Water Resources Management and Economy

University of Anbar– College of Engineering

Dams & Water Resources Engineering Department – 4th stage

Asst. Prof. Dr. Sadeq Oleiwi Sulaiman

Course Objectives

- Introduction to
  - Water resource systems
  - Planning, design, and operation
  - Application of
  - Economic principles (Cost – Benefit and Microeconomic analysis)
  - Operations research (linear and nonlinear optimization, and simulation modeling)

to various surface and ground water resource allocation problems

Course Outcomes

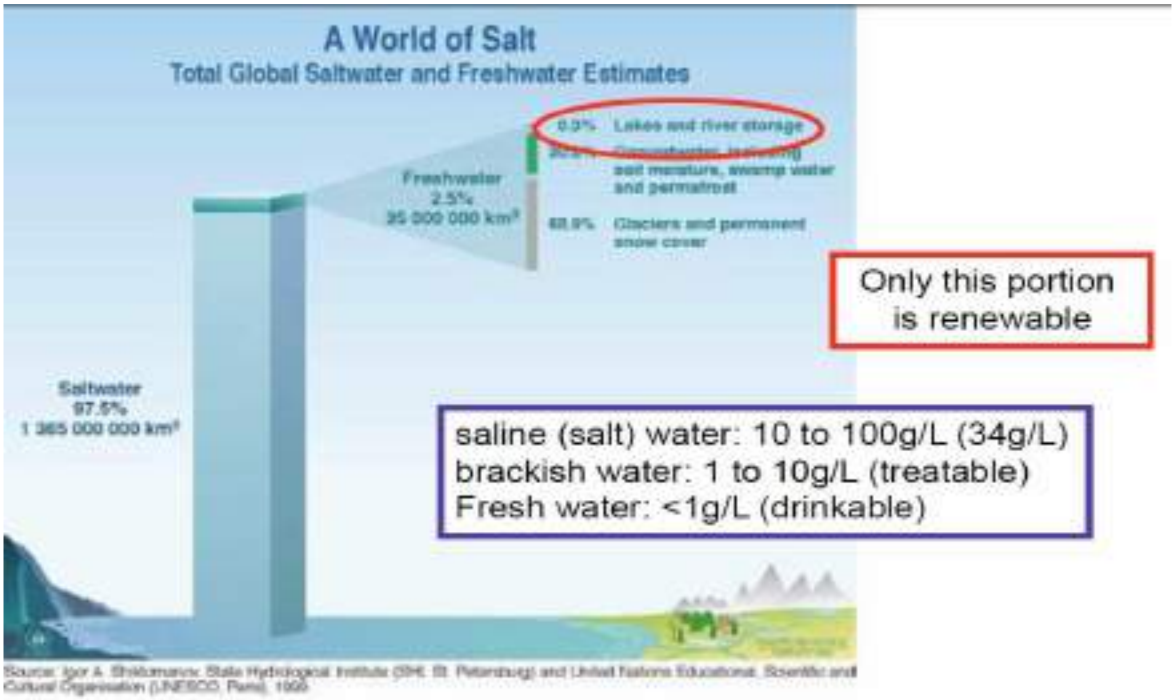
- Students should
  - Be able to develop and solve various types of water resources planning and management (WRPM) models
  - Understand the advantages and limitations of modeling methods and algorithms used in WRPM
  - Understand and appreciate how models can be used in WRPM
  - Understand and critically evaluate literature in WRPM

Course Topics

- Planning and management issues:
  - Institutional objectives and constraints
  - Identification and evaluation of alternatives
  - Advantages and limitations of modeling
- Economic Analysis:
  - Use of cost-benefit and microeconomic analysis
- Modeling:
  - Application of models, solution methods
- Integrated River Basin Planning

Ch. 1 - Water Availability

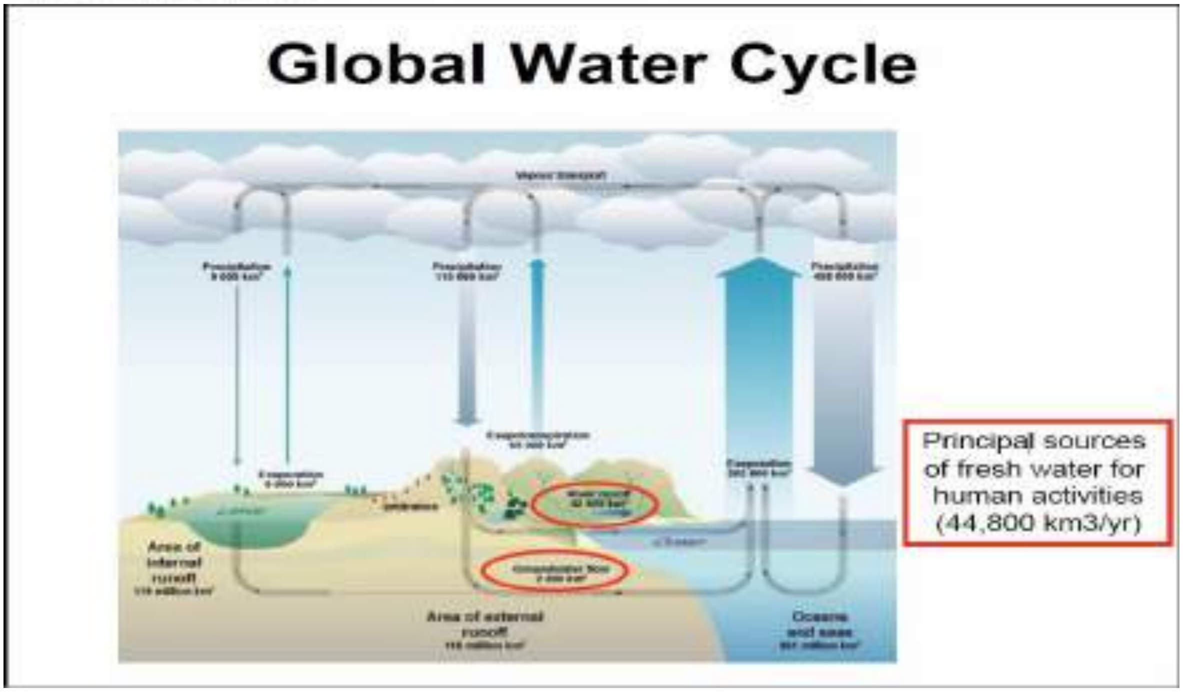
Global Water Resources



Only this portion is renewable

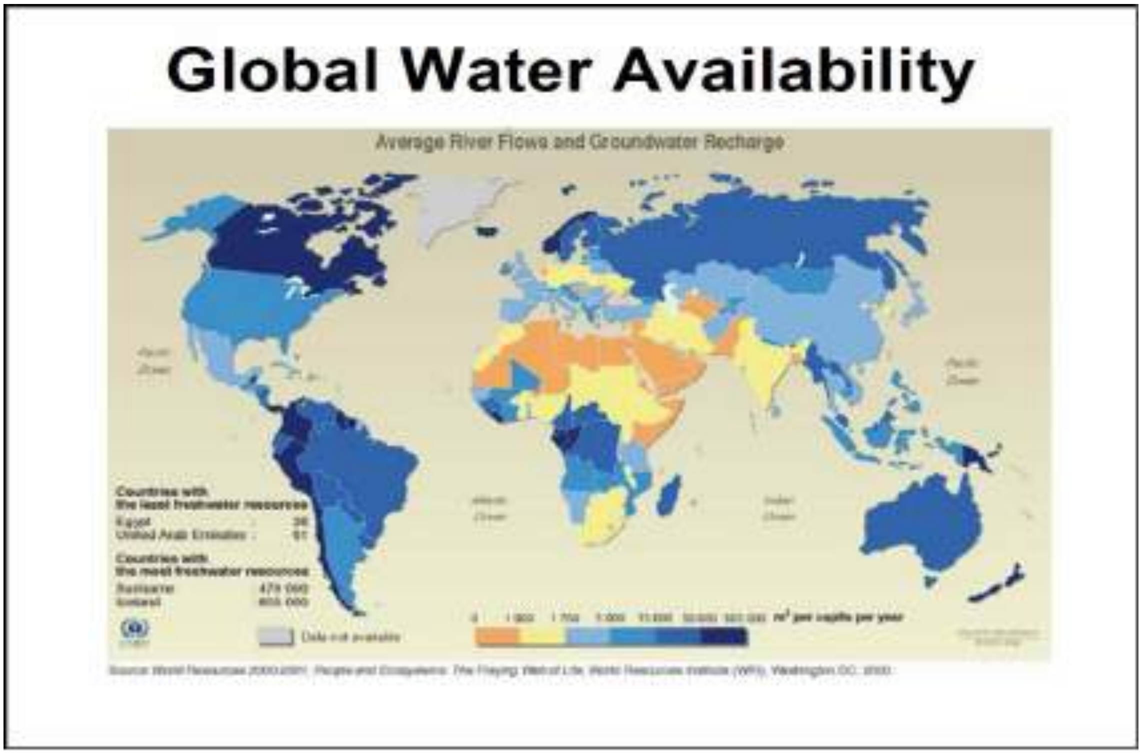
saline (salt) water: 10 to 100g/L (34g/L)
brackish water: 1 to 10g/L (treatable)
Fresh water: <1g/L (drinkable)

Source: Igor A. Shiklomanov, State Hydrological Institute (SHI, St. Petersburg) and United Nations Educational, Scientific and Cultural Organisation (UNESCO, Paris), 1999

