(a)

CH. 5

Continouse functions	31
open fun., Closed fung Homeomorphism	32
Restruction function $f|_A$	34
Projection maps	35
Hereditary, Topological property	37
Indentification Top	38
Retraction fun.	40
General prod., Projection	41 (*)
Axiom of Choice, Choise fun, x_0-proj maps	41 (*)
Tychonoff prod. Top, Basic open sets	42 (*)
in each Top	45
Empedding, Evaluation map, Separate point	47 (*)
Quotient space, Quotient Top, Strong Top	49
Partition (Decomposition)	51
Saturated	52

CH. 6

1) Separation Axioms T_0, T_1, T_2
2) Retract of X
3) Regular space, T_3 (T_1+reg), Completely Haushoff ($T_{2\frac{1}{2}}$)
Completly regular, Tychonoff space ($T_{3\frac{1}{2}}$) (C.r + T...
4) Zero dimintion
5) Cube
6) Normal, ~~T_4~~ (T_1+Normal)
7) $\boxed{\text{Jone's Lemma}}$
8) $\boxed{\text{Urysohn's Lemma}}$, $\frac{k}{2^n}$ = dyatic number
9) Extension
10) Tietze's Extension Th.
11) Shrinkable, Cover, Open Cover, Point finite cover
12) Completly Normal, T_5 (C.N + T_1)
13) F_σ, G_δ, Perfect space, Perfect Normal, Urysohn Space
14) Collection wise Normal
15) T_i space [illegible] Long [illegible]
16) Axiom of Countablity, Local base, [illegible]
17) First Ctble
18) Second Ctble
17) Separable
18) Lindelof (Exc (132) Lindlof + refinable ...
19) Perfect maps

-C-

16 ε-net + Exc (sep, see 44), c-cpt, cpt) (142) 114
17 cptification + embedded + Tych. + L-cpt Th 120

18 Locally cpt (Exc on L.c 127) 120

19 Compactifications (Embedding 47) 123
20 One point cptification 124

21 Stone-Cech Compactification 126
(β)- embedded 126

22 127

23 Locally finite (45) ≠ Refinement (118) (see 145) → 130
24 para cpt 131
25 Sequentially cpt 136

Ch. 9 Metric Spaces 75 137
Complet metric space 71 144
Cachy, converg, Baire's Th. 144
Completion 145
Metrizable 145
Stare 145
bary centric refinement 145
Partial order ...

CH-7

Ch. 8

1 Axiom of choice + Tychonoff product space + Basic open sets $\boxed{41+42}$
~~Dittorian space~~ Weak Topology 45 — embedding of X in Y 46, 47
Quotient space 49, Partition 51
T_0, T_1, T_2 spaces ~~83~~ 53
Regular, $T_{2\frac{1}{2}}$, T_3, Completeregular, $T_{3\frac{1}{2}}$ 59 | Cube 64
Normal, T_4, Complete Normal, T_5 66
Urysohn's Lemma 68, Jones Lemma 67
Tietze's Extension Th 70
Completely Normal 71
Perfect space + Perfectly Normal Topology 73
First ctble 75 (Axioms of ctbility)
Second ctble 78
Separable space 80
indep space 81
N.7 Sequence 85
Net 88
Filters 91

بسم الله الرحمن الرحيم
بكر احمد بكر العدل
نبوي . تلعفر
الرقم ....
التاريخ ....

Ch 8 Connected space 94 + separated 95
Components + Local cpt. 100
Local conn. 102
Compact 105
σ-Compact 110
Countable compact 114
Locally compact 120
Compactifications 123
C^*, C embedding 126
127
~~Paracompact~~ Paracompact 137 — Locally finite 130 + 45
Metric space 137

Chapter (3)

Dfn ① Let $X \neq \phi$ be set. $\boxed{\text{A topology on } X}$ is a collection τ of subsets of X i.e. $(\tau \subseteq P(X))$ $\ni$

① $\phi, X \in \tau$
② if $U, V \in \tau$ then $U \cap V \in \tau$
③ if $U_\alpha \in \tau$, $\forall \alpha \in \Delta$ then $\bigcup_{\alpha \in \Delta} U_\alpha \in \tau$

(X, τ) is a topological space

Type of Topology

$\boxed{\text{Trival Topology}}$ = (indiscrete Topology) = (X, τ_{ind}) is X is arbitrary set & $\tau = \{\phi, X\}$ is a topology on X is called the indiscrete Topology

$\boxed{\text{discrete topology}}$: X is arbitrary set, $\tau = P(X)$ is a topology on X is called discrete Top

ex ① Let $X = \{a, b\}$: $P(X) = \{\{a\}, \{b\}, \{a,b\}, \phi\} = \tau_{dis}$

② Let $X = \{a, b, c\}$ Try this statement is Top or not.

(i) $\tau = \{\phi, X, \{a\}\}$ ✓
(ii) $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ ✓
(iii) $\tau = \{\phi, X, \{a\}, \{b\}\}$ X
(iv) $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ ✓

(Notes) The elements of τ are called τ open sets

2//

be arbitrary, $\tau_{cofinite} = \{U \subseteq X : U^c \text{ is finite}\} \cup \phi$

called (**cofinite** Topology)

$X \in \tau$ because $X^c = \phi$ finite $\Rightarrow X \in \tau$

$U, V \in \tau$

if $U \cap V = \phi$ then $U \cap V \in \tau$

if $U \cap V \neq \phi$ then $U \neq \phi$ & $V \neq \phi$

so U^c & V^c are finite

$(U \cap V)^c = \underbrace{\underset{finite}{U^c} \sqcup \underset{finite}{V^c}}_{finite} = X - U \cap V = X - U \cup X - V = (U \cap V)^c$

$\therefore U \cap V \in \tau$

Let $U_\alpha \in \tau$ $\forall \alpha \in \Delta$

if $\bigcup_{\alpha \in \Delta} U_\alpha = \phi \Rightarrow \bigsqcup_{\alpha \in \Delta} U_\alpha \in \tau$

if $\bigsqcup_{\alpha \in \Delta} U_\alpha \neq \phi \Rightarrow U_{\alpha_0} \neq \phi$ for some α_0

$X - U_{\alpha_0}$ is finite

$X - \bigsqcup_{\alpha \in \Delta} U_\alpha = \bigsqcap_{\alpha \in \Delta} (X - U_\alpha) \subseteq X - U_{\alpha_0}$ (finite)

(ii) c بطريقة اخرى

we may assume $U_\alpha \neq \phi$ $\therefore U_\alpha^c$ is finite

$(\bigsqcup_{\alpha \in \Delta} U_\alpha)^c = \bigsqcap_{\alpha \in \Delta} U_\alpha^c$ is finite

$\therefore \bigcup U_\alpha \in \tau$

Theorem Let τ be Top on X, Let $U_1, U_2, \dots U_n \in \tau$ then $\bigsqcap_{i=1}^{n} U_i \in \tau$

Proof by induction let $n = 2$ ✓, assume it's true for $n = k$, $k > 2$

Take $(n = k+1)$ $U_1 \cap U_2 \cap \dots \cap U_k \cap U_{k+1} = \underbrace{(U_1 \cap \dots \cap U_k)}_{\in \tau} \cap \underbrace{U_{k+1}}_{\in \tau}$

(4) X arbitrary, $\tau = \{U \subseteq X,\ U^c \text{ is countable}\} \cup \{\phi\}$

is called co-countable topology

finite البرهان نفس

(5) $X = R$, $\tau_s = \tau_u = \{U \subseteq R : \text{if } \forall x \in U \text{ then } \exists (a,b) \ni x \in (a,b) \subseteq U\} \cup \{\phi\}$

is called (stander or Euclidean or Usual Top on R

Proof

(i) $\phi, \mathbb{R} \in \tau_u$

(ii) Let $U, V \in \tau_u$ assume $U \cap V \neq \phi$

Let $x \in U \cap V$ then $x \in U$ & $x \in V$

$\therefore \exists (a_1, b_1)$ & $(a_2, b_2) \ni x \in (a_1, b_1) \subseteq U$ & $x \in (a_2, b_2) \subseteq V$

$a_1 \quad a_2 \quad b_1 \quad b_2$

Let $a = \max\{a_1, a_2\}$ & $b = \min\{b_1, b_2\}$

$x \in (a,b) = (a_1, b_1) \sqcap (a_2, b_2) \subseteq U \cap V$ so $U \cap V \in \tau$

(iii) Let $U_\alpha \in \tau$ $\forall \alpha \in \Delta$ (1) if $\bigcup_{\alpha \in \Delta} U_\alpha = \phi \Rightarrow \bigcup_{\alpha \in \Delta} U_\alpha \in \tau$

(2) if $\bigsqcup U_\alpha \neq \phi$ then $x \in U_{\alpha_0}$ for some $\alpha_0 \in \Delta$

$\Rightarrow x \in \bigsqcup_{\alpha \in \Delta} U_\alpha$ then $\exists (a,b) \ni x \in (a,b) \subseteq U_{\alpha_0} \subseteq \bigsqcup U_\alpha$

Hence τ is Top on R

(7)

Notes each open interval in $\mathbb{R}$ is an open set in

⑥ Let $X=R$, if $a\in R$ then $L_a=\{x\in R : x<a\}=(-\infty,a)$ is called Left ray topology where
$T_{L.r}=\{\phi, R\}\cup\{L_a : a\in R\}$

⑦ Let $X=R$ if $a\in R$ then $T_{r}=\{\phi, R, (a,\infty) : a\in R\}$ is called Right ray Topology on $R \equiv R_a$

Proof ⑥

① $\phi, R\in \tau$

② Let $U,V\in\tau$
Let $U\cap V\neq\phi\neq R$, $U\neq\phi\neq V$ & $U\neq R\neq V$
$U=L_a$ for some $a\in R$
$V=L_b$ for some $b\in R$
$U\cap V=L_a\cap L_b=(-\infty,a)\cap(-\infty,b)=(-\infty,c)$
where $c=\min\{a,b\}$
$\therefore U\cap V\in\tau$

③ Let $U_\alpha\in\tau$ $\forall\alpha\in\Delta$ & $\{U_\alpha\neq R\neq\phi$ $\forall\alpha\in\Delta\}$ & $\bigcup_{\alpha\in\Delta}U_\alpha\neq R$
$U_\alpha=L_{x_\alpha}$ $\forall\alpha\in\Delta$
if $\{x_\alpha:\alpha\in\Delta\}$ is unbounded above
claim $\bigcup U_\alpha=\mathbb{R}$ to show this $\subseteq$ is trival
$\supseteq$ let $y\in R$ $\exists x_{\alpha_0}$ $\ni y<x_{\alpha_0}$
$\Rightarrow y\in L_{x_{\alpha_0}}\subseteq\bigcup U_\alpha$
$\Rightarrow \bigcup_{\alpha\in\Delta}U_\alpha=R$
if $\{x_\alpha:\alpha\in\Delta\}$ is bounded above $\therefore$ has supremum say C
claim $\bigcup_{\alpha\in\Delta}U_\alpha=L_C$

to show this $\subseteq$: Let $y\in\bigcup_{\alpha\in\Delta}U_\alpha$
$\Rightarrow y\in U_{\alpha_0}$ for some $\alpha_0\in\Delta\Rightarrow y\in L_{x_{\alpha_0}}$
where $x_{\alpha_0}\leq C$ $\Rightarrow y<C\Rightarrow y\in L_C$

$\supseteq$: Let $y\in L_C\Rightarrow y<C\Rightarrow\exists\alpha_0\in\Delta\ni y<x_{\alpha_0}<C$
then $y\in L_{x_{\alpha_0}}=U_{\alpha_0}\subseteq\bigcup_{\alpha\in\Delta}U_\alpha$ $\Rightarrow \bigcup_{\alpha\in\Delta}U_\alpha=L_C$

Hence τ is a Top on R
Called Left Ray Topology

* ((Notes))

① α is an upper bounded for A iff $\alpha\geq a$, $\forall a\in A$
② α is an Lower bounded for A iff $\alpha\leq a$ $\forall a\in A$

③ $U(A)$ = The set of all upper bounds for A
④ $L(A)$ = The set of all Lower bounds for A

⑤ The smallest (minimum) element of $U(A)$ is called the Least upper bounded of A ($Lub(A)=\sup A$)
⑥ The greatest element of $L(A)$ is called the greatest Lower bound of A ($glb(A)=\inf A$)

⑦ B is the maximum element of A iff ① $B\in U(A)$
② $B\in A$

⑧ γ is the minimum element of A iff ① $\gamma\in L(A)$
② $\gamma\in A$

example ① $A=(-1,1)\cup\{17,18\}$
$\max A=18$ min A does not exist $U(A)=[18,\infty)$
$L(A)=(-\infty,-1]$
$\sup A=18$
$\inf A=-1$

Dfn Let τ_1 & τ_2 be two topologies for X if $\tau_1 \subset \tau_2$

(i.e) each element of τ_1 is also a member of τ_2

$\therefore$ τ_1 is said to be smallest than τ_2

& τ_2 " " to be larger than τ_1

if neither $\tau_1 \subset \tau_2$ nor $\tau_2 \subset \tau_1$ is true then τ_1 & τ_2 are said to be not comparable غير متساويين

or in comparable

متحققات

Let τ_I = indescrete Topology & τ_2 = desecret Top & τ is a Topology on X then $\tau_1 \subseteq \tau \subseteq \tau_2$

$\boxed{\tau_{l.r} \subsetneq \tau_S}$ because $L_a \in \tau_{l.r} \Rightarrow L_a = (-\infty, a) \in \tau_S$

but $(a,b) \in \tau_S \not\Rightarrow (a,b) \notin \tau_{l.r}$

$\boxed{\tau_{cof} \subsetneq \tau_S}$ Proof

Let $U \in \tau_{cof}$: $U \neq \phi \Rightarrow U^c$ is finite & $U^c = \{x_1, \ldots x_n\}$

Take $x \in U$

Let $\epsilon_i = |x - x_i|$; $i = 1, 2 \ldots n$

Let $\epsilon = \min\{\epsilon_i, i=1, \ldots n\} \Rightarrow$

$x \in (x-\varepsilon, x+\varepsilon) \subseteq U \Rightarrow U \in \tau_S$

if not

$\exists x_k \in (x-\varepsilon, x+\varepsilon) \Rightarrow \epsilon_k = |x - x_k| < \varepsilon$ #

with $\varepsilon < \varepsilon_k$ because $\varepsilon = \min$

&

$(a,b) \in \tau_S$ but $(a,b) \notin \tau_{cof}$

(14) τ_{cof} & $\tau_{l.r}$ They have no relation in between

$\tau_{cof} \nsubseteq \tau_{l.r}$

((Theorms)) + Ex

(1) Let $\{\tau_\alpha : \alpha \in \Delta\}$ be a family of topologies on a set X then $\tau = \bigcap_{\alpha \in \Delta} \tau_\alpha$ is a topology on X

Proof (i) $\phi, X \in \tau_\alpha$, $\forall \alpha \in \Delta \rightarrow \phi, X \in \bigcap_{\alpha \in \Delta} \tau_\alpha$

(ii) Let $U, V \in \bigcap_{\alpha \in \Delta} \tau_\alpha$ then $U, V \in \tau_\alpha, \forall \alpha \in \Delta$

then $U \cap V \in \tau_\alpha, \forall \alpha \in \Delta \Rightarrow U \cap V \in \bigcap_{\alpha \in \Delta} \tau_\alpha$

(iii) Let $U_\beta \in \bigcap_{\alpha \in \Delta} \tau_\alpha$, $\forall \beta \in \Delta'$

then $U_\beta \in \tau_\alpha, \forall \alpha \in \Delta$ & $\forall \beta \in \Delta'$ then $\bigcup_{\beta \in \Delta'} U_\beta \in \tau_\alpha \ \forall \alpha \in \Delta$

then $\bigcup_{\beta \in \Delta'} U_\beta \in \bigcap_{\alpha \in \Delta} \tau_\alpha$

