, we get,

$$x_3 = 1$$
$$x_2 = \frac{11\times 1}{53} - \frac{170}{53} = -3$$
$$x_1 = \frac{3}{2} - \frac{3}{10}x_2 - \frac{4}{10}x_3 = \frac{3}{2} - \frac{3}{10}\times(-3) - \frac{4}{10}\times 1 = 2$$

Therefore, the required solution by Crout's method (LU decomposition method) is $x_1 = 2, \;\; x_2 = -3, \;\; x_3 = 1.$

Example 5. Solve the following system of linear equations by Crout's Method (LU factorization or decomposition method):

$$9x_1 + 3x_2 + 3x_3 + 3x_4 = 24$$
$$3x_1 + 10x_2 - 2x_3 - 2x_4 = 17$$
$$3x_1 - 2x_2 + 18x_3 + 10x_4 = 45$$
$$3x_1 - 2x_2 + 10x_3 + 10x_4 = 29$$

Solution: The given system of equation can be written in matrix form as

$$\begin{pmatrix} 9 & 3 & 3 & 3 \\ 3 & 10 & -2 & -2 \\ 3 & -2 & 18 & 10 \\ 3 & -2 & 10 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 24 \\ 17 \\ 45 \\ 29 \end{pmatrix}$$

Let $\begin{pmatrix} 9 & 3 & 3 & 3 \\ 3 & 10 & -2 & -2 \\ 3 & -2 & 18 & 10 \\ 3 & -2 & 10 & 10 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Comparing, we get,

$$l_{11} = 9, \ \ l_{21} = 3, \ \ l_{31} = 3, \ \ l_{41} = 3;$$

$$l_{11}u_{12} = 3 \Rightarrow u_{12} = \frac{1}{3}. \text{ Similarly, } u_{13} = u_{14} = \frac{1}{3};$$

$$l_{21}u_{12} + l_{22} = 10 \Rightarrow l_{22} = 10 - l_{21}u_{12} = 10 - 3 \times \frac{1}{3} = 9$$

$$l_{21}u_{13} + l_{22}u_{23} = -2 \Rightarrow u_{23} = \frac{-2 - l_{21}u_{13}}{l_{22}} = -\frac{1}{3}$$

$$l_{21}u_{14} + l_{22}u_{24} = -2 \Rightarrow u_{24} = \frac{-2 - l_{21}u_{14}}{l_{22}} = -\frac{1}{3}$$

$$l_{31}u_{12} + l_{32} = -2 \Rightarrow l_{32} = -2 - l_{31}u_{12} = -3$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 18 \Rightarrow l_{33} = 18 - l_{31}u_{13} - l_{32}u_{23} = 16$$

$$l_{31}u_{14} + l_{32}u_{24} + l_{33}u_{34} = 10 \Rightarrow u_{34} = \frac{10 - l_{31}u_{14} - l_{32}u_{24}}{l_{33}} = \frac{1}{2}$$

$$l_{41}u_{12} + l_{42} = -2 \Rightarrow l_{42} = -2 - l_{41}u_{12} = -3$$

$$l_{41}u_{13} + l_{42}u_{23} + l_{43} = 10 \Rightarrow l_{43} = 10 - l_{41}u_{13} - l_{42}u_{23} = 8$$

$$l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + l_{44} = 10$$

$$\Rightarrow l_{44} = 10 - l_{41}u_{14} - l_{42}u_{24} - l_{43}u_{34} = 4$$

Therefore, we get

$$L = \begin{pmatrix} 9 & 0 & 0 & 0 \\ 3 & 9 & 0 & 0 \\ 3 & -3 & 16 & 0 \\ 3 & -3 & 8 & 4 \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{-1}{3} & \frac{-1}{3} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Forward substitution gives

$$9y_1 = 24 \Rightarrow y_1 = \frac{8}{3}$$

$$3y_1 + 9y_2 = 17 \Rightarrow y_2 = 1$$

$$3y_1 - 3y_2 + 16y_3 = 45 \Rightarrow y_3 = \frac{5}{2}$$

$$3y_1 - 3y_2 + 8y_3 + 4y_4 = 29 \Rightarrow y_4 = 1$$

Thus, $y = \begin{pmatrix} 8/3 \\ 1 \\ 5/2 \\ 1 \end{pmatrix}$ and $U\tilde{x} = y$ gives

$$\begin{pmatrix} 1 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & -1/3 & -1/3 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 8/3 \\ 1 \\ 5/2 \\ 1 \end{pmatrix}$$

By back substitution we get,

$$x_4 = 1$$

$$x_3 + \frac{1}{2}x_4 = \frac{5}{2} \Rightarrow x_3 = 2$$

$$x_2 - \frac{1}{3}x_3 - \frac{1}{3}x_4 = 1 \Rightarrow x_2 = 2$$

$$x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 = \frac{8}{3} \Rightarrow x_1 = 1$$

Therefore, the required solution by Crout's method is

$$x_1 = 1,\ x_2 = 2,\ x_3 = 2,\ x_4 = 1$$

(5)

10.2 ITERATIVE METHODS FOR SOLVING LINEAR SYSTEMS

As a numerical technique, Gaussian elimination is rather unusual because it is *direct*. That is, a solution is obtained after a single application of Gaussian elimination. Once a "solution" has been obtained, Gaussian elimination offers no method of refinement. The lack of refinements can be a problem because, as the previous section shows, Gaussian elimination is sensitive to rounding error.

Numerical techniques more commonly involve an iterative method. For example, in calculus you probably studied Newton's iterative method for approximating the zeros of a differentiable function. In this section you will look at two iterative methods for approximating the solution of a system of n linear equations in n variables.

The Jacobi Method

The first iterative technique is called the **Jacobi method,** after Carl Gustav Jacob Jacobi (1804–1851). This method makes two assumptions: (1) that the system given by

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\
\vdots \qquad \vdots \qquad\quad\; \vdots \quad &\;\; \vdots\\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

has a unique solution and (2) that the coefficient matrix A has no zeros on its main diagonal. If any of the diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$ are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal.

To begin the Jacobi method, solve the first equation for x_1, the second equation for x_2, and so on, as follows.

