DEPARTMENT OF MATHEMATICS-COLLEGE OF EDUCATION FOR PURE SCIENCE-UNIVERSITY OF ANBAR

ON SEPARATION AXIOMS (T_0, T_1, T_2, $T_{2\frac{1}{2}}$ T_3, $T_{3\frac{1}{2}}$, T_4, AND T_5) AND RELATIONSHIPS AMONG THEM

A graduation project is submitted to the department of mathematics in partial fulfillment of the requirements for the degree of Bachelor of Science in Mathematics

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Definition (1-1) [1]

Let X be a non-empty set. Then the collection T of sub sets of X is called **Topology** for X if T satisfies the following axioms:-

1- X and Ø ∈ T.

2- If A_1 and A_2 are any two sets in T. then $A_1 \cap A_2 \in T$.

3- If $\{A_\alpha : \alpha \in \Delta\}$ be an arbitrary collection of sets in T then $\cup \{A_\alpha : \alpha \in \Delta\}$ is in T.

Remark (1-1) [6]

If T is topology on X. Then (X, T) is called Top-space.

Remark (1-2) [6]

In a topological space (X.T). The members of T are called open sets.

So: in a topological space (X.T):-

1- Ø. X

2- The intersection of finite collection of open sets is open.

3- Arbitrary (in finite) union of open sets is open.

Examples:

Let X = {a, b, c} consider the following collection of subset of X:

$T_1 = \{Ø, X, \{a\}\}$ and $T_2 = \{Ø, X, \{a\}, \{a, c\}\}$.

It's clear that each one of above collections or families are topology or X

$T_3 = \{Ø, X, \{a\}, \{c\}\}$ not topology on X because $\{a\} \in T_3$ and $\{c\} \in T_3$ But $\{a\} \cup \{c\} = \{a, c\} \notin T_3$.

Some types of topological space

First: Let $X \neq \emptyset$. The collection $T_i = \{\emptyset, X\}$ is topology and it known indiscrete topology.

The pair: (X, T_i) is called Top-sp

Second: $X \neq \emptyset$ and T_d is collection of all possible subsets of X. then T_d is topology for X. (i. e) $T_d = \{power (X) = \{P(X)\}$

Third: Let $X \neq \emptyset$ and $T^* = \{U: X\text{-}U \text{ is finite}\} \cup \{\emptyset\}$

(i.e) T* consist of Ø and all non-empty subsets of X whose complement are finite.

Then (X , T_c) is called co-finite Top.

Fourth: Let $X \neq \emptyset$ and $T^c = \{U: X\text{-}U \text{ countable}\} \cup \{\emptyset\}$

Then (X, T^c) is called co-countable Topological space..

Fifth: Let X = R be all a real numbers and Let T_u be a family consisting of Ø and all non-empty subsets G of R which have the following property:-

$\{\forall \ x \in G\}$ open interval I_x such that $X \in I_x \subseteq G$, Then (X, T_u) is called usual Topological space.

Comparison of Topologies

Definition (1-2) [6]

Let T_1 and T_2 be any two topologies for a set $X \neq \emptyset$:-

1) If every open set in T_1 is open set in T_2 then we write $T_1 \subset T_2$ and say :

T_1 is **coarser or weaker or smaller** than T_2 or T_2 is **finer or stronger or longer** than T_1.

2) If either $T_1 \subset T_2$ or $T_2 \subset T_1$ we say that T_1 and T_2 **comparable** otherwise we say not comparable.

Definition (1-3) [2]

Let (X, T) be a topology space a subset F of X is said to be **closed** if the complement $F^c \in T$

Intersection and union of open and closed set

Theorem (1-1) [6]

1- The intersection of a finite collection of open sets is open .

2- The intersection of finite collection of open sets not necessarily open set.

3- The union of in finite the collection of open sets is open.

Theorem (1-2) [6]

1- The union of finite collection of closed sets is closed.

2- The union of in finite collection of closed sets not necessarily closed set.

3- The intersection of in finite collection of closed sets is closed.

Definition (1-4) [6]

A topological space (X,T) is called **door space** . If every subset of X is either open or closed.

Definition (1-5) [6]

Let (X, T) be a topological space and let x ∈ X . Then a subset N of X is said to be:-

T-neighborhood or neighborhood of x if there exists open set G such that x ∈ G ⊆ N.