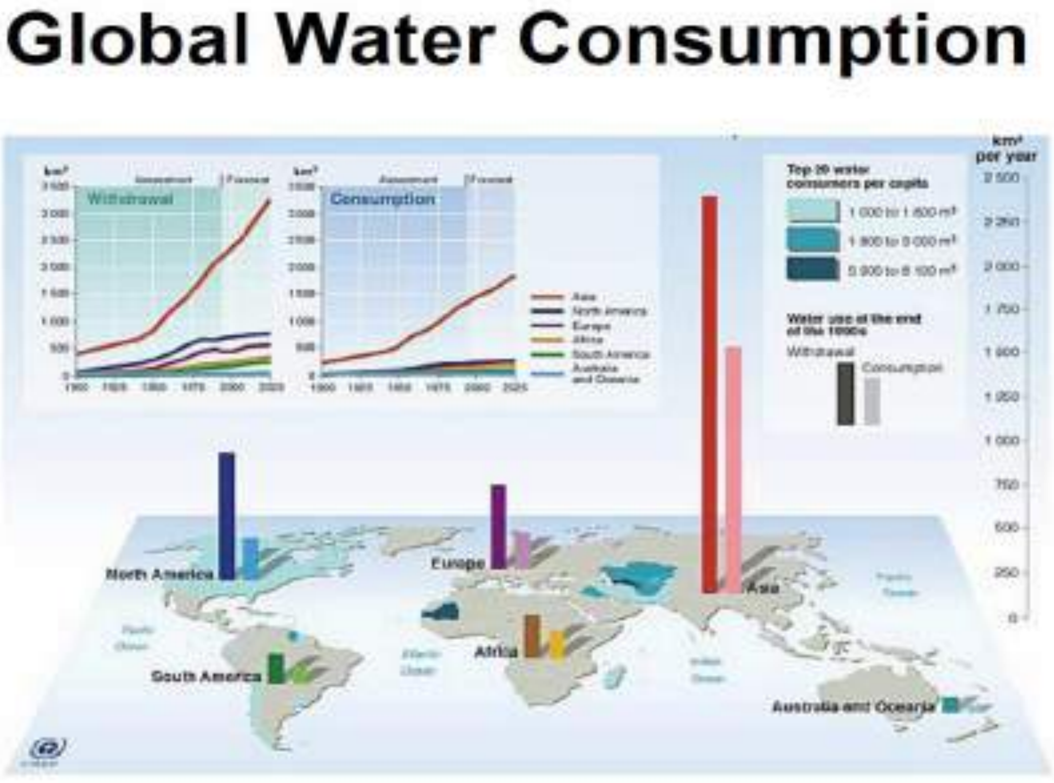


Global Water Cycle

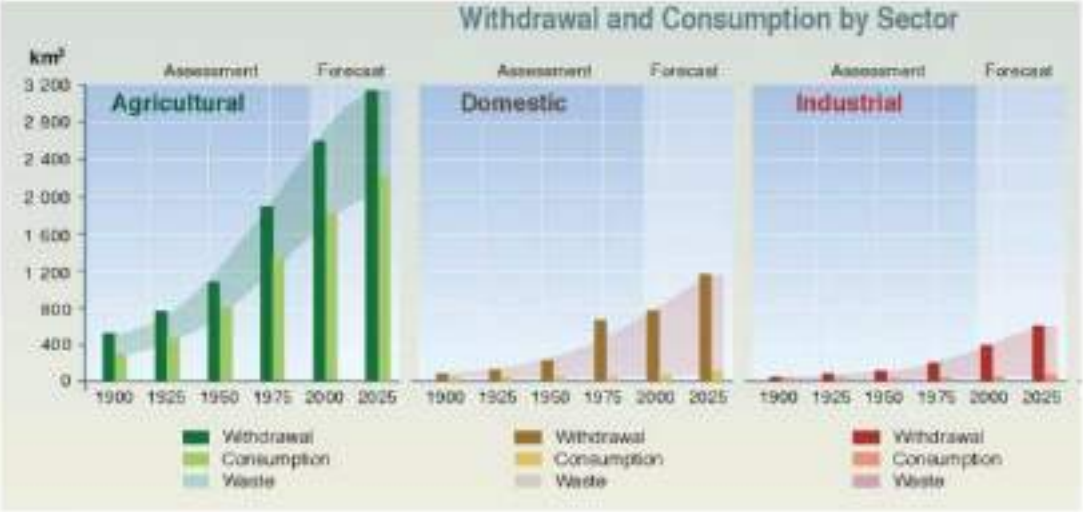
Principal sources of fresh water for human activities (44,800 km3/yr)



Source: World Resources 2000-2001, People and Ecosystems: The Fraying Web of Life, World Resources Institute (WRI), Washington DC, 2000.



Global Water Use



Domestic Water Use

- Survival = 5 L/day
- Drinking, cooking, bathing, and sanitation = 50 L
  - United States = 250 to 300 L (Includes yard watering)
  - Netherlands = 104 L
  - Somalia = 9 L

  - 100-600 L/c/d* (high-income)
  - 50-100 L/c/d (low-income)
  - 10-40 L/c/d (water scarce)

* L/c/d = liters per person per day

Water Stress Index

- Based on human consumption

  linked to population growth

- Domestic requirement:

  100 L/c/d = 40 m3/c/yr

- Associated agricultural, industrial & energy need:

  20 x 40 m3/c/yr = 800 m3/c/yr

- Total need:

  840 m3/c/yr

  about 1000 m3/c/yr

Water Stress Index

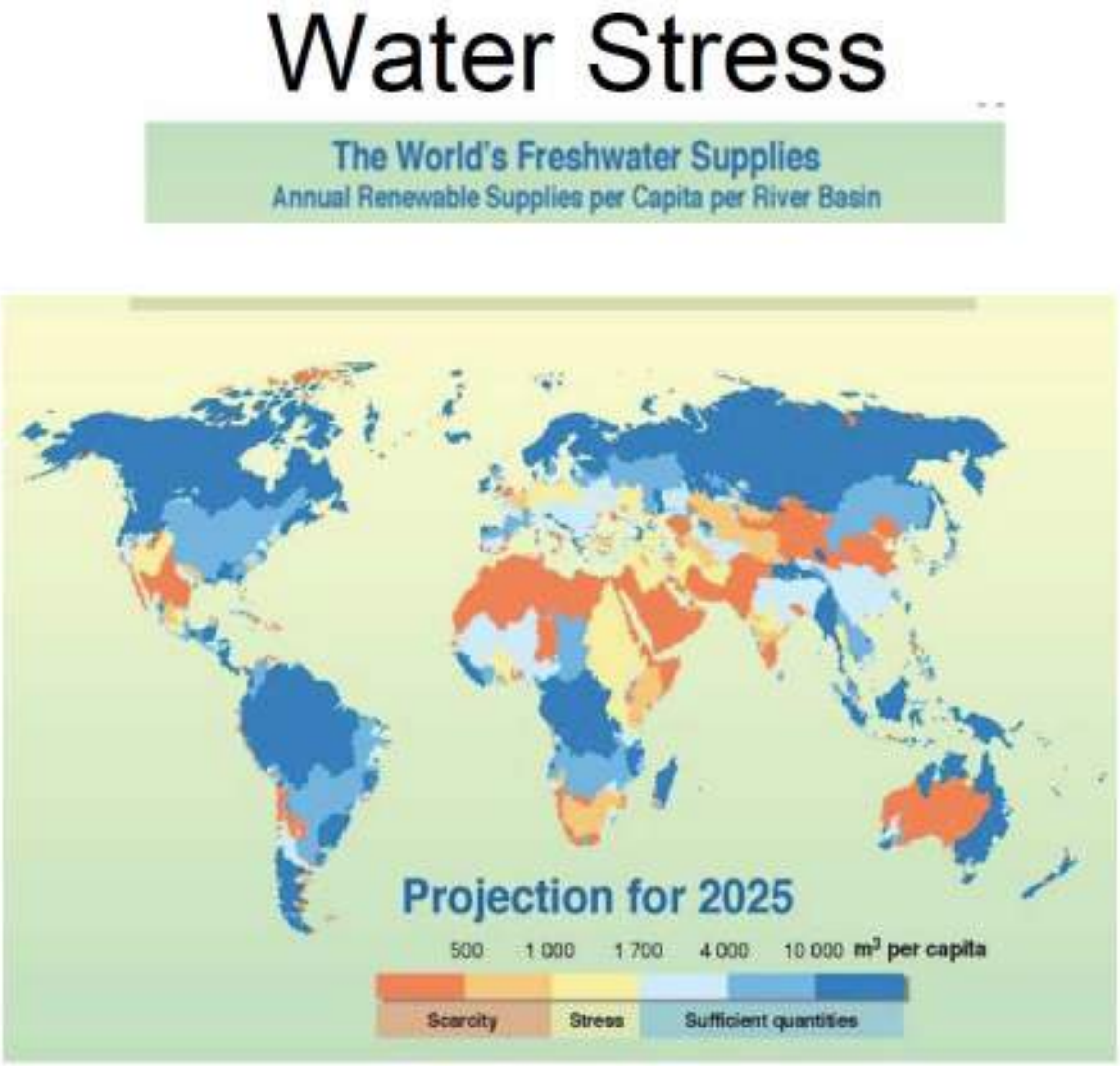
- **Water availability below 1,000 m3/c/yr**

  chronic water related problems impeding development and harming human health

Water sufficiency: >1700 m3/c/yr

Water stress: <1700 m3/c/yr

Water scarcity: <1000 m3/c/yr

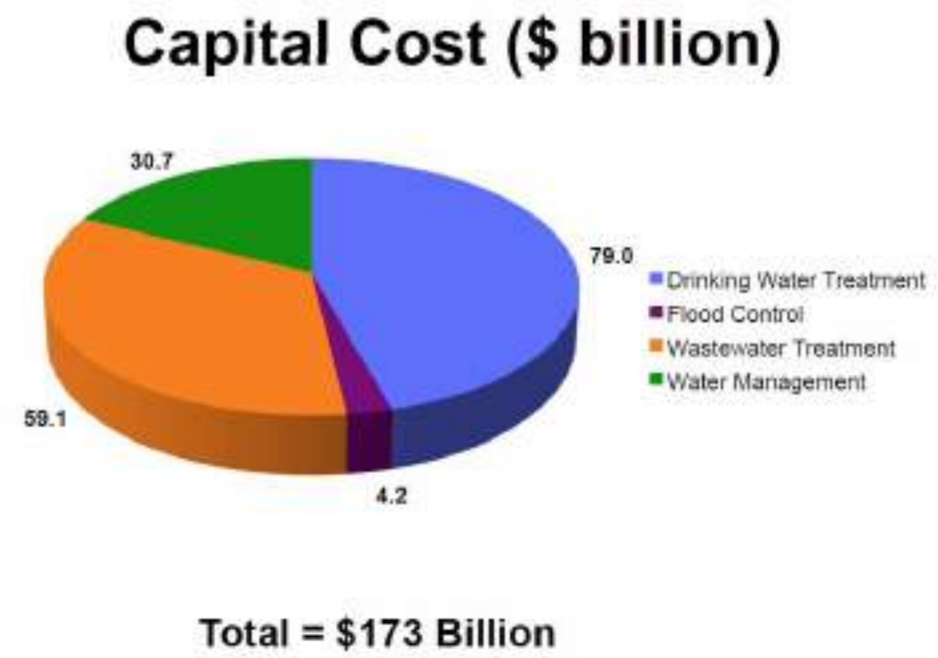


Water Planning

- State Water Plan provides for development, management, and conservation of water resources and preparation for and response to drought conditions, in order that sufficient water will be available at a reasonable cost to ensure public health, safety, and welfare; further economic development; and protect the agricultural and natural resources of the entire state
- Steps:

– Describe the regional water planning area.