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n)\\
x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n)\\
&\;\;\vdots\\
x_n &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1})
\end{aligned}$$

Then make an *initial approximation* of the solution,

$(x_1, x_2, x_3, \ldots, x_n),$ **Initial approximation**

and substitute these values of x_i into the right-hand side of the rewritten equations to obtain the *first approximation*. After this procedure has been completed, one **iteration** has been performed. In the same way, the second approximation is formed by substituting the first approximation's x-values into the right-hand side of the rewritten equations. By repeated iterations, you will form a sequence of approximations that often **converges** to the actual solution. This procedure is illustrated in Example 1.

EXAMPLE 1 ***Applying the Jacobi Method***

Use the Jacobi method to approximate the solution of the following system of linear equations.

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Continue the iterations until two successive approximations are identical when rounded to three significant digits.

Solution To begin, write the system in the form

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}x_2 - \tfrac{3}{5}x_3 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}x_1 - \tfrac{1}{9}x_3 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}x_1 - \tfrac{1}{7}x_2. \end{aligned}$$

Because you do not know the actual solution, choose

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0 \qquad \text{Initial approximation}$$

as a convenient initial approximation. So, the first approximation is

$$\begin{aligned} x_1 &= -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200 \\ x_2 &= \tfrac{2}{9} + \tfrac{3}{9}(0) - \tfrac{1}{9}(0) \approx 0.222 \\ x_3 &= -\tfrac{3}{7} + \tfrac{2}{7}(0) - \tfrac{1}{7}(0) \approx -0.429. \end{aligned}$$

Continuing this procedure, you obtain the sequence of approximations shown in Table 10.1.

TABLE 10.1

Itration

n	0	1	2	3	4	5	6	7
x_1	0.000	−0.200	0.146	0.192	0.181	0.185	0.186	0.186
x_2	0.000	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0.000	−0.429	−0.517	−0.416	−0.421	−0.424	−0.423	−0.423

Because the last two columns in Table 10.1 are identical, you can conclude that to three significant digits the solution is

$$x_1 = 0.186, \qquad x_2 = 0.331, \qquad x_3 = -0.423.$$

For the system of linear equations given in Example 1, the Jacobi method is said to **converge.** That is, repeated iterations succeed in producing an approximation that is correct to three significant digits. As is generally true for iterative methods, greater accuracy would require more iterations.

The Gauss-Seidel Method

You will now look at a modification of the Jacobi method called the Gauss-Seidel method, named after Carl Friedrich Gauss (1777–1855) and Philipp L. Seidel (1821–1896). This modification is no more difficult to use than the Jacobi method, and it often requires fewer iterations to produce the same degree of accuracy.

With the Jacobi method, the values of x_i obtained in the nth approximation remain unchanged until the entire $(n + 1)$th approximation has been calculated. With the Gauss-Seidel method, on the other hand, you use the new values of each x_i as soon as they are known. That is, once you have determined x_1 from the first equation, its value is then used in the second equation to obtain the new x_2. Similarly, the new x_1 and x_2 are used in the third equation to obtain the new x_3, and so on. This procedure is demonstrated in Example 2.

EXAMPLE 2 ***Applying the Gauss-Seidel Method***

Use the Gauss-Seidel iteration method to approximate the solution to the system of equations given in Example 1.

Solution The first computation is identical to that given in Example 1. That is, using $(x_1, x_2, x_3) = (0, 0, 0)$ as the initial approximation, you obtain the following new value for x_1.

$$x_1 = -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200$$

Now that you have a new value for x_1, however, use it to compute a new value for x_2. That is,

$$x_2 = \tfrac{2}{9} + \tfrac{3}{9}(-0.200) - \tfrac{1}{9}(0) \approx 0.156.$$

Similarly, use $x_1 = -0.200$ and $x_2 = 0.156$ to compute a new value for x_3. That is,

$$x_3 = -\tfrac{3}{7} + \tfrac{2}{7}(-0.200) - \tfrac{1}{7}(0.156) \approx -0.508.$$

So the first approximation is $x_1 = -0.200$, $x_2 = 0.156$, and $x_3 = -0.508$. Continued iterations produce the sequence of approximations shown in Table 10.2.

TABLE 10.2

n	0	1	2	3	4	5
x_1	0.000	−0.200	0.167	0.191	0.186	0.186
x_2	0.000	0.156	0.334	0.333	0.331	0.331
x_3	0.000	−0.508	−0.429	−0.422	−0.423	−0.423

Note that after only five iterations of the Gauss-Seidel method, you achieved the same accuracy as was obtained with seven iterations of the Jacobi method in Example 1.

Neither of the iterative methods presented in this section always converges. That is, it is possible to apply the Jacobi method or the Gauss-Seidel method to a system of linear equations and obtain a divergent sequence of approximations. In such cases, it is said that the method **diverges.**

EXAMPLE 3 *An Example of Divergence*

Apply the Jacobi method to the system

$$\begin{aligned} x_1 - 5x_2 &= -4 \\ 7x_1 - x_2 &= 6, \end{aligned}$$

using the initial approximation $(x_1, x_2) = (0, 0)$, and show that the method diverges.

Solution As usual, begin by rewriting the given system in the form

$$\begin{aligned} x_1 &= -4 + 5x_2 \\ x_2 &= -6 + 7x_1. \end{aligned}$$

Then the initial approximation (0, 0) produces

$$\begin{aligned} x_1 &= -4 + 5(0) = -4 \\ x_2 &= -6 + 7(0) = -6 \end{aligned}$$

as the first approximation. Repeated iterations produce the sequence of approximations shown in Table 10.3.

TABLE 10.3

n	0	1	2	3	4	5	6	7
x_1	0	−4	−34	−174	−1244	−6124	−42,874	−214,374
x_2	0	−6	−34	−244	−1244	−8574	−42,874	−300,124

For this particular system of linear equations you can determine that the actual solution is $x_1 = 1$ and $x_2 = 1$. So you can see from Table 10.3 that the approximations given by the Jacobi method become progressively *worse* instead of better, and you can conclude that the method diverges.

The problem of divergence in Example 3 is not resolved by using the Gauss-Seidel method rather than the Jacobi method. In fact, for this particular system the Gauss-Seidel method diverges more rapidly, as shown in Table 10.4.