(2) The union of two topology on X need not be a Topological space on X

Solution

Counter example :- Let $X = \{a, b, c\}$

$\tau_1 = \{\phi, X, \{a\}\}$

$\tau_2 = \{\phi, X, \{b\}\}$

but $\tau_1 \cup \tau_2 = \{\phi, X, \{a\}, \{b\}\}$ is not Topology on X

for $\{a\} \in \tau_1$ $\{b\} \in \tau_2 \not\Rightarrow \{a,b\} \in \tau_1 \cup \tau_2$ Q.E.D

NOTES

(1) $\bigcup_{i \in I} (-\infty, a_i) = (-\infty, c)$ where $c = \begin{cases} \sup\{a_i : i \in I\} & \text{if exist} \\ \infty & \text{if the sup does not exist} \end{cases}$

(2) $\bigcup_{i \in I} (a_i, \infty) = (c, \infty)$ where $c = \begin{cases} \inf\{a_i : i \in I\} & \text{if exist} \\ -\infty & \text{if it does not exist} \end{cases}$

5. Let $(X,\mathcal{T})$ be a topological space & Let $\mathcal{U} \subseteq X$ then $\mathcal{U}$ is open iff $\forall x \in \mathcal{U}\ \exists$ an open set $V(x) \ni x \in V(x) \subseteq \mathcal{U}$

Proof

$\Rightarrow$ Let $\mathcal{U}$ be open & $x \in \mathcal{U}$ then take $V_x = \mathcal{U}$. It is clear that V_x is open & $x \in V_x \subseteq \mathcal{U}$

$\Leftarrow$

Claim $\mathcal{U} = \bigsqcup_{x \in \mathcal{U}} V_x$ لذلك ان $\mathcal{U}$ (open) لابد V_x كلها open .. اتحادهم يساوي لذلك لازم يتحقق هنا

$\supseteq$ trivale

$\subseteq$ if $t \in \mathcal{U}$ then $t \in V_t \subseteq \bigsqcup_{x \in \mathcal{U}} V_x \Rightarrow \mathcal{U} = \bigsqcup_{x \in \mathcal{U}} V_x$

---

4. $\mathcal{T}$ is discrete topology iff $\{x\} \in \mathcal{T}$, $\forall x \in X$

Proof: $\Rightarrow$ Let $\mathcal{T}_{dis} = P(X)$ so $x \in X$ then $\{x\} \subseteq X$ & hence $\{x\} \in \mathcal{T}$ $\forall x \in X$

$\Leftarrow$

Try $P(X) = \mathcal{T}$

Let $\{x\} \in \mathcal{T}$, $\forall x \in X$

Let $A \subseteq X$ if $A = \phi$ then $A \in \mathcal{T}$

if $A \neq \phi$ then $A = \bigsqcup_{x \in A} \{x\}$ (open set) so $A \in \mathcal{T}$

& hence $P(X) \subseteq \mathcal{T}$

but $\mathcal{T} \subseteq P(X)$ always.

there for $\mathcal{T} = P(X)$

---

5. Let the set under consideration be N, $\forall n \in N$ define $\mathcal{U}_n = \{n, n+1, \ldots\}$ & Let $\mathcal{T}$ consist of ϕ, N & all subsets $\mathcal{U}_n$ of N. Proof that $\mathcal{T}$ is a Top for N

Proof

$X = N$ & $\mathcal{T} = \{\phi, N, \{\mathcal{U}_n : n \in N\}\}$ is a Top on N

1. $\phi, N \in \mathcal{T}$
2. Let $\mathcal{U}, V \in \mathcal{T}$ (assume $\phi \neq \mathcal{U}$, $V \neq N$)
then $\mathcal{U} = \mathcal{U}_m$ & $V = \mathcal{U}_n$ for some $n, m \in N$
Let $K = \max\{m, n\}$ $\Rightarrow \mathcal{U} \cap V = \mathcal{U}_m \cap \mathcal{U}_n = \mathcal{U}_K \in \mathcal{T}$
3. Let $\phi \neq G_\alpha \in \mathcal{T}$, $\forall \alpha \in \Delta$ (assume $\phi \neq \bigcup G_\alpha \neq N$)
then $G_\alpha = \mathcal{U}_{n_\alpha}$ for some $n_\alpha \in N$. Let $n_0 = \min\{n_\alpha : \alpha \in \Delta\}$
$\Rightarrow \bigsqcup_{\alpha \in \Delta} G_\alpha = \mathcal{U}_{n_0}$ So $(N, \mathcal{T})$ is Top-space

---

* 6. Let $(X, \mathcal{T})$ be a top-space then

1. The finite $\sqcap$ of open set is open?
2. The arbitrary $\sqcup$ of open set is open?
3. The finite $\sqcup$ of closed set is closed?
4. The arbitrary $\sqcap$ of closed set is closed?

Proof 1. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots \mathcal{U}_n$ be open sets then $\mathcal{U}_i \in \mathcal{T}$ $i=1,2,\ldots n$
$\Rightarrow \bigcap_{i=1}^{n} \mathcal{U}_i \in \mathcal{T}$ ie ($\bigcap_{i=1}^{n} \mathcal{U}_i$ is open)

2. Let $\mathcal{U}_\alpha$ $\forall \alpha \in \Delta$ be open then $\mathcal{U}_\alpha \in \mathcal{T}$, $\alpha \in \Delta$ $\Rightarrow \bigcup_{\alpha \in \Delta} \mathcal{U}_\alpha \in \mathcal{T}$
$\therefore \bigcup \mathcal{U}_\alpha$ is open

4. C_α be closed $\forall \alpha \in \Delta$ $\therefore$ $C_\alpha = (X - \mathcal{U}_\alpha)$ $\forall \alpha$ where $\mathcal{U}_\alpha$ is open)
$\bigcap_{\alpha \in \Delta} C_\alpha = \bigcap_{\alpha \in \Delta} (X - \mathcal{U}_\alpha) = X - \bigcup_{\alpha \in \Delta} \mathcal{U}_\alpha$ $\therefore$ $\bigcap C_\alpha$ is closed

3. Let $\{C_i, i=1, \ldots, n\}$ be closed sets to show $\bigcup_{i=1}^{n} C_i$ is closed
Since C_i $i=1 \ldots n$ is closed $\Rightarrow \{C_i^c, i=1 \ldots n\}$ be open $\in \mathcal{T}$
$\bigcap_{i=1}^{n} C_i^c \in \mathcal{T} \Rightarrow (\bigcap_{i=1}^{n} C_i^c)^c$ is closed $\Rightarrow \bigcup_{i=1}^{n} C_i$ closed

(3) Let $(X, \mathcal{T})$ be a topological space & Let $\mathcal{U} \subseteq X$ then $\mathcal{U}$ is open iff $\forall x \in \mathcal{U}\ \exists$ an open set $V(x)$ $\ni x \in V(x) \subseteq \mathcal{U}$

Proof

$\rightarrow$ Let $\mathcal{U}$ be open

& $x \in \mathcal{U}$ then take $V_x = \mathcal{U}$. It is clear that V_x is open & $x \in V_x \subseteq \mathcal{U}$

$\leftarrow$

Claim $\mathcal{U} = \bigsqcup_{x \in \mathcal{U}} V_x$ (open) لازم أثبت أن $\mathcal{U}$ .. اتحاد V_x open .. اخذ عنصر من داخل هذا

$\supseteq$ trivale

(5) & $t \in \mathcal{U}$ then $t \in V_t \subseteq \bigsqcup_{x \in \mathcal{U}} V_x \Rightarrow \mathcal{U} = \bigsqcup_{x \in \mathcal{U}} V_x$

(4) $\mathcal{T}$ is discrete topology iff $\{x\} \in \mathcal{T}$, $\forall x \in X$

Proof: $\rightarrow$ Let $\mathcal{T}_{dis} = P(X)$ so $x \in X$ then $\{x\} \subseteq X$

hence $\{x\} \in \mathcal{T}$ $\forall x \in X$

$\leftarrow$

Try $P(X) = \mathcal{T}$

Let $\{x\} \in \mathcal{T}$, $\forall x \in X$

Let $A \subseteq X$ & $A = \phi$ then $A \in \mathcal{T}$

& $A \neq \phi$ then $A = \bigsqcup_{x \in A} \{x\}$ (open set) so $A \in \mathcal{T}$

hence $P(X) \subseteq \mathcal{T}$

but $\mathcal{T} \subseteq P(X)$ always.

therefore $\mathcal{T} = P(X)$

(5) Let the set under consideration be N, $\forall n \in N$ define $U_n = \{n, n+1, \ldots\}$ & Let $\mathcal{T}$ consist of ϕ, N & all subsets U_n of N. Proof that $\mathcal{T}$ is a Top for N.

Proof

$X = N$ & $\mathcal{T} = \{\phi, N, \{U_n : n \in N\}\}$ is a Top on N

(1) $\phi, N \in \mathcal{T}$

(2) Let $U, V \in \mathcal{T}$ (assume $\phi \neq U$, $V \neq N$)

then $U = U_m$ & $V = U_n$ for some $n, m \in N$

Let $K = \max\{m, n\}$ $\Rightarrow U \cap V = U_m \cap U_n = U_K \in \mathcal{T}$

(3) Let $\phi \neq G_\alpha \in \mathcal{T}$, $\forall \alpha \in \Delta$ (assume $\phi \neq \bigcup G_\alpha \neq N$)

then $G_\alpha = U_{n_\alpha}$ for some $n_\alpha \in N$. Let $n_0 = \min\{n_\alpha : \alpha \in \Delta\}$

$\Rightarrow \bigsqcup_{\alpha \in \Delta} G_\alpha = U_{n_0}$ So $(N, \mathcal{T})$ is Top-space

(6) Let $(X, \mathcal{T})$ be a top-space then

(1) The finite $\sqcap$ of open set is open?

(2) The arbitrary $\sqcup$ of open set is open?

(3) The finite $\sqcup$ of closed set is closed?

(4) The arbitrary $\sqcap$ of closed set is closed?

Proof (1) Let $U_1, U_2, \ldots U_n$ be open sets then $U_i \in \mathcal{T}$ $i = 1, 2, \ldots n$

$\Rightarrow \bigcap_{i=1}^{n} U_i \in \mathcal{T}$ i.e. ($\bigcap_{i=1}^{n} U_i$ is open)

(2) Let U_α $\forall \alpha \in \Delta$ be open then $U_\alpha \in \mathcal{T}$, $\alpha \in \Delta \Rightarrow \bigcup_{\alpha \in \Delta} U_\alpha \in \mathcal{T}$

$\therefore \bigcup U_\alpha$ is open

(4) C_α be closed $\forall \alpha \in \Delta$ $\therefore C_\alpha = X - U_\alpha$ $\forall \alpha$ where U_α is open)

$\bigcap_{\alpha \in \Delta} C_\alpha = \bigcap_{\alpha \in \Delta} (X - U_\alpha) = X - \bigsqcup_{\alpha \in \Delta} U_\alpha$ $\therefore \bigcap C_\alpha$ is closed

(3) Let $\{C_i, i = 1, \ldots, n\}$ be closed sets to show $\bigsqcup_{i=1}^{n} C_i$ is closed.

Since C_i $i = 1, \ldots, n$ is closed $\Rightarrow \{C_i^c, i = 1, \ldots, n\}$ be open $\in \mathcal{T}$

$\bigcap_{i=1}^{n} C_i^c \in \mathcal{T} \Rightarrow (\bigcap_{i=1}^{n} C_i^c)^c$ is closed $\Rightarrow \bigcup_{i=1}^{n} C_i$ closed $\square$

Notes ① arbitrary union of closed sets need not be closed.

Counter example →

(R, T_s), let $\{r_i\}$ is closed $\forall r_i \in Q$ but $Q = \bigsqcup_{i=1}^{\infty} \{r_i\}$ [illegible]

since $\forall x$ can't be contained in (a,b) which lies entirely in $R - Q$ then $R - Q$ is not open $\Rightarrow$ & so Q is not closed.

② The union of two closed set is closed.

Proof

Let C_1, C_2 be closed i.e. ($C_1 = X - U_1$ & $C_2 = X - U_2$, U_1, U_2 is open)

$C_1 \cup C_2 = (X - U_1) \cup (X - U_2) = \underbrace{X - \underbrace{(U_1 \cap U_2)}_{\text{open}}}_{\text{closed}}$ $\therefore C_1 \cup C_2$ is closed

③ The arbitrary $\sqcap$ of open set need not be open

Counter example:-

$X = R$, $\tau = T_{l.r}$ - $U_n = (-\infty, \frac{1}{n})$ $n \in N$ is open

but $\bigcap_{n=1}^{\infty} U_n = (-\infty, 0]$ is not open

Dfn Let (X, τ) be topological space a subset C of X is called be $\boxed{\text{Closed}}$ if it's complement is open set

i.e. $C \subseteq X$ closed iff $\boxed{X - C}$ is open.

Examples

① in (R, T_u) the $[a,b]$ closed because $R - [a,b] = \overset{\text{open}}{\overbrace{\overset{\text{open}}{(-\infty, a)} \cup \overset{\text{open}}{(b, \infty)}}}$

② Let $A = [0,1] = \{x : 0 \le x \le 1\}$ is closed because

$R / A = \{x : x > 1\} \sqcup \{x : x < 0\}$

③ Let $a \in R$, $\{a\}$ is closed However in (R, T_s) its so

$R - \{a\} = (-\infty, a) \cup (a, \infty)$ is open

Dfn Let τ be a topological space on X, $U \subseteq X$ is said to be open $\Leftrightarrow$ $U \in \tau$

⑦ Between any two distinct real number a & b there is a rational number. i.e. ($a < b$, $\exists$ $c \in Q$, $\ni a < c < b$)

Proof [illegible]

since $a < b$ so $b - a > 0$ then $\lim_{n \to \infty} n(b-a) = +\infty$

then $\exists n_0 \in N \ni n_0(b-a) > 10$ [illegible]

$n_0 b - n_0 a > 1.0$

$\exists \boxed{m} \in Z \ni n_0 a < m < n_0 b$

$a \le \frac{m}{n_0} \le b$ but $\frac{m}{n_0} \in Q$

$\therefore \exists c \in Q$: $a < c < b$

⑧ $U \ne \phi \in T_s$ iff U is union of open intervals.

Proof → let $U \in T_s$ for $x \in U$ $\exists$ an open interval I_x

$\ni x \in I_x \subseteq U$

Claim $U = \bigsqcup_{x \in U} I_x$

($\supseteq$) trivial

($\subseteq$) let $t \in U \rightarrow t \in I_t \subseteq \bigsqcup_{x \in U} I_x \rightarrow t \in \bigsqcup_{x \in U} I_x$

$\therefore U \subseteq \bigsqcup_{x \in U} I_x$

$\therefore U = \bigsqcup_{x \in U} I_x$

⟵

clear because the union of arbitrary open set is open

⑨ For R if $\{U_\alpha : \alpha \in \Delta\}$ is any family of disjoint open interval then $\{U_\alpha : \alpha \in \Delta\}$ is countable

Proof [illegible]

Define $h : U_\alpha \longrightarrow b_\alpha \in Q$ by $h(\alpha) = b_\alpha$

& $\alpha \ne \beta$ then $b_\alpha \ne b_\beta$. Then h is 1-1 $\therefore |\Delta| \le |Q| = \aleph_0$ $\therefore \Delta$ is countable

$\square$

7

$(\mathbb{R}, T_s)$, $A \subseteq \mathbb{R}$, A is an interval iff whenever $x, y \in A$ & $x < t < y$ then $t \in A$ $t \in \mathbb{R}$

Proof

$\Rightarrow$ obvious

$\Leftarrow$ Consider the case A is bounded below & above

Let $a = glb(A)$ & $b = Lub(A)$ (inf, sup)

claim $(a,b) \subseteq A \subseteq [a,b]$

Let $t \in (a,b) \Rightarrow a < t < b \Rightarrow \exists\, x, y \in A \ni a \le x \le t \le y \le b$
$\Rightarrow t \in A$

So A either (a,b), $[a,b)$, $(a,b]$ or $[a,b]$.