- Quantify current and projected population and water demand
- Evaluate and quantify current water supplies
- Identify surpluses and needs
- Evaluate water management strategies and prepare plans to meet the needs
- Recommend regulatory, administrative, and legislative changes; and
- Adopt the plan, including the required level of public participation.



Sustainable Management of Water Resources

Water resource systems that are designed and managed to fully contribute to the needs of society, now and in the indefinite future, while protecting their cultural, ecological and hydrological integrity.

- ASCE sustainable water management workshop, 1997

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Operations Research

The first formal activities of Operations Research (OR) were initiated in England during World War II, when a team of British scientists set out to make scientifically based decisions regarding the best utilization of war materiel. After the war, the ideas advanced in military operations were adapted to improve efficiency and productivity in the civilian sector.

What Is Operations Research?

Imagine that you have a 5-week business commitment between Baghdad (BAG) and Arbil (ARB). You fly out of Baghdad on Mondays and return on Wednesdays of the same week. A regular round-trip ticket costs $400, but a 20% discount is granted if the dates of the ticket span a weekend. A one-way ticket in either direction costs 75% of the regular price. How should you buy the tickets for the 5-week period?

We can look at the situation as a decision-making problem whose solution requires answering three questions:

1. What are the decision alternatives?
2. Under what restrictions is the decision made?
3. What is an appropriate objective criterion for evaluating the alternatives?

Three alternatives are considered:

1. Buy five regular BAG - ARB - BAG for departure on Monday and return on Wednesday of the same week.
2. Buy one BAG - ARB, four ARB - BAG- ARB, that span weekends, and one ARB - BAG.
3. Buy one BAG - ARB - BAG to cover Monday of the first week and Wednesday of the last week and four ARB - BAG- ARB, to cover the remaining legs. All tickets in this alternative span at least one weekend.

The restriction on these options is that you should be able to leave **BAG** on Monday and return on Wednesday of the same week.

An obvious objective criterion for evaluating the proposed alternative is the price of the tickets. The alternative that yields the smallest cost is the best. Specifically, we have

Alternative 1 cost = 5 x 400 = $2000

Alternative 2 cost = 0.75 x 400 + 4 x (0.8 x 400) + 0.75 x 400 = $1880

Alternative 3 cost = 5 x (0.8 x 400) = $1600

Thus, you should **choose alternative 3**.

Though the preceding example illustrates the **three main components of an Operations Research model** 1- alternatives, 2- objective criterion, and 3- constraints, situations differ in the details of how each component is developed and constructed.

To illustrate this point, consider forming a maximum-area rectangle out of a piece of wire of length L inches. What should be the width and height of the rectangle?

In contrast with the tickets example, the number of alternatives in the present example is not finite; namely, the width and height of the rectangle can assume an infinite number of values. To formalize this observation, the alternatives of the problem are identified by defining the width and height as continuous (algebraic) variables.

Let

w = width of the rectangle in inches

h = height of the rectangle in inches

Based on these definitions, the constraints of the situation can be expressed verbally as

1. Width of rectangle + Height of rectangle = Half the length of the wire
2. Width and height cannot be negative

These restrictions are translated algebraically as

1. $2(w + h) = L$
2. $w \geq 0, h \geq 0$

The only remaining component now is the objective of the problem; namely, maximization of the area of the rectangle. Let Z be the area of the rectangle, then the complete model becomes:

$$\text{Maximize } Z = w*h$$

subject to

$$2(w+ h) = L \qquad w, h \geq 0$$

The optimal solution of this model is $w = h = \frac{L}{4}$, which calls for constructing a square shape.

Based on the preceding two examples, the general OR model can be organized in the following general format:

> Maximize or minimize Objective Function subject to Constraints

A solution of the mode is feasible if it satisfies all the constraints. It is optimal if, in addition to being feasible, it yields the best (maximum or minimum) value of the objective function.

Ch.2 - Modeling with Linear Programming

Chapter Guide: This chapter concentrates on model formulation and computations in linear programming (LP). It starts with the modeling and graphical solution of a two-variable problem which, though highly simplified, provides a concrete understanding of the basic concepts of LP and lays the foundation for the development of the general simplex algorithm in Chapter 3. To illustrate the use of LP in the real world, applications are formulated and solved in the areas of urban planning, currency arbitrage, investment, production planning and inventory control, gasoline blending, manpower planning, and scheduling.

Mathematical Formulation of LP model

Step 1. Study the given situation, find the key decision to be made. Hence, identify the decision variables of the problem.

Step 2. Formulate the objective function to be optimized.

Step 3. Formulate the constraints of the problem.

Step 4. Add non-negativity restrictions.

The objective function, the set of constraints, and, the non-negativity restrictions together form an LP model.

TWO-VARIABLE LP MODEL

This section deals with the graphical solution of a two-variable LP. Though two-variable problems hardly exist in practice, the treatment provides concrete foundations for the development of the general simplex algorithm.

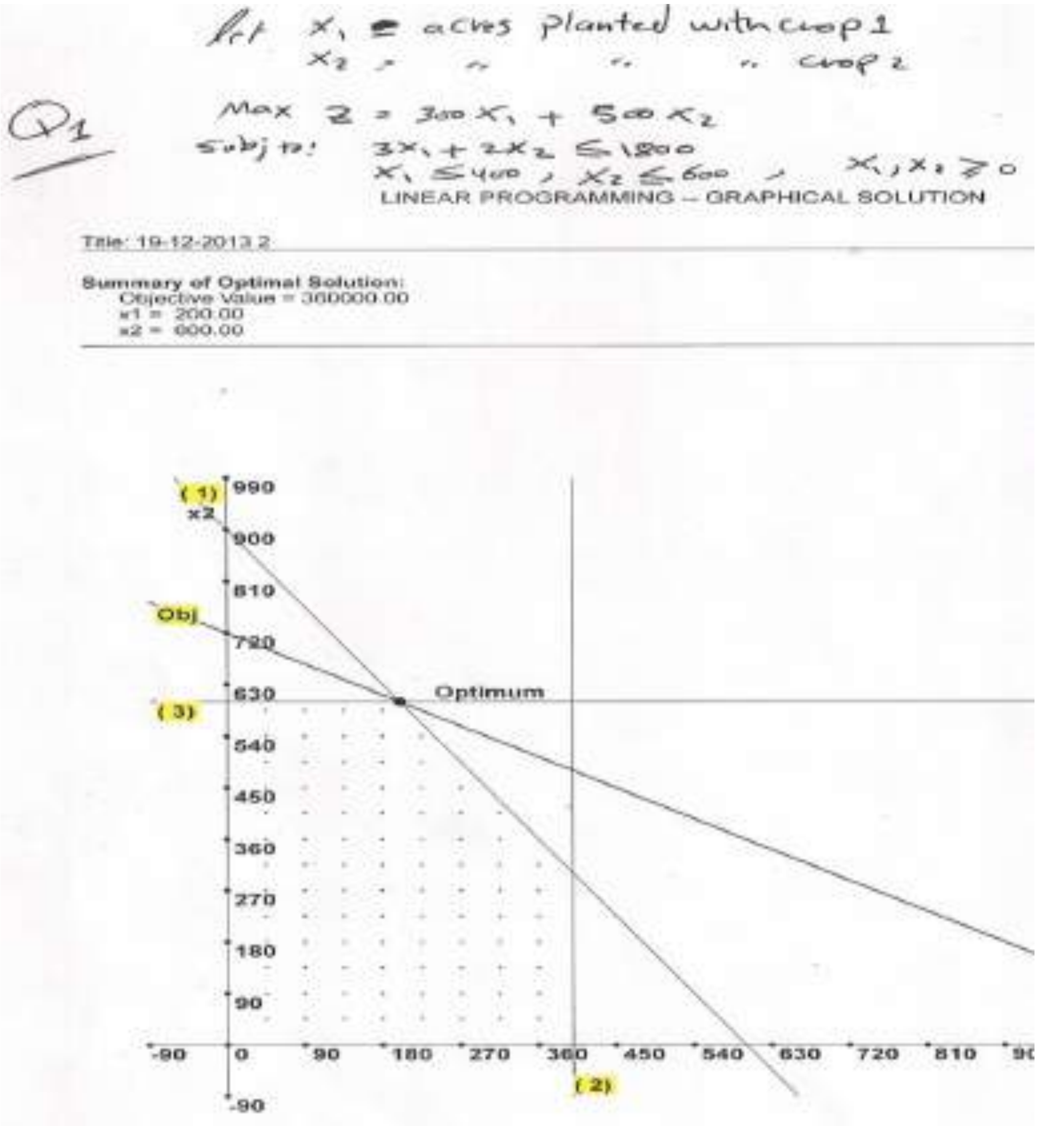
Example 1:- A farm has 1800 acre-feet of water available annually. Two crops are considered for which annual irrigation water requirements are 3 acre-feet/acre and 2 acre-feet/acre, respectively. For various reasons, no more than 400 acres can be planted in crop 1, and no more than 600 acres can be allocated to crop 2. Estimated profits are 300 $ per acre planted in crop 1, and 500 $ per acre planted in crop 2. Determine how many acres to plant in each crop to maximize profits. (Formulate and Graphically solve a linear programming model).

Sol:

The LP model, as in any OR model, has three basic components.

1. Decision variables that we seek to determine.
2. Objective (goal) that we need to optimize (maximize or minimize).
3. Constraints that the solution must satisfy.

The proper definition of the decision variables is an essential first step in the development of the model. Once done, the task of constructing the objective function and the constraints becomes more straightforward.



let x_1 = acres planted with crop 1
x_2 " " " " crop 2

Q1

Max $Z = 300x_1 + 500x_2$
subj to: $3x_1 + 2x_2 \leq 1800$
$x_1 \leq 400$, $x_2 \leq 600$, $x_1, x_2 \geq 0$

LINEAR PROGRAMMING – GRAPHICAL SOLUTION

Title: 19-12-2013 2

Summary of Optimal Solution:
Objective Value = 360000.00
x1 = 200.00
x2 = 600.00

Example 2.

