DEPARTMENT OF MATHEMATICS-COLLEGE OF EDUCATION FOR PURE SCIENCE-UNIVERSITY OF ANBAR

ON CONTINUOUS MAPPINGS AND TOPOLOGICAL EQUIVALENT

A graduation project is submitted to the department of mathematics in partial fulfillment of the requirements for the degree of Bachelor of Science in Mathematics

BY

HEBA HAMMAD AHMED

SUPERVISOR

ALAA MAHMOOD FARHAN AL-JUMAILI

IRAQ-ANBAR

2017-2018

Chapter one-الفصل الأول

Introduction-المقدمة

يتألف هذا الفصل من مجموعة من التعاريف الاساسية المهمة في الفضاءات التبلوجية والتي تم اخذها من مؤلفات وكتب لها فيمها ولها وزنها من بين كتب علم التبولوجيا .

من هذه التعاريف التي تناولناها في بحثنا هذا هو تعريف التبولوجي على مجموعة غير خالية مثل (X) ثم قدمنا تعاريف اخرى مثل تعريف المجموعة المفتوحة والمغلقة وبعض الخواص المتعلقة بهذه المجموعات وكذلك تعريف النقاط الداخلية لمجموعة نقاط الغاية وانفلاق المجموعة اضافة الى تعاريف اخرى وامثله توضح هذه المفاهيم وذلك بحاجتنا اليها عند دراسة موضوع الاستمرارية والتشاكل في الفضاءات التبولوجية والذي هو موضوع بحثنا هذا اضافه الى بعض المبرهنات الاساسية التي توضع المفاهيم اعلاه.

((1-1)) Definition [1]

Let X be a non empty set [$X \neq \emptyset$] Collection T of a Sub Set of X is called ((Topology)) on X if it Satisfies the following conditions:-

1- $\emptyset, X \in T$.

2- The inter Section of any finite collection of elements of T is element of T.

((i. e)) if $A \in T$ and $B \in T$ then $A \cap B \in T$.

3- The union of and in finite collection of elements of T is element of T.

((1-2)) Definition [1]

The elements of Tare called open.

((1-3)) Examples

Let $X = \{1, 2\}$ and $T_1 = \{ \emptyset , X , \{1\} \}$

$T_2 = \{\emptyset, X\}$, $T_3 = \{ \emptyset , X , \{2\} \}$

$T_4 = \{\emptyset, X, \{1\}, \{2\}\}$

The above are distinct topologies on Same Set X because T_1, T_2, T_3, T_4 Satisfied Conditions of Top-Space

2- Let $X = \{a, b, c\}$ and $T = \{ \emptyset , X , \{ c \} , \{ b \} \}$

Then it clear that is not topology on X because:-

$\{ c \} \in T$ and $\{ b \} \in T$ but $\{ c \} V \{ b \} = \{ b , c \} \notin T$

3- let $X = \{ 1 , 2 , 3 , 4 , 5 \}$ and $T = \{ \emptyset , X , \{ 1 , 2 , 3 \} \} , \{ 2 , 3 , 5 \} , \{ 1 , 5 \}$. T is not topology space because:- $\{ 1 , 2 , 3 \} \in T$ and $\{ 2 , 3 , 5 \} \in T$ and $\{1,2,3\} \cap \{ 2 , 3 , 5 \} = \{ 2 , 3 \}$and $\{ 2 , 3 \} \notin T$

((1-4)) Definition [1]

Let X be any Set more than X has at last two topologies on X defined as following :-

Ti = { Ø , X } is always a topology on X and is called in discrete topology or trivial topology .

((1-5)) Definition [1]

The family P(X) the Set of all Sub Sets X is Called Discrete topology on X topology is Stronger than any other topology defined on X.

((1-6)) Remarks

1- (X , T_i) = (X , T_d) if X has one element

2- (X , T_d) is Called Discrete topology.

((1-7)) Definition [2]

Let X = R be the set of all Real number and let Ty be a family consisting of Ø and all non – empty Subset G of R which have the following property { ∀ X ∈ G ∃ open interval I_x Such that :- X ∈ I_x ⊆ G } Then (R , T_u) is Called usual Topology.

((1-8)) Definition [2]

Co – finite Topology: - Let X be infinite Set and Let:-

Tc = { U : X - U is finite } ⋃ { Ø } , Then T_c is Topology on a Set X.

((1-9)) Definition [2]

Let (X , T) be topology Space a sub Set E of X is Called Closed iff X – E is open ((i . e)) E closed iff Ê is open.

((1-10)) Example [3]

Let = { a, b, c, d } and T = { Ø , X , {a} , {b} , {a, b} }

Then E_1 = { a , b , c } is not Closed because :-

E_1^c = { d } is not open then E_1 is not closed

If E_2 = { b, c, d } Then E_2 is closed because :

$E_2^c = \{a\}$ is open then E_2 is closed

((1-11)) Theorem [3]

If T_1 and T_2 are topologies on the Same Set X Then:-

1- $T_1 \cap T_2$ is Topology on X.

2- - $T_1 \cup T_2$ is not topology on X

((1-12)) Proposition [4]

Let (X , T) is Topology space Then **:-**

1- Ø , X are open sets

2- The intersection of a finite number of open Sets is open Set .

3- The union of in finite number open Sets is open set

((1-13)) Propositions [3]

1- Ø , X are closed Sets

2- The union of finite number of closed Sets is closed set

3- The intersection of in finite number closed sets is closed set.

((1-14)) Definition [3]

Let (X , T) be a topology , P ∈ X any open Set containing P is Called Neighbourhood .

((1-15)) Definition [3]

The Set of all neighbourhoods of appoint X ∈ X is called the neigh bourhood System of X and denoted by N_x .

((1-16)) Example [3]

Let X = { a, b, c } and let Tis Topology on X and T = { Ø , X , **{**a**}** , **{**b**} , {**a**,** b**}** , **{**b, c**}** } Find The (nhd) system of appoints a , b and c .