IF I_α is an open interval $\forall \alpha \in \Delta$ & $\bigcap_{\alpha \in \Delta} I_\alpha \neq \phi$

then $\bigsqcup_{\alpha \in \Delta} I_\alpha$ is an open interval

Proof. Let $p \in \bigcap_{\alpha \in \Delta} I_\alpha$ & consider $\boxed{\text{the case}}$ when $\bigsqcup_{\alpha \in \Delta} I_\alpha$ is bounded above but not below Let $(b = Lub(\bigsqcup_{\alpha \in \Delta} I_\alpha))$

claim $\bigsqcup_{\alpha \in \Delta} I_\alpha = (-\infty, b)$

$\subseteq$ Let $x \in \bigcup I_\alpha \Rightarrow x \le b \Rightarrow x \in (-\infty, b]$, However; $b \notin \bigsqcup I_\alpha$ suppose $b \in \bigsqcup_{\alpha \in \Delta} I_\alpha$

then $b \in I_{\alpha_0}$ for some $\alpha_0 \in \Delta$ $\exists\, (c,d) \ni b \in (c,d) \subseteq I_{\alpha_0} \subseteq \bigcup I_\alpha$ #

$\supseteq$

Let $x \in (-\infty, b) \Rightarrow x < b \Rightarrow \exists\, y \in \bigcup_{\alpha \in \Delta} I_\alpha \ni x < y$

moreover; $\exists\, z \in \bigsqcup I_\alpha \ni z < x$ so $y \in I_{\alpha_0}$ & $z \in I_{\alpha_1}$

for some $\alpha_0, \alpha_1 \in \Delta$ — $x \le p \Rightarrow z < x \le p$, $z, p \in I_{\alpha_1}$
$\Rightarrow x \in I_{\alpha_1}$

— $p \le x \Rightarrow p \le x < y$, $p, y \in I_{\alpha_0}$
$\Rightarrow x \in I_{\alpha_0}$

(12) Let $\mathcal{U} \neq \phi \in T_s \Leftrightarrow \mathcal{U}$ is the countable union of disjoint open intervals.

Proof $\Leftarrow$ obvious

$\Rightarrow$ Let $\phi \neq \mathcal{U} \in T_s$ for $x \in \mathcal{U}$, take I_x = the union of all open intervals that contain x & contained in $\mathcal{U}$

Let $x, y \in \mathcal{U}$ claim $I_x = I_y$ or $I_x \cap I_y = \phi$

Suppose $I_x \cap I_y \neq \phi$ then $I_x \sqcup I_y = I$ is an open interval

However; $I_x \subseteq I \subseteq I_x \Rightarrow I_x = I$ $(x \in I \subseteq \mathcal{U})$

The Family $= \{I_x : x \in \mathcal{U}\}$ is a family of disjoint open interval & so it is countable

Since $\mathcal{U} = \bigsqcup_{x \in I} I_x$ $(t \in \mathcal{U} \Rightarrow t \in I_t \subseteq \bigcup_{x \in \mathcal{U}} I_x)$

There fore; $\mathcal{U}$ is the countable union of disjoint open intervals

(13) IF $a \& b \in \mathbb{R}$ & $a \neq b$ $\boxed{\text{Prove}}$ $\exists$ disjoint open sets contain a & b respectively?

Solution

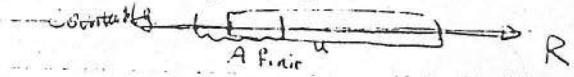
$a, b \in \mathbb{R}$ $(a \neq b)$ $\exists\, V, \mathcal{U}$ $\ni a \in \mathcal{U}$, $b \in V$ & $\mathcal{U} \cap V = \phi$

assume $a < b$ choose $c_1, c_2, c_3 \in \mathbb{R}$

$\ni c_1 < a < c_2 < b < c_3$

Take $\mathcal{U} = (c_1, c_2)$ & $V = (c_2, c_3)$

(14) Let $\mathcal{U} \in T_s$ & $A \subset \mathbb{R}$ (finite) $\boxed{\text{Prove}}$ $(\mathcal{U} - A)$ (open)



Proof

Suppose $\mathcal{U} \cap A = \{x_1, x_2, \ldots, x_n\}$ & $\mathcal{U} / A \cap \mathcal{U} = \{\mathcal{U} - \{x_1, \ldots, x_n\}$

$\mathcal{U}$ = countable $\bigcup$ of disjoint open intervals, $\forall\, x_i \in$ one of this interval

then Let $I_{x_i} = (a,b) \Rightarrow I_{x_i} = (a, x_i) \cup (x_i, b)$

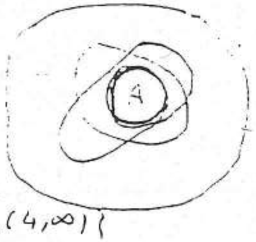
& so on again $\mathcal{U}/A$ = countable $\bigcup$ of disjoint open intervals

$\mathcal{U}/A \in T_s$ — open $\square$

(1) Closure

Dfn Let (X,τ) be a topological space & $A \subset X$ then (the closure of A) is the intersection of all closed sets in X which contain A

i.e. $\overline{A} = \sqcap \{ F \underset{closed}{\subseteq} X ,\ A \subseteq F \}$



Example (1) Let $(R, \{\phi, R, (c,\infty), (4,\infty)\})$
what is $\overline{(2,3)}$
∴ open $\{R, \phi, (c,\infty), (4,\infty)\}$
closed $\phi, R, [-\infty, c], (-\infty, 4]$

$\overline{(2,3)} = R \sqcap (-\infty, 4] = (-\infty, 4]$

Property of closure

(1) $\overline{A}$ is always a closed set
(2) $A \subseteq \overline{A}$ always
(3) if C is any closed set containing $A \Rightarrow \overline{A} \subset C$??

Theorm & Ex

(1) $\overline{A}$ is the smallest closed set containg A

Proof
since $A \subset \overline{A}$ then if any closed set contain A is contain $\overline{A} \Rightarrow \overline{A}$ is smallest.

(2) $\overline{\phi} = \phi$ & $\overline{R} = R$ & $\boxed{\overline{\overline{A}} = \overline{A}}$

(3) C is closed iff $\overline{C} = C$

Proof
$\Rightarrow$ C is closed & containg C so $\overline{C} \subseteq C$ (becuse smalle...)
but $C \subseteq \overline{C}$ always true so $\overline{C} = C$
$\Leftarrow$ suppose $\overline{C} = C$ ∴ $\overline{C}$ is closed always then C is close...

(4) if $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$

Proof
since $A \subseteq B$ & $B \subseteq \overline{B}$ then $A \subseteq \overline{B}$
since $\overline{B}$ is always closed so $\overline{A} \subseteq \overline{B}$

(5) $\boxed{\overline{A \cup B} = \overline{A} \cup \overline{B}}$

Proof
$(\subseteq)$ $A \subseteq \overline{A}$ & $B \subseteq \overline{B} \Rightarrow A \cup B \subseteq \overline{A} \cup \overline{B}$ [since the union of finite closed ...]
∴ $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$ (becuse closed)
∴ $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ —— (1)
$(\supseteq)$ $A \subseteq A \cup B$ & $B \subseteq A \cup B$ $\Rightarrow$ $\overline{A} \subseteq \overline{A \cup B}$, $\overline{B} \subseteq \overline{A \cup B}$ $\Rightarrow \overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$ —— (2)
from (1) & (2) get $\boxed{\overline{A \cup B} = \overline{A} \cup \overline{B}}$

Q.E.D

(6) Let (X,τ) be Top-space & let $A \subseteq X$ then
$\boxed{x \in \overline{A} \Leftrightarrow \forall \text{open } U \ni x \ \& \ U \cap A \neq \phi}$

Proof
$\Rightarrow$ let $x \in \overline{A}$ & suppose $\exists\ U \ni x$ (open) $\ni U \cap A = \phi$
then $\boxed{A \subseteq X - U}$ so $\overline{A} \subseteq X - U$. since $x \in \overline{A} \Rightarrow x \in X - U$
Hence $x \in U \sqcap X - U = \phi$ #
$\Leftarrow$ Let $x \in X$ be given $\ni U \cap A \neq \phi$ $\forall$ open $U \ni x$
Suppose $x \notin \overline{A} \Rightarrow x \in X - \overline{A}$ (open) $\Rightarrow x \in (X - \overline{A}) \sqcap A \subseteq (X - A) \cap A = \phi$

$(X, T_{cof}) \equiv (X, T_{dis})$

9

Prove that cofinite topology on a finite set X is the same as the discrete topology on X.

Solution

Since $X = \{x_1, x_2, \ldots x_n\}$ be finite

& $T_{cof} = \{\phi, X, U\}$ where $((X - U^c = \text{finite}))$

So $U_i = X - V_i$ Where V_i is finite $\forall\ V_i \in P(X)$

but X is finite then U = every finite set of X

so $T = P(X)$

Let $A \subseteq R$ have an upper bound

Proof that the least upper bound of A belongs to $\overline{A}$

ie ($p = Lub(A)$ then $p \in \overline{A}$)

Proof

Suppose U is an open set $\ni p \in U$

then $\exists (a,b) \ni p \in (a,b) \subseteq U$

then $\exists\ x \in A \ni a < x \leq p < b \Rightarrow x \in (a,b)$

then $x \in A \cap U \neq \phi \Rightarrow x \in \overline{A}$

$A \subset X$ & T_1, T_2 Two Topology on $X \ni T_1 \subseteq T_2$

then $\overline{A}^{T_2} \subseteq \overline{A}^{T_1}$

Proof

Let $x \in \overline{A}^{T_2}$ to show $x \in \overline{A}^{T_1}$

Let $U \in T_1 \subseteq T_2$ & $x \in U$ then $U \in T_2$

& $x \in U$ then $U \cap A \neq \phi$

$\Rightarrow x \in \overline{A}^{T_1}$

Q.E.D

10

Notes

(1) Give an example of a Collection of opensets in a space (X,T) whose $\cap$ is not open

example Let $(-\frac{1}{n}, \frac{1}{n}) \in T_s\ \forall n \in N$ & $\bigcap_{n \in N} (\frac{-1}{n}, \frac{1}{n}) = \{0\} \notin T_s$ not open

(2) Given example of collection of closed set in Top whose $\cup$ is not closed.

ex $T_{L.r} = \{[x, \infty),\ x \geq a$ for some fixed $a \in R\}$

$Q = \{r_1, r_2, \ldots, r_n, \ldots\} = \bigcup_{i=1}^{\infty} \{r_i\}$ not closed

(3) Is there a set upon which the discrete & indiscrete Top are equal?

$X = \{a\}$ $T_{dis} = \{\phi, \{a\} = X\}$ = indis

(4) Given example of a Top on an infinite set which has only a finite number of elements

(R, T) : $T = \{\phi, R, \{0\}, \{x : x \geq 0\}, \{x : x \leq 0\}\}$

(N, T) : $T = \{\phi, N, N_1 = \{a : a \in N, a \text{ even}\}, U_2 = \{b, b \in N, b \ldots$

(5) Consider the set of real numbers $A = \{x : 0 < x < 1\} \cup \{2\}$ in R

Describe $\overline{A}$ for the following Top on R

(a) strong (b) cofinite (c) left ray (d) Discrete

Solve $A = (0,1) \cup \{2\}$

a. T_s $A \subseteq \overline{A}$ $\forall x < 0$ $x \notin \overline{A}$ $x = 0 \in U \Rightarrow 0 \in (a,b) \subseteq U \Rightarrow 0 \in \overline{A}$

$x = 1 \in U \Rightarrow 1 \in (a,b) \subseteq U \Rightarrow 1 \in \overline{A}$

$\forall\ x \in (1,2)$, $x \notin \overline{A}$ $\forall x > 2$, $x \notin \overline{A}$ so $\overline{A} = [0,1] \cup \{2\}$

b. $T = T_{cof}$ closed set containing A: R $\therefore \overline{A} = R$

c. closed containg A in $T_{L.r}$: $R \cup [a, \infty) : a \leq 0$ $\therefore \bigcap_{a \leq 0} [a, \infty) = [0, \infty)$ $\therefore \overline{A} = [0, \infty)$

d. T_{dis} $\overline{A} = A$ closed

(2) Topology Induced by function

(1) Let $f: X \longrightarrow Y$ be any function and suppose X has a topology T_X then the collection $T_Y = \{V : V \subseteq Y \;\&\; f^{-1}(V) \in T_X\}$ is a Top on Y
$= T_Y(f, T_X)$

Proof

$\phi \in T_Y$ because $\phi \subseteq Y \;\&\; f^{-1}(\phi) = \phi \in T_X$
$Y \in T_Y$ because $Y \subseteq Y \;\&\; f^{-1}(Y) = X \in T_X$

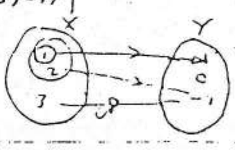
(2) Let $A, B \in T_Y$ ... Try $A \cap B \in T_Y$
$A \in T_Y \Rightarrow A \subseteq Y \;\&\; f^{-1}(A) \in T_X$
$B \in T_Y \Rightarrow B \subseteq Y \;\&\; f^{-1}(B) \in T_X$
$A \cap B \subseteq Y \;\&\; f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \in T_X \quad \therefore A \cap B \in T_Y$

(3) Let $A_\alpha \in T_Y$, $\alpha \in \Delta$ Try $\bigsqcup_{\alpha\in\Delta} A_\alpha \in T_Y$
since $A_\alpha \in T_Y \Rightarrow A_\alpha \subseteq Y \;\&\; f^{-1}(A_\alpha) \in T_X \quad \forall \alpha \in \Delta$
now $\bigsqcup_{\alpha\in\Delta} A_\alpha \subseteq Y \;\&\; f^{-1}(\bigsqcup_{\alpha\in\Delta} A_\alpha) = \bigsqcup_{\alpha\in\Delta} f^{-1}(A_\alpha) \in T_X$
$\therefore \bigsqcup_{\alpha\in\Delta} A_\alpha \in T_Y \quad \therefore T_Y$ is a top on Y

Q.E.D

Example

Let $X = \{1,2,3\}$ & $T_X = \{\phi, X, \{1\}, \{1,2\}\}$
$Y = \{1, 0, -1\}$ $\quad f = \{(1,1), (2,-1), (3,-1)\}$
Find $T_Y(f, T_X) =$



(1) $\phi \in Y$ & $f^{-1}(\phi) = \phi \in T_X$
ϕ

$T_Y(f, T_X) = \{\phi, \{1\}, \{c\}, \{0,1\}, \{1,-1\}, \{1,0,-1\}\}$
$f^{-1}(-1) = \{2,3\} \notin T_X$, $f^{-1}(\{-1,0\}) = \{2,3\} \notin T_X$
$f^{-1}\{-1,1\} = X \in T_X$
$f^{-1}(\{0,1\}) = \{1\} \in T_X$
$f^{-1}(\{1,0,-1\}) = \{1,2,3\} \in T_X$

(2) Let $f: X \longrightarrow Y$ be any function & Let Y have a top T_Y then collection $T_X = \{f^{-1}(U) : U \in T_Y\}$ is a top. for X this top is called the Topology induced on X by f & (Y, T_Y) is denoted by $T_X(f, T_Y)$

Proof

(1) $\phi \in T_X$ because $\phi = f^{-1}(\phi)$ & $\phi \in T_Y$
$X \in T_X$ because $X = f^{-1}(Y)$ & $Y \in T_Y$