Definition (1-6) [6]

The set of all neighborhoods of a point x ∈ X is called the **neighborhood system** of x and denoted by N_x.

Definition (1-7) [3]

Let (X, T) be a topological space. Let x ∈ X and let N_x be the T – neighborhood system of X. Then the sub family β_X of N_x is called **local base** of x if for each N ∈ N_x ∃ B ⊆ B_x such that X ∈ B ⊆ N.

Definition (1-8) [6]

Let (X, T) be a topology space. a sub family β of T is said to be form a **base** for T if for each open set G and each x ∈ G ∃ a member B in β such that x ∈ B ⊆ G .

Limit points and closure of sets

Definition (1-9) [4]

Let (X, T) be a topology space and let $A \subseteq X$ A point $x \in X$ is called **adherent point** or contact point of A if every open set containing X. Contains at least one point of A

Definition (1-10) [6]

A point $x \in X$ is called a limit point or accumulation point of A or a **cluster point** of A if and only if every open set containing x contains at least on point of A other than x .

Remark (1-2) [6]

The set of all limit points of A is called the derived set of A and will denoted by À or D_r (A)

Theorem (1-3) [5]

Let (X, T) be a topology space and let $A \subseteq X$. Then A is closed if and only if $À \subseteq A$. or $D(A) \subseteq A$.

Theorem (1-4) [5]

Let (X, T) be a topological space and let A and B be any subset of X then:

1- $\emptyset^{-} = \emptyset$ or $D(\emptyset) = \emptyset$

2- If $A \subseteq B \Rightarrow D(A) \subseteq D(B)$

3- $D(A \cap B) \subseteq D(A) \cap D(B)$

4- $D(A \cup B) = D(A) \cup D(B)$

Definition (1-11) [2]

Let (X,T) be a topological space and let $A \subseteq X$, then the intersection of all closed sets of A is called the **closure** of A and denoted by $\bar{A}$ or C/(A).

Theorem (1-5) [2]

Let (X, T) be a topological space and let $A \subseteq X$. Then $\bar{A}$ is the smallest closed of A {contains A}.

Theorem (1-6) [4]

Let (X, T) be topological space and let $A \subseteq X$ then A is closed if and only if $\bar{A} = A$.

Theorem (1-7) [3]

Let (X, T) be a topological space and let A and B be a subsets of X then:-

1- $\overline{\emptyset} = \emptyset$ and $\overline{X} = X$ and $\overline{\overline{A}} = \overline{A}$

2- If $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$

3- $(\overline{A \cap B)} \subseteq (\overline{A} \cap \overline{B})$

4- $(\overline{A \cup B)} \subseteq (\overline{A} \cup \overline{B})$.

Interior, Exterior and Boundary of sets

Definition (1-12) [6]

Let (X, T) be a topological space and let $A \subseteq X$, a point $x \in A$ is said to be a **interior point** of A if and only if A is a neighborhood of x.

Remark (1-3) [6]

The sets of all interior points of A is called the **interior** of A and denoted by int (A) or Å .

Theorem (1-8) [1]

Let (X, T) be a topological space and let $A \subseteq X$, then:-

1- Å is the largest open subsets contained in A.

2- A is open if and only if and only if Å = A or int (A) = A.

Theorem (1-9) [5]

Let (X,T) be a topological space and let A and B be any subsets of X , then :-

1- $\emptyset^\circ = \emptyset$, $X^\circ = X$ and $(A^\circ)^\circ = A^\circ$.

2- If $A \subseteq \backslash b \Rightarrow A^\circ \subseteq B^\circ$.

3- $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Definition (1-13) [6]

Let (X, T) be a topological space and let $A \subseteq X$ a point $x \in X$ is called an **exterior point** of A if and only if it is an interior point of A^c .

Remark (1-4) [6]

The set of all exterior points of A is called the exterior of A and denoted by ext (A).