Solution:-

T^c = { Ø , X , {b, c} , {a, c} , {c} , {b} }

1- N_a = { {a} {a, b} {a, c}, X}

2- N_b = { {b} {a, b} {b, c}, X}

3- N_c = { {b, c}, X}

((1-17)) Definition [3]

Let E ⊆ X . P ∈ E is called in terior point of E iff ∃ Neighborhood N_p of pin X Such that P ∈ N_p ⊆ E .

((1-18)) Remarks :-

1- The Set of all interior Point of E is called interior of E denoted by E^0 or int(E) or i(E).

2- Evidently we define:. A^0= { E ∈ T : E ⊆ A}

((1-19)) Example :-

Let X = {a, b, c, d, e} and Let T = { Ø , X , {a} , {c, d} , {a, c, d} , {b, c, d, **e}** } is Topology on X.

Let A = { a , b ,e } , B = { a , c , d } Then: A^0= {a} , B^0 = {a, c, d}

((1-20)) Theorem [5]

A Sub Set A of topology space (X, T) is open iff $A^0 = A$.

((1-21)) Theorem [5]

Let (X, T) be a topology space and A ⊆ B Then:

1- $\emptyset^0 = \emptyset$

2- $X^0 = X$

3- if A ⊆ B Then $A^0 \subseteq B^0$

4- $A^{0^0} = A^0$

((1-22)) Theorem [5]

In any topology space (X , T) Then $(A \cap B)^0 = A^0 \cap B^0$

((1-23)) Definition [1]

Let E ⊆ X , P ∈ X P is called Limit Point of E iff every neighborhood (N_p) of P contains at Least one Point of E different From P.

((i . e)) $N_p - \{ P \} \cap E \neq \emptyset$.

((1-24)) Remark

The set of all limit Point of E is called Derived Set of E and denoted by D(E).

((1-25)) Definition [3]

Let (X , T) be topology space and Let E ⊆ X the closure of E the inter section of all closed Sets in X which Contain E and denoted by $\bar{E}$ or cl (E)

((i . e)) $\bar{E}$ = ⋂ { f ⊆ X : f closed : E ⊆ f }

((1-26)) Remarks

1- E = the smallest closed Sets containing E.

2-if E ⊆ A then A is closed if $\bar{E}$ ⊆ A

((1-27)) Example :-

Let X = { a, b, c, d, e } and T={ Ø , X , {a} , {a, b} , {a, c, d} , {a ,b, c, d} }

Find The closure of following Sets :-

1- $\{\bar{a}\}$

2- $\{\bar{b}\}$

3- $\{\bar{c}\}$

4- $\{\overline{c, e}\}$

- The closed Sets are: - = { X , Ø , {b, c, d, e} , {c, d, e} , {b, e} , {e} , {c, d}}.

1- $\{\bar{a}\}$ = X

2- $\{\bar{b}\}$ = {b, e}.

3- $\{\bar{c}\}$ = {c, d}.

4- $\{\overline{c, e}\}$ = {c , d, e}.

((1-28)) Theorem [3]

Let (X , T) be topology space and Let E is sub Set of X then :-

1-$\bar{E}$ is smallest closed Set Containing E.

2- E is closed iff $\bar{E} = E$.

((1-29)) Theorem [1]

Let (X , T) be topology space and A, B are Sub Sets of X then :

1- $\bar{\emptyset} = \emptyset$

2- $\bar{X} = X$

3- if A ⊆ B then $\bar{A}$ ⊆ B

4- $\bar{\bar{A}} = A$

5- $\overline{A \cup B} = \bar{A} \cup \bar{B}$

((1-30)) Example [3]

Let X = {a, b, c, d, e} and Let

T= {Ø , X , {a, b} , {a, c, d} , {a, b, c, d} , {a, b, e}} &A = { c , e }

Then: D (A) = {d}

((1-31)) Definition [3]

A point P ∈ X in atopology space (X , T) is Said to be boundary point of E ⊆ X iff every N_P of has anon intersection with E and E^c

((1-32)) Remarks

1- The set of all boundary point of E is called boundary of E denoted b(E) 2- b (E) = $[E^{o} \cup (E)^{c})^{o}]^{c}$

((1-33)) Example

Let X = {a, b, c, d, e} and Let T = {Ø , X , {b} , {c, d} , {a, b, c, d} , {b, c, d} , {a, c, d} , {a, b, c, d}} and Let A = { c } Then: b (A) = { a, c, d, e}.

((1-34)) Definition [1]

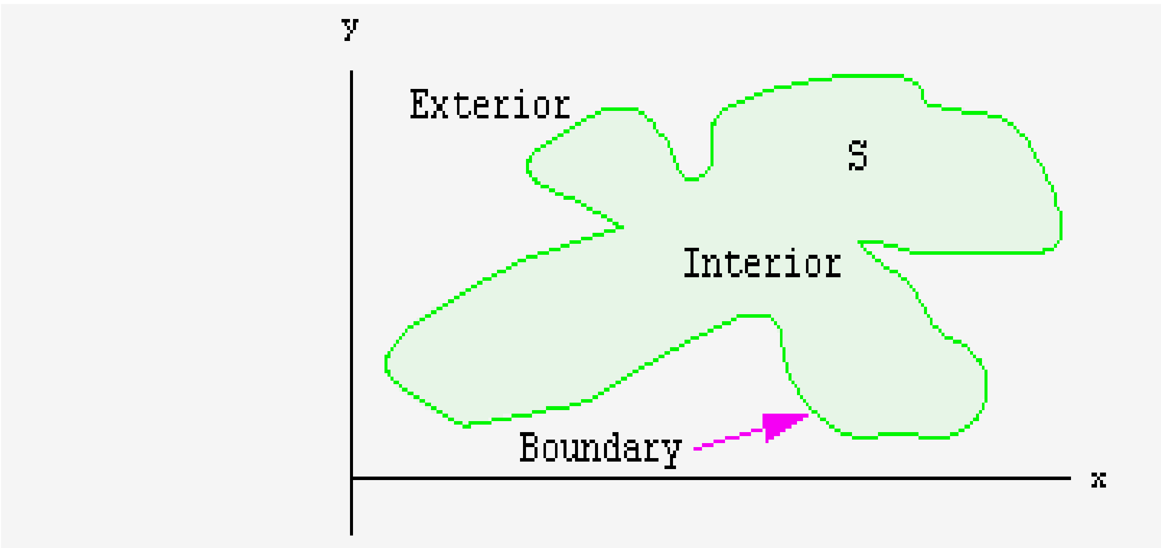
A point P ∈ X is an exterior point of E ⊆ X iff ∃ open Set u ∈ T containing p and u ∩ E = Ø.