(2) let $A, B \in T_X$ Try $A \cap B \in T_X$
since $A \in T_X$ then $\exists\, U_1 \in T_Y \ni A = f^{-1}(U_1)$
$\quad B \in T_X$ then $\exists\, U_2 \in T_Y \ni B = f^{-1}(U_2)$
$\therefore A \cap B = f^{-1}(U_1) \cap f^{-1}(U_2) = f^{-1}(U_1 \cap U_2)$
but $U_1 \cap U_2 \in T_Y$ so $A \cap B \in T_X$

(3) Let $A_\alpha \in T_X$ $\alpha \in \Delta$ Try $\bigsqcup_{\alpha\in\Delta} A_\alpha \in T_X$
$\exists\, U_\alpha \in T_Y : A_\alpha = f^{-1}(U_\alpha)$, $\alpha \in \Delta$
$\bigsqcup_{\alpha\in\Delta} A_\alpha = \bigsqcup_{\alpha\in\Delta} f^{-1}(U_\alpha) = f^{-1}(\bigsqcup_{\alpha\in\Delta} U_\alpha)$
But $\bigsqcup_{\alpha\in\Delta} U_\alpha \in T_Y$
so
$\bigcup_{\alpha\in\Delta} A_\alpha \in T_X \quad \therefore T_X(f, T_Y)$ is topology on X

Relative Topology

11

Defn Let (X,τ) be top-space & $A \subseteq X$

$j : A \hookrightarrow X$ — The inclusion map بالإنجليزي

$j(x) = x \quad , x \in A$ Define

$T_A = \{ j^{-1}(U) : U \in \tau \} = \{ U \cap A : U \in \tau \}$ is called the relative topology on A

(A, T_A) the sub space top on (X, τ)

Let (A, T_A) be a subspace of (X,τ) then $V \in T_A \Leftrightarrow V = U \cap A$ where $U \in \tau$

Proof let $V \in T_A \Leftrightarrow V = j^{-1}(U)$ for some $U \in \tau$

$\Leftrightarrow V = U \cap A$ for some $U \in \tau$

example

$X = \{1,2,3,4,5\}$ & $A = \{1,4,5\}$

$T_X = \{ \phi, X, \{1\}, \{1,2\}, \{3,5\}, \{1,2,3,5\}, \{1,3,5\} \}$

$T_A = \{ \phi, A, \{1\}, \{5\}, \{1,5\} \}$

Let (A, T_A) be a sub space of (X,τ) then C is closed in A iff $C = F \cap A$ where F is closed in X ($F \subseteq X$, F closed)

Proof Let C be closed in $A \Leftrightarrow A - C$ is open in A

$\Leftrightarrow A - C = U \cap A$ where U is open in X

$\Leftrightarrow C = A - U \cap A \Leftrightarrow C = A \cap (X - U) = A \cap F$

$\Leftrightarrow C = A \cap F$ where F is closed in X ☒

---

(6) If $A \in \tau$ then $T_A \subseteq \tau$ then In other words every open subset of an open subspace A of X is open in X كالتالي

Proof

Given: A is an open set in (X,τ), U is open in (A, T_A) (i.e $U \subseteq A$ & $U \in T_A$) **Try** U is open in (X,τ)

Since $U \subseteq A$ & $A \subseteq X \Rightarrow U \subseteq X$

Now $U \in T_A \Rightarrow \exists\, V \in \tau : U = V \cap A$

But $A \in \tau$ so $V \cap A \in \tau \quad \therefore U \in \tau$ ☒

---

(7) Let (A, T_A) be a subspace of (X,τ) & let $B \subseteq A$ then $\overline{B}^{A} = A \cap \overline{B}^{X}$

Proof

$A \cap \overline{B}^{X} = A \cap \{ \cap \{F : F \text{ closed in } X \text{ \& } B \subseteq F\} \}$

$= \cap \{ A \cap F : F \text{ closed \& } B \subseteq F \text{ in } X \}$

$= \overline{B}^{A}$

---

(8) Prove that every closed subset of a closed sub space is closed in X.

i.e (X,τ) is a Top-space, (A, T_A) is a closed subspace of (X,τ) [A closed]

C closed in $A \Rightarrow C$ closed in X

$\Rightarrow C = \underbrace{F}_{\text{closed}} \cap A$ where F closed in X

Note - A-closed, C is an open subset (A, T_A) is not neccesary open in (X,τ)

example (R, T_s), (A, T_A) open subspace ($A = [0,1)$ & $C = [0, \frac{1}{2}]$ closed)

$\therefore [0, \frac{1}{2}] \cap A$ but not closed in R.

(X,τ) is a top-space. Prove that a subspace of subspace is subspace of (X,τ).

Let (X,τ) be Top-space, $A \subseteq X$, (A,τ_A) is subspace (X,τ), $B \subseteq A$ & (B,τ_{AB}) denote the subspace on B of (A,τ_A)

Try: $V \in \tau_{AB} \Leftrightarrow V = U \cap B$ where $U \in \tau$

$\Leftrightarrow V = G \cap B$; $G \in \tau_A$

$\Leftrightarrow V = (U \cap A) \cap B$ where $U \in \tau$

$\Leftrightarrow V = U \cap B$ where $U \in \tau$.

$\Rightarrow$

③ Interior & Exterior & Boundary of a set

Dfn ① Interior point of A: Let (X,τ) be a topological space $A \subseteq X$, a point $x \in A$ is an interior point of A if $\exists$ an open set $U \ni x \in U \subseteq A$

② The set of all interior points of A is called (Interior of A) denote by $Int(A) = A^{\circ}$

Propertes of Int(A)

① $A^{\circ} \subseteq A$ & $Int(\phi) = \phi$ & $Int(X) = X$

② $Int(A)$ is open for any $A \subseteq X$.

Proof

since $\forall x \in A^{\circ} \Rightarrow x$ is an enterior point of A then $\exists\, U$ (open) $\ni x \in U \subseteq A$ then $x \in U \subseteq A^{\circ} \Rightarrow A^{\circ}$ is open

③ U is open iff $U = Int\, U$

Proof

$\Rightarrow$ let U is open to show $U = Int\, U$

(1) $U^{\circ} \subseteq U$ always

(2) ($\supseteq$) let $x \in U$ then $x \in U$ (open) $\subseteq U$ so $x \in U^{\circ}$

$\therefore U \subseteq U^{\circ}$ $\therefore U = U^{\circ}$

$\Leftarrow$

since U° is always open then U is open because $U = U^{\circ}$

Int(A) = The union of all open sets contained in A
$= \bigcup \{U : U \subseteq A, \text{open}\}$

Proof
Let U be open & $U \subseteq A$, If $t \in U$ then $t \in U \subseteq A$
$\therefore t$ is interior point of A so $U \subseteq A^\circ$
$\Rightarrow \bigcup U \subseteq A^\circ$

$x \in A^\circ \Rightarrow x$ is an interior then $\exists x \in U \subseteq A$, open
then $x \in \bigcup\{U \subseteq A, \text{open}\}$ so $A^\circ \subseteq \bigcup U$

Int(A) is the largest open set contained (in) A

Proof
$A^\circ \subseteq A$ & A° is open
Let G be any open set $\ni G \subseteq A$ (Try) $G \subseteq A^\circ$
Let $x \in G \Rightarrow x \in G(\text{open}) \subset A$ $x \in A^\circ$ $\therefore G \subseteq A^\circ$

$A \subseteq B$ then $A^\circ \subseteq B^\circ$

Proof
$A^\circ \subseteq A \subseteq B$ but $B^\circ \subset B$ (& B° is largest open by (5))
$\therefore A^\circ \subset B^\circ$

$(A \cap B)^\circ = A^\circ \cap B^\circ$

Proof
(1) $A \cap B \subset A$ then $(A \cap B)^\circ \subseteq A^\circ$
$A \cap B \subset B$ then $(A \cap B)^\circ \subseteq B^\circ$
$\therefore (A \cap B)^\circ \subseteq A^\circ \cap B^\circ$

[illegible]

(2) $\left.\begin{array}{l} A^\circ \subseteq A \\ B^\circ \subseteq B \end{array}\right\} \Rightarrow \underbrace{(A^\circ \cap B^\circ)}_{\text{open}} \subseteq A \cap B$ then $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$
$\therefore (A \cap B)^\circ = A^\circ \cap B^\circ$

(8) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

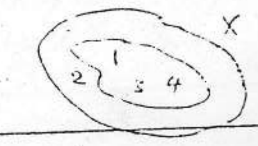
(9) $(A \cup B)^\circ \neq A^\circ \cup B^\circ$
Counterexample: Let (R, T_s) be Top space
& Q: rational no's & T: irrational no's
$R = Q \cup T$
$(Q \cup T)^\circ = R^\circ = R$
However $Q^\circ = \phi$ & $T^\circ = \phi \Rightarrow Q^\circ \cup T^\circ = \phi$
$\therefore (Q \cup T)^\circ \neq Q^\circ \cup T^\circ$

[illegible] (14)

Dfn (2) a point p is called an exterior of A
if $\exists G$ open $\ni p \in G \subseteq A^c$
(or) (X, τ) be Top-space $A \subseteq X$ $\boxed{Ext(A) = Int(X-A)}$

* Dfn (3) a point p is called $\boxed{\text{a boundary point}}$ $\boxed{\text{of } A}$ if $\forall$ open G containing p we have $G \cap A \neq \phi$
& $G \cap A^c \neq \phi$

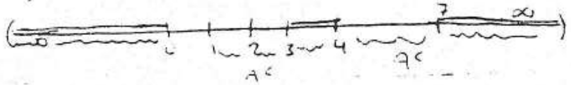
Example (1)
Let $X = \{1,2,3,4\}$, $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$
Let $A = \{1,3,4\}$ $A^c = \{2\}$



Interior point of $A = 1 \in \underset{\text{open}}{\{1\}} \subseteq A$
exterior point of $A = 2 \in \{2\} \subseteq A^c$
boundary point of $A = 3, 4$

[illegible]

Example (2) Let $(\mathbb{R}, \tau_{\ell.r})$, $A = (-\infty, 0) \cup [3,4] \cup [7, \infty[$



$A^{\circ} = (-\infty, 0)$

$\text{Ext}\,A = \phi$ $\quad Bd\,A = [0, \infty)$

---

(11) $\text{Int}(A)$ & $\text{Ext}(A)$ & $Bd(A)$ are mutually disjoint.

Proof

$A^{\circ} \subseteq A$
$\text{Ext}\,A \subseteq A^{c}$ $\Big\} \Rightarrow A^{\circ} \cap \text{Ext}(A) \subseteq A \cap (X-A) = \phi$

$\therefore A^{\circ} \cap \text{Ext}(A) = \phi$

$A^{\circ} \cap Bd(A) = \phi$

if $x \in A^{\circ}$ then $\exists\, U$ open $\ni x \in U \subseteq A$ then $U \cap (X-A) = \phi$

then $x \notin \overline{X-A}$ then $x \notin Bd(A)$

Let A be a subset of the space X then

$$\boxed{Bd(A) = \overline{A} \cap \overline{(X-A)}}$$

Proof (⊆) Let $x \in Bd(A)$ i.e $\exists\, U$ open $x \in U$

$U \cap A \neq \phi$ & $U \cap X-A \neq \phi$

$x \in \overline{A}$ & $x \in \overline{X-A}$

$\therefore x \in \overline{A} \cap \overline{(X-A)}$

$\therefore Bd(A) \subseteq \overline{A} \cap \overline{(X-A)}$

الآن

(⊇) $x \in \overline{A} \cap \overline{(X-A)} \Rightarrow x \in \overline{A}$ & $x \in \overline{(X-A)}$

then ∀ U open containg x

$U \cap A \neq \phi$ & $U \cap (X-A) \neq \phi$

$\therefore$ ie $x \in Bd(A)$

$\overline{A} \cap \overline{(X-A)} \subseteq Bd(A)$

$$\therefore \boxed{Bd(A) = \overline{A} \cap \overline{(X-A)}}$$

---

(12) $\text{Int}(A) \sqcup \text{Ext}(A) \cup Bd(A) = X$

Proof

(⊆) obvious

(⊇) Let $x \in X$

case (1) $x \in A$ suppose $x \notin A^{\circ}$ then whenever U is open & contins x then $U \not\subseteq A$ i.e $U \cap X-A \neq \phi$

also $U \cap A \neq \phi$ ∀ open U with $x \in U$

So $x \in Bd(A)$

case (2) $x \notin A$

S.m

---

(13) Let (X, τ) be a Top-space $A \subseteq X$

A is open $\iff A \cap Bd(A) = \phi$

Proof

$\rightarrow$ Suppose A is open then $A = A^{\circ}$

since $A^{\circ} \cap Bd(A) = \phi$ therefore $A \cap Bd(A) = \phi$

$\leftarrow$

let $A \cap Bd(A) = \phi$ since $A \subseteq X = A^{\circ} \cup \text{Ext}(A) \cup Bd(A)$

then $A \subseteq A^{\circ}$ — (1) Thus $A = \text{Int}\,A \therefore A$ is open.

always $A^{\circ} \subseteq A$ — (2)

---

(14) Let (X, τ) be Top-space, $A \subseteq X$ (A is closed $\iff Bd(A) \subseteq A$)

Proof

$\rightarrow$ Let A is closed then $A = \overline{A}$ But $Bd(A) = \overline{A} \cap \overline{X-A} \subseteq \overline{A}$

i.e $Bd(A) \subseteq A$

$\leftarrow$

Let $Bd(A) \subseteq A$ then $X-A \cap Bd(A) = \phi$

so $Bd(X-A) \cap (X-A) = \phi$

Hence $X-A$ is open & so A is closed.

⊠

$\overline{A} = A \cup Bd(A) = Int(A) \cup Bd(A)$ For any $A \subseteq X$
① ②

Proof ①

($\subseteq$) let $x \in \overline{A}$ & suppose $x \notin A$ & U(open) $\ni x \in U$, then $U \cap A \neq \phi$ & $U \cap (X - A) \neq \phi$
so $x \in Bd(A)$

($\supseteq$) obvious $A \cup Bd(A) \subseteq \overline{A}$
$\therefore \overline{A} = A \cup Bd(A)$

② ($\supseteq$) trivial
($\subseteq$) $A \subseteq X = A^\circ \cup Ext(A) \cup Bd(A)$
& since $A \cap Ext(A) = \phi$
$\therefore A \subseteq Int(A) \cup Bd(A)$
$\therefore A = Int(A) \cup Bd(A)$

For any A, $\boxed{Bd(A) \subseteq \overline{A}}$

Proof
let $x \in Bd(A)$ to show $x \in \overline{A}$
suppose not $x \in \overline{A}$ so $x \in (\overline{A})^c$ open set
so $(\overline{A})^c \cap A \neq \phi$ & $(\overline{A})^c \cap A^c \neq \phi$
we know that $A \subseteq \overline{A}$ is always true
so $A \cap (\overline{A})^c = \phi$ #
$\therefore x \in \overline{A}$ $\therefore Bd(A) \subseteq \overline{A}$

$\boxed{A^\circ = A - Bd(A)}$

Proof ($\subseteq$)
let $x \in A^\circ \Rightarrow x \in A$ & $x \notin Bd(A) \Rightarrow x \in A / Bd(A)$
$\Rightarrow A^\circ \subseteq A / Bd(A)$

($\supseteq$) let $x \in A / Bd(A) \Rightarrow x \in A$ & $x \notin Bd(A)$
then $\forall U \underset{open}{\subseteq} X \ni x \in U$ then $U \cap A = \phi$
or $U \cap X - A = \phi$

but $U \cap A = \phi$ (# to $x \in U \cap A$) so
$U \cap X - A = \phi$. then $x \in U \subseteq A$ so $x \in A^\circ$
$\therefore A / Bd(A) \subseteq A^\circ$
$\therefore A^\circ = A / Bd(A)$

⑱ Let $(X, \mathcal{T})$ be Top-space, $A \subseteq X$ then
$Bd(A) = \phi \Leftrightarrow A$ both open & closed