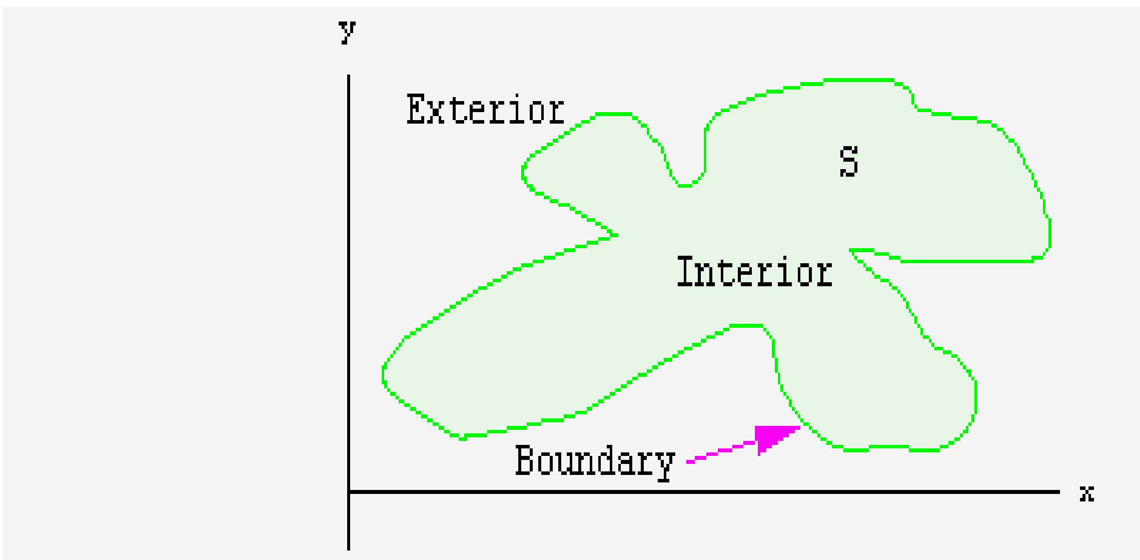
Definition (1-14) [6]

Let (X, T) be a topological space and let A ⊆ X. A point x ∈ X is called **boundary point** or frontier point of A if and only if :-

Every open set containing x intersects both A and A^c or A and cl (A).

Remark (1-5) [6]

The set of all boundary point is called the boundary of A written as bd (A) or Fr (A)



Definition (1-14) [2]

Let (X, T) be a topological space and let A ⊆ X, then A is said to be 1:

1- Everywhere dense if $\overline{A}$ = X .

2- Nowhere dense if ext.(A) = x .

3- Dense in itself if $\overline{A}$ ⊆ A (i.e.) every limit point of A is in A.

4- Dense relative to another set B , if B ⊆ $\overline{A}$.

Definition (1-15) [2]

A topological space (X, T) is said to be **separable** if and only if there exists a countable dense subset A of X.

Definition (1-16) [6]

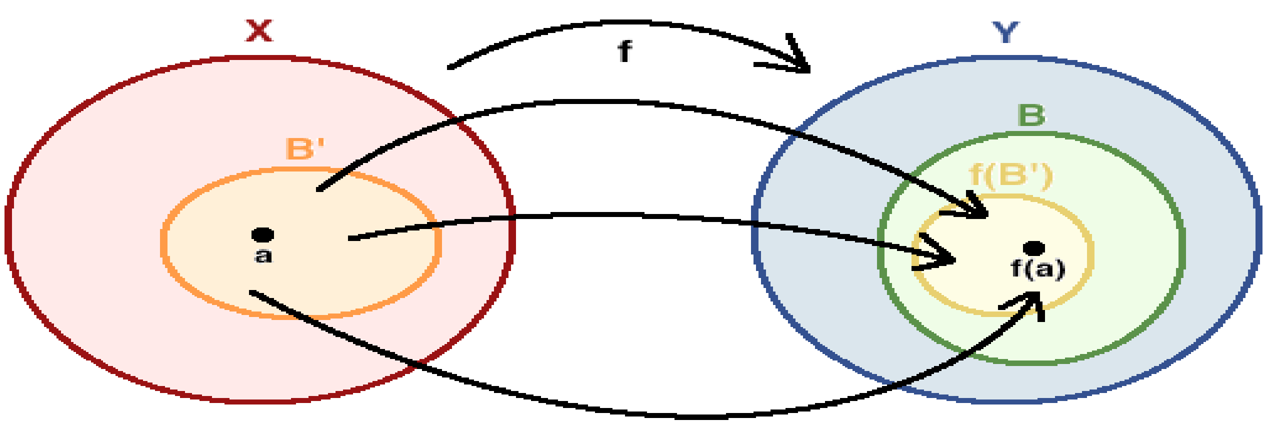
Let (X, T) be a topological space and let $Y \subseteq X$, then The collection $T_y = \{G \cap Y : G \in T\}$ is topology on X.