A Company produces both interior and exterior paints from two raw materials, M1 and M2. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum availability (tons) per day
	Exterior paint	Interior paint	
Raw material, M1	6	4	24
Raw material, M2	1	2	6
Profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons. The Company wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit. (Formulate and Graphically solve a linear programming model).

<u>Solution:</u>

The LP model, as in any **OR** model, has three basic components.

1- Decision variables that we seek to determine.
2- Objective (goal) that we need to optimize (maximize or minimize).
3- Constraints that the solution must satisfy.

The variables of the model are defined as

xl = Tons produced daily of exterior paint
x2 = Tons produced daily of interior paint

Total profit from exterior paint = $5X_1$ (thousand) dollars
Total profit from interior paint = $4X_2$ (thousand) dollars

The objective of the company is

Maximize **$Z = 5x1 + 4x2$**

Next, we construct the constraints that restrict raw material usage and product demand. The raw material restrictions are expressed verbally as

(Usage of a raw material by both paints) ≤ (Maximum raw material availability)

$6X1 + 4X2 \leq 24$ (Raw material M1)
$Xl + 2X2 \leq 6$ (Raw material M2)

The first demand restriction stipulates that the excess of the daily production of interior over exterior paint, x2 - x1, should not exceed 1 ton, which translates to

$x2 - x1 \leq 1$ (Market limit)

The second demand restriction stipulates that the maximum daily demand of interior paint is limited to 2 tons, which translates to

$X2 \leq 2$ (Demand limit)

An implicit (or "understood-to-be") restriction is that variables x1 and x2 cannot assume negative values. The nonnegative restrictions, $x1 \geq 0$, $x2 \geq 0$, account for this requirement.
The complete Company model is:

Maximize $Z = 5X1 + 4X2$

subject to

$6x1 + 4X2 \leq 24$ (1)
$X1 + 2x2 \leq 6$ (2)
$- x1 + x2 \leq 1$ (3)
$x2 \leq 2$ (4)
$X1,\ X2 \geq 0$ (5)

Any values of x1 and x2 that satisfy all five constraints constitute a feasible solution. Otherwise, the solution is infeasible.

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Step 1. Determination of the Feasible Solution Space:

First, we account for the non-negativity constraints $x1 \geq 0$ and $x2 \geq 0$. In Figure 2.1, the horizontal axis x1 and the vertical axis x2 represent the exterior- and interior-paint variables, respectively. Thus, the non-negativity of the variables restricts the solution-space area to the first quadrant that lies above the xl-axis and to the right of the x2-axis.

To account for the remaining four constraints, first replace each inequality with an equation and then graph the resulting straight line by locating two distinct points on it. For example, after replacing $6x1 + 4x2 \leq 24$ with the straight line $6x1 + 4x2 = 24$, we can determine two distinct points by first setting $xl = 0$ to obtain $x2 = 6$ and then setting $x2 = 0$ to obtain $x1 = 4$. Thus, the line passes through the two points (0, 6) and (4, 0), as shown by line (1) in Figure 2.1.

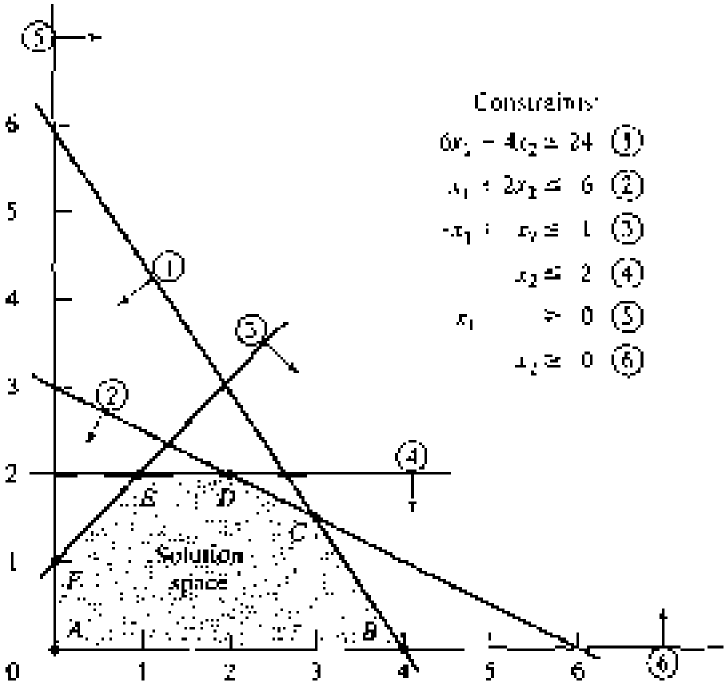


FIGURE 2.1 Feasible space of the Reddy Mikks model

Step 2. Determination of the Optimum Solution:

The feasible space in Figure 2.1 is delineated by the line segments joining the points A, B, C, D, E, and F. Any point within or on the boundary of the space ABCDEF is feasible. Because the feasible space ABCDEF consists of an infinite number of points, we need a systematic procedure to identify the optimum solution.

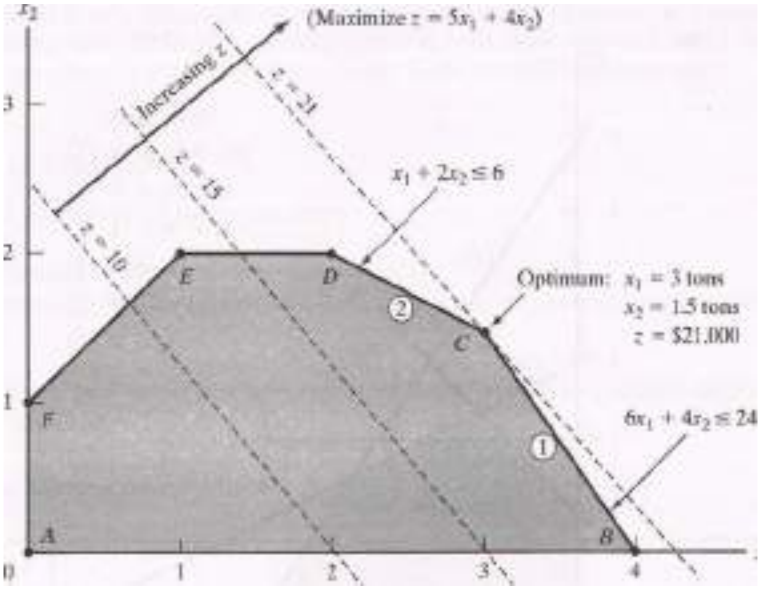
The determination of the optimum solution requires identifying the direction in which the profit function z = 5x1 + 4x2 increases (recall that we are maximizing z). We can do so by assigning arbitrary increasing values to z. For example, using z = 10 and z = 15 would be equivalent to graphing the two lines 5.x1 + 4x2 = 10 and 5x1 + 4x2 = 15. Thus, the direction of increase in z is as shown Figure 2.2. The optimum solution occurs at C, which is the point in the solution space beyond which any further increase will put z outside the boundaries of ABCDEF.

The values of x1 and x2 associated with the optimum point C are determined by solving the equations associated with lines (1) and (2)—that is,

6x1 + 4x2 = 24

x1 + 2x2 = 6

The solution is x1 = 3 and x2 = 1.5 with z = 5* 3 + 4 * 1.5 = 21. This calls for a daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint. The associated daily profit is $21,000.

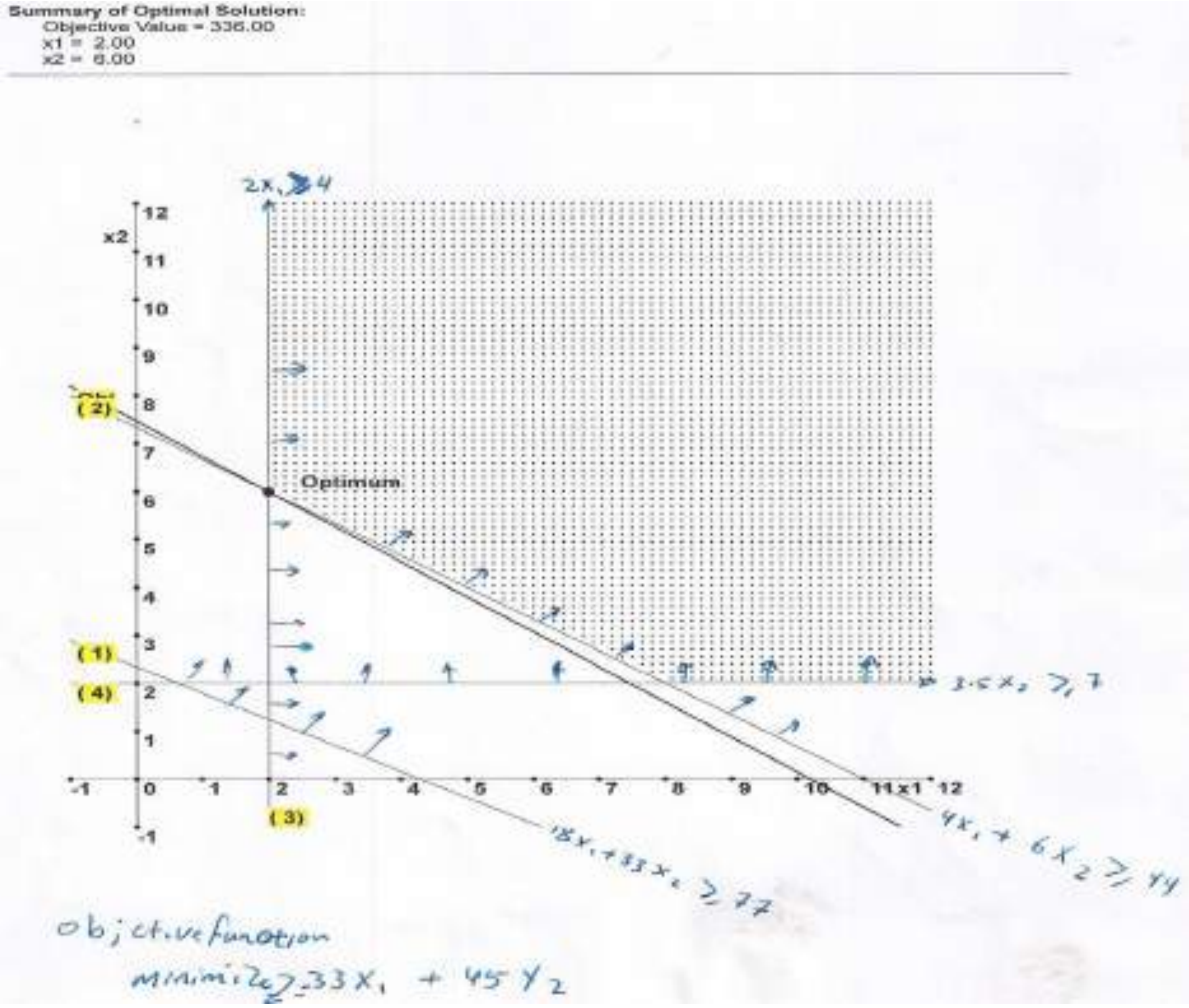


Example: Solve the following problem using Graphical method:

Minimize Z= 33X1+45X2

Subject to:

$18X1+33X2 \geq 77 \quad 4X1+6X2 \geq 44 \quad 2X1 \geq 4 \quad 3.5X2 \geq 7 \quad X1,X2 \geq 0$



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Summary of Optimal Solution:
    Objective Value = 336.00
    x1 = 2.00
    x2 = 6.00
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Self –Test:

A farm need at least 800 m^3 of water daily. The water is provided by two nearby wells, and have the following properties:

	TDS (ppm)	Nitrate (ppm)	Cost \$/ m^3
Well 1	980	125	0.3
Well 2	300	20	0.9

The special requirements of the crop in the farm are at most 600 ppm for TDS, and at least 50 ppm for Nitrate. The farm directorate wishes to determine the daily mixture of water from the two wells to obtain daily minimum cost. (use graphical method)

Quiz 12-8-2015

Two types of crops A & B can be grown in particular irrigation area each year. Each unit quantity of two types of crops can be sold for a price and requires units of water, units of land, units of fertilizer, and units of labor as shown in table below.

(a) Structure a linear programming model for estimating the quantities of each of the two crops that should be produced in order to maximize total income.

(b) Solve the problem graphically.

	REQUIREMENTS PER UNIT OF		
Resource	Crop A	Crop B	Maximum Available Resources
Water	2	3	60
Land	5	2	80
Fertilizer	3	2	60
Labor	1	2	40
Unit Price (1000 \$)	30	25	

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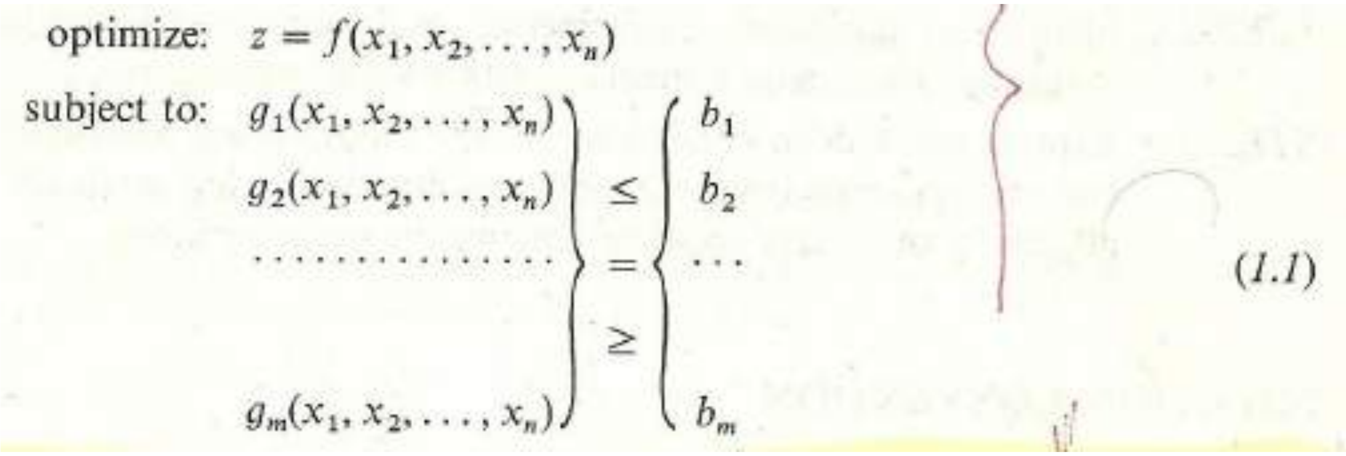
Linear Program Model in Equation Form: (Basic Concepts)

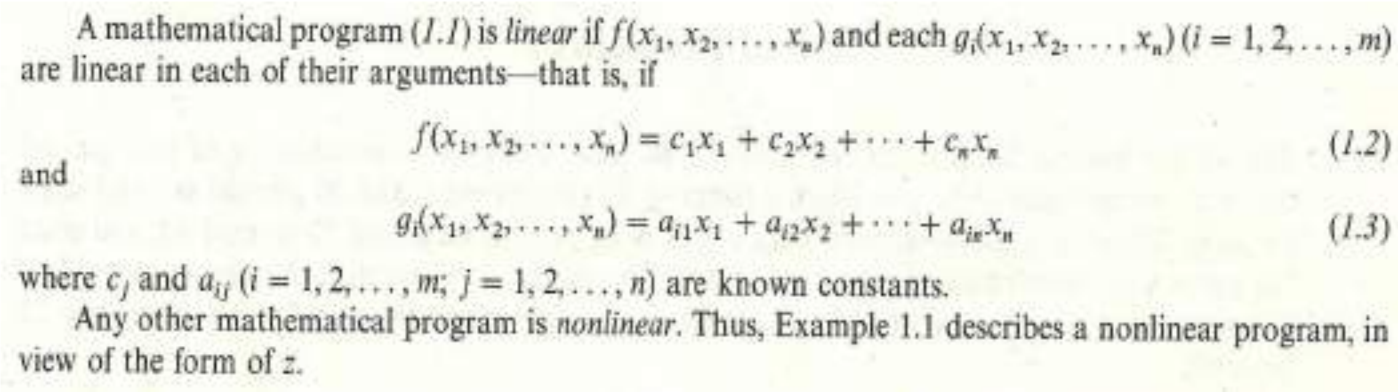
A *mathematical program* is an optimization problem in which the objective and constraints are given as mathematical functions and functional relationships. Mathematical programs treated in linear programs have the form:



$$\begin{array}{ll} \text{optimize:} & z = f(x_1, x_2, \ldots, x_n) \\ \text{subject to:} & \left.\begin{array}{l} g_1(x_1, x_2, \ldots, x_n) \\ g_2(x_1, x_2, \ldots, x_n) \\ \cdots\cdots\cdots\cdots\cdots \\ \\ g_m(x_1, x_2, \ldots, x_n) \end{array}\right\} \begin{array}{c} \le \\ = \\ \ge \end{array} \left\{\begin{array}{l} b_1 \\ b_2 \\ \cdots \\ \\ b_m \end{array}\right. \end{array} \qquad (1.1)$$

Each of the *m* constraint relationships in (1 .1) involves one of the three signs: ≤ , = , ≥

LINEAR PROGRAMS



A mathematical program (*1.1*) is *linear* if $f(x_1, x_2, \ldots, x_n)$ and each $g_i(x_1, x_2, \ldots, x_n)$ $(i = 1, 2, \ldots, m)$ are linear in each of their arguments—that is, if

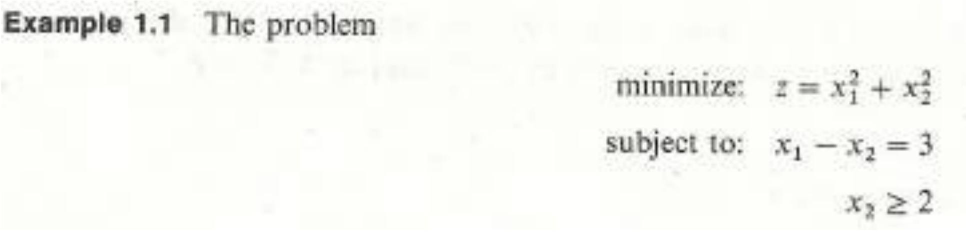
$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \qquad (1.2)$$

and

$$g_i(x_1, x_2, \ldots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \qquad (1.3)$$

where c_j and a_{ij} $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ are known constants.

Any other mathematical program is *nonlinear*. Thus, Example 1.1 describes a nonlinear program, in view of the form of z.



Example 1.1 The problem

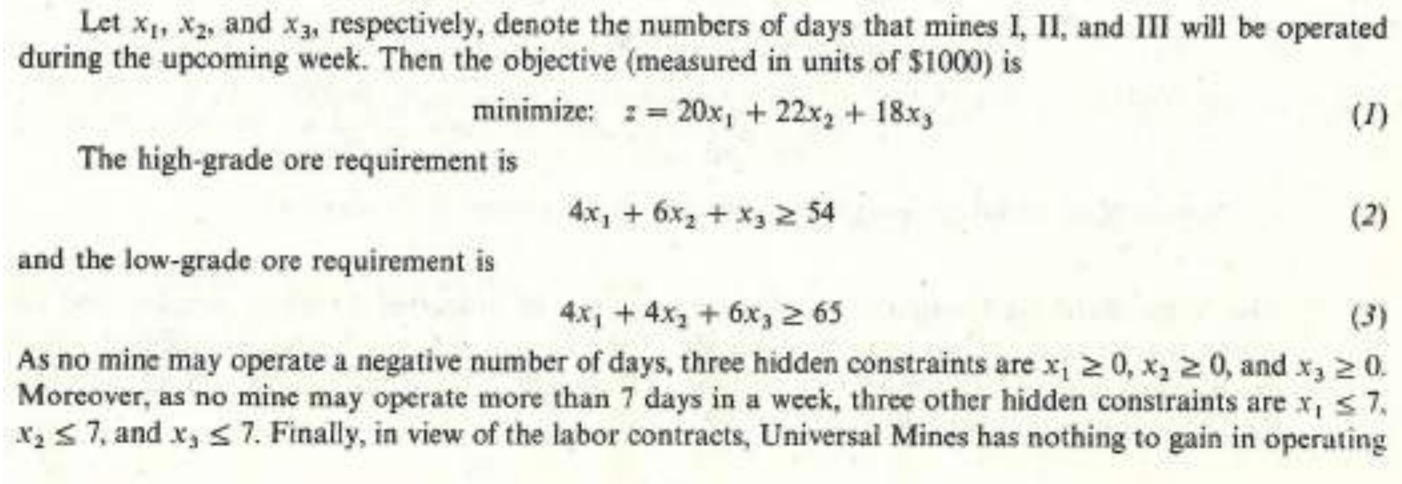
$$\begin{array}{rl} \text{minimize:} & z = x_1^2 + x_2^2 \\ \text{subject to:} & x_1 - x_2 = 3 \\ & x_2 \ge 2 \end{array}$$

(Linear: no powers, exponentials or product terms)

Example 3:- Universal Mines Inc. operates three mines in West Virginia. The ore from each mine is separated into two grades before it is shipped; the daily production capacities of the mines, as well as their daily operating costs, are as follows:

	High-Grade Ore, tons/day	Low-Grade Ore, tons/day	Operating Cost, 1000$/day
Mine I	4	4	20
Mine II	6	4	22
Mine III	1	6	18

Universal has committed itself to deliver 54 tons of high-grade ore and 65 tons of low-grade ore by the end of the week. It also has labor contracts that guarantee employees in each mine a full day's pay for each day or fraction of a day the mine is open. Determine the number of days each mine should be operated during the upcoming week if Universal Mines is to fulfill its commitment at minimum total cost.