Proof
$\Rightarrow$ $\overline{A} = A \cup Bd(A) = A \cup \phi = A \Rightarrow A$ closed
also $A \subseteq X = A^\circ \cup Ext(A) \cup Bd(A) \subseteq A^\circ \cup Ext(A)$
$\therefore A \subseteq Int A \subseteq A \Rightarrow A = A^\circ$ open

$\Leftarrow$ suppose A is clopen
$\boxed{Bd(A) = \overline{A} - A^\circ} = A - A = \phi$

× ⑲ Let $A, B \subseteq X$ & suppose $Bd(A) \cap Bd(B) = \phi$
Prove $\boxed{A^\circ \cup B^\circ = (A \cup B)^\circ}$

Proof
($\subseteq$) obvious
($\supseteq$) let $x \notin A^\circ \cup B^\circ$ then $x \notin A^\circ$ & $x \notin B^\circ$
we Recall $X = A^\circ \cup Bd(A) \cup Ext(A)$
& $X = B^\circ \cup Bd(B) \cup Ext(B)$
Case ① $x \in Bd(A)$ & $x \in Ext(B)$, $\forall$ open $U \ni x$
$\Rightarrow U \cap (X - A) \neq \phi$ & $\exists U$ open $\ni x \in U \subseteq X - B$
suppose $\exists G$(open) $\ni x \in G \cap U \subseteq G \subseteq A \cup B$
But $(G \cap U) \cap B = \phi \Rightarrow G \cap U \subseteq A$ # so $x \notin (A \cup B)^\circ$

Case ② — $x \in Ext(A)$ & $x \in Bd(B)$

Case ③ — $x \in Ext(A)$ & $x \in Ext(B)$

$\Rightarrow \exists$ u open $\ni x \in u \subseteq X - A$ & $\exists V$ (open) $\ni x \in V \subseteq X - B$

Suppose $\exists G$ open $\ni x \in G \subseteq A \cup B$ then $x \in G \cap u \cap V \cap G$
$\subseteq G \subseteq A \cup B$ #

$\{A_\alpha : \alpha \in \Delta\}$ is a family of subsets of a space (X, τ)

① $\bigcup_{\alpha \in \Delta} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha \in \Delta} A_\alpha}$

② $\overline{\bigcap_{\alpha \in \Delta} A_\alpha} \subseteq \bigcap_{\alpha \in \Delta} \overline{A_\alpha}$

Proof ① $\forall \alpha \in \Delta$ ∴ $A_\alpha \subseteq \bigcup A_\alpha$

then $\overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha \in \Delta} A_\alpha}$

$\Rightarrow \bigcup_{\alpha \in \Delta} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha \in \Delta} A_\alpha}$

② $\bigcap_{\alpha \in \Delta} A_\alpha \subseteq A_\alpha$ $\forall \alpha \in \Delta$ $\Rightarrow \overline{\bigcap_{\alpha \in \Delta} A_\alpha} \subseteq \overline{A_\alpha}$ $\forall \alpha \in \Delta$

then $\overline{\bigcap_{\alpha \in \Delta} A_\alpha} \subseteq \bigcap_{\alpha \in \Delta} \overline{A_\alpha}$

$A \subseteq X$ Prove $Bd(A) = \overline{A} - A^\circ$

Proof $\overline{A} = A^\circ \cup Bd(A)$; $A^\circ \cap Bd(A) = \phi$

Let $x \in \overline{A} - A^\circ \Longleftrightarrow x \in A^\circ \cup Bd(A)$ & $x \notin A^\circ$

$\Longleftrightarrow x \in A^\circ$ or $x \in Bd(A)$ & $x \notin A^\circ$]

$\Longleftrightarrow x \in A^\circ$ & $x \notin A^\circ$] or [$x \in Bd(A)$ & $x \notin A^\circ$]

$\Longleftrightarrow x \in Bd(A)$

(22) examples:

① Let (R, τ) be topl space & $A = \{x \in R ; 0 < x \leq 1\} = (0,1]$

Describe A° & $Ext\,A$ & $Bd(A)$ if the topology on R is

(a) Strong Top $T = T_S$ $A^\circ = (0,1)$ $Bd(A) = \{0,1\}$

Let $t \in U$ open then $t \in (a,b) \subseteq U$ then $a < t < b$

Let $c = \min\{b,1\}$ then $a < t < c \leq b$ so $\forall t < c$

$\Rightarrow t \in A$

ie $Ext(A) = Int(A^c)$ $(-\infty,0] \cup (1,\infty)) = (-\infty,0) \cup (1,\infty)$

(b) $T = T_{L.r}$ $A^\circ = \phi$ $(\forall a \in (0,1],\ (-\infty,a) \not\subseteq (0,1])$

$Ext(A) = Int(-\infty,0] \cup (1,\infty) = (-\infty,0)$

$Bd(A) = [0,\infty)$

(c) $T = T_{disc}$ $A^\circ = (0,1]$

$Ext(A) = Int((-\infty,0] \cup (1,\infty)) = (-\infty,0] \cup (1,\infty)$

$Bd(A) = \phi$

(d) $T = T_{ind}$ $A^\circ = \phi$

$Ext(A) = Int((-\infty,0] \cup (1,\infty)) = \phi$

$Bd(A) = R$

② (R, T_{cof}) & $A = \{1,2,\ldots\}$

$A^\circ = \phi$ Let $n \in A$ & suppose $\exists$ U (open) $\ni n \in U \subseteq A \Rightarrow \underset{\text{infinite}}{A^c} \subseteq \underset{\text{finite}}{U^c}$ #

$Bd(A) = R$ Let $x \in R$ & Let $x \in U \subseteq R$ open

① $U \cap A \neq \phi$ (if $U \cap A = \phi$ then $\underset{\text{inf}}{A} \subseteq \underset{\text{finite}}{R - U}$ # !!

② $U \cap (R - A) \neq \phi$

if $(U \cap (R-A)) = \phi$ then $U \subseteq A$ # !!

(4) Cluster point & Isolated

Let (X,τ) be Top space, $A \subseteq X$, $p \in X$, p is a cluster point or a accumulation point of A iff every openset containing p contains at least one point of A different from $p_{(X,T)}$

$\forall G$ open : $p \in G$ we have $G \cap A - \{p\} \neq \phi$

$A' = \{x \in X$ is a cluster point of $A\}$ is called **the derived** set of A (or) the set of all cluster points of A

: Let (X,τ), $A \subseteq X$, a point x is called Isolated point of A if $x \in A$ But x is not a cluster point of A then $\exists$ U open $\ni x \in U$ & $U \cap A - \{x\} = \phi$ then $U \cap A = \{x\}$ denoted by $I(\ )$

Let (X,τ) be Top-space & $A \subseteq X$ then A is dense in (X,τ) iff $\overline{A} = X$

A is **perfect set** iff $A' = A$

Examples:

$A = Q$ Let $x \in R$ take $U = \{x\}$

$Q - \{x\} = \phi \Rightarrow x$ is not cluster point of Q

any real no. is not cluster point

(2) Let $X = \{1,2,3\}$ & $\tau = \{\phi, \{1,2\}, \{3\}, X\}$ & $A = \{3\}$

Find (1) $A' = \{1\}$ (2) $A^\circ = \{3\}$ (3) $Ext\,A = (A^c)^\circ = \phi$

(4) $I(A) = \{2,3\}$ (5) $Bd(A) = \{1,2\}$

(6) $\overline{A} = A \cup Bd(A) = Int(A) \cup Bd(A) = \{1,2,3\} = X$

(7) A is dense.

(3) Let (R, τ_s), $A = \{\frac{1}{n}, n = 1,2,\dots\}$

- 0 is a cluster point of A

Let U be open & $0 \in U$ $\exists$ $\varepsilon > 0 \ni 0 \in (-\varepsilon, \varepsilon) \subseteq U$

choose $n \in N \ni n > \frac{1}{\varepsilon}$ then $\frac{1}{n} < \varepsilon$ then $\frac{1}{n} \in (-\varepsilon,\varepsilon) \cap A \subseteq U \cap A$

so $\frac{1}{n} \in U \cap A - \{0\} \neq \phi$

- $x \in R$, $x \neq 0$ is not a cluster points

(1) $x \in A \Rightarrow x = \frac{1}{k}$ for some $k \in N$

$\frac{1}{k} \in U = (\frac{1}{k+1}, \frac{1}{k-1}) \Rightarrow U \cap A = \{\frac{1}{k}\} \Rightarrow U \cap A - \{\frac{1}{k}\}$

(2) $x \notin A$ then $\exists\, n \in N \ni \frac{1}{n+1} < x < \frac{1}{n}$

$U = (\frac{1}{n+1}, \frac{1}{n}) \cap A = \phi \Rightarrow U \cap A - \{x\} = \phi$

(4) Let (R, τ_{cof}), what are the cluster-point of $A = ($
$\mathbb{Q}$ / $(0,1$

solve

Let $x \in R$ & let U (open) & $x \in U$

- $U \cap A \neq \phi$ (suppose $U \cap A = \phi$ the $\underset{\text{inf}}{A} \subseteq \underset{\text{finite}}{R - U}$ #

- $U \cap A - \{x\} \neq \phi$

Suppose $U \cap A - \{x\} = \phi$

put $V = U - \{x\}$ (open)

$\phi \neq A \cap V \subseteq U \cap A - \{x\}$ $\quad (U - \{x\} = U \cap \{R - \{x\})$

Hence $A' = R$

Theorms + Ex

Let D be subset of a space (X,τ) then the following statement are equivalent

1. D is dence
2. If B is closed + $D \subseteq B \Rightarrow B = X$
3. If U (open) + $U \neq \phi$ then $U \cap D \neq \phi$
4. $Int(X-D) = \phi$

Proof

(1→2) Let $B \underset{clod}{\subseteq} X$ + $D \subseteq B$ then $X = \overline{D} \subseteq \overline{B} = B \subseteq X$ (since Dense)

So $B = X$

(2→3) Suppose $U \cap D = \phi$ for some $U \neq \phi \underset{open}{\subseteq} X$

then $D \subseteq X - U$ (closed) #

because by (2) $X - U = X \Rightarrow U = \phi$

(3→4)

Suppose $Int(X-D) \neq \phi$

Put $U = Int(X-D)$ then U is open + nonempty subset of X then $U \cap D = \phi$ #

(4→1) Suppose $\overline{D} \neq X$ then $X - \overline{D} \neq \phi$

But $\underset{open}{X-\overline{D}} \subseteq X - D$ so $\phi \neq X - \overline{D} \subseteq Int(X-D) = \phi$

#

(2) Let $A \subseteq B$ then $Ext(A) \supseteq Ext(B)$

Proof: $A \subseteq B \Rightarrow B^c \subset A^c \Rightarrow Int(B^c) \subseteq Int(A^c)$

$\Rightarrow Ext(B) \subseteq Ext(A)$

(3) Let $A \subseteq B$ then $Bd(A) \nsubseteq Bd(B)$

$\boxed{\text{Counter example}}$

Let $A = (-1,1)$ + $B = (-2,2)$

$Bd(A) = \{-1, 1\}$ $Bd(B) = \{-2, 2\}$ but $Bd(A) \nsubseteq Bd(B)$

(4) Proof that A is perfect iff its closed + has no Isoluted point

Proof

$\Rightarrow$ Since A is perfect $A = A'$ + hence

$\boxed{\overline{A} = A \cup A' = A}$ Thus A is closed

Since $I(A) \cap A' = \phi$ + $A' = A$

therefor $I(A) \cap A = \phi$ but $\boxed{I(A) \subset A}$

So $I(A) \cap A = I(A) = \phi$

$\Leftarrow$

to show $A' = A$

(1) Let $x \in A' \Rightarrow x \in A' \cup A = \overline{A} = A$ since A closed

$x \in A$ $\therefore A' \subseteq A$

(2) Let $x \in A$ since $I(A) = \phi$ $x \notin I(A)$

$\exists U \ni x \in U$ + $U \cap A - \{x\} \neq \phi \Rightarrow x \in A'$

$\therefore A \subset A'$ then $A = A'$

(5) For any two subsets A + B of a space X if $A \subset B$ then $A' \subset B'$

Proof

Let $x \notin B' \Rightarrow \exists U$ (open) $\ni x \in U$ + $U \cap B - \{x\} = \phi$

But $A \cap U - \{x\} \subseteq B \cap U - \{x\} = \phi$ $\Rightarrow A \cap U - \{x\} = \phi$

Thus $x \notin A'$

A & B are subsets of the space X. Prove $(A \cup B)' = A' \cup B'$

Proof

$A \subseteq A \cup B$ then $A' \subseteq (A \cup B)'$, $B \subseteq A \cup B$ then $B' \subseteq (A \cup B)'$, then $A' \cup B' \subseteq (A \cup B)'$

$(A \cup B)' \subseteq A' \cup B'$

Let $x \notin A' \cup B'$ then $x \notin A'$ & $x \notin B' \Rightarrow \exists\, U(\text{open}) \ni x \in U$ but $U \cap A - \{x\} = \phi$ also $\exists\, V(\text{open}) \ni x \in V$ but $V \cap B - \{x\} = \phi$ sets $G = U \cap V$ open

$G \cap (A \cup B) - \{x\} = (G \cap A) \cup (G \cap B) - \{x\} = \phi$

So $x \notin (A \cup B)'$

Q.E.D

Let A be a subset of space X. Prove that

$x \in A' \Leftrightarrow x \in \overline{A - \{x\}}$

$A' \subseteq \overline{A / \{x\}}$

$\Rightarrow$ Suppose $x \notin \overline{A / \{x\}}$ then $x \in (\overline{A/\{x\}})^c_{\text{open}}$

but $\underbrace{(\overline{A - \{x\}})^c}_{\text{open}} \cap \underbrace{(\overline{A - \{x\}})}_{\text{set}} = \phi \Rightarrow x \notin A'$ #

$\Leftarrow$

Suppose $x \notin A'$ then $\exists\, G$ open $\ni x \in G$ & $G \cap A - \{x\} = \phi$ then $A - \{x\} \cap G = \phi \Rightarrow A - \{x\} \subseteq G^c$

$\overline{A - \{x\}} \subseteq \overline{G^c} = G^c$ (because G^c is closed)

but $x \in \overline{A/\{x\}}$

then $x \in G^c$ & (ie $x \notin G$) #

(8) Let (X, τ) be Top-space, D dense in X, $G \subseteq_{\text{open}} X$ $\overline{D} = X$

$\boxed{\overline{D \cap G} = \overline{G}}$

Proof

$(\subseteq)$ $D \cap G \subseteq G \Rightarrow \overline{D \cap G} \subseteq \overline{G}$

$(\supseteq)$ Let $x \in \overline{G}$ & let $x \in U(\text{open})$ then $U \cap G \neq \phi$.

$\Rightarrow \underset{\text{open}}{(U \cap G)} \cap D \neq \phi$ (since D is dense)

then

$U \cap (G \cap D) \neq \phi \Rightarrow x \in \overline{D \cap G} \Rightarrow \overline{G} \subseteq \overline{D \cap G}$

by (1) & (2) $\therefore \overline{D \cap G} = \overline{G}$

(9) A is closed $\Leftrightarrow$ A contains all of its cluster point

i.e. A closed $\Leftrightarrow A' \subseteq A$

Proof

$\Rightarrow$ Let $x \in A'$ ie (x is cluster point) IF U open

then $U \cap A - \{x\} \neq \phi$ ie $U \cap A \neq \phi \Rightarrow x \in \overline{A} = A$

$\therefore A' \subseteq A$

$\Leftarrow$

Let $A' \subseteq A$ & $x \in X - A$ then $x \notin A'$ ($\exists\, U \ni$,

$\ni U \cap A - \{x\} = \phi \Rightarrow U \cap A = \phi \Rightarrow U \subseteq X - A$

Thus $X - A$ is open & so A is closed.

(10) IF (X, τ) be Top-space, $A \subseteq X$ then $\overline{A} = A \cup A'$

Proof

$(\subseteq)$ Let $x \in \overline{A}$ (assum $x \notin A$) Let $U \ni x(\text{open}) \Rightarrow U \cap A \neq \phi$.