((1-35)) Definition [1]

1- The Set of all exterior point is called exterior of E. 2- $E^{e} = (E^{c})^{o}$

((1-36)) Example

Let X = {a, b, c, d, e} and T = {Ø, X, {b} , {c, d} , { b, c, d} , {a, c, d} , {a, b, c, d}} and Let A = { c } Then A^{e} = { b }.



((1-37)) Definition [3]

Let $X \neq \emptyset$ be anon empty Set and let T_2 be topology X if every open in T_1 is also open Set in T_2 that $T_1 \subseteq T_2$ then We Say that T_1 is Smaller or weaker or coarser Than T_2 or T_2 is longer or strong or finer then T_1.

((1-38)) Example

Let $X = \{a, b, c\}$ and Let $T_1 = \{\emptyset, X, \{a\}\}$ and $T_2 = \{\emptyset, X, \{a\}, \{a, b\}\}$ and Let $T_3=\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ Then We have $T_1 \subseteq T_2 \subseteq T_3$

((i . e)) T_3 is finer then T_1 as well and finer than T_2.

((1-39)) Theorem [3]

Let T_1 and T_2 be topology on a same Set X then $T_1 = T_2$ iff T_1 is finer then T_2 and T_2 finer then T_1.

((1-40)) Examples [3]

1- The discrete topology (X , T_d) is largest topology an anon empty Set.

2- The indiscrete topology (X , T_i) is the Smallest topology on anon empty Set.

3- Let $X = \{a, b, c\}$ and let $T_1 = \{\emptyset, X, \{a\}\}$

$T_2 = \{\emptyset, X, \{a\}, \{a, b\}\}$ & $T_3=\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$

Then: $Ti \subseteq T_1 \subseteq T_2 \subseteq T_3 \subseteq T_d$

((1-41)) Definition [5]

Let (X , T) be atop-SP and $E \subseteq R$ Then :- E is Said to be Dense or every where Dense in X if $\overline{E} = X$.

((1-42)) Examples

Let (N ,T^c) be a co-finite topology and let E ⊆ N Such that E = { 2, 4, 6, 8}

Then $\overline{E}$ = N thus E is Dense in N.

((1-43)) Definition [1]

E is Said to be dense in it Self If every point in E is limit point {E,⊆, D**,{**E**}**}

((1-44)) Example [3]

Let (X , T_u) be a usual Topology space E ⊆ X ∋ E = [0,1] thin E is dense in it Self .

((1-45)) Definition [3]

Let (X , T) be a topology space Then X is Said to Separable if there Exist E ⊆ X Such that E is countable and Dense.

((i. e)) X is Separable iff X has Countable dense Set.

((1-46)) Example [3]

R the real number Set in R Then R is a Separable Set in R.

((1-47)) Definition [4]

Let (X , T) be topology Space and let Y be a Sub Set of X the Relative topology is the collection T given by :

T* ={G ∩ Y:G ∈ T}The topological Space (Y,T*) is called Sub Space

((1-48)) Example [3]

Let X = { 1, 2, 3, 4, 5 } and Let Y be a sub Set of X Such that

Y = {1, 4, 5} and T= {Ø, X, {1}, {3, 4},{1, 3, 4},{2, 3, 4, 5}}

Then T*= {G ∩ Y: G ∈ T}

T*= {Ø,Y, [1], [4], [1,4], [4,5]} , There Fore (Y, T*) is Subspace of (X , T)

((1-49)) Definition [3]

A topology Space (X , T) is Said to be Separable iff There exists a countable dense Sub Set A of X.

((1-50)) Example [3]

The Usual topological Spas ((R , T_u)) is Separable Since The Set Q of all rational numbers is countable Subset of R. ((s. t)) $\bar{Q}$ = R

((1-51)) Definition [3]

A property of Topological space is called or is said to be hereditary Property if it is satisfied by every Subspace of the given space.

Chapter Two - الفصل الثاني

- الاستمرارية في الفضاءات التبولوجية

(The Continuity in Topological spaces)

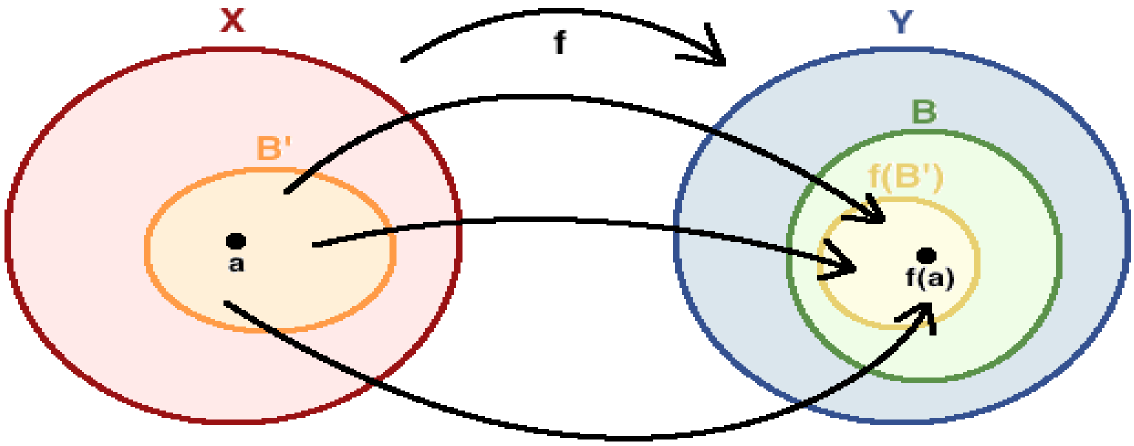
<u>INTRODUCTION</u>

المقدمـــــة

ان مفهوم الاستمرارية يبين صنفاً من الدوال ذا اهميه خاصه ليس فقط في دراسة الرياضيات نفسهاً بل حتى في الاستخدامات العديدة في الهندسة والفيزياء حيث ان هذا الصنف من الدوال دوراً مهما , فالاستمرارية من مفاهيم الرياضيات الاساسية ذات المدلول الهندسي المباشر على مخطط الدالة وقولنا ان الدالة مستمرة في نقطة ما يضن ان مخططها في تلك النقطة متصل مع بقيه اجزاءه . وسنقدم في هذا الفصل مفهوم استمراريه الدوال في الفضاءات التبولوجيه بشكل عام وتقدير مبرهنات مهمة توضح هذا المفهوم في هذه الفضاءات . كما تضمن هذا الفصل دراسة موضوع الفضاءات الجزئية او ما تسمى بالفضاءات النسبية ودراسة مفهوم الاستمرارية في هذا الفضاء وتقديم اهم الخواص المتعلقة بهذا الموضوع .