TABLE 10.4

n	0	1	2	3	4	5
x_1	0	−4	−174	−6124	−214,374	−7,503,124
x_2	0	−34	−1224	−42,874	−1,500,624	−52,521,874

With an initial approximation of $(x_1, x_2) = (0, 0)$, neither the Jacobi method nor the Gauss-Seidel method converges to the solution of the system of linear equations given in Example 3. You will now look at a special type of coefficient matrix A, called a **strictly diagonally dominant matrix,** for which it is guaranteed that both methods will converge.

Definition of Strictly Diagonally Dominant Matrix

An $n \times n$ matrix A is **strictly diagonally dominant** if the absolute value of each entry on the main diagonal is greater than the sum of the absolute values of the other entries in the same row. That is,

$$\begin{aligned} |a_{11}| &> |a_{12}| + |a_{13}| + \cdots + |a_{1n}| \\ |a_{22}| &> |a_{21}| + |a_{23}| + \cdots + |a_{2n}| \\ &\vdots \\ |a_{nn}| &> |a_{n1}| + |a_{n2}| + \cdots + |a_{n,n-1}|. \end{aligned}$$

EXAMPLE 4 ***Strictly Diagonally Dominant Matrices***

Which of the following systems of linear equations has a strictly diagonally dominant coefficient matrix?

(a)
$$\begin{aligned} 3x_1 - x_2 &= -4 \\ 2x_1 + 5x_2 &= 2 \end{aligned}$$

(b)
$$\begin{aligned} 4x_1 + 2x_2 - x_3 &= -1 \\ x_1 \qquad + 2x_3 &= -4 \\ 3x_1 - 5x_2 + x_3 &= 3 \end{aligned}$$

Solution (a) The coefficient matrix

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$$

is strictly diagonally dominant because $|3| > |-1|$ and $|5| > |2|$.

(b) The coefficient matrix

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & 2 \\ 3 & -5 & 1 \end{bmatrix}$$

is not strictly diagonally dominant because the entries in the second and third rows do not conform to the definition. For instance, in the second row $a_{21} = 1$, $a_{22} = 0$, $a_{23} = 2$, and it is not true that $|a_{22}| > |a_{21}| + |a_{23}|$. Interchanging the second and third rows in the original system of linear equations, however, produces the coefficient matrix

$$A' = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -5 & 1 \\ 1 & 0 & 2 \end{bmatrix},$$

and this matrix is strictly diagonally dominant.

The following theorem, which is listed without proof, states that strict diagonal dominance is sufficient for the convergence of either the Jacobi method or the Gauss-Seidel method.

Theorem 10.1
Convergence of the Jacobi and Gauss-Seidel Methods

If A is strictly diagonally dominant, then the system of linear equations given by $A\mathbf{x} = \mathbf{b}$ has a unique solution to which the Jacobi method and the Gauss-Seidel method will converge for any initial approximation.

In Example 3 you looked at a system of linear equations for which the Jacobi and Gauss-Seidel methods diverged. In the following example you can see that by interchanging the rows of the system given in Example 3, you can obtain a coefficient matrix that is strictly diagonally dominant. After this interchange, convergence is assured.

EXAMPLE 5 ***Interchanging Rows to Obtain Convergence***

Interchange the rows of the system

$$\begin{aligned} x_1 - 5x_2 &= -4 \\ 7x_1 - x_2 &= 6 \end{aligned}$$

to obtain one with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to four significant digits.

Solution Begin by interchanging the two rows of the given system to obtain

$$\begin{aligned} 7x_1 - x_2 &= 6 \\ x_1 - 5x_2 &= -4. \end{aligned}$$

Note that the coefficient matrix of this system is strictly diagonally dominant. Then solve for x_1 and x_2 as follows.

$$\begin{aligned} x_1 &= \tfrac{6}{7} + \tfrac{1}{7}x_2 \\ x_2 &= \tfrac{4}{5} + \tfrac{1}{5}x_1 \end{aligned}$$

Using the initial approximation $(x_1, x_2) = (0, 0)$, you can obtain the sequence of approximations shown in Table 10.5.

TABLE 10.5

n	0	1	2	3	4	5
x_1	0.0000	0.8571	0.9959	0.9999	1.000	1.000
x_2	0.0000	0.9714	0.9992	1.000	1.000	1.000

So you can conclude that the solution is $x_1 = 1$ and $x_2 = 1$.

Do not conclude from Theorem 10.1 that strict diagonal dominance is a necessary condition for convergence of the Jacobi or Gauss-Seidel methods. For instance, the coefficient matrix of the system

$$\begin{aligned} -4x_1 + 5x_2 &= 1 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

is not a strictly diagonally dominant matrix, and yet both methods converge to the solution $x_1 = 1$ and $x_2 = 1$ when you use an initial approximation of $(x_1, x_2) = (0, 0)$. (See Exercises 21–22.)

SECTION 10.2 ❑ EXERCISES

In Exercises 1–4, apply the Jacobi method to the given system of linear equations, using the initial approximation $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$. Continue performing iterations until two successive approximations are identical when rounded to three significant digits.

1. $$\begin{aligned} 3x_1 - x_2 &= 2 \\ x_1 + 4x_2 &= 5 \end{aligned}$$

2. $$\begin{aligned} -4x_1 + 2x_2 &= -6 \\ 3x_1 - 5x_2 &= 1 \end{aligned}$$

3. $$\begin{aligned} 2x_1 - x_2 \phantom{{}+x_3} &= 2 \\ x_1 - 3x_2 + x_3 &= -2 \\ -x_1 + x_2 - 3x_3 &= -6 \end{aligned}$$

4. $$\begin{aligned} 4x_1 + x_2 + x_3 &= 7 \\ x_1 - 7x_2 + 2x_3 &= -2 \\ 3x_1 \phantom{{}-7x_2} + 4x_3 &= 11 \end{aligned}$$

5. Apply the Gauss-Seidel method to Exercise 1.

6. Apply the Gauss-Seidel method to Exercise 2.

7. Apply the Gauss-Seidel method to Exercise 3.

8. Apply the Gauss-Seidel method to Exercise 4.

In Exercises 9–12, show that the Gauss-Seidel method diverges for the given system using the initial approximation $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$.