Let x_1, x_2, and x_3, respectively, denote the numbers of days that mines I, II, and III will be operated during the upcoming week. Then the objective (measured in units of $1000) is

$$\text{minimize:} \quad z = 20x_1 + 22x_2 + 18x_3 \tag{1}$$

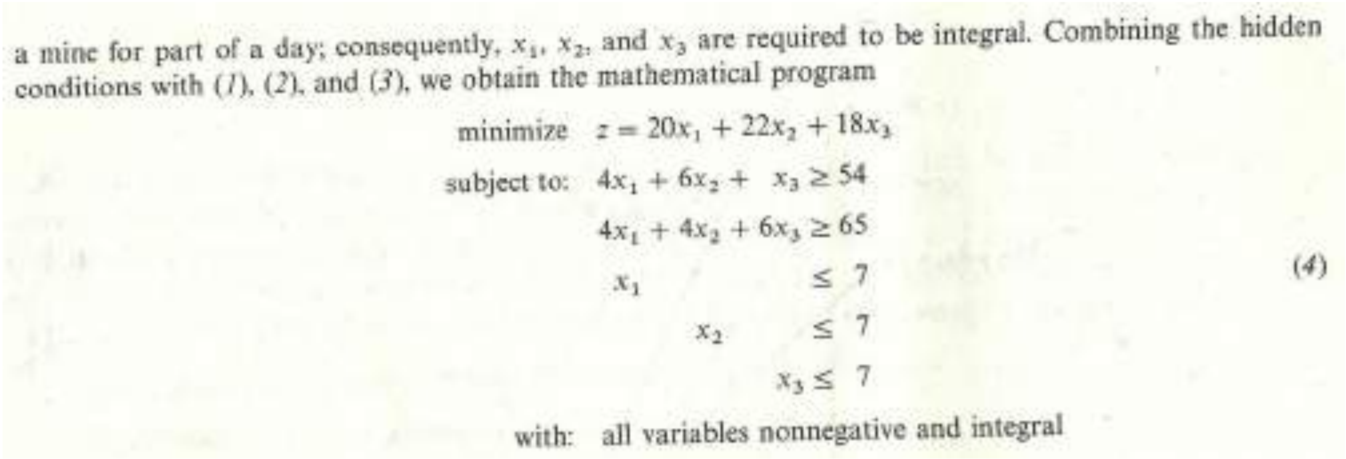
The high-grade ore requirement is

$$4x_1 + 6x_2 + x_3 \geq 54 \tag{2}$$

and the low-grade ore requirement is

$$4x_1 + 4x_2 + 6x_3 \geq 65 \tag{3}$$

As no mine may operate a negative number of days, three hidden constraints are $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. Moreover, as no mine may operate more than 7 days in a week, three other hidden constraints are $x_1 \leq 7$, $x_2 \leq 7$, and $x_3 \leq 7$. Finally, in view of the labor contracts, Universal Mines has nothing to gain in operating



a mine for part of a day; consequently, x_1, x_2, and x_3 are required to be integral. Combining the hidden conditions with (1), (2), and (3), we obtain the mathematical program

$$\begin{array}{rl}
\text{minimize} & z = 20x_1 + 22x_2 + 18x_3 \\
\text{subject to:} & 4x_1 + 6x_2 + x_3 \geq 54 \\
 & 4x_1 + 4x_2 + 6x_3 \geq 65 \\
 & x_1 \leq 7 \\
 & x_2 \leq 7 \\
 & x_3 \leq 7 \\
\text{with:} & \text{all variables nonnegative and integral}
\end{array} \tag{4}$$

Standard Linear Program Form

To initialize the method for solving linear programs involving many variables, one must transform all inequality constraints into equalities and must know one feasible, nonnegative solution.

Any **LP** can be transformed into **Standard Form**

Minimize $Z = c1\ x1 + c2\ x2 + \ldots\ldots. + cn\ xn$

subject to $a11\ x1 + a12\ x2 + \ldots\ldots.. + a1n\ \ xn = b1$

$a21\ x1 + a22\ x2 + \ldots\ldots. + a2n\ xn = b2$

.

.

am1 x1 + am2 x2 + ……. + **a**mn **x**n = bm

and $x1 \geq 0$, $x2 \geq 0$ ……… $xn \geq 0$

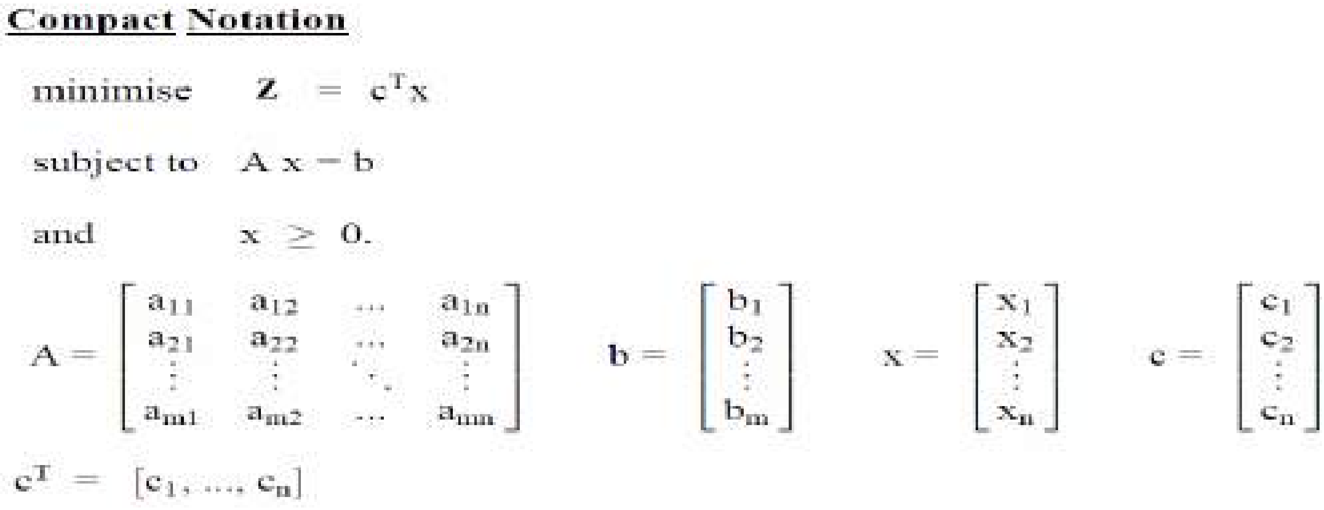
bi, cj, aij : fixed real constants; xi; i =0, ..., n: real numbers, to be determined.

We assume that bi ≥ 0 (each equation may be multiplied by -1 to achieve this).

For example:

The constraint $2Xl - 3x2 + 4X3 \leq -5$ is multiplied by -1 to obtain $-2Xl + 3x2 - 4X3 \geq 5$, which has a nonnegative right-hand side.

Any variable not already constrained to be nonnegative is replaced by the difference of two new variables which are so constrained.



Compact Notation

minimise $\quad Z = c^T x$

subject to $\quad A\,x = b$

and $\quad x \geq 0.$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$c^T = [c_1, \ldots, c_n]$$

SLACK VARIABLES AND SURPLUS VARIABLES

A linear constraint of the form **Σ aij xj $\leq$ bi** can be converted into an equality by adding a new, nonnegative variable to the left-hand side of the inequality.

Such a variable is numerically equal to the difference between the right- and left-hand sides of the inequality and is known as a **slack variable**.

It represents the waste involved in that phase of the system modeled by the constraint.
In previous Example 2. The first constraint in Problem

$$6X1 + 4X2 \leq 24$$

The left-hand side of this inequality models the Usage of a raw material **M1** in tons by both paints, while the right-hand side is the Maximum raw material **M1** availability. This inequality is transformed into the equation:

$$6X1 + 4X2 + X3 = 24 \quad \ldots. \ldots. \ldots \ldots \quad X3 \geq 0$$

by adding the slack variable **X3** to the left-hand side of the inequality. Here **X3** represents the number of assembly tons available to the manufacturer but not used.

(لغرض تحويل المتباينات الاقل من او تساوي الى معادلات نضيف متغير راكد الى الطرف الايسر تكون قيمته اكبر من او تساوي صفر ويمثل الكمية الباقية وغير المستعملة في الانتاج او الفرق بين الطرفين الايمن والايسر للمعادلة ولايغير هذا المتغير من طبيعة القيد او دالة الهدف فمثلا للقيد اعلاه

(let x1=3, x2 = 1 then x3 = 2) or if (x1=2, x2 =3 then x3=0)

وفي الزمن صفر اي قبل بداية الانتاج عندما X1=0 و X2 =0 تكون قيمة X3=24 أي ان الكمية غير المستعمله هي كل الكمية)

A linear constraint of the form **Σ aij xj $\geq$ bi** can be converted into an equality by subtracting a new, nonnegative variable from the left-hand side of the inequality. Such a variable is numerically equal to the difference between the left- and right-hand sides of the inequality and is known as a *surplus variable*.

It represents excess input into that phase of the system modeled by the constraint.
The first constraint in Example 3. is:

$$4Xl + 6X2 + X3 \geq 54$$

The left-hand side of this inequality represents the combined output of high-grade ore from three mines, while the right-hand side is the minimum tonnage of such ore required to meet contractual obligations. This inequality is transformed into the equation

$4 Xl + 6X_2 + X_3 - X_4 = 54$ …. …. … … **$X_4 \geq 0$**

by subtracting the surplus variable X_4 from the left-hand side of the inequality. Here X4 represents the amount of high-grade ore mined over and above that needed to fulfill the contract.