$\Rightarrow x \notin U \cap A$ so $U \cap A - \{x\} \neq \phi$ Thus $x \in A' \subseteq A \cup A'$

$(\supseteq)$ Let $x \notin \overline{A}$ then $x \notin A \Rightarrow \exists\, U \ni x(\text{open})$ & $U \cap A = \phi \Rightarrow U \cap A - \{x\} = \phi$

So $x \notin A'$

$\mathcal{B}$ is base for T

1. $\mathcal{B} \subseteq \mathcal{T}$
2. $\forall\, U \in T \iff U = \bigcup\{B : B \in \mathcal{B}\}$

Chapter (4)

"Bases, sub Base & Products"

1 - Base

Dfn ① **Bases**:- Let (X, τ) be a topological space. a Base for τ is a collection β of subsets of X $\ni$ ① $\beta \subseteq \tau$
② $\forall\, U \in \tau$ iff U is a union of elements of β

Examples:-

① (R, T_s) be strong Top ① Take β_1 the family of all open intervals is β Base because ① $\beta \subseteq T_s$
② $U \in T_s \iff U$ is a union of open interval ✓

② Take $\beta_2 = \{(a,b) : a, b \in R ; a < b\}$ ...

(to show) β_2 is Base ① $\beta_2 \subseteq T_s$
② $\beta_2 \subseteq U \Leftarrow$
$\Rightarrow$ Let $U \in T_s$, $\forall\, x \in U \Rightarrow \exists\, I_x$ (open interval)
$\ni x \in I_x \subseteq U$ so $\exists$ an open interval (a_x, b_x)
$\ni (a_x, b_x) \subseteq I_x \subseteq U \Rightarrow U = \bigcup_{x \in U} (a_x, b_x)$

② Let $(R, T_{L.r})$ be toplogical space & $T_{L.r} = \{\phi, R, (-\infty, a) : a \in R$ show that this is Base or not

①- $\beta_1 = \{(-\infty, a] : a \in R\}$ not Base because $\beta_1 \not\subseteq \tau$
② $\beta_2 = \{(-\infty, a) : a \in R\}$ is Base
③ $\beta_3 = T_{L.r}$ is Base [τ is a Base for itself]
④ $\beta_4 = \{(-\infty, a) : a \in \mathbb{Q}\}$ is Base because

Proof: For $n \in N$, p, $p - \frac{1}{n}$ are two distinct real number so $\exists$
$\exists\, r_n \in \mathbb{Q} \ni p - \frac{1}{n} < r_n < p$ so $\lim_{n \to \infty} r_n = p$ hence
$(-\infty, p) = \bigcup_{n=1}^{\infty} (-\infty, r_n)$

$\mathcal{B}$ is base for $\mathcal{T}$

① $\mathcal{B} \subseteq \mathcal{T}$

② if $U \in \mathcal{T} \Longleftrightarrow U = \bigcup \{B : B \in \mathcal{B}\}$

Chapter (4)

" Bases, sub Base & Products "

1- Base

Dfn ① Bases :- Let (X, τ) be a topological space. A Base for τ is a collection $\mathcal{B}$ of subsets of X $\ni$

① $\mathcal{B} \subseteq \tau$

② if $U \in \tau$ iff U is a union of elements of $\mathcal{B}$

Examples :-

① (R, T_s) be strong Top ① Take $\mathcal{B}_1$ the family of all open intervals is $\mathcal{B}$ Base because ① $\mathcal{B} \subseteq T_s$

② $U \in T_s \Leftrightarrow U$ is a union of open interval

② Take $\mathcal{B}_2 = \{ (a,b) : a, b \in R ; a < b \}$ ...

(to show) $\mathcal{B}_2$ is Base ① $\mathcal{B}_2 \subseteq T_s$

② $\mathcal{B}_2 \subseteq U$ ⇐

$\Rightarrow$ let $U \in T_s$ & $x \in U \Rightarrow \exists I_x$ (open interval)

$\ni x \in I_x \subseteq U$ so $\exists$ an open interval (a_x, b_x)

$\ni (a_x, b_x) \subseteq I_x \subseteq U \Rightarrow U = \bigcup_{x \in U} (a_x, b_x)$

② Let $(R, T_{L.r})$ be toplogical space & $T_{L.r} = \{ \phi, R, (-\infty, a) : a \in R$

show that this is Base or not

①- $\mathcal{B}_1 = \{ (-\infty, a] : a \in R \}$ not Base because $\mathcal{B}_1 \not\subseteq \mathcal{T}$

② $\mathcal{B}_2 = \{ (-\infty, a) : a \in R \}$ is Base

③ $\mathcal{B}_3 = T_{L.r}$ is Base [$\mathcal{T}$ is a Base for itself]

④ $\mathcal{B}_4 = \{ (-\infty, a) : a \in Q \}$ is Base because

Proof For $n \in N$, p, $p - \frac{1}{n}$ are two distinct real number so $\exists$

$\exists r_n \in Q \ni p - \frac{1}{n} < r_n < p$ so $\lim_{n \to \infty} r_n = p$ hence

$(-\infty, p) = \bigcup_{n=1}^{\infty} (-\infty, r_n)$

Notes: In general $\{\tau - X\}$ is Base? NO

Counter example

Let $X = \{1,2\}$ & $\tau = \{\phi, \{1\}, \{1,2\}\}$ to show $\tau - X$ is not Base take $\beta = \{\phi, \{1\}\}$ not Base since $\{1,2\} \in \tau$ & $\{1,2\} \neq \bigsqcup_{B \in \beta} B$

Theorems & Exe

① Let (X,τ) be a Top-space & β a Base for τ then $\phi \neq \mathcal{U}$ is open iff $\forall x \in \mathcal{U}, \exists B \in \beta \ni x \in B \subseteq \mathcal{U}$

Proof: $\Rightarrow$ Suppose $\mathcal{U}$ is open & Let $x \in \mathcal{U}$ since β is Base for τ & $\mathcal{U} \in \tau$

so $\mathcal{U} = \bigsqcup_{B_x \in \beta} B_x$, since $x \in \mathcal{U}$ so $x \in \bigsqcup_{B_x \in \beta} B_x$

so $x \in B_{x_0}$ for some $B_{x_0} \in \beta$

Thus $x \in B_{x_0} \subseteq \mathcal{U}$

$\Leftarrow$

$\forall x \in \mathcal{U}, \exists B_x \in \mathcal{U}$ thus $\mathcal{U} = \bigsqcup_{x \in B_x \subseteq \mathcal{U}} B_x$

But $B_x \in \beta \subseteq \tau$

then B_x is open so $\bigsqcup B_x$ is arbitrary is open $= \mathcal{U}$ ∎

② Let (A, τ_A) be a subspace of (X,τ) & Let β be a Base of X then $\beta_A = \{A \cap B : B \in \beta\}$ is Base of (A, τ_A)

Proof

حسب تعريف

First $\beta_A \subseteq \tau_A$

Let $\mathcal{U} \in \beta_A$ then $\mathcal{U} = B \cap A$ for some $B \in \beta$

since $B \in \tau$ because Base Hence $B \in \tau$ then $\mathcal{U} \in \tau_A$

② Let $\mathcal{U} \in \tau_A$, $\mathcal{U} \neq \phi$ let $x \in \mathcal{U}$ Thus $\mathcal{U} = G \cap A$ for some $G \in \tau$

$x \in \mathcal{U}$ then $x \in G$ & $x \in A$ But $G \in \tau$, $G \neq \phi$

β is a base for τ then $\exists B \in \beta$, $x \in B \subseteq G$ Thus

$x \in B \cap A \subseteq G \cap A = \mathcal{U}$ Thus $B \cap A \in \beta_A$

(*) $\tau(\beta) \doteq$ The Topology generated by the Base β

③ Let β be a family of subsets of set X then β is Base for (some) Topology on X

iff

① $\forall x \in X, \exists B \in \beta : x \in B$ $\left(\text{i.e } \bigsqcup_{B \in \beta} B = X\right)$

② if $B_1, B_2 \in \beta$ & $x \in B_1 \cap B_2$ then $\exists B_3 \in \beta \ni x \in B_3 \subseteq B_1 \cap B_2$

Proof $\Rightarrow$

suppose β is a Base for some Top. τ on X

1) To show Let $x \in X$ & $X \in \tau$ (open), β is a base for τ then $\exists B \in \beta : x \in B \subseteq X$ (i.e) X is a union elements of β $X = \bigsqcup_{B \in \beta} B$

2) Let $B_1, B_2 \in \beta$ & $x \in B_1 \cap B_2$ since $B_1 \cap B_2$ is open then $\exists B_3 \in \beta \ni x \in B_3 \subseteq B_1 \cap B_2$

$\Leftarrow$

Let $\beta \subseteq P(X)$ & Satisfy ① & ② Req β is a base for some Top on

take $\tau = \{\mathcal{U} \subseteq X : \mathcal{U}$ is union of a subcollection of $\beta\} \cup \phi$

1. $X \in \tau$ because $X = \bigsqcup_{B \in \beta} B$ & $\phi \in \tau$ because $\phi = \bigsqcup_{\alpha \in \phi} B$

2. Let $u, v \in \tau$ if $u \cap v = \phi \Rightarrow u \cap v \in \tau$ if not $u \cap v \neq \phi$ then $x \in \mathcal{U} \cap V$

then $x \in \mathcal{U}$ & $x \in V$ therefore $\exists B_1, B_2 \in \beta$

$\ni x \in B_1 \subseteq U$ & $x \in B_2 \subseteq V$ So $x \in B_1 \cap B_2 \subseteq U \cap V$

then $\exists B_x \in \mathcal{B} \ni x \in B_x \subseteq B_1 \cap B_2 \subseteq U \cap V$

is clear that $U \cap V = \sqcup \{B_x : x \in U \cap V\}$

Hence $U \cap V \in \tau$

Let $U_\alpha \in \tau$ $\forall \alpha \in \Delta$ then $\bigcup_{\alpha\in\Delta} U_\alpha$ is a union of unions of subcollection of $\mathcal{B}$. Thus $\bigsqcup_{\alpha\in\Delta} U_\alpha$ is a union of subcollection of $\mathcal{B}$. Hence $\bigsqcup_{\alpha\in\Delta} U_\alpha \in \tau$

assume $\bigsqcup_{\alpha\in\Delta} U_\alpha \neq \phi$

Let $x \in \bigcup U_\alpha \Rightarrow x \in U_{\alpha_0}$ for some $\alpha_0 \in \Delta$

$\exists B_x \ni x \in B_x \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$

$\therefore \bigcup_{\alpha\in\Delta} U_\alpha = \sqcup \{B_x : x \in \bigsqcup_{\alpha\in\Delta} U_\alpha\}$

Hence $\mathcal{B} \subseteq \tau$

Examples:

Let $X = \{1,2,3\}$ & $\mathcal{B} = \{\{1,2\}, \{1,3\}\}$ Can we find a Topology τ on X for which $\mathcal{B}$ is Base?

No since Let $\tau = \{\phi, X, \{1,2\}, \{1,3\}, \{1\}\}$

$\{1\} \notin \mathcal{B}$ حيث تخلف الشرط الثاني لا يتحقق

$\exists \{1\} \in \mathcal{B}$

Let $X = \{1,2,3\}$ & $\mathcal{B} = \{\{1\}, \{2\}\}$ Can we fined a topology on X for which $\mathcal{B}$ is a Base?

No Because $X \neq \{1\} \cup \{2\}$

$X = \{1,2,3,4\}$ & $\mathcal{B} = \{\{1\}, \{2\}, \{3,4\}\}$ is Base? Yes

because $\mathcal{B} \in P(X)$ ① $\bigsqcup_{B\in\mathcal{B}} B = X$

$\tau = \{\phi, \{1\}, \{2\}, \{3,4\}, \{1,2\}, \{1,3,4\}, \{2,3,4\}, X\}$

∴ جميع الشروط تتحقق

---

④ Let $X = R$ & $\mathcal{B} = \{(-\infty, c], [c, \infty)\}$ is Base

No because $(-\infty, c] \cap [c, \infty) = \{c\} \notin \mathcal{B}$

$\therefore \mathcal{B}$ is not Base for any Top on X

---

⑤ $X = R$, $\mathcal{B} = \{(-\infty, c], [c, \infty), \{c\}\}$ Base for some Topo

$\tau(B) = \{\phi, (-\infty, c], [c, \infty), \{c\}, R\}$

---

⑥ Let $R = X$ & $B = \{[a,b) : a, b \in R, a < b\}$

is called Sorgenfrey Line

① $\mathcal{B}$ is Base for some Topology on R

② $T_S \subseteq \tau$

Proof

① to show $\mathcal{B}$ is Base by Theorm

(i) $\sqcup\{B : B \in \mathcal{B}\} = R$ to show this

($\subseteq$) cleare $\sqcup\{B : B \in \mathcal{B}\} \subseteq R$

($\supseteq$) Let $x \in R$ choose $b > x$ take $[x, b) \in \mathcal{B}$

then $x \in [x, b) \subseteq \sqcup\{B : B \in \mathcal{B}\}$

$\sqcup\{B : B \in \mathcal{B}\} = R$

(ii) Let $B_1, B_2 \in \mathcal{B}$ & $x \in B_1 \cap B_2$ Take $B_1 = [a_1, b_1)$, $B_2 = [a_2, b_2)$

$B_1 \cap B_2 = [a_1, b_1) \cap [a_2, b_2) = [\underbrace{\max\{a_1, a_2\}}_{a}, \underbrace{\min\{b_1, b_2\}}_{b})$

Take $B_3 = B_1 \cap B_2$ then $x \in B_3 \subseteq B_1 \cap B_2$

② $T_S \subseteq \tau_{Sorg}$ becouse if $x \in U \in T_S$ then $\exists (a,b) \ni$ $x \in (a,b) \subseteq U$ But $x \in [x, b) \subseteq (a,b) \subseteq U$ $\Rightarrow U \subseteq \tau_{S}$

but $T_S \neq \tau_S$ because $[a,b) \in \tau$ but not $[a,b) \in T_S$

Dfn (X, τ) is called Zero-dimensional iff τ has a Base consisting clopen sets

Two Base are said to be equivalent iff They generat the same Topology on X i.e. $T(B_1)=T(B_2)$

$\mathcal{B}_1$ & $\mathcal{B}_2$ are equivalent iff ① Given any $B_1 \in \mathcal{B}_1$ & $x \in B_1$ $\exists B_2 \in \mathcal{B}_2 \ni x \in B_2 \subseteq B_1$

② Given any $B_2 \in \mathcal{B}_2$ & any $x \in B_2$ $\exists B_1 \in \mathcal{B}_1 \ni x \in B_1 \subseteq B_2$

Let $\mathcal{B}_1$ & $\mathcal{B}_2$ be equivalent bases Say both are base of $\mathcal{T}$ then $\mathcal{B}_1 \subseteq \mathcal{T}$ & $\mathcal{B}_2 \subseteq \mathcal{T}$
$B_1 \in \mathcal{B}_1$ & $x \in B_1$ (open) $\exists B_2 \in \mathcal{B}_2 \ni x \in B_2 \subseteq B_1$ & vis versa.