((2-1-1)) Definition [3]

Let(X, T) and (Y, T) be topological spaces and Let f: X → Y be Map Then *f* is said to **continuous** at x ∈ X iff for each U open in Y (*f* (x) ∈ U) ∃ an open set V in x containing x (x ∈ V) such that *f* (V) ⊆ U.



The function f is said to be continuous at the point **a in X** if there exists local bases $\mathcal{B}_a$ of **a** and $\mathcal{B}_{f(a)}$ of **f(a)** such that for every **B** in $\mathcal{B}_{f(a)}$ there exists a **B'** in $\mathcal{B}_a$ such that **f(B')** ⊆ **B**.

(Note that f(B') is the image of B' under f, i.e., the set of all points f(x) in Y such that x is in B'.)

((2-1-2)) Remark [3]

If the mapping *f* continuous at each x ∈ X then the mapping is called continuous mapping

((2-1-3)) Example [3]

Let X = {1, 2, 3} and T = {∅ , X, {1}, {2}, {1,2}}

Let Y = {a, b} and T={∅ , Y, {a}} and

f : X → Y defined as *f* (1)= a ,*f* (2) = *f* (3)=b

G = X → Y defined g (1) = b , g (2)= g (3) = a

Then *f* is continuous mapping but g is not continues mapping.

((2-1-4)) Example [3]

Let f: (X, T) → (X, T) be a constant mapping then f is continuous.

Proof:-

Let f: X→ Y defend by f (x) ∈ c , ∀ x ∈. X

Let U be open subset in Y then:

$$f^{-1}(\mathrm{U}) = \int_{\emptyset \text{ if } C \in U}^{X \text{ if } C \in U}$$

Since ∅ and X are open subset then f is continuous.

((2-1-5)) Example [3]

Let x= { a, b, c, d} and T={∅,X,{a}, {a,b},{b,c,d} , {b}}and f: (X,T)→ (X,T) be a mapping by : f(a)= ,f (c)=b , f (b)=d , f (d)=c Then

1- f if not continuous .

2- f continuous at point d.

3- f not continues at point c.

((2-1-6)) Definition [3]

A mapping f: (X, T) → (Y, T*) is open mapping iff U is open in x then f (U) is open in Y.

((2-1-7)) Example [3]

Let (X,T)be topology space and let Y={a, b, c} and T ={ ∅ , Y, {a}, {a, c}} Then a mapping f:X→ y defined as :f (x)= , ∀ x ∈ X is open.

((2-1-8)) Definition [2]

A mapping $f:(x,y) \to (Y,T^*)$ is closed iff E is closed in X then f(E)closed in Y.

((2-1-9)) Example [3]

Let (X ,T) be a topology and Y={a, b, c}and T={Ø ,Y, {a},{a, c}then mapping $f: x \to y$ defined $f(x) = b; \forall x \in X$ is closed

[Results on continuous mapping in Topology spaces]

((2-1-10)) Theorem [3]

Let (X, T) and (Y, T^*) be topology space let $f: x \to y$ then f is continuous iff the in veres image under f of every open set in y is open x.

Proof:-

Let f be continuous and let H be any open satiny if $f^{-1}(H) = \emptyset$, it is clearly open so let $f^{-1}(H) \neq \emptyset$, let $x \in f^{-1}(H)$

Then $f(x) \in H$ Bycontinuity of f ,∃ an open set Gin X such that $x \in G$ and $f(G) \subseteq H$. consequently $x \in G \subseteq f^{-1}(H)$

This show that $f^{-1}(H)$ is nhd of each of its points and therefore, it is open in X.

Conversely , let the inverse image under f of every open set in y be open in X, then in order to show that if is continues it is sufficient to show that it is continues at an arbitrary point $x \in X$ let H be any open set in y such that $f(x) \in H$. Then $x \in f^{-1}(H)$ .by hypothesis $f^{-1}(H)$ is an open set in X Now, if we set $f^{-1}(H) = G$, Then h is an open set in X contain x such that $f(G)= f[f^{-1}(H)] \subseteq H$. This Show that f is continues at each point of X.

2-1-11" Theorem [1]

Let (X, T) and (Y, T) topology space and let f: X $\rightarrow$Y Then F is continuous iff for each x $\in$ X The inverse image under F of every T – nhd of f (x) is T-nhd of X.

Proof:-

Let f be continuous and let x $\in$ X let M be and T – nhd of f (x) then $\ni$ an open set Hiny Such that f(x) $\in$ H $\subseteq f^{-1}$(M) Since f is continuous and His T open , So f^{-1}(H) is T- open this Show that f^{-1}(M) is a T – nhd of X.

Conversely , let the inverse image under f of every T-nhd of f(x) be a T-nhd of X let H be any open set in Y note is f^{-1}(H) = $\emptyset$, it is clearly open So let f^{-1}(H) $\neq \emptyset$ and let x $\in f^{-1}$(H) then f (x) $\in$ H, This show that His a T- nhd of f (x). So by hypothesis ; f (H) is aT- nhd of X .Thus f^{-1}(H) is a T-nhd of each of its points and there its open, so it follows that in verse image under f of every open Sub of Y is an open Sub Set of X. Hence f is continuous.

"2-1-12" Theorem [1]

Let (X, T) and (Y,T) be topology space and f: X $\longrightarrow$ Y then f is continuous iff the inverse image under f of every closed Subset of Y is a closed sub set of X.