Definition (1-17) [6]

A property of topological space is called or said to be a **hereditary property** if it is satisfied by every sub spaces of the given space

Definition (1-18) [6]

Let (X, T) and (Y,T) be a topological spaces and let $F : X \to Y$, then F is said to be **continuous** at a point $x \in X$ if and only if for each T^*- open set H containing F(x) $\exists$ a T-open set G containing X such that : $F(G) \subseteq H$.



The function f is said to be continuous at the point **a in X** if there exists local bases $\mathcal{B}_a$ of **a** and $\mathcal{B}_{f(a)}$ of **f(a)** such that for every **B** in $\mathcal{B}_{f(a)}$ there exists a **B'** in $\mathcal{B}_a$ such that **f(B')** $\subseteq$ **B**.

(Note that f(B') is the image of B' under f, i.e., the set of all points f(x) in Y such that x is in B'.)

Remark (1-6) [6]

The mapping F is said to be continuous if and only if is continuous at each points of X.

Theorem (1-10) [3]

Let (X, T) and (Y,T) be a topological spaces and let F: X → Y then F is continuous if and only if the inverse image under F of every open set in Y is open in X.

Definition (1-19) [6]

Let (X, T) and (Y, T^*) be a topological spaces and let F: X → Y, then F is said to be

1- **Open mapping** if and only if the image under F of every T-open set in X is T^* - open in Y.

2- **Closed mapping** if and only if the image under F of every T-closed set in X is T^*-closed in Y.

3- **Bi-continuous mapping** if and only If F is open and continuous.

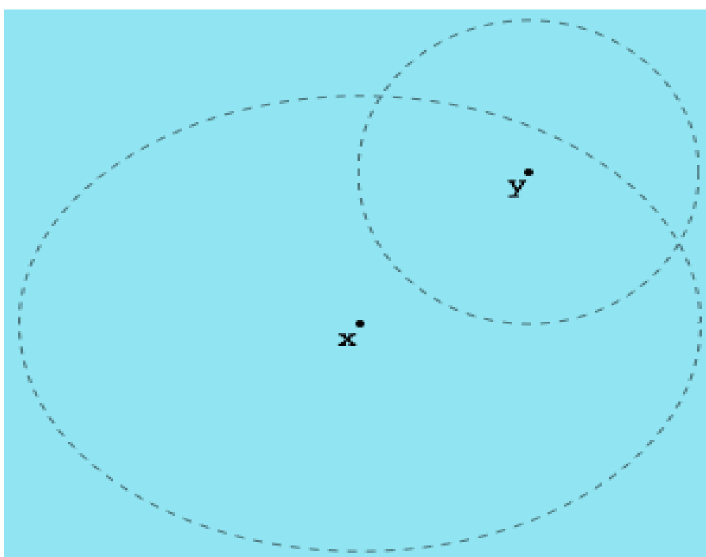
SEPARATION AXIOMS (T_0, T_1, T_2, $T_{2\frac{1}{2}}$ T_3,$T_{3\frac{1}{2}}$, T_4, AND T_5) AND RELATIONSHIPS AMONG THEM

T_0 property and spaces

A topological space **X** has the $\mathbf{T_0}$ property if there exists an open set which separates any two distinct points: if **x** and **y** are distinct points of **X**, there exist an open set which contains one but not the other. Let me be more explicit. A topological space X has the $\mathbf{T_0}$ property if, for any two distinct points x and y in **X**, either there exists an open set **M(x)** containing **x** which does not contain **y**, or there exists an open set **N(y)** containing **y** which does not contain **x**.

NOTE: that the space **X** is an open set containing **x**, but it contains y, and vice versa.

Here's a picture of $\mathbf{T_0}$, showing an open set containing y that does not contain **x**. A $\mathbf{T_0}$ space is sometimes, but rarely do I think, called Kolmogorov.