(لغرض تحويل المتباينات الاكبر من او تساوي الى معادلات نطرح متغير فائض من الطرف الايسر تكون قيمته اكبر من او تساوي صفر ويمثل الكمية المنتجة الزائدة عن الحاجة للطرف الايمن او الفرق بين الطرفين الايمن والايسر للمعادلة ولايغير هذا المتغير من طبيعة القيد او دالة الهدف فمثلا

(let x1=5, x2 = 5, X3=5 then x4 = 1) or if (x1=5, x2 = 5, X3=4 then x4 = 0)

وفي الزمن صفر اي قبل بداية الانتاج عندما X1=0 و X2 =0 و X3 =0 تكون قيمة X4= -54 وهذا يخالف شرط عدم السلبية لذا لابد من اضافة متغير موجب يمثل الحل الاولي في حالة عدم وجود متغير راكد)

(على الاغلب عندما تكون دالة الهدف تعظيم تكون القيود اقل من او تساوي وعندما تكون تصغير تكون القيود اكبر من او تساوي ولامانع ان تكون هناك قيود تجمع كل الاشارات ≤ و ≥ و = في دالة هدف واحدة)

GENERATING AN INITIAL FEASIBLE SOLUTION

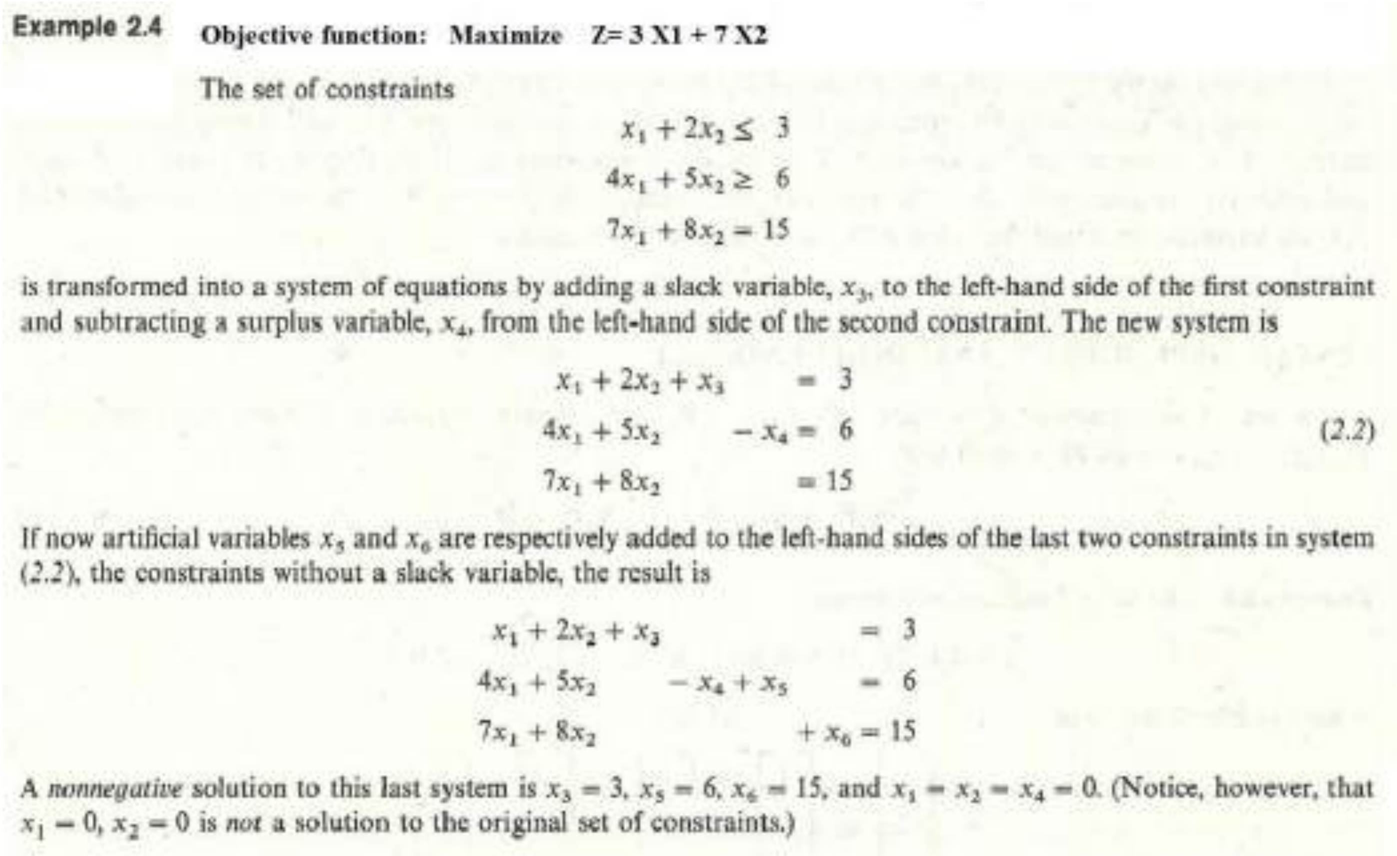
After all linear constraints (with nonnegative right-hand sides) have been transformed into equalities by introducing **slack** and **surplus** variables where necessary, add a new variable, called an **artificial variable,** to the left-hand side of each constraint equation that does not contain a slack variable.

Each constraint equation will then contain either one slack variable or one artificial variable.

A nonnegative initial solution to this new set of constraints is obtained by setting each slack variable and each artificial variable equal to the right-hand side of the equation in which it appears and setting all other variables, including the surplus variables, equal to zero.

بعد تحويل المتباينات الى معادلات باضافة متغيرات راكدة او طرح متغيرات فائضة ولغرض ايجاد الحل الاولي نقوم باضافة متغير يسمى المتغير الصناعي للطرف الايسر من كل معادلة قيد لاتحوي متغير راكد (مثل قيود الاكبر من او يساوي او قيود المساواة) أي ان معادلات القيود يجب ان تحوي اما متغير راكد او متغير صناعي.

وبذلك يصبح الحل الاولي غير السالب لمجموعة معادلات القيود هو ان كل متغير راكد او متغير صناعي في الطرف الايسر من المعادلة يكون مساوي للطرف الايمن منها وتكون بقية المتغيرات بما فيها المتغيرات الفائضة تساوي صفرا



Example 2.4 **Objective function: Maximize $Z= 3 X1 + 7 X2$**

The set of constraints

$$\begin{aligned} x_1 + 2x_2 &\le 3 \\ 4x_1 + 5x_2 &\ge 6 \\ 7x_1 + 8x_2 &= 15 \end{aligned}$$

is transformed into a system of equations by adding a slack variable, x_3, to the left-hand side of the first constraint and subtracting a surplus variable, x_4, from the left-hand side of the second constraint. The new system is

$$\begin{aligned} x_1 + 2x_2 + x_3 \quad\quad &= 3 \\ 4x_1 + 5x_2 \quad\quad - x_4 &= 6 \\ 7x_1 + 8x_2 \quad\quad\quad\quad &= 15 \end{aligned} \tag{2.2}$$

If now artificial variables x_5 and x_6 are respectively added to the left-hand sides of the last two constraints in system (2.2), the constraints without a slack variable, the result is

$$\begin{aligned} x_1 + 2x_2 + x_3 \quad\quad\quad\quad\quad &= 3 \\ 4x_1 + 5x_2 \quad\quad - x_4 + x_5 \quad\quad &= 6 \\ 7x_1 + 8x_2 \quad\quad\quad\quad\quad + x_6 &= 15 \end{aligned}$$

A *nonnegative* solution to this last system is $x_3 = 3$, $x_5 = 6$, $x_6 = 15$, and $x_1 = x_2 = x_4 = 0$. (Notice, however, that $x_1 = 0$, $x_2 = 0$ is *not* a solution to the original set of constraints.)

PENALTY COSTS

The introduction of slack and surplus variables alters neither the nature of the constraints nor the objective.

Accordingly, such variables are incorporated into the objective function with zero coefficients.

Artificial variables, however, do change the nature of the constraints. Since they are added to only one side of an equality, the new system is equivalent to the old system of constraints if and only if the artificial variables are zero.

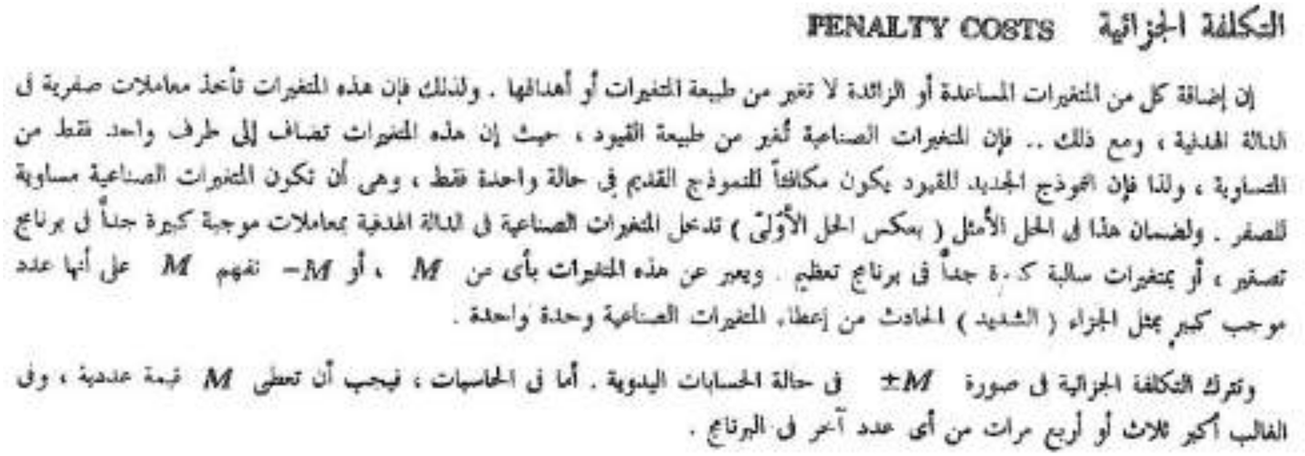
To guarantee such assignments in the optimal solution (in contrast to the initial solution), artificial variables are incorporated into the objective function with very large positive coefficients in a minimization program or very large negative coefficients in a maximization program.