Proof:-

let f be continuous and let K be any closed sub set of Y then (Y-k) is an open sub set of y so by continuity of f, f^{-1}(Y-k) is an open sub set of X But, f^{-1}(Y^{-1}(k)) is open and therefore f^{-1}(k) is closed

Conversely: - let the inverse image under f of every closed sub set Y then (Y-H) is closed and therefore by hypothesis f^{1}(Y-H) is closed But f^{-1}(Y-H) = f^{-}

^{1}(H) = X-f^{-1}(H). So X. f^{-1}(H) is closed and therefore f^{-1}(H) is open. Thus the inverse image under f of every open sub set of Y is an open sub set of X. This show that f is continuous.

<u>"2-1-13" Theorem [1]</u>

Let X, Y and Ƶ be any three Top – Spaces and let f: X ⟶ Y and g: Y ⟶ Ƶ be cont mappings Then the composite gof : X⟶ Ƶ is continuous.

Proof:-

Let H be any open sub set in Ƶ, we must prove that $(gof)^{-1}$(H) is open sub set in X. Since g is cont ⟶ g^{-1}(H) is open sub set in Y and Since f is cont ⟶ $f^{-1}(g^{-1}(H)))$ is open sub set in X So, $f^{-1}(g(H)) = (f^{-1} o g^{-1})(H) = (gof^{-1})(H)$ is open in X. Thus The inverse image under (gof) of every open sub set of Ƶ is open sub set of X.

<u>"2-1-14" Theorem [1]</u>

Now $A \subseteq f^{-1}[f(A)] \subseteq f^{-1}[\overline{f(A)}]$ ⟶ [$f(A) \subseteq \overline{f(A)}$ Let (X, T) and (Y,T*) be a topological spaces and Let f: X ⟶ Y Then f is continuous iff for every sub set A of X $(\bar{A}) \subseteq \overline{f(A)}$

Proof:-

Let f be continuous and let $A \subseteq X$. Then $\overline{f(A)}$ is a closed set in Y. so by continuity of f, $f^{-1}[\overline{f(A)}]$ is closed in X

$\therefore\ f^{-1}[\overline{f(A)}] = f^{-1}[f(A)]$ …. (1)

$\Longrightarrow \bar{A} \subseteq f^{-1}[\overline{f(A)}] \quad = f^{-1}[\overline{f(A)}]$

$\Longrightarrow f(A) \subseteq f\overline{(A)}$

Conversely:- Let $f\ (\bar{A}) \subseteq \overline{f}\ (A)$ for every $A \subseteq X$ let K be any closed set in Y, then $f = K$ Now $f^{-1}(K)$ is a sub set of X and by the given hypo thesis $f\,[f^{-1}(R)] \subseteq f\,[f^{-1}(R)] \subseteq f = f$ OR $f^{-1}(R) \subseteq f^{-1}(R)$

but $f^{-1}(K) \subseteq f^{-1}(K)$ ∴ $f^{1}(K) = f^{-1}(K)$ This show that (K) is a closed set in X.

"2-1-15" Theorem [3]

Let (X, T) and (Y,T*) be a Topological space and Let f: X→ Y Then f is continuous iff for every $B \subseteq Y$; $f^{-1}(B) \subseteq f^{-1}(\overline{B})$

Proof:-

Let f be continuous and let $B \subseteq Y$. Then $\overline{\mathbf{B}}$ being closed, by continuity of f, $f^{-1}(\overline{B})$ is closed ∴ $f^{-1}(B) = f^{-1}(\overline{B})$. Now, $B \subseteq \overline{B} \Longrightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B})$

$\Longrightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B})$.

Conversely:- Let $f^{-1}(B) \subseteq f^{-1}(\overline{B})$ for every $B \subseteq Y$.

Now, let K be a closed sub set of Y so that $\overline{f} = K$.

Now, by hypothesis, $f^{-1}(K) \subseteq f^{1}(\overline{K}) = f^{-1}(K)$. But, $f^{-1}(K) \subseteq f^{1}(K)$

∴ $f^{-1}(K) = f^{-1}(K)$ showing that $f^{-1}(K)$ is closed thus the inverse image under f of every closed sub set of Y is a closed sub set of X Hence f is continuous.

((2-1-16)) Theorem [1]

Let (X, T) and (Y,T*) be topology space and let $f: X \to Y$ then f is continuous iff for every $B \subseteq Y$, $\{f^{1}(B)\}^{\circ} \subseteq f^{-1}(B^{\circ})$

Proof:-

Let f be continues and let B$\subseteq$ y .then B° being open, by continuity of f, $f^{-1}(B^\circ)$ is consequently $\{ f^{-1}(B^\circ) \subseteq f^{-1}(B) \Longrightarrow \{ f^{-1}(B^\circ) \}^\circ \subseteq \{ f^{-1}(B) \}^\circ$

$\Longrightarrow \{ f^{-1}(B^\circ) \subseteq \{ f^{-1}(B) \}^\circ \therefore \{ f^{-1}(B)\}^\circ \supseteq f^{-1}(B^\circ)$ for every B$\subseteq$ Y

Conversely:- Let $\{ f^{-1}(B) \}^\circ \supseteq f^{-1}(B^\circ)$ for every B $\subseteq$ Y

Let H be any open sub set of Y , so that $H^\circ = H$

$\therefore$ By given hy phthisis $f^{-1}(H) \subseteq \{ f^{-1}(H)\}^\circ$ Or

$f^{-1}(H) \} \subseteq \{ f^{-1}(H) \}^\circ$ But $\{f^{-1}(H) \}^\circ \subseteq \{ f^{-1}(H) \therefore \{f^{-1}(H) \}^\circ = f^{-1}(H)$

Showing that f-1(H) is open Thus, the inverse image under f of every open sub set of Y is on open subset of X Hence f is continuous

((2-1-17)) Theorem [2]

Let X, Y and Z be any Three topological spaces Let $f: X \to Y$ and $g: Y \to Z$ be continues mapping then, the composite mapping (gof) :(X→ Z is continuous

Proof:-

Let H be any set open z. then by continuity of g, $g^{-1}(H)$ is open in Y And by continuity of $f^{-1}\{g^{-1}(H) \}$ is open in X

So , $f^{-1} \{ g^{-1}(H) \} = (f^{-1} o g^{-1}) (H)$ is open in x. Thus the inverse image under (g of) of every open subset of Z is an open subset of X. Hence of is continuous.