T_0-Space

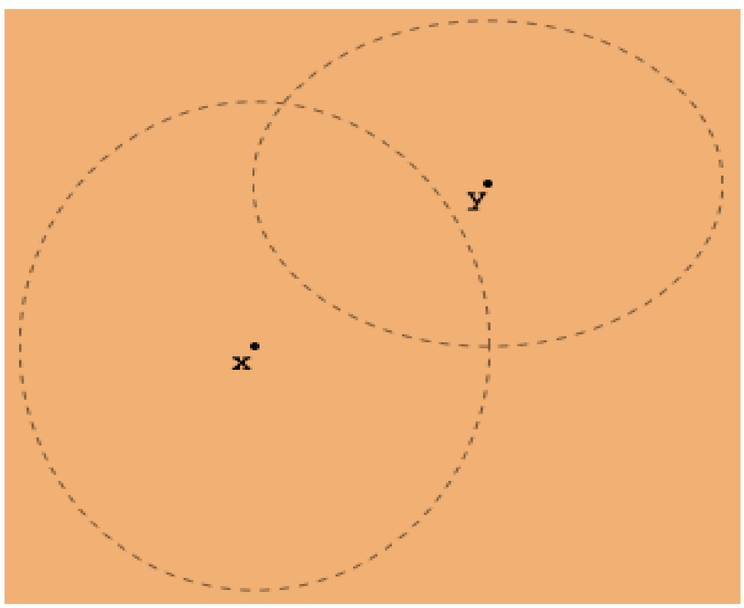
T_1 property and spaces

A topological space **X** has the $\mathbf{T_1}$ property if **x** and y are distinct points **of X**, there exists an open set **M(x)** which contains **x but not y, <u>and</u>** an open set **N(y)** which contains y but not **x**.

One crucial property of a T_1 space is that points (singleton sets) are closed.

This time each point has an open set which contains it but not the other.

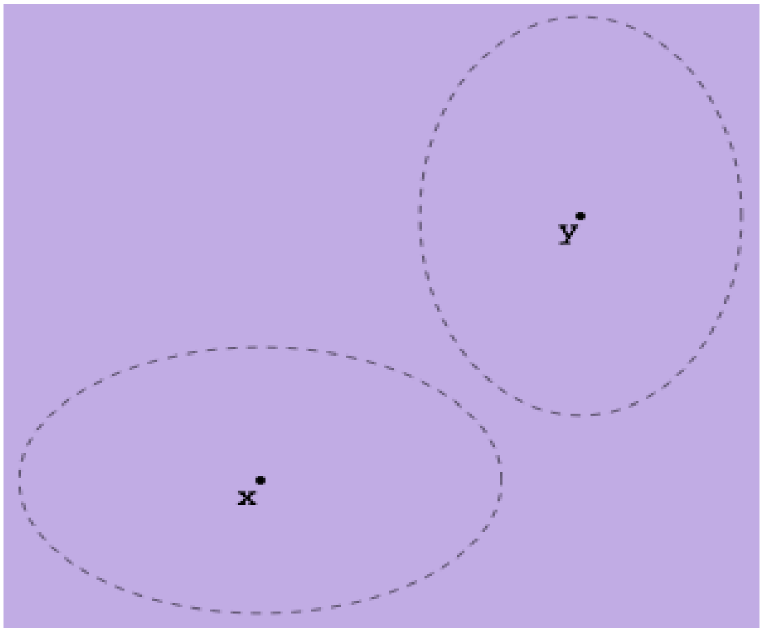
NOTE: that we did not assert that the two open sets do not intersect, merely that their intersection contains neither **x nor y**. (That's the next property.) Here's a picture of $\mathbf{T_1}$, showing open sets which intersect, but their intersection, as we require, does not contain **x** or **y**. A $\mathbf{T_1}$ space is sometimes, but again rarely, I think, called **Frechet**.



T_1-spaces

T_2 property and spaces

A topological space **X** has the **T_2** property if **x** and y are distinct points of **X**, there exist **disjoint** open sets **M(x)** and **N(y)** containing **x and y** respectively. Here's a picture of **T_2**. A **T_2** space is almost always, in my experience, called **Hausdorff**. **One crucial property of a Hausdorff space is that limit points are unique**. (No, I haven't defined a limit point. That's another interesting subject.)