9. $$\begin{aligned} x_1 - 2x_2 &= -1 \\ 2x_1 + x_2 &= 3 \end{aligned}$$

10. $$\begin{aligned} -x_1 + 4x_2 &= 1 \\ 3x_1 - 2x_2 &= 2 \end{aligned}$$

11. $$\begin{aligned} 2x_1 - 3x_2 \phantom{{}-10x_3} &= -7 \\ x_1 + 3x_2 - 10x_3 &= 9 \\ 3x_1 \phantom{{}+3x_2} + x_3 &= 13 \end{aligned}$$

12. $$\begin{aligned} x_1 + 3x_2 - x_3 &= 5 \\ 3x_1 - x_2 \phantom{{}-x_3} &= 5 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

In Exercises 13–16, determine whether the matrix is strictly diagonally dominant.

13. $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$

14. $\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 12 & 6 & 0 \\ 2 & -3 & 2 \\ 0 & 6 & 13 \end{bmatrix}$

16. $\begin{bmatrix} 7 & 5 & -1 \\ 1 & -4 & 1 \\ 0 & 2 & -3 \end{bmatrix}$

17. Interchange the rows of the system of linear equations in Exercise 9 to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to two significant digits.

18. Interchange the rows of the system of linear equations in Exercise 10 to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to two significant digits.

19. Interchange the rows of the system of linear equations in Exercise 11 to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to two significant digits.

20. Interchange the rows of the system of linear equations in Exercise 12 to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to two significant digits.

In Exercises 21 and 22, the coefficient matrix of the system of linear equations is not strictly diagonally dominant. Show that the Jacobi and Gauss-Seidel methods converge using an initial approximation of $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$.

21. $$\begin{aligned} -4x_1 + 5x_2 &= 1 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

22. $$\begin{aligned} 4x_1 + 2x_2 - 2x_3 &= 0 \\ x_1 - 3x_2 - x_3 &= 7 \\ 3x_1 - x_2 + 4x_3 &= 5 \end{aligned}$$

In Exercises 23 and 24, write a computer program that applies the Gauss-Siedel method to solve the system of linear equations.

23. $$\begin{aligned} 4x_1 + x_2 - x_3 &= 3 \\ x_1 + 6x_2 - 2x_3 + x_4 - x_5 &= -6 \\ x_2 + 5x_3 - x_5 + x_6 &= -5 \\ 2x_2 + 5x_4 - x_5 - x_7 - x_8 &= 0 \\ -x_3 - x_4 + 6x_5 - x_6 - x_8 &= 12 \\ -x_3 - x_5 + 5x_6 &= -12 \\ -x_4 + 4x_7 - x_8 &= -2 \\ -x_4 - x_5 - x_7 + 5x_8 &= 2 \end{aligned}$$

24. $$\begin{aligned} 4x_1 - x_2 - x_3 &= 18 \\ -x_1 + 4x_2 - x_3 - x_4 &= 18 \\ -x_2 + 4x_3 - x_4 - x_5 &= 4 \\ -x_3 + 4x_4 - x_5 - x_6 &= 4 \\ -x_4 + 4x_5 - x_6 - x_7 &= 26 \\ -x_5 + 4x_6 - x_7 - x_8 &= 16 \\ -x_6 + 4x_7 - x_8 &= 10 \\ -x_7 + 4x_8 &= 32 \end{aligned}$$

.Newton-Raphson Method

It is based on linearization of the nonlinear continuous function f(x). That is, the zero of f(x) is approximated by the zero of the tangent line of f(x).

Graphical Derivation of Newton-Raphson Method

Assume we have the **nonlinear continuous** function y = f(x) = 0 shown in Fig.1 and it is required to find its root (r). Let x_0 be the initial estimate of r

$$\tan\theta = \left.\frac{dy}{dx}\right|_{(x=x_0)} = f'(x = x_0) = \frac{f(x_0)}{x_0 - x_1}$$

Solve for x_1

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Follow the same procedure to find x_2

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

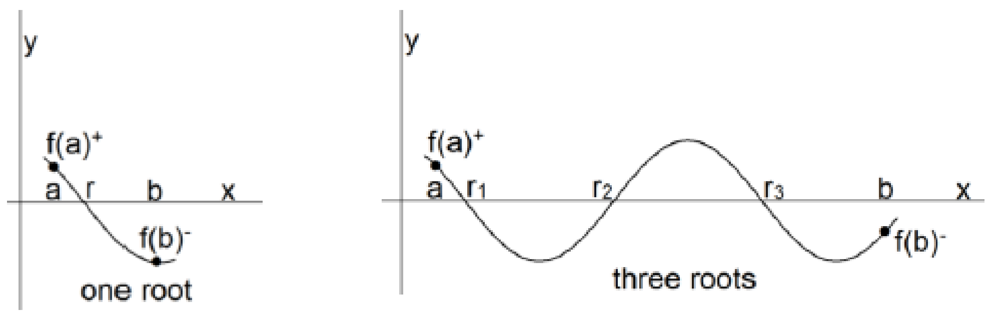
In general use the first approximation to get a second, the second to get a third, and so on, using the following numerical scheme

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad i = 0, 1, 2, \ldots \qquad f'(x_i) \neq 0 \qquad (1)$$

The stopping criterion is: $|x_{i+1} - x_i| \leq T_x$ where T_x is the tolerance for x

1. **Open Methods:** Sometimes we need to find the root of an equation near a point **x**. Use $x_0 = x$.
2. **Bracketing methods:** Sometimes we need to find the root(s) of an equation in an interval [a, b]. Examine the sign of f(x) at the ends of the interval [a, b]. There are two cases:

- If f (a) f (b) < 0, then there is one root or odd number of roots.



- If f (a) f (b) > 0, then there are no roots, even number of roots, or multiple equal roots.