"2-2-1" Theorem [3]

Let (X, T) be a Top – SP and let Y $\subseteq$ X Then the collection:-

$T_y = \{G \cap Y: G \in T\}$ is Topology on X.

Proof:-

(1) $\emptyset \in T$ And $\emptyset \cap Y = \emptyset \Longrightarrow \emptyset \in T_y$.

$X \in T$ And $X \cap Y = Y \Longrightarrow Y \in T_y$.

(2) let H_1 and H_2 be any tow sets in T_{y}, we must prove That

$H_1 \cap H_2 \in T_y$.

Since $H_1 \in T_1 \Longrightarrow H_1 = G_1 \cap Y$ for some $G_1 \in T$

Since $H_2 \in T_y \Longrightarrow H_2 = G_2 \cap Y$ for some $G_2 \in T$

So , $H_1 \cap H_2 = (G_1 \cap Y) \cap (G_2 \cap Y)$

$= (G_1 \cap G_2) \cap Y \in T_y$

$\therefore H_1 \cap H_2 \in T_y$

(3) let $\{ H_x : \ltimes \in \Delta\}$ beany family of setsin T_y.

We must prove that $\cup \{H_{x:} \ltimes \in \Delta\} \in T_y$.

Since $\{ H_x : \ltimes \in \Delta\} \in T_y$. so that for each $\ltimes \in \Delta$

$\ni$ A set $G \ltimes \in T$ "S – t " $H_x = G \ltimes \cap Y \Longrightarrow \cup \{ H_x: \ltimes \in \Delta\} = \cup \{ G_x \cap Y : \ltimes \in \Delta\} = [\cup \{G_x : \ltimes \in \Delta\}] \cap Y \in T_y$ Therefore $\therefore \cup \{ H_x: \ltimes \in \Delta\} \in T_y$.

<u>"2-2-2" Theorem [3]</u>

Let (X, T) and (Y,T^*) be topology space and let f: $X \longrightarrow Y$ be continuous let $A \subseteq X$ Then the restriction f_A of f to A is $T_A – T$ continuous

proof:-

let H be and T^* - open sub set of Y then $f_A^{-1}(H) = A \cap f^{-1}(H)$

Note, by continuity of f , f^{-1} is T- open and therefore:

A $\cap f^{-1}$(H) is T_A – open. Consanguinity, f_A^{-1} (H) is T_A – open Thus the inverse may under f_A of every T – open sub set of Y is a T_A – open sub set of A.

<u>"2-2-3" Theorem [1]</u>

Let (X, T) and (Y, T^*) be topology space and f: X $\longrightarrow$ Y be one – one and continuous. Then f maps every dense in itself subset of X on to dense in itself subset of Y

Proof:-

Let A be dense init self subset of X , Then every point of A is a limit point of A . Let Y $\in f$(A). Then f being one- one $\exists$ aunigue X$\in$ A such Y = f(x) new let N be T-nhd of f(x) then by continuity of f, f^{-1}(N) is a T – nhd of X But. X$\in$ A being a limit point of Af^{-1}(N) must contain at least a point Ƶ $\neq$ X of A.

Now Ƶ $\in f^{-1}$(N) $\Rightarrow f$ (Ƶ) $\in$ (N).

AISO Ƶ $\neq$ X $\Rightarrow f$ (Ƶ) $\neq f$ (X) = Y

Thus N contains at least one point f((Ƶ) of f(A) defferent fromy. This shows thaty is a limit point of f(A) thus each point of f(A) is a limit point of f(A) . Hence f(A) is dense in itself.

<u>"2-2-4" Theorem [3]</u>

E every continuous image of separable space is separable.

Proof:-

Let (X,T) be separable and let (Y, T^*) be topological space let be a continuous mapping of X on to Y Now X being separable ∃ a countable subset A of X Such that $\overline{A} = X$ ∴ Y = f (X) = f ($\bar{A}$) ⊆ $\overline{f(A)}$

So, $\overline{f(A)} = Y$. ∴ f (A) ⊆ Y , always. Also f (A) is countable dense subset of Y , Hence (Y, T^*) is separable.

Chapter Three - الفصل الثالث

- التشاكل في الفضاءات التبولوجية

The homeomorphism in topological Spaces

INTRODUCTION

المقدمـــــة

يعتبر مفهوم التشاكل او التكافؤ التبولوجي من المفاهيم المهمة في هذا الفصل ثم دراسة مفهوما مهما لاقل اهمية عن مفهوم الاستمرارية جدا فيعلم التوبولوجيا وتكمن اهمية هذا المفهوم في كونه أنه بعض الصفات التبولوجية مثل كون المجموعة مفتوحة او مغلقة هي صفات تبولوجية تنذل بفعل التشاكل التبولوجي وذلك كون التشاكل يلعب دورا رئيسيا ومهما في انتقال الصفات التبولوجية من فضاء تبولوجي او فضاء تبلوجي اخر مثل الترابط. وقد تمكنا في هذا الفصل من دراسة مفهوم التشاكل وأهم خواصه وصفاته التي يتمتع بها الفصل الثالث يتألف من بندين رئيسيين تطرقنا في البند الاول لمفهومي الدوال المفتوحة والمغلقة ودراسة خواصها لدورها البارز بالنسبة لمفهومي الاستمرارية والتشاكل التبولوجي وأهم النتائج المتعلقة بهذا المفهوم التكافؤ التبولوجي بين الفضاءات التبولوجية.

"3-1-1" Definition [3]

Let (X, T) and (Y, T^*) be Top – spaces and let *f*: X⟶ Y. Then *f* is said to be:-

1 – Open mapping iff the image under *f* if every open set in X is open set in Y.

2 – Closed mapping iff the image under *f* of every closed set in X is closed set in Y.

3 – bi-continuous mapping iff *f* open and continuous.