Try; $T(\mathcal{B}_1)=T(\mathcal{B}_2)$

Clearly take $u \in T(\mathcal{B}_1)$ then u is union of elements of $\mathcal{B}_1$ each of which is union of elements of $\mathcal{B}_1$ which itself is union of element of $\mathcal{B}_2$ & Hence
$u \in T(B_2)$
$T(B_2) \subseteq T(B_1)$ (Vis versa)

Example Let $(X=R)$ & ① $\mathcal{B}_1=\{(a,b) : a,b \in R, a<b\}$
$\mathcal{B}_2=\{(p,q) : p,q \in Q, p<q\}$
$\mathcal{B}_3=\{(c,d) : c,d \in R-Q, c<d\}$
$\mathcal{B}_4=\{\{x\} : x \in R\}$

① $\mathcal{B}_1$ & $\mathcal{B}_2$ are equivalent base
② $\mathcal{B}_2$ & $\mathcal{B}_3$ are equivalent base
③ $\mathcal{B}_1$ & $\mathcal{B}_4$ are not $\leq$ $\leq$
since let $a \in \mathcal{B}_4$ but $\nexists u \in \mathcal{B}$, $\ni x \in u \subseteq B_1$

⑤ Let X be a set & $\mathcal{T}$, $\mathcal{T}'$ Topology on X
$\forall x \in X$ generated by $B(x)$ & $\forall x \in X$ generated by $B'(x)$
Show $\mathcal{T} \subseteq \mathcal{T}' \Leftrightarrow \forall x \in X, \forall B \in \mathcal{B}(x)$ $\exists B' \in \mathcal{B}'(x) \ni x \in B' \subseteq B$

Proof $\Rightarrow$
Let $\mathcal{T} \subseteq \mathcal{T}'$ & let $x \in X$ & $B \in \mathcal{B}(x)$ then $x \in B^{\circ} \in \mathcal{T} \subseteq \mathcal{T}'$
$\exists B' \in B'(x) \ni B' \subseteq B^{\circ} \subseteq B$
$\Leftarrow$ Let $u \in \mathcal{T}$, if $x \in u$ $\exists B \in B(x) \ni x \in B \subseteq u$
by assumption $\exists B' \in B'(x) \ni x \in B' \subseteq B \subseteq u$
$\therefore u \in \mathcal{T}'$ $\therefore \mathcal{T} \subset \mathcal{T}'$

⑥ Let $(X,\mathcal{T}_1)$ be a Top-space where Base β_1 & $(Y,\mathcal{T}_2)$ be Top-space whose Base $\mathcal{B}_2$ then
$\mathcal{B}=\{u \times V : u \in \mathcal{B}_1$ & $V \in \mathcal{B}_2\}$ is a Base for some Topology on $\underline{X \times Y}$
ملاحظة لا يكون ان تشكل $\mathcal{B}_1 \times \mathcal{B}_2$
This Top (generated by $\mathcal{B}$) i.e $T[\mathcal{B}]$ is called the product Topology of $(X,\mathcal{T}_1)$ & $(Y,\mathcal{T}_2)$ $\doteq T_{product}$

Proof

① Let $(x,y) \in X \times Y \Rightarrow x \in X$ & $y \in Y$ Since X, Y (are) in $(X,\mathcal{T}_1)$ & $(Y,\mathcal{T}_2)$
$\exists B_1 \in \mathcal{B}_1 \ni x \in B_1 \subseteq X$ & $\exists B_2 \in \mathcal{B}_2 \ni x \in B_2 \subseteq Y$
then $(x,y) \in \beta_1 \times \beta_2 \subseteq X \times Y$ & $B_1 \times B_2 \in \mathcal{B}_{product}$

② Let $\mathcal{U}, V \in \mathcal{B}_{pro}$ & $(x,y) \in \mathcal{U} \cap V$
Try $\exists w \in \mathcal{B}_{prod} \ni (x,y) \in W \subseteq \mathcal{U} \cap V$
So $\mathcal{U} = \mathcal{U}_1 \times V_1$ & $V = U_2 \times V_2$ for some $u_1 \in \mathcal{B}_1$ & $v_1 \in \mathcal{B}_2$ & $u_2 \in \mathcal{B}_1$ & $v_2 \in \mathcal{B}_2$
But
$(x,y) \in \mathcal{U} \cap V$ so $(x,y) \in \mathcal{U}_1 \times V_1$ & $(x,y) \in U_2 \times V_2$

Hence $x \in U_1$ & $y \in V_1$ & $x \in U_2$, $y \in V_2$

$\Rightarrow x \in U_1 \cap U_2$ & $y \in V_1 \cap V_2$

Since $U_1, U_2 \in \mathcal{B}_1 \subseteq T_1$ $\ni$ $V_1, V_2 \in \mathcal{B}_2 \subseteq T_2$

$\Rightarrow U_1 \cap U_2$ is open in (X, T_1), $V_1 \cap V_2$ is open in (Y, T_2)

$\exists B_1 \in \mathcal{B}_1 \ni x \in B_1 \subseteq U_1 \cap U_2$

$\exists B_2 \in \mathcal{B}_2 \ni y \in B_2 \subseteq V_1 \cap V_2$

$(x,y) \in B_1 \times B_2 \subseteq (U_1 \times V_1) \cap (U_2 \times V_2) = U \cap V$

$\therefore \mathcal{B}_{prod}$ is a base for some Top on $X \times Y$

Let $X \neq \phi$ be a set & $\mathcal{B}$ a collection of subsets of X obeying ① & ② in The ③

Prove that $T(\mathcal{B})$ is the smallest topology on X containing $\mathcal{B}$

Proof

Note that $\mathcal{B} \subseteq T(\mathcal{B})$ & Let T_1 be Top on X & $\mathcal{B} \subseteq T_1$

Try $T(\mathcal{B}) \subseteq T_1$

$U \in T(\mathcal{B}) \Rightarrow U$ is union of element of $\mathcal{B}$, & hence

U is union of elements of T_1 thus $U \in T_1$

Let (X, T) be Top-space & $\mathcal{B}$ a base for T.

Prove D subset of X is dence $\Leftrightarrow$ each nonempty element of $\mathcal{B}$ contains a point of D

Proof

$\Rightarrow$ Let D be dence & $B \neq \phi \in \mathcal{B}$ then $B \in T$ so $B \cap D \neq \phi$

$\Leftarrow$

Let $U \neq \phi \in T$ then U contains a points say x then $\exists B_x \in \mathcal{B}$

$\ni x \in B_x \subseteq U$ But $B_x \cap D \neq \phi$ given so $U \cap D \neq \phi$

$\therefore D$ is dence in X

---

② (Sub base) — finite product

Dfn $\boxed{\text{Sub base}}$:— Let (X, T) be a top-space & a sub base of T is a family $S \subseteq X$ such

① $S \subseteq T$

② The collection all finite $\cap$ of elements of S together with X forms a base for T

OR $\mathcal{B}(S) = \{B \subseteq X : B$ is an $\cap$ of finite subsets of $S\}$ is a base of

[illegible Arabic] sub base [illegible Arabic]

Examples

① Let (X, T_S) be Top-space & $S = \{(-\infty, a) : a \in R\} \cup \{(a, \infty)\}$

$S \subseteq T_S$ & S is a sub base of T_S & $\{(a,b) : a, b \in R, a < b\} \subseteq \mathcal{B}(S)$

---

② $X = R$, $T = T_u$ show ① $S = \{(x, \infty) : x \in R\}$ is subb

① $S \subseteq T$

② $\mathcal{B}(S) = \{R, (x, \infty) : x \in R\}$ not a base for T_S

$\Rightarrow S$ is not a sub base for T_S ([illegible Arabic] X [illegible Arabic] Basis)

---

② $S = \{(a, \infty), (-\infty, a) : a \in R\}$

$\therefore \mathcal{B}(S) = \{R, \phi, (-\infty, a), (a, \infty), (a, b) : a, b \in R\}$

Yes $\mathcal{B}(S)$ is a base for T_u then S is a sub base for T_u

---

③ $X = R^2$, $T = T_S$ & $S = \{R \times (a, b), (c, d) \times R\}$ is subbase for T_S. Since

$\mathcal{B} = \{R \times (a,b), (c,d) \times R, R \times R, [illegible]\}$

قال دكتور ... فوري ...

[illegible] $S = \{A : A \subseteq \mathbb{Z}$ & A contains exactly 2 elements$\}$
[illegible] S a base for some top & fined $\mathcal{B}(S)$ & $T(\mathcal{B}(S))$

[illegible] $x \in \mathbb{Z} \Rightarrow \{x, x+1\} \in S$ & $x \in \{x, x+1\}$
[illegible] $\in S$, let $x \in U \cap V$ — Try $\exists W \in S \ni x \in W \subseteq U \cap V$
[illegible] $; y \neq x$
[illegible] $, x \neq z$
S not Base $\{$ since $\{1,2\}, \{2,3\} \in S$
[illegible] $\cap \{2,3\}$?? ($\notin S$ [illegible])
[illegible] x, all singletons of X, all doubletons [illegible]
$= T_{dis}$

(3) Finite Product of Top space

$A \times B = \{(x,y) : x \in A$ & $y \in B\}$
$A \times B \times C = \{(x,y,z) : x \in A$ & $y \in B$ & $z \in C\}$
$\prod X_i = X_1 \times X_2 \times X_3 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) : x_i \in X \quad i = 1, 2, \ldots, n\}$

Theorms & Ex

[illegible] (X_i, T_i) $i = 1, 2, \ldots n$ be a topological space then
$\mathcal{B} = \{\prod U_i = U_1 \times U_2 \times \cdots \times U_n : U_i \in T_i \quad i = 1, 2, \ldots, n\}$
[illegible] for some topology on $\prod_{i=1}^{n} X_i$

[illegible] $\cup \mathcal{B} = \prod_{i=1}^{n} X_i$
[illegible] $\in \mathcal{B}$ then $B_1 = \prod_{i=1}^{n} U_i$ & $B_2 = \prod_{i=1}^{n} V_i$ where $U_i, V_i \in T_i$ [illegible]
[illegible] $\prod U_i \cap \prod_{i=1}^{n} V_i = \prod_{i=1}^{n} (U_i \cap V_i)$

(4) Let (X_1, T_1) & (X_2, T_2) be two topological space & $A_1 \subseteq X_1$ & $A_2 \subseteq X_2$ then

(1) $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$
(2) $Int(A_1 \times A_2) = Int(A_1) \times Int(A_2)$
(3) $(A_1 \times A_2)' = (\overline{A_1} \times A_2') \sqcup (A_1' \times \overline{A_2})$
(4) $Bd(A_1 \times A_2) = Bd(A_1) \times \overline{A_2} \sqcup (\overline{A_1} \times Bd(A_2))$

Proof (1) to show $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$
($\subseteq$) Suppose $(x,y) \notin \overline{A_1} \times \overline{A_2}$ so $x \notin \overline{A_1}$ or $y \notin \overline{A_2}$
(say $x \notin \overline{A_1}$) $\exists\ U \in X_1$ open $\ni x \in U$ & $U \cap A_1 = \phi$
take $W = U \times X_2$ then W is open in $X_1 \times X_2$
& $W \cap (A_1 \times A_2) = U \times X_2 \cap (A_1 \times A_2) = (U \cap A_1) \times (X_2 \cap A_2) = \phi$
So $(x,y) \notin \overline{A_1 \times A_2}$
Hence $\overline{A_1 \times A_2} \subseteq \overline{A_1} \times \overline{A_2}$
($\supseteq$) suppose $(x,y) \notin \overline{A_1 \times A_2}$ $\exists\ W \subseteq X_1 \times X_2$ open
$\ni W \cap (A_1 \times A_2) = \phi$ (provided that $(x,y) \in W$)
$\exists$ a basic open set $U \times V \ni (x,y) \in U \times V \subseteq W$
So $(U \times V) \cap (A_1 \times A_2) = (U \cap A_1) \times (V \cap A_2) = \phi$
if $U \cap A_1 = \phi$ then $x \notin \overline{A_1}$ & hence $(x,y) \notin \overline{A_1} \times \overline{A_2}$
if $V \cap A_2 = \phi$ then $y \notin \overline{A_2}$ & again $(x,y) \notin \overline{A_1} \times \overline{A_2}$
Thus $\overline{A_1} \times \overline{A_2} \subseteq \overline{A_1 \times A_2}$
$\therefore \overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$

(2) Similarly $(A_1 \times A_2)^\circ = A_1^\circ \times A_2^\circ$

Proof
($\subseteq$) $(x,y) \in (A_1 \times A_2)^\circ \Longrightarrow \exists\ G$ open $\ni (x,y) \in G \subseteq$ [illegible]
$\exists$ basic open set $U \times V \ni (x,y) \in U \times V \subseteq G \subseteq A_1 \times A_2$

then $x \in U \subseteq A_1$ & $y \in V \subseteq A_2$

then $x \in A_1^{\circ}$ & $y \in A_2^{\circ}$

then $(x,y) \in A_1^{\circ} \times A_2^{\circ}$

⊇)

Let $(x,y) \in A_1^{\circ} \times A_2^{\circ}$ then $x \in A_1^{\circ}$ & $y \in A_2^{\circ}$

∃ U open ∋ $x \in U \subseteq A_1$ & ∃ V open ∋ $y \in V \subseteq A_2$

then $(x,y) \in \underbrace{U \times V}_{open} \subseteq A_1 \times A_2 \Rightarrow (x,y) \in (A_1 \times A_2)^{\circ}$

Let $\{(X_k, T_k) : k = 1, \dots, n\}$ & $A_k \subseteq X_k$ $k = 1, 2, \dots, n$

then $Int\left(\prod_{k=1}^{n} A_k\right) = \prod_{i=1}^{n} Int(A_k)$

Proof ⊇) $A_k^{\circ} \subseteq A_k$ $\forall k = 1, 2, \dots, n$

$\prod_{k=1}^{n} A_k^{\circ} \subseteq \prod_{k=1}^{n} A_k$ then $\prod_{k=1}^{n} A_k^{\circ} \subseteq Int\left(\prod_{k=1}^{n} A_k\right)$

⊆)

Let $(x_1, \dots, x_n) \in Int\left(\prod_{i=1}^{n} A_k\right)$ then ∃ a basic open set

$\prod U_k$ ∋ $(x_1, \dots, x_k) \in \prod_{k=1}^{n} U_k \subseteq \prod_{k=1}^{n} A_k$

$x_k \in U_k \subseteq A_k$ $\forall k = 1, \dots, n$

then $x_k \in (A_k)^{\circ}$ & hence $(x_1, \dots, x_n) \in \prod_{k=1}^{n} Int(A_k)$

$Int\left(\prod A_k\right) \subseteq \prod_{k=1}^{n} Int(A_k)$ —— (2)

Let $\{(X_k, T_k) : k = 1, \dots, n\}$, $A_k \subseteq X_k$ $k = 1, \dots, n$

then $\left(\prod_{k=1}^{n} A_k\right)' = (A_1' \times \overline{A_2} \times \dots \times \overline{A_n}) \cup (\overline{A_1} \times A_2' \times \overline{A_3} \times \dots \times \overline{A_n}) \cup \dots$

$\cup (\overline{A_1} \times \overline{A_2} \dots \times \overline{A_{n-1}} \times A_n')$

Proof

$n = 2$

$(A_1 \times A_2)' \overset{?}{=} (A_1' \times \overline{A_2}) \cup (\overline{A_1} \times A_2')$

Recall

$x \in A' \Leftrightarrow x \in \overline{A - \{x\}}$ [illegible]

∴

$(x_0, y_0) \in (A_1 \times A_2)' \Leftrightarrow (x_0, y_0) \in \overline{A_1 \times A_2 - \{(x_0, y_0)\}}$

$\Leftrightarrow (x_0, y_0) \in \overline{A_1 - \{x_0\}} \times \dots$

$\Leftrightarrow (x_0, y_0) \in \overline{A_1 - \{x_0\}} \times \overline{A_2} \cup \overline{A_1} \times \dots$

Suppose it is true for n

to show $n+1$

$\prod_{k=1}^{n+1} A_k \cong \left(\prod_{k=1}^{n} A_k\right) \times A_{n+1}$

$\therefore \left(\left(\prod_{k=1}^{n} A_k\right) \times A_{n+1}\right)' = \left(\left(\prod_{k=1}^{n} A_k\right)' \times \overline{A_{n+1}}\right) \cup \left(\overline{\prod_{k=1}^{n} A_k} \times A_{n+1}'\right)$

$= (A_1' \times \overline{A_2} \times \dots \times \overline{A_{n+1}}) \cup \text{——} \cup (\overline{A_1} \times \dots \times A_{n+1}')$

Q.E.D

3 Neighborhoods & Neighborhoods base

nbd: Let (X,τ) be top. space & $x \in X$, a nbd of x is a subset $U \subseteq X \ni x \in U^{\circ} \subseteq U$

nbd system at x - مجموعة جوارات x

$\mathcal{U}(x)$ = is family of all nbds of point x
$= \{U : U$ is nbd of $x\}$ is called nbd system at x.