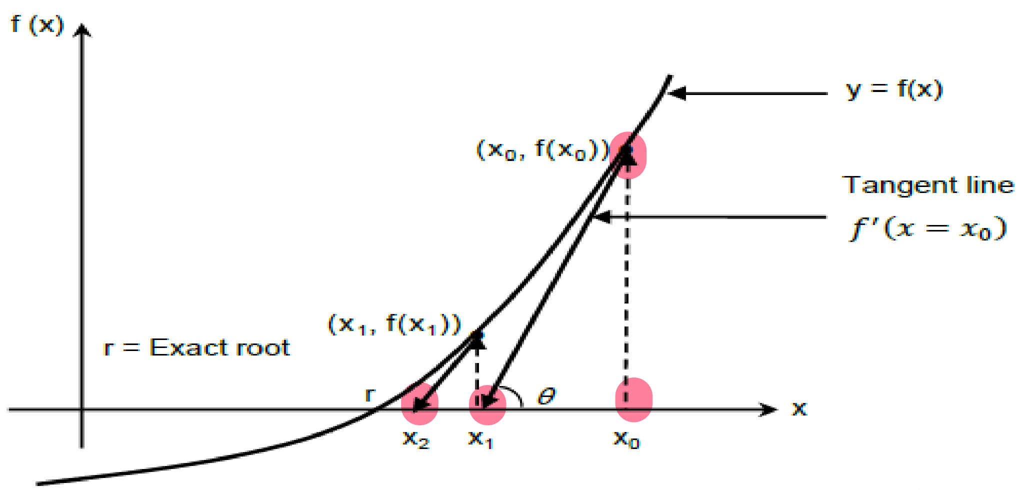
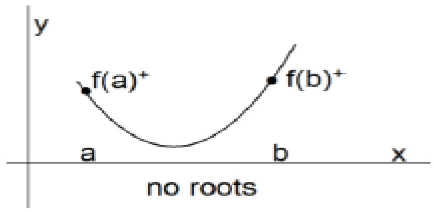
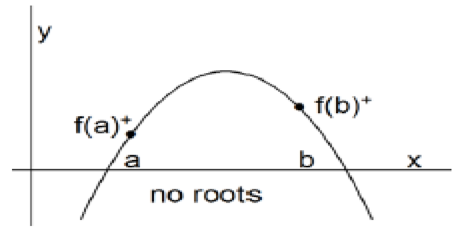
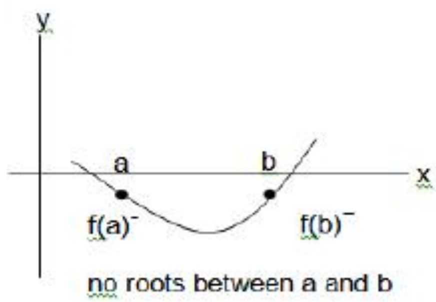
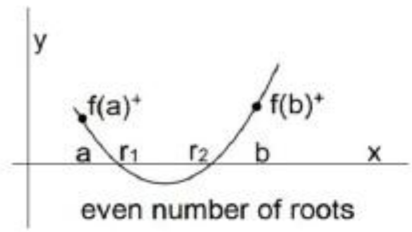


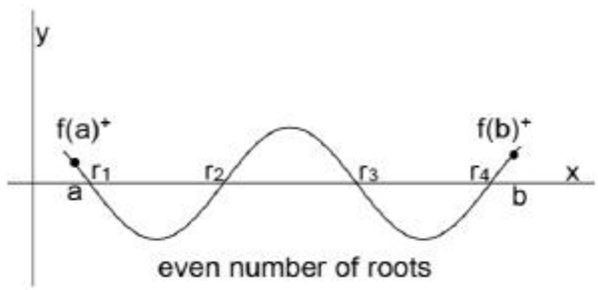
Figure 1: Geometrical illustration of Newton-Raphson method

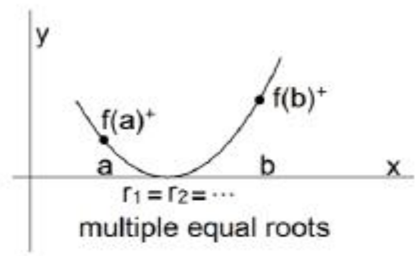












Example 1: Use NR method to find the real root of $x^3 - x - 1 = 0$ correct to 5 decimal places (dp) in the interval $[-4, 4]$. Choose $\Delta x = 1$.

Solution

Determine the position of the root

x	−4	−3	−2	−1	0	1		2	3	4
$f(x) = x^3 - x - 1$	−	−	−	−	−	−	↓ r	+	+	+

Determine the initial point x_0. The best initial point is the point that makes the value of f(x) closer to zero.

x	$1(x_1)$	$2(x_2)$	$1.5(x_3 = (x_1 + x_2)/2)$
$f(x) = x^3 - x - 1$	−1	5	**0.875**

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{(x_i)^3 - x_i - 1}{3(x_i)^2 - 1}$$

i	x_i (x old)	x_{i+1} (x new)	$E_i = \|x_{i+1} - x_i\|$	Notes
0	1.50000(x_0)	1.34783(x_1)	E_0 = 0.15217 >T_x	–
1	1.34783(x_1)	1.32520(x_2)	E_1 = 0.02263 >T_x	$E_1 < E_0$ ok
2	1.32520(x_2)	1.32472(x_3)	E_2 = 0.00048 >T_x	$E_2 < E_1$ ok
3	1.32472(x_3)	**1.32472**(x_4)	E_3 = 0.00000 =T_x	$E_3 < E_2$ ok

The root is r = 1.32472 and $f_{(x = 1.32472)}$ = 0.00001 to 5dp

Example 2: Use NR method to estimate the positive **abscissa** (x-coordinate) of the intersection point of $f_1(x) = \sin x$ and $f_2(x) = x^2$ correct to 4dp.

Solution

The intersection point (points) of $f_1(x)$ and $f_2(x)$ is the exact root (roots) of the equation $f(x) = f_1(x) - f_2(x) = 0$ and vice versa.