"3-1-2" Example [3]

Let (X, T) and (Y, T^*) be a topology spaces: where

$Y = \{a, b, c\}$ And $T^* = \{ \emptyset, Y, \{a\}, \{a, c\}\}$ then a mapping *f*: $X \longrightarrow Y$ defined as:

$f(x) = a, \forall x \in X$ Is open since for any u is T-open set, we have:

$$f(u) = \begin{cases} \emptyset & \text{When } u = \emptyset \\ \{a\} & \text{When } u \neq \emptyset \end{cases}$$

And each on of $\emptyset$ and {a} is T^* - open set.

$\{f(u)$is open in y $\forall$ u open Subset in X$\}$

"3-1-3" Example [3]

Let (X, T) and (Y, T^*) be topology spaces and let $Y = \{a, b, c\}$ and $T^* = \{\emptyset, Y, \{a\}, \{a, c\}\}$

Then the mapping *f*: $X \longrightarrow Y$ defined as:- $f(x) = b, \forall x \in X$, is closed mapping since for any *f* is T – closed set ,

$$f(F) = \begin{cases} \emptyset & \text{When } F = \emptyset \\ \{b\} & \text{When } F \neq \emptyset \end{cases}$$

And each one of $\emptyset$ and $\{b\}$ is T^* - closed

$\{f(F)$ is closed in Y $\forall$ F closed sub set in X.$\}$

"3-1-4" Theorem [1]

Let (X, T) and (Y, T^*) be topology space let f:X $\longrightarrow$ Y Then f isopen iff $f(\mathring{A}) \subseteq [f(A)]°$ for every A $\subset$X

proof:-

Let f be open, Then A° being open it follows that $f(A)°$ is open consequently, $[f(A)°]° = f(A°)$

Now. $A° \subseteq A \Longrightarrow f(A)° \subseteq f(A)$

$\Longrightarrow [f(A)° \subseteq [f(A)]°$

$\Longrightarrow f(A)° \subseteq [f(A)]°$

Conversely:- let $f(A°) \subseteq [f(A)]°$ for every A $\subset$X

Let A be an open subset of X so that $A° = A$

$\therefore f(A)° \subseteq [f(A)]° \Longrightarrow f(A) \subseteq [f(f)]° \therefore [A° = A$ But $[f(A)]° \subseteq f(A)$

$\therefore [f(A)]° = f(A)$. This show that $f(A)$ is open when every A is open.

"3-1-5" Theorem [3]

Let (X,T) and (Y, T)* be topology space let f: X $\rightarrow$ Y then f is closed iff $\overline{f(A)} \subseteq f\left(\overline{A}\right)$ for every A $\subset$X.

proof:-

Let f be closed and let A $\subset$X. then $\overline{A}$ being closed $f\left(\overline{A}\right)$ is therefore closed consequently $\overline{f(\bar{A})} = f(\bar{A})$

Now , $A \subseteq \bar{A} \Longrightarrow f(A) \subseteq f(\bar{A})$

$$\Longrightarrow \overline{f(A)} \subseteq \{f(\bar{A})\}$$

Hence, $\overline{f(A)} \subseteq f(\bar{A})$ for very A$\subset$X

Let A be a closed subset of X then A = A$^-$

$\therefore \overline{f(A)} \subseteq f(\bar{A}) \Longrightarrow \overline{f(A)} \subseteq f(A)$ $[\therefore \bar{A} = A]$

But , $f(A) \subseteq f(A)$ Therefore $\overline{f(A)} = f(A)$.

This show that f (A) is closed, when every so is A. Hence f is a closed mapping.

"3-2-1" Definition [1]

Two topological spaces (X, T) and (Y, T*) are closed homeomorphic if there exits: One – to – one and onto function f: X $\longrightarrow$ y such that f and f^{-1} are continuous and the function f is called homeomorphism.

"3-2-2" Example [3]

Let $X = \{a, b, c, d\}$ and $T = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$

And $Y = \{a, b, c, d\}$ and T^{*} $\{\emptyset, y, \{c\}, \{d\}, \{c, d\}\}$

And f: $X \longrightarrow Y$ defined as:

f (a) = a , f (b) = b , f (c) = c, f (d) = d. is (X, T)

And (Y, T^{*}) are homeomorphic?

1 – f is one – to – one and onto But f is not continuous since $\{c\} \in T^{*}$ But

$f^{-1}\{c\} = \{c\} \notin T$. Therefore f is not homeomorphic.

"3-2-3" Example [3]

Let $X = \{a, b, c, d\}; T = \{\emptyset, y, \{c\}, \{d\}, \{c, d\}\}$ and g: $(X, T) \rightarrow (Y, T)^{*}$ such that:- g (a) = d , g (b) = c , g (c)= b , g(d) = a. is

(X, T) and (Y, T^{*}) are homeomorphic?

Sol:-

(1) And (2) are clear g is one – to – one and onto.

(3) Is g continuous?

(*) $Y \in T^{*} \rightarrow g^{-1}(y) = X \in T$.

(*) $g^{-1}\{\emptyset\} = \emptyset \in T$

(*) $g^{-1}\{c\} = \{b\} \in T*$, $g^{-1}\{d\} = \{a\} \in T$ and

(*) $g^{-1}\{c, d\} = \{a, b\} \in T$. So g is continuous,

(4) is g^{-1} continuous?

(*) $(g^{-1})^{-1}$ {a} = g {a} = {d} ∈ T^*

(*) $(g^{-1})^{-1}${∅} = ∅ ∈ T^*

(*) $(g^{-1})^{-1}$ {b} = g {b} = {c} ∈ T

(*) $(g^{-1})^{-1}${X} = Y ∈ T^*.

(*) $(g^{-1})^{-1}$ {a, b} = g {a, b} = {d, c} ∈ T^*

Since g is one – to – one, onto, g and g^{-1} are continuous.

So, g is homeomorphism. Therefore (X, T) and (Y, T^*) are homeomorphic

<u>"3-2-4" Theorem [3]</u>

let (X,T) and (Y, T^*) be topology space let *f* be a one – one mapping of on to Y then the following statements are all equivalent to one another:-

(i) *f* is open continuous.