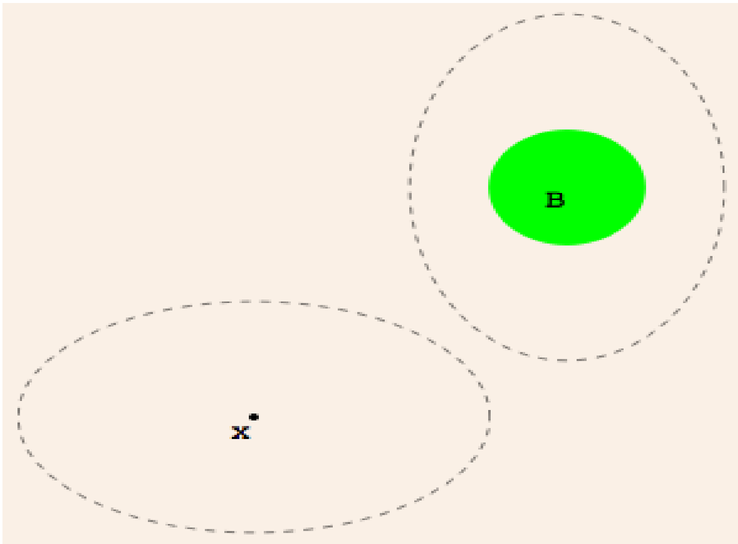


T_2-spaces

T_3 and regular

Now we look at separating **sets** instead of points, still separating them by open sets of some kind. First we separate a point and a closed set. (A set **A** in **X** is closed if its complement **X – A** is open; the closure of **A** ($\bar{A}$), is the smallest closed set containing **A**.) A topological space **X** has the **T_3** property if there exist disjoint open sets which contain any closed set and any point not in the set: for any closed set $\boldsymbol{B}$ and any point $\boldsymbol{x} \notin \boldsymbol{B}$, there exist disjoint open sets containing **x** and $\boldsymbol{B}$ respectively.

Here's $\mathbf{T_3}$. This time I use uppercase ("$\boldsymbol{B}$") and color to denote the closed set.



$\mathbf{T_3}$-spaces

It is crucial that the following set and topology (shown earlier as "an intermediate example") is $\mathbf{T_3}$ but not $\mathbf{T_1}$ (the problem is that the point is not closed):

X = {a, b, c} . $\boldsymbol{T} = \{\emptyset, \boldsymbol{X}, \{\boldsymbol{a}\}, \{\boldsymbol{b}, \boldsymbol{c}\}\}$

This is why and where we need to combine properties in order to get especially worthwhile topological spaces. (Yes, we can study $\mathbf{T_3}$, $\mathbf{T_4}$, and $\mathbf{T_5}$ spaces per se. it is more fruitful to study $\mathbf{T_3} + \mathbf{T_1}$, $\mathbf{T_4} + \mathbf{T_1}$, and $\mathbf{T_5} + \mathbf{T_1}$)

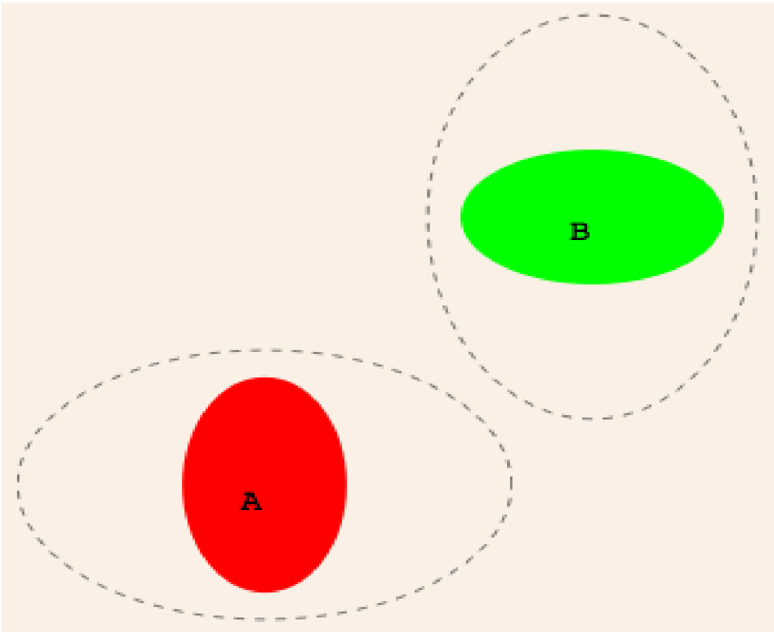
We say that a space is **regular** if it is $\mathbf{T_1}$ and $\mathbf{T_3}$.