Determine the root position

x	0	0.5		1
$f(x) = \sin x - x^2$	0	+	↓ **r**	–

Determine the initial point x_0. The best initial point is the point that makes the value of f(x) closer to zero.

x	0.5(x_1)	1(x_2)	0.75 ($x_3 = (x_1 + x_2)/2$)
$f(x) = \sin x - x^2$	0.229	– 0.159	0.119

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\sin x_i - {x_i}^2}{\cos x_i - 2x_i}$$

i	x_i (x old)	x_{i+1} (x new)	$E_i = \lvert x_{i+1} - x_i \rvert$	Notes
0	0.7500(x_0)	0.9051(x_1)	$E_0 = 0.1551$	–
1	0.9051(x_1)	0.8777(x_2)	$E_1 = 0.0274$	$E_1 < E_0$ OK
2	0.8777(x_2)	0.8767(x_3)	$E_2 = 0.0010$	$E_2 < E_1$ OK
3	0.8767(x_3)	**0.8767** (x_4)	$E_2 = 0.0000$	$E_3 < E_2$ OK

Euler's Method from Taylor Series

The approximation used with Euler's method is to take only the first two terms of the Taylor series:

$$f(a+h) = f(a) + hf'(a)$$

In general form:

new value = old value + step size × slope

If $f(a+h) = y_{i+1}$ and $f(a) = y_i$ as well as $f'(a) = y_i'$

then

$$y_{i+1} = y_i + hy_i'$$

with $i = 0, 1, 2, \ldots, N-1$.

Euler's (Forward) Method

Alternatively, from step size $h = x_{i+1} - x_i$ and rearrange to $x_{i+1} = x_i + h$ we use the Taylor series to approximate the function $f(x_{i+1}) = y_{i+1}$ around $f(x_i) = y_i$ with step size $h = x_{i+1} - x_i$. Taking only the first derivative:

$$y_{i+1} = y_i + f(x_i, y_i)h$$

where $f(x_i, y_i)$ is the differential equation evaluated at x_i and y_i.

This formula is referred to as Euler's forward method, or explicit Euler's method, or Euler-Cauchy method, or point-slope method.

Example

Obtain a numerical solution of the differential equation

$$\frac{dy}{dx} = 3(1 + x) - y$$

given the initial conditions that $x = 1$ when $y = 4$, for the range $x = 1$ to $x = 2$ with intervals of 0.2.

Solution:

$$\frac{dy}{dx} = y' = 3(1 + x) - y$$

with $x_0 = 1$ and $y_0 = 4$, then $y'_0 = 3(1 + 1) - 4 = 2$.

By Euler's method: $y_1 = y_0 + hy'_0$

$$y_1 = 4 + (0.2)(2) = 4.4$$

At $x_1 = x_0 + h = 1 + 0.2 = 1.2$ and $y_1 = 4.4$ where

$$y'_1 = 3(1 + x_1) - y_1$$

$$y'_1 = 3(1 + 1.2) - 4.4 = 2.2$$

then

$$y_2 = y_1 + hy'_1$$

$$y_2 = 4.4 + (0.2)(2.2) = 4.84$$

If the step by step Euler's method is continued, we can present the results in a table:

x_i	y_i	y'_i
1	4	2
1.2	4.4	2.2
1.4	4.84	2.36
1.6	5.312	2.488
1.8	5.8096	2.5904
2	6.32768	

MATLAB Implementations

```
function [x,y] = EulerForward(f,xinit,xend,yinit,h)

% Number of iterations
N = (xend-xinit)/h;

% Initialize arrays
% The first elements take xinit and yinit, correspondingly,
% the rest fill with 0s.
x = zeros(1, N+1);
y = zeros(1, N+1);

x(1) = xinit;
y(1) = yinit;

for i=1:N
    x(i+1) = x(i)+h;
    y(i+1) = y(i) + h*feval(f,x(i),y(i));
end

end
```

```
function dydx = EulerFunction(x,y)

dydx = 3*(1+x)-y;

end
```

```
a = 1; b = 2; ya = 4; h = 0.2;

N = (b-a)/h;
t = a:h:b;

[x,y] = EulerForward('EulerFunction', a, b, ya, h);

ye = y; % Numerical solution from using Euler's forward method
yi = 3*t+exp(1-t); % Exact solution

hold on;
plot(t,yi,'r','LineWidth', 2);
plot(t,ye,'b','LineWidth', 2); hold off;
box on;
```

Fifth lecture

4..1Lagrange Interpolating polynomials

Determining a polynomial of degree 1 that passes through the distinct point

$(x_0, y_0)\, and\, (x_1, y_1)$ is the same as approximating a function f for which

$f(x_0) = y_0\ and\ f(x_1) = y_1$ by means of first-degree polynomial interpolating or agreeing with . The values of f at the given points. We first define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad and \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

And note that

$$L_0(x_0) = \frac{x_0 - x_1}{x_0 - x_1} = 1,\ L_0(x_1) = \frac{x_1 - x_1}{x_1 - x_0} = 0$$

$$L_1(x_0) = \frac{x_0 - x_0}{x_0 - x_1} = 0,\ L_1(x_1) = \frac{x_1 - x_0}{x_1 - x_0} = 1$$

We then define

$$p(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

This gives

$$p(x_0) = 1.f(x_0) + 0.f(x_1) = f(x_0) = y_0$$
$$and \quad p(x_1) = 0.f(x_1) + 1.f(x_1) = f(x_1) = y_1$$

So p is the unique linear function passing through $(x_0, y_0)\, and\, (x_1, y_1)$

Example 1: Determine the linear Lagrange interpolating polynomial that passes through the points (2,4) and (5,1) with $f(x_0) = 4\, and\ \ f(x_1) = 1$

Solution:

$$p(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 5}{2 - 5} = \frac{(x - 5)}{3}$$

$$and \quad L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 2}{5 - 2} = \frac{x - 2}{3}$$

$$p(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - 5}{-3}.4 + \frac{x - 2}{3}.1 = -x + 6$$

n th Lagrange interpolating polynomial

$$p_n(x) = L_{n,0}(x)f(x_0) + L_{n,1}(x)f(x_1) + ... + L_{n,\mathrm{n}}(x)f(x_n) = \sum_{i=0}^{n} L_{n,\mathrm{k}}(x)f(x_k)$$