(ii) *f* is homeomorphism .

(iii) *f* is closed and continuous.

Proof:- (i) ⟹ (ii) let *f* be a one – one open and continuous mapping of X onto Y then by definition it is a homeomorphism so (i) ⟹ (ii)

(ii) ⟹ (iii) : let *f* be homeomorphism. Then it is a one – one continuous open mapping of X onto Y. Let *f* beany closed subset of X then (X-*f*) is open

Now *f* being open it follows that *f* (X-*f*) is open But

f (X-*f*) = *f* (X) – *f* (F) = Y-*f* (F)

Thus Y-*f* (F) is open and therefor, *f* (F) is closed. This show that *f* is closed and continuous so (ii) ⟹ (iii)

(iii) $\Longrightarrow$ (i) : let f be closed and continuous let G be an open subset of X then X – G is closed and being closed f (X-G) is therefore, closed

But, f (X-G) = f (X) – f (G) = Y-f (G)

Thus, Y. f (G) is closed and therefore, f (G) is open this show that f is closed and continuous.

So (iii) $\Longrightarrow$ (i) Thus , (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) Hence all the given statements are equivalent to on another

"3-2-5" Theorem [3]

Let (X, T) and (Y, T^*) be topology space let $f : X \longrightarrow Y$ be a one – one mapping of X on to y then f is a homeomorphism iff $f(\bar{A})$ for every $A \subset X$

proof:-

Let f be homeomorphism. Then f is a one – one continuous and closed mapping of X onto Y.

Let $A \subset X$ then by continuity of f, we have $f(\bar{A}) \subseteq f(A)$

Also, f being closed we have $f(\bar{A}) \subseteq f(A)$, hence $f(\bar{A}) = f(\bar{A})$

Conversely:- let $f : X \longrightarrow y$ such that is f is one – one onto and for every $A \subset X$, let $f(\bar{A}) = \overline{f(A)}$ Then $f(\bar{A}) \subseteq f(\bar{A})$ and $\overline{f(A)} \subseteq f(\bar{A})$. But these results show that f is continuous and closed f is one – one onto also, so it a homeomorphism.

"3-2-6" Theorem [1]

Let (X, T) and (Y, T^*) be topology space let f: X $\longrightarrow$ Y Be a one – one mapping of X onto Y then f is a homeomorphism iff $f\{A\}° = \{f(A)\}°$ for every $A \subset X$.

proof:-

Let f be homeomorphism. Then f is are one-one continuous and open mapping of X onto Y let $A \subseteq X$.

Then f being open we have $f(A°) \subseteq \{f\ (A\}° \ldots\ldots..(1)$

Also f being continuous and on to and $f(A) \subset Y$ we have $f^{-1}[(f(A)]° = A°$

$[f^{1}(B°) \subseteq \{f^{-1}(A)]°$ for every $B \subset y$

Or $[f(A)° \subseteq f\ (A)°$…….. (2)

Thus for (1) and (2) we get $f(A)° = [f(A)]°$

Conversely:-

Let f be one-one mapping of X onto y such that $f(A)° = [f(A)]°$ for every $A \subset X$ Then $f(A)° \subseteq [f(A)]°$ and $[f(A)]° \subseteq f(A)°$

But $f(A)° \subseteq [f(A)]°$ for every $A \subset X$ implies that f is open agin let $B \subset X$. Such that $B = f(A)$ or $A = f^{-1}(B)$

Now,

$[f(A)]° \Longrightarrow f(A°) \Longrightarrow f^{-1}[f(A)°] \subseteq [f^{-1}[f(A°)] = A°$

$\Longrightarrow f^{-1}(B°) \subseteq [f^{-1}(B)]°$

$\therefore \{A = f^{-1}(B)$ and $f(A)=B\}$

is homeomorphism

$\therefore (X, T) \approx (Y, T^{*}) \Longrightarrow (Y, T^{*}) \approx (X, T)$

(iii) Transitivity:-

Let $(X,T) \approx (Y,T^{*})$ and $(Y,T^{*}) \approx (Ƶ,T^{*})^{*}$ and let f and g be the corresponding homeomorphisms Then f is one-one onto, T^{*}-continuous and f^{-1} is T^{*}-T continuous Also g is one-one onto T^{*}-T^{**}continuous and g^{-1} is T^{**}-T^{*} continuous we claim that the composite mapping *gof*: $X \longrightarrow Ƶ$ is a homeomorphisms since the composite of tow continuous mapping beings being continuous it follows that got is T,T^{**} continuous more-over g^{-1} is T^{**},T^{*} continuous and f^{-1} is T^{*}-T continuous.

$\Longrightarrow f^{-1} o g^{-1}$ is T^{**}-T continuous

$\Longrightarrow (gof)^{-1}$ is T^{**}-T continuous. Thus *gof* is homeomorphisms and then $(X,T) \approx (Ƶ, T^{**})$.

Hence the relation of homeomorphism on the set of all topology space is an equivalence relation. This shows that $f^{-1}(B°) \subseteq \{f^{-1}(B)\}°$ for every $B \subset y$ so f is continuous. Thus, f is a one – one continuous open mapping of X on to Y Hence f is a homeomorphism.

"3-2-7" Theorem [3]

The relation of homeomorphism on the set of all topological spaces is equivalence.

Proof:-

This relation satisfies the following properties:

(i)-Reflexivity let (X, T) be any topology space then the identity mapping I: $X \longrightarrow X: I(X) = X$.

Is clearly one-one onto, I is continuous for if $G \in T$, Then $I^{-1}(G)= G \in T$.

Also I is open, for if $G \in T$, Then $I(G) = G \in T$.

Thus I is homeomorphism. $Therefore\ (X,T) \approx (X,T)$

(ii) Symmetry: - let $(X,T) \approx (y,T)^*$ and let f be the corresponding homeomorphism. Then f is one-one onto T-T* continuous and open. Now f is one-one onto $\Longrightarrow f^{-1}$ is one-one onto f is open $\Longrightarrow f^{-1}$ is T-T* continuous $\Longrightarrow (f^{-1})^{-1}$ is T-T* continuous Thus show that the mapping $f^{-1} : Y \longrightarrow X$.