(In fact, we can show that if a space is $\mathbf{T_0}$ and $\mathbf{T_3}$, then it is $\mathbf{T_2}$, hence $\mathbf{T_1}$, hence $\mathbf{T_1}$ and $\mathbf{T_3}$. this means we could have defined a space as regular if it is $\mathbf{T_0}$ and $\mathbf{T_3}$. Of course, $\mathbf{T_1}$ and $\mathbf{T_3}$ immediately implies $\mathbf{T_0}$ and $\mathbf{T_3}$, so the two possible definitions of "regular" are equivalent.)

Although I used "normal" and "T_4" in the introductory discussion, the alternative terminology appears here as well, It applies to all subscripts **3** and higher. Where I say that a topological space is regular iff it is T_1 and T_3 other people use regular to refer to my T_3 property, and say a topological space is T_3 iff T_1 and regular. Whereas the progression of the earlier separation axioms kept tightening the requirements on the open sets whose existence we asserted, here we just replaced a point by a closed set. That would be a refinement of the earlier property if points themselves were closed sets. **But** that's T_1, and that's why we want to study spaces which are both T_1 and T_3.

T_4 and normal

Now we separate two closed sets instead of a point and a closed set. A topological space X has the T_4 property if there exist disjoint open sets which contain any two disjoint closed sets: for any disjoint closed sets **A** and **B**, there exist disjoint open sets containing **A** and **B** respectively.



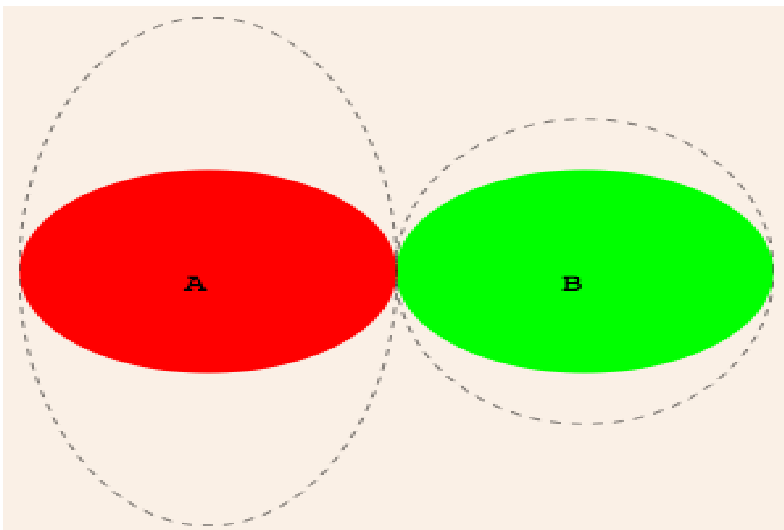
T_4-spaces

I should mention that a bad property of T_4 spaces is that T_4 is not hereditary: not every subspace of T_4 is T_4. We say that a space is **normal** if it is $\mathbf{T_1}$ and $\mathbf{T_4}$. We still have the analogous: not very subspace of a normal space is normal.

T_5 and completely normal

Two subsets **A** and **B** of topological space are **separated** if $\bar{A} \cap B = \emptyset = A \cap \bar{B}$.

A topological space **X** has the $\mathbf{T_5}$ property if there exist disjoint open sets which contain any two separated sets: for any separated sets **A** and **B**, there exist disjoint open sets containing **A** and **B** respectively.



T_5-spaces

Should mention that an alternative equivalent definition of T_5 is that: a space is T_5 iff every subspace is T_4. It corrects the problem with $\mathbf{T_4}$.

We say that a space is **completely normal** if it is T_5 and T_1**. We have the analogous: a space is completely normal iff every subspace is normal**. It corrects the problem with normal, too.