Where

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)...(x-x_{k-1})(x-x_{k+1})...(x-x_n)}{(x_k-x_0)(x_k-x_1)...(x_k-x_{k-1})(x_k-x_{k+1})...(x_k-x_n)}$$

$$foreachk = 0,1,2,...,n$$

Example 2:

a) use number (called nodes) $x_0 = 2, x_1 = 2.75\ and\ \ x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = \frac{1}{x}$

b)use this polynomial to approximate $f(3) = \frac{1}{3}$

solution :

a)we first determined the coefficient polynomials $L_0(x), L_1(x)\,and\ \ L_2(x)$

$$p_2(x) = \sum_{i=0}^{2} f(x_k)\mathrm{L}_k(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)} = \frac{1}{3}(x-2.75)(x-4)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-2)(x-4)}{(2,75-2)(2.75-4)} = \frac{1}{0.9375}(x-2)(x-4)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-2)(x-2.75)}{(4-2.)(4-2.75)} = \frac{1}{2.5}(x-2.)(x-2.75)$$

$$p_2(x) = \sum_{i=0}^{2} f(x_k)\mathrm{L}_k(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$p_2(x) = \frac{1}{2}.\frac{1}{3}(x-2.75)(x-4) + \frac{1}{2.75}.\frac{1}{0.9375}(x-2)(x-4) + \frac{1}{4}.\frac{1}{2.5}(x-2.)(x-2.75)$$

$$p_2(x) = \frac{x^2-4x-2.75x+11}{6} + \frac{x^2-4x-2x+8}{2.5781} + \frac{x^2-2.75x-5.5}{10}$$

Example3: Use the numbers (called nodes) $x_0 = 0, x_1 = 1\ and\ \ x_2 = 2$ to find the second Lagrange interpolating polynomial where ($y_0 = 1, y_1 = 5\,and\ y_0 = 29$)

$$p_2(x)=\sum_{i=0}^{2} f(x_k)\,\mathrm{L}_k(x)=L_0(x)f(x_0)+L_1(x)f(x_1)+L_2(x)f(x_2)$$

$$L_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{(x-1)(x-2)}{(0-1)(0-2)}=\frac{1}{2}(x-1)(x-2)$$

$$L_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}=\frac{(x-0)(x-2)}{(1-0)(1-2)}=-x(x-2)$$

$$L_1(x)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}=\frac{(x-0)(x-1)}{(2-0)(2-1)}=\frac{1}{2}x(x-1)$$

$$p_2(x)=\sum_{i=0}^{2} f(x_k)\,\mathrm{L}_k(x)=L_0(x)f(x_0)+L_1(x)f(x_1)+L_2(x)f(x_2)$$

$$p_2(x)=(1).\frac{1}{2}(x-1)(x-2)-(5).x(x-2)+(29).\frac{1}{2}x(x-1)$$

$$p_2(x)=\frac{x^2-3x+2}{2}-5x^2-10x+\frac{29x^2-29x}{2}$$

$$p_2(x)=10x^2-6x+1$$

we can check

$$p_2(x_0)=10(0)-6(0)+1=1$$

$$p_2(x_1)=10(1)^2-6(1)+1=5$$

$$p_2(x)=10(2)^2-6(2)+1=29$$

Example 4: Find $f(1.5)$ by use Lagrange interpolating polynomial with points

$x_0==1.3, x_1=1.6$ *and* $x_2=1.9$ (

$$y_0=0.62008, y_1=0.45540 \text{ and } y_2=0.28181$$

Solution:

$$p_2(x)=\sum_{i=0}^{2} f(x_k)\,\mathrm{L}_k(x)=L_0(x)f(x_0)+L_1(x)f(x_1)+L_2(x)f(x_2)$$

$$L_0(1.5)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{(1.5-1)(1.5-2)}{(0-1)(0-2)}$$

$$L_1(1.5)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}=\frac{(1.5-0)(1.5-2)}{(1-0)(1-2)}$$

$$L_2(1.5)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}=\frac{(1.5-0)(1.5-1)}{(2-0)(2-1)}$$

$$p_2(x)=\sum_{i=0}^{2} f(x_k)\,\mathrm{L}_k(x)=L_0(x)f(x_0)+L_1(x)f(x_1)+L_2(x)f(x_2)$$

$$p_2(x)=(0.62008)\frac{(1.5-1)(1.5-2)}{(0-1)(0-2)}+(0.45540)\frac{(1.5-0)(1.5-2)}{(1-0)(1-2)}+(0.28181)\frac{(1.5-0)(1.5-1)}{(2-0)(2-1)}$$

$$p_2(x)=0.51128$$

H0mwork

1)find approximation for f(2.3) from tables

x	1.1	1.7	3.0
y	10.6	15.2	20.3

2)

x	1.1	1.7	3.0	4.2	5
y	10.6	15.2	20.3	25.2	39.1

4.2: Forward difference

If $f(x)=y$ is a function whose values at known points (n+1). As $x_0, x_1=x_0+h,\dots,x_n=x_{n-1}+h$

We can denotes for forward difference as Δ the first forward difference at point x

$$\Delta f(x)=f(x+h)-f(x)$$

$$\Delta y_0=y_1-y_0$$

$$\Delta y_1=y_2-y_1$$

$$\Delta y_2=y_3-y_2$$

.

.

.

$$\Delta y_i=y_{i+1}-y_i \qquad i=0,1,\dots,n-1$$

The second forward difference find as

$$\Delta^2 y_i=\Delta y_{i+1}-\Delta y_i$$

$$\Delta^2 y_0=\Delta(\Delta y_0)=\Delta(y_1-y_0)=\Delta y_1-\Delta y_0$$

$$\Delta^2 y_1=\Delta(\Delta y_1)=\Delta(y_2-y_1)=\Delta y_2-\Delta y_1$$

We can obtained the forward difference for power k

$$\Delta^k y_i=\Delta^{k-1}(\Delta y_i)=\Delta^{k-1}y_{i+1}-\Delta^{k-1}y_i \quad i=0,1,2,..,n-k$$

We can obtained the forward difference for tables

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

Note: when the polynomial for order n the row difference (n+1) and the next row will be zero

Note: power =numbers point -1

Example write forward difference , table of the function $f(x) = x^3 \; with \; (x = 0,1,2,3,4 and\, 5)$

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$
0	**0**					
		1				
1	**1**		**6**			
		7		**6**		
2	8		**12**		**0**	
		19		**6**		**0**
3	**27**		**18**		**0**	
		37		**6**		
4	64		**24**			
		61				
5	125					

Homework

write forward difference , table of the function $f(x) = x^3 - 2x + 1 with \ (x = 1,2,3,4,5 and \ 6)$

4.3: Backward difference

If $f(x) = y$ is a function whose values at known points (n+1). As $x_0, x_1 = x_0 + h, ..., x_n = x_{n-1} + h$

We can denotes for forward difference as Δ the first forward difference at point x

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla y_i = y_i - y_{i-1} \qquad {\scriptstyle i=0,1,...,n-1}$$

$$\nabla^2 y_i = \nabla(\nabla y_i) = \nabla(y_i - y_{i-1}) = (\nabla y_i - \nabla y_{i-1})$$

.