Properties of nbd system

The nbd system $\mathcal{U}(x)$ at x in a topological space X has following properties

- N_1: if $U \in \mathcal{U}(x) \Rightarrow x \in U$
- N_2: if $U_1, U_2 \in \mathcal{U}(x) \Rightarrow U_1 \cap U_2 \in \mathcal{U}(x)$
- N_3: if $U \in \mathcal{U}(x)$ then $\exists V \in \mathcal{U}(x) \ni$ whenever $y \in V$ then $U \in \mathcal{U}(y)$
- N_4: if $U \in \mathcal{U}(x)$ & $U \subseteq W$ (superset) then $W \in \mathcal{U}(x)$
- N_5: a subset $G \subseteq X$ is open $\Leftrightarrow$ G contains a nbd of each of it's point.

Proof

N_1 by defn.

N_2 $x \in U_1^{\circ}$ & $x \in U_2^{\circ} \Rightarrow x \in U_1^{\circ} \cap U_2^{\circ} = (U_1 \cap U_2)^{\circ}$

$\therefore x \in (U_1 \cap U_2)^{\circ} \subset U_1 \cap U_2 \quad \therefore x \in U_1 \cap U_2 \in \mathcal{U}(x)$

N_3 $U \in \mathcal{U}(x)$ Take $V = U^{\circ} \in \mathcal{U}(x)$ & $y \in V$ then $y \in U^{\circ} \subset U$
$\therefore U \in \mathcal{U}(y)$ $\therefore U$ is a nbd of y

N_4 $U \in \mathcal{U}(x) \Rightarrow x \in U^{\circ} \subseteq W^{\circ} \Rightarrow x \in W^{\circ} \Rightarrow W \in \mathcal{U}(x)$

N_5 Suppose G is open, let $x \in G = G^{\circ} \Rightarrow G \in \mathcal{U}(x)$ & $G \subseteq G$.
Conversely, $\forall x \in G$ then $\exists U \in \mathcal{U}(x) \ni x \in U \subseteq G$ Thus G is open

---

(Q) Show that if each point $x \in X$ has assigned a collection $\mathcal{U}(x)$ of subsets of X satisfying N_1 through N_4 then the collection
$\tau = \{G \subset X : \forall x \in G \ni x \in U \subseteq G$ for some $U \in \mathcal{U}(x)\}$ is
a Topology for X in which the nbd system at each x is $\mathcal{U}(x)$
opensets تحدد لنا $\mathcal{U}(x)$ أي تعين top.

pf. Let us call a subset $G \subseteq X$ to be open $\Leftrightarrow \forall x \in G \ \exists U \in \mathcal{U}(x)$
$\ni x \in U \subseteq G$

لإثبات أن المجموعة τ تمثل تبولوجي نطبق شروط التبولوجي، فإذا كانت لكل x إذن يجب التحقق من الشروط التالية

1- ϕ, X are open & $X \in \tau$ because $\forall x \in X \ \exists U \in \mathcal{U}(x) \ni x \in U \subset X$
Suppose $\phi \notin \tau \Rightarrow \exists x \in \phi$ which doesn't hold [illegible] $U \in \mathcal{U}(x) \ni$ [illegible]

2- Let G_1, G_2 be open & (i) if $G_1 \cap G_2 = \phi$ is open $\in \tau$
(ii) if $G_1 \cap G_2 \neq \phi \Rightarrow x \in G_1 \cap G_2 \Rightarrow x \in G_1$ & $x \in G_2$
then $\exists U_1, U_2 \in \mathcal{U}(x) \ni x \in U_1 \subseteq G_1$ & $x \in U_2 \subseteq G_2$
then $x \in U_1 \cap U_2 \subseteq G_1 \cap G_2 \Rightarrow G_1 \cap G_2$ is open.

3. Let G_α be open set $\forall \alpha \in A$ & $\bigcup_{\alpha \in A} G_\alpha = \phi$ trivial
& $\bigcup_{\alpha \in A} G_\alpha \neq \phi$ let $x \in \bigcup_{\alpha \in A} G_\alpha \Rightarrow x \in G_{\alpha_0}$ for some $\alpha_0 \in A$
then $\exists U \in \mathcal{U}(x) \ni x \in U \subseteq G_{\alpha_0} \therefore G_{\alpha_0} \subseteq \bigcup_{\alpha \in A} G_\alpha$

(X, τ), let $x \in X$ To prove that $\mathcal{U}(x)$ is the only nbd system: نثبت أن

$\Rightarrow$
Let W be a nbd of x (i.e. $x \in W^{\circ}$ open) hence $\exists U \in \mathcal{U}(x) \ni x \in U \subseteq W$
So $W \in \mathcal{U}(x)$

$\Leftarrow$ let $U \in \mathcal{U}(x)$, let $G = \{y : U \in \mathcal{U}(y)\}$ [illegible] then $x \in G$ &
& $G \subseteq U$ we show G is open. Let $s \in G \Rightarrow U \in \mathcal{U}(s)$ for some
& $U \in \mathcal{U}$ by (N_3). $\exists W \in \mathcal{U}(s) \ni$ whenever $y \in W$ then
$U \in \mathcal{U}(y)$ thus $s \in W \subseteq G$ therefore G is open

Nbd Base at x

(X, T) be top-space & Nbd system $\mathcal{U}(x)$

Nbd Base at X = a sub family $B(x) \subseteq \mathcal{U}(x) \ni U \in \mathcal{U}(x)$ iff $B \subseteq U$ for some $B \in B(x)$ }

$\mathcal{U}(x) = \{U \subseteq X ; B \subseteq U$ for some $B \in B(x)\}$

Properties of Nbd Base at x

Let be a topological space & for each $x \in X$, Let $B(x)$ be a nbd base then

- if $B \in B(x)$ then $x \in B$
- if $B_1, B_2 \in B(x)$ then $\exists B_3 \in B(x) \ni B_3 \subseteq B_1 \cap B_2$
- if $B \in B(x)$ then $\exists B_0 \in B(x) \ni$ whenever $y \in B_0$ then $\exists W \in B(y) \ni W \subseteq B$

Furthermore

G is open iff $\forall x \in G \ \exists B \in B(x) \ni x \in B \subseteq G$

Proof

$B_1, B_2 \in \mathcal{U}(x)$ by (N_2) $B_1 \cap B_2 \in \mathcal{U}(x) \Rightarrow \exists B_3 \in B(x) \ni B_3 \subseteq B_1 \cap B_2$

Let $B \in B(x)$ so $B \in \mathcal{U}(x)$

$\Rightarrow$ when ever $y \in V$ then $B \in \mathcal{U}(y)$ then $\exists B_0 \in B(x)$

$\ni y \in B_0$ then $\exists W \in B(y) \ni W \subseteq B$

$\Rightarrow \exists U \in \mathcal{U}(x) \ni x \in U \subseteq G \ \exists B \in B(x) \ni x \in B \subseteq U \subseteq G$

Since $\forall x \in G \ \exists B \in B(x) \ni x \in B \subseteq G$ & $G \subseteq \mathcal{U}(x)$

$\Rightarrow G$ is open

Exercise

(1) The Sorgenfrey Line. $\forall x \in R$ we let $B(x) = \{[x, a) \ a \in R, x < a\}$

(a) show can generate a Topology in R.

(b) what is relative of T to T_u in R

(c) are the sets $[x, a)$ $a > x$ — T-open

Solution

(a) (NB_1) if $B \in B(x)$ then $B = [x, a)$ for some $a \in R \Rightarrow x \in B$

(NB_2) if $B_1, B_2 \in B(x)$ then $B_1 = [x, a)$ & $B_2 = [x, b)$ $\in B_x$

then $B_1 \cap B_2 = [x, c)$ where $c = \min(a, b)$ take $B_3 = [x, c)$

(NB_3) let $B \in B(x)$ say $B = [x, a)$ for some $a \in R$ choose

$B_0 = B = [x, a)$ Now if $y \in B_0$ then $[y, a) \in B(y)$

& $[y, a) \subseteq B$ so $B_1 = [y, a)$ $\therefore (R, \tau)$ is Topology

(b) to show $T_u \subsetneq T_{sor}$

if $V \in T_u$ if $x \in V$ $\exists a, b \in R \ni x \in (a, b) \subseteq V$

so $x = [x, b) \subseteq (a, b) \subseteq V$ then $V \in T_{sor}$ $\therefore T_u \subseteq T_s$

but Converse $T_{sor} \nsubseteq T_u$ let $x \in T_{sor}$ not

Can show $[a, b)$ open in T_u since $[a, b)$ is closed

(c) $G = [a, b)$ is open if Contains a nbd of each $x \in G$

let $x \in G$ then $[x, b) \subseteq G$ + $[x, b) \in B(x)$ so it is a nbd base of x & it is nbd of X

or

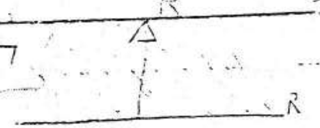
$[x, a)$ is $T(B(x))$ open

let $y \in [x, a)$ take $\boxed{B \in B(x)}$ $\ni B = [x, a)$

then $y \in B \subseteq [x, a)$ by condition ③ T-open

24

Moore plan (Niemycki plane): Γ



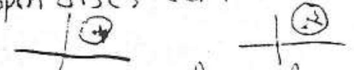
... the closed half plan

$\Gamma = X = \{(x,y) : y \geq 0\}$ in R^2

..., let $B(x)$, For $x=(s,t)$ with $t>0$ مناطق ادائية

... $B(x)$ be the family of all open discs centered at x

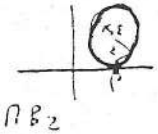
... above the x-axis



... then $B(p)$ contains elements of the form

... where B is an open disc tagent to the x-axis

... $p=(x,0)$ or $B(p) = \{S(x,\varepsilon),\varepsilon) \cup \{p\} : \varepsilon>0\}$



... $\overset{\text{by def}}{\Longrightarrow} x \in B$

... $B_2 \in B(x)$ then $\exists B_3 \in B(x) \ni B_3 \subseteq B_1 \cap B_2$

... تقاطع الجوارات (C)

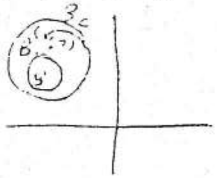
$B_3 = B_1 \cap B_2$

$B_1 = D \cup \{x\}$

$B_2 = D_2 \cup \{x\}$

... $B_1 \cap B_2$

... $\exists B_0 \in B(x) \ni$ whenever $y \in B_0$ $\exists w \subseteq B_y$ we



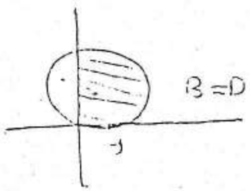
... $y \in B_0$ $\exists B_1 \in B(y)$ $\ni B_1 \subseteq B$

... $x \in$ ○

... $\in B_0 \Rightarrow \exists B_1 \in B(y)$

... take $B_1 = B$

... case (1)



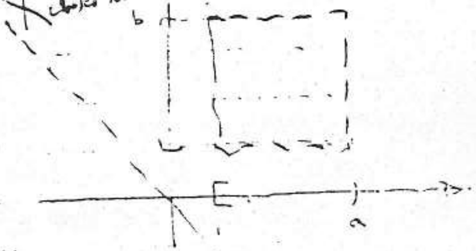
$B = D \cup \{x\}$

(3) The product space $E \times E$ is called Sorgenfrey plan in we let

$X = \{(x,-x) \in E \times E, x \in R\}$ then show that

(1) X is closed

(2) the relative Topology on X is the discreate Topology



تمرين

$E \times E$ Sorgenfrey space

Solution

(1) to show X is closed is sufficient to show

$E \times E - X$ is open

Let $y \in E \times E - X$ — they $y = (x_1, y_1)$

take $B = [x_1, x_1+\varepsilon) \times [y_1, y_1+\varepsilon)$ contains (x_1,y_1)

$y \in B \subseteq \underbrace{E \times E - X}_{\text{open}}$ $\Longrightarrow$ X is closed

(2) The relative top on X is discrete Topology

Since every point in X is open it can be written as the $\sqcap$ of open set in $E \times E$ with X

$X \sqcap \underbrace{[x,c) \times [-x,\varepsilon)}_{\text{open in } E\times E} = \{(x,-x)\}$

Q.E.D

$x \in R$ Let $B(x) = \{ [a,b] ,\ a,b \in R ,\ a < x < b \}$
- how can generate Top τ on R.
- How does τ compare with τ_u.
- are the set $[a,b]$, $a < b$, τ-open.

Solution

- Let $B \in B(x)$ then $B = [a,b]$ $\ni a,b \in R$
  $a < x < b \Rightarrow x \in B$
- Let $B_1, B_2 \in B(x)$ then $B_1 = [a_1,b_1]$ + $B_2 = [a_2,b_2]$
  $\therefore B_1 \cap B_2 = [\max\{a_1,a_2\}, \min\{b_1,b_2\}] = B_3$
  then $\boxed{B_3} = B_1 \cap B_2 \subseteq B_1 \cap B_2$
- Let $B \in B(x)$ then $B = [a,b]$ $\ni a < x < b$
  take $B_1 = B = [a,b] \Rightarrow B_1 \in B(x)$
  Let $y \in B_1 = [a,b] \Rightarrow a < y < b$ take
  $w = B_1$ + $w \in B(y)$

to prove $\tau_u \subseteq \tau$
since $\forall a,b \in R$, $U = (a,b) \in \tau_u$ + Let U open set in
τ_u if $x \in U$ + $\exists [a,b] \in \tau \ni x \in [a,b]$
, $a < x < b$

Let $U = (a,b)$, $x \in (a,b) \subseteq [a,b] \neq U$
$\therefore \tau_u \subsetneqq \tau$

$[a,b]$ is not τ-open
Let $x \in [a,b]$ $\nexists B \in B(x) \ni x \in B \subseteq [a,b]$

but if defn of $B(x) = \{ [a,b] , a,b , a \le x \le b \}$
given discrete Top + open.

مشكل

(4) **Smirnov Topology** :- Let $C = \{ \frac{1}{n} , n = 1,2,\dots \}$ For $x \in$
+ $x \neq 0$ we let $B(x) = \{ (a,b) , a < x < b \}$
$B(0) = \{ (a,b) - C , a < 0 < b \}$

$\therefore B = \bigcup_{x \in B} B(x)$ هي طريقة البحث التي أختبر صحة أو خطأ

Solve

$NB(1)$ $B \in B(x)$ then $B = (a,b)$: $a < x < b \Rightarrow x \in B$
$NB(2)$ let B_1 + $B_2 \in B(x)$ then $B_1 = (a_1,b_1)$ + $B_2 = (a_2,b_2)$
then $B_1 \cap B_2 = (\max(a_1,a_2), \min\{b_1,b_2\})$ open
(CY) $B_1 \cap B_2 = \phi$ $\exists B_3 \in B(x)$ $\ni B_3 \subseteq B_1 \cap B_2$
(NB_3) let $B \in B(x)$ take $B = B_1$ let $y \in B_1$ +
① $y \neq 0 \Rightarrow y \in B = (a,b)$ ~~...~~ $\in B(y)$
So we can take $B_2 = B_1 = B$
② if $y = 0$ $B_2 = (a,b) - C = B - C \in B(0)$
$\therefore$ $\tau_{smirnov}$ is Top
+ $\tau_u \subseteq \tau_{sim}$ but $\tau \nsubseteq \tau_{sime}$