Consider the two open intervals **A = (0, 1/2) and B = (1/2, 1)** with the usual topology of the real line. The sets do not intersect: $\boldsymbol{A} \cap \boldsymbol{B} = \emptyset$, but the closed intervals, their closures, do: $\overline{\boldsymbol{A}} = [\mathbf{0}, \frac{1}{2}]$

$\overline{\boldsymbol{B}} = [\frac{1}{2}, \mathbf{1}]$ and $\overline{\boldsymbol{A}} \cap \overline{\boldsymbol{B}} = \{\frac{1}{2}\}$, Nevertheless, **A** and **B** are separated, because $\overline{\boldsymbol{A}} \cap \boldsymbol{B} = \emptyset = \boldsymbol{A} \cap \overline{\boldsymbol{B}}$.

A and **B** have the T_5 property because **A** and **B** themselves are disjoint open sets. All of those properties, T_0 thru T_5, asserted the existence of open sets, sometimes satisfying additional conditions.

$3\frac{1}{2}$ *And Completely Regular*

We have an intermediate property which is described differently.

Given two disjoint subsets **A** and **B** of a space **X,** a **Urysohn function** for **A** and **B** is a continuous function $\boldsymbol{f}: \boldsymbol{X} \longrightarrow [\mathbf{0}, \mathbf{1}]$ such that ***f*** **(A) = 0** and ***f*** **(B) = 1.**

Urysohn's Lemma, then, says that if **A** and **B** are disjoint closed subsets of a T_4 space, then there exists a Urysohn function for **A** and **B**.

A topological space **X** has the $\mathbf{3}\frac{\mathbf{1}}{\mathbf{2}}$ property if there exist a real-valued continuous function which separates an open set from any point not in it: **(i.e.)** for each open set $\boldsymbol{U} \subset \boldsymbol{X}$ and each x not in **U**, there exist a Urysohn function ***f*** for **x** and **U.**

We say that a space is **completely regular** (or **Tychonoff**) if it is $\mathbf{3\frac{1}{2}}$ and $\mathbf{T_1}$.

Implications of the properties

At this point, thanks to adding $\mathbf{T_1}$ to the definitions, we can show (**!**)

Completely normal ⇒ **normal** ⇒ **completely regular** ⇒ **regular** ⇒ $\boldsymbol{T_{2\frac{1}{2}}}$ ⇒ $\mathbf{T_2}$ ⇒ $\mathbf{T_1}$ ⇒ $\mathbf{T_0}$

The implications among the $\mathbf{T_i}$ properties (for **i** > $\mathbf{2\frac{1}{2}}$) are not so pretty.

Note that a Urysohn space was not in that list. Instead of the subsequence completely regular ⇒ regular ⇒ $\boldsymbol{T_{2\frac{1}{2}}}$

We could have written completely **regular** ⇒ **Urysohn** ⇒ $\boldsymbol{T_{2\frac{1}{2}}}$.

But there is no inclusion relationship between Urysohn and regular. We have two beautiful inclusions, if we omit either regular or Urysohn, but not if we include both.

This is the second reason why I decided to follow Steen & Seebach and use T's for the properties and names for the combinations. If we did it the other way, with names for the properties and T's for the combinations, we could write

$\mathbf{T_5}$ ⇒ $\mathbf{T_4}$ ⇒$\boldsymbol{T_{3\frac{1}{2}}}$⇒ $\mathbf{T_3}$ ⇒ $\boldsymbol{T_{2\frac{1}{2}}}$⇒$\mathbf{T_2}$ ⇒ $\mathbf{T_1}$ ⇒ $\mathbf{T_0}$, or, more elegantly,

Ti ⇒ **Tj** for **i** > **j**, with **i, j** in **{0, 1, 2,** $\boldsymbol{T_{2\frac{1}{2}}}$**, 3,** $\boldsymbol{T_{3\frac{1}{2}}}$**, 4, 5}**

But then we've left Urysohn spaces out in the cold. Since the theorem is no longer pretty, I chose to use the shorter **Ti** to denote a property, and write, for example, **normal** = $\mathbf{T_1 + T_4}$.

I first saw them the other way: $\mathbf{T_4}$ = **normal** + $\mathbf{T_1}$, etc.

And it is possible that I would not have been so struck by them without the lovely **Ti** $\Rightarrow$ **Tj for, i > j**. (Adamson emphasizes that he chooses this convention because of the simplicity of that statement.) Nevertheless, I have presented them the other way. The fact is, if you're studying someone else's work, you may have to adopt their terminology as long as you're there.