.

.

$$\nabla^k y_i = \nabla^{k-1}(\nabla y_i) = \nabla^{k-1}(y_i - y_{i-1}) = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1})$$

We can obtained the forward difference for tables

x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$	$\nabla^5 y_i$
x_5	y_5					
		∇y_4				
x_4	y_4		$\nabla^2 y_3$			
		∇y_3		$\nabla^3 y_2$	$\nabla^4 y_1$	
x_3	y_3		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_1$		$\nabla^5 y_0$
x_2	y_2		$\nabla^2 y_1$		$\nabla^4 y_0$	
		∇y_1		$\nabla^3 y_0$		
x_1	y_1		$\nabla^2 y_0$			
		∇y_0				
x_0	y_0					

Example write Backward difference , table of the function $f(x) = x^3 - 2x - 1 with\ (x = 1,2,3,4,5 and\ 6)$

Solution

x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$	$\nabla^5 y_i$
1	**-2**					
		5				
2	**3**		**12**			
		17				
				6		
3	**20**		**18**		**0**	
		35		**6**		**0**
4	**55**		**24**		**0**	
		59		**6**		
5	**114**		**30**			
		89				
6	**203**					

Homework

1) **Write the backward difference table for function** $f(x_i) = y_i$ be having like polynomial $(1 + x + 2x^2)$ over [0,4] with $\Delta x = h = 1$

2) **Write the backward difference table to** $x = (0(1))4)\ and\ \ y = (3,6,11,18,12)$

4.4 <u>Newton-Gregory Forward Interpolating Formula</u>

In this formula for interpolation at the beginning of the givens values it uses forward operators

$$f(x) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)\Delta^2 f(x_0)}{2!} + \frac{p(p-1)(p-2)\Delta^3 f(x_0)}{3!}$$

$$+ \frac{p(p-1)(p-2)(p-3)\Delta^4 f(x_0)}{4!} + \ldots + \frac{p(p-1)(p-2)(p-3)\ldots(p-n+1)\Delta^n f(x_0)}{n!}$$

where $p = \frac{x_p - x_0}{h}$

Example: write Newton-Gregory Forward Interpolating (N-G I.P.) that fit the following table

x	0	1	2	3	4	5
y	-5	1	9	25	55	105

Solution :

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$
0	**-5**					
		6				
1	**1**		**2**			
		8		**6**		
					0	
2	9		**8**			**0**
		16		**6**		
3	**25**		**14**		**0**	
		30		**6**		
4	55		**20**			
		50				
5	105					

$$\because\ p=\frac{x_p-x_0}{h}=\frac{x_p-x_0}{h}=\frac{x_p-0}{1}=x_p \Rightarrow p=x_p$$

$$f(x_p)=f(x_0)+p\Delta f(x_0)+\frac{p(p-1)\Delta^2 f(x_0)}{2!}+\frac{p(p-1)(p-2)\Delta^3 f(x_0)}{3!}$$

$$+\frac{p(p-1)(p-2)(p-3)\Delta^4 f(x_0)}{4!}+\frac{p(p-1)(p-2)(p-3)(p-4+1)\Delta^5 f(x_0)}{5!}$$

$$f(x_p)=p^3-2p^2+7p-5$$

Homework

1)Find the (N-G I.P.) of degree two which takes the following values

$(0,0.25),(0.5,-1.5),(1,-1.75),(1.5,-0.5),(2.5,6.5),(3,12.25)$

($f(x_p)=3p^2-5p+0.25$)

x	1	2	3	4	5	6

2)write N.G.I.P. that fite the following

y	1	8	27	64	125	216

And find y at x=1.24

Solution

$x_0 = 1$

$x_p = 1.24$

$$p = \frac{x_p - x_0}{h} = \frac{1.24 - 1}{1} = 0.24$$

4.5 Newton-Gregory Backward Interpolating Formula

This formula used to find f(x) at end the table

$$f(x) = f(x_0) + p\nabla f(x_0) + \frac{p(p+1)\nabla^2 f(x_0)}{2!} + \frac{p(p+1)(p+2)\nabla^3 f(x_0)}{3!}$$

$$+ \frac{p(p+1)(p+2)(p+3)\nabla^4 f(x_0)}{4!} + \ldots + \frac{p(p+1)(p+2)(p+3)\ldots(p+n-1)\nabla^n f(x_0)}{n!}$$

Where $p = \frac{x_p - x_0}{h}$,

Example find the value y when x=1.35 and x=1.05 and

x	1	1.1	1.2	1.3	1.4
y	2.7183	3.0042	3.3201	3.6699	4.0552

Solution

x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$	$\nabla^5 y_i$
1	2.7183					
		0.2859				
1.1	3.0042	**0.3159**	**0.0300**	**0.0033**		
1.2	3.3201	**0.3492**	**0.0333**	**0.0034**	**0.0001**	
1.3	3.6699	**0.3859**	**0.0367**			

1.4	4.0552					

$$p = \frac{x_p - x_0}{h} = \frac{1.35 - 1.4}{0.1} = \frac{-0.05}{0.1} = -0.5$$

$$f(x) = f(x_0) + p\nabla f(x_0) + \frac{p(p+1)\nabla^2 f(x_0)}{2!} + \frac{p(p+1)(p+2)\nabla^3 f(x_0)}{3!}$$

$$+ \frac{p(p+1)(p+2)(p+3)\nabla^4 f(x_0)}{4!}$$

$$f_p(x) = 4.0552 + (-0.5)(0.3859) + \frac{(-0.5)(0.5)(0.0367)}{2!} + \frac{(-0.5)(0.5)(1.5)(06034)}{3!}$$

$$+ \frac{(-0.5)(0.5)(1.5)(2.5)(0.001)}{4!}$$