These coefficients, denoted by either **M or - M,** where **M** is understood to be a large positive number, represent the (severe) penalty incurred in making a unit assignment to the artificial variables.

In hand calculations, penalty costs can be left as ± M. In computer calculations, M must be assigned a numerical value, usually a number three or four times larger in magnitude than any other number in the program.

So, for the previous Example the objective function become:

Maximize : Z= 3X1 + 7X2 + 0X3 – MX5 – MX6



PENALTY COSTS التكلفة الجزائية

إن إضافة كل من المتغيرات المساعدة أو الزائدة لا تغير من طبيعة المتغيرات أو أهدافها . ولذلك فإن هذه المتغيرات تأخذ معاملات صفرية في الدالة الهدفية ، ومع ذلك .. فإن المتغيرات الصناعية تغير من طبيعة القيود ، حيث إن هذه المتغيرات تضاف إلى طرف واحد فقط من المتساوية ، ولذا فإن النموذج الجديد للقيود يكون مكافئاً للنموذج القديم في حالة واحدة فقط ، وهي أن تكون المتغيرات الصناعية مساوية للصفر . ولضمان هذا في الحل الأمثل (بعكس الحل الأوّليّ) تدخل المتغيرات الصناعية في الدالة الهدفية بمعاملات موجبة كبيرة جداً في برنامج تصغير ، أو بمتغيرات سالبة كبيرة جداً في برنامج تعظيم . ويعبر عن هذه المتغيرات بأي من M ، أو −M نفهم M على أنها عدد موجب كبير يمثل الجزاء (الشديد) الحادث من إعطاء المتغيرات الصناعية وحدة واحدة .

وتترك التكلفة الجزائية في صورة ±M في حالة الحسابات اليدوية . أما في الحاسبات ، فيجب أن تعطى M قيمة عددية ، وفي الغالب أكبر ثلاث أو أربع مرات من أي عدد آخر في البرنامج .

Example 5: Put the following program in standard matrix form:

maximize: z = Xl + X2
subject to: Xl + 5x2 ≤ 5
2Xl + X2 ≤ 4
with: Xl and X2 nonnegative

sol:

Adding slack variables X3 and *X4* , respectively, to the left-hand sides of the constraints, and including these new variables with zero cost coefficients in the objective, we have
maximize: z = Xl + X 2 + 0X3 + 0X4
subject to: Xl + *5x2* + X3 = 5
2Xl + X2 + X4 = 4
with: all variables nonnegative

Since each constraint equation contains a slack variable, no artificial variables are required; an initial feasible solution is X3 = 5, X4 = 4, x 1 = X2 = 0. System of above equations is in the standard **LP** form if we define:

$$\mathbf{X} \equiv [x_1, x_2, x_3, x_4]^T \qquad \mathbf{C} \equiv [1, 1, 0, 0]^T$$

$$\mathbf{A} \equiv \begin{bmatrix} 1 & 5 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} \equiv \begin{bmatrix} 5 \\ 4 \end{bmatrix} \qquad \mathbf{X}_0 \equiv \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

Example 5: Put the following program in standard form:

minimize: z = 80x1 + *60x 2*
subject to: 0.20x1 + 0.32x2 $\geq$ 0.25
Xl + x 2 = 1
with: Xl and X2 nonnegative

sol:

To convert the first constraint into an equality, subtract a surplus variable X3 from the left-hand side. The second constraint is already an equation, then:

minimize: z = 80x1 + 60x 2 +0x3
subject to: 0.20x1 + 0.32x2 - X3 = 0.25
Xl + x 2 = 1
with: Xl , X2, and X3 nonnegative

Since both first and second constraints does not contain a slack variable, add an artificial variable X4, and X5 to its left-hand side. Both new variables are included in the objective function, the artificial variable with a very large positive cost coefficient, yielding the program

minimize: Z = 80x1 + 60x2+ 0x3 + MX4 + MX5
subject to: 0.20x1+ 0.32x2 - X3 +X4 = 0.25
x1 + x2 + X5 = 1

with: all variables nonnegative

This program is in standard form, with an initial feasible solution :

X4 = 0.25, X5 = 1, X1 = X2 = X3 =0.

equations is in the standard **LP** form if we define:

X= [X1 X2 X3 X4 X5]^T C= [80 60 0 M M]^T

$$A = \begin{bmatrix} 0.2 & 0.32 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 0.25 \\ 1 \end{bmatrix} \quad X0 = \begin{bmatrix} X4 \\ X5 \end{bmatrix} \quad C0 = \begin{bmatrix} M \\ M \end{bmatrix}$$

Linear Programming: The Simplex and the Dual Simplex Methods

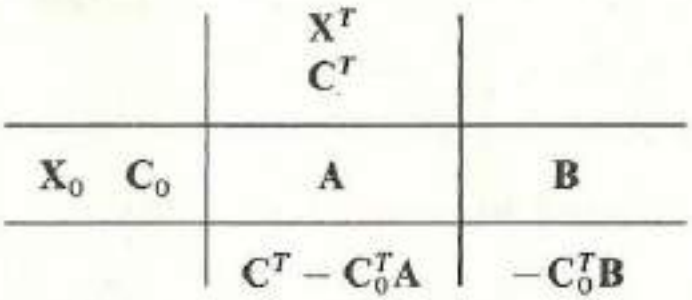
THE SIMPLEX TABLEAU

The ***simplex method*** is a matrix procedure for solving linear programs in the standard form:
optimize: , $Z = C^T X$

subject to: $AX = B$

with : $X \geq 0$

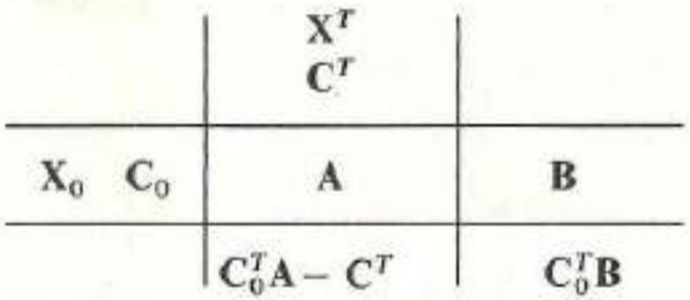
where $B \geq 0$ and a basic feasible solution Xo is known. Starting with Xo, the method locates successively other basic feasible solutions having better values of the objective, until the optimal solution is obtained. For minimization programs, the simplex method utilizes Tableau 3-1, in which Co designates the cost vector associated with the variables in Xo .



		X^T C^T	
X_0	C_0	A	B
		$C^T - C_0^T A$	$-C_0^T B$

Tableau 3-1 For minimization programs

For maximization programs, Tableau 3-1 applies if the elements of the bottom row have their signs reversed



		X^T C^T	
X_0	C_0	A	B
		$C_0^T A - C^T$	$C_0^T B$

Tableau 3-1 For maximization programs

Example 3.1: Put the following standard matrix form of linear program in simplex tableau where the objective function is minimize.

$$\mathbf{X} \equiv [x_1, x_2, x_3, x_4, x_5, x_6]^T \qquad \mathbf{C} \equiv [1, 2, 3, 0, 0, M]^T$$

$$\mathbf{A} \equiv \begin{bmatrix} 3 & 0 & 4 & 1 & 0 & 0 \\ 5 & 1 & 6 & 0 & 0 & 0 \\ 8 & 0 & 9 & 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{B} \equiv \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} \qquad \mathbf{X}_0 \equiv \begin{bmatrix} x_4 \\ x_2 \\ x_6 \end{bmatrix}$$

initial feasible solution $x_4 = 5, x_2 = 7, x_6 = 2, x_1 = x_3 = x_5 = 0$

Sol:

For the minimization program of Problem $\mathbf{C}_0 = [0, 2, M]^T$. Then,

$$\mathbf{C}^T - \mathbf{C}_0^T\mathbf{A} = [1, 2, 3, 0, 0, M] - [0, 2, M]\begin{bmatrix} 3 & 0 & 4 & 1 & 0 & 0 \\ 5 & 1 & 6 & 0 & 0 & 0 \\ 8 & 0 & 9 & 0 & -1 & 1 \end{bmatrix}$$

$$= [1, 2, 3, 0, 0, M] - [10 + 8M, 2, 12 + 9M, 0, -M, M] = [-9 - 8M, 0, -9 - 9M, 0, M, 0]$$

$$-\mathbf{C}_0^T\mathbf{B} = -[0, 2, M]\begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} = -14 - 2M$$

and Tableau 3-1 becomes

		x_1	x_2	x_3	x_4	x_5	x_6	
		1	2	3	0	0	M	
x_4	0	3	0	4	1	0	0	5
x_2	2	5	1	6	0	0	0	7
x_6	M	8	0	9	0	-1	1	2
		$-9-8M$	0	$-9-9M$	0	M	0	$-14-2M$

A TABLEAU SIMPLIFICATION

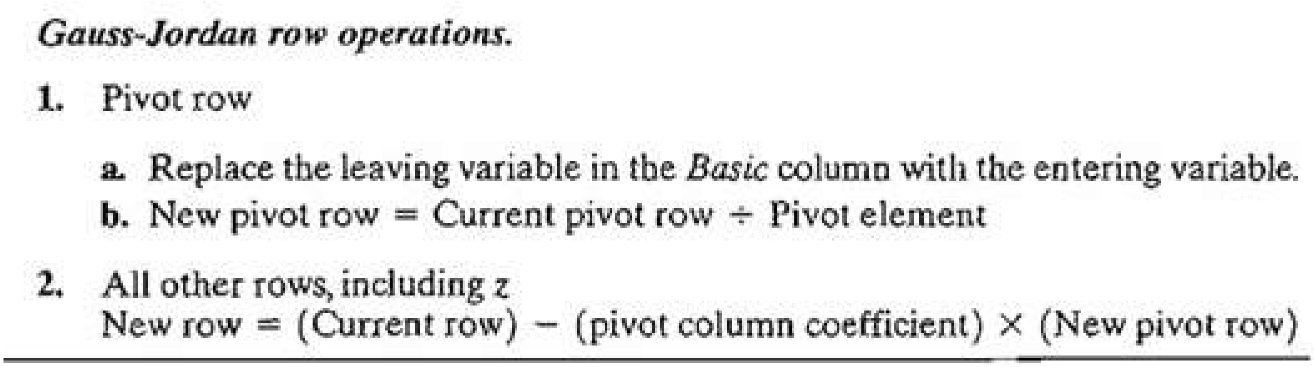
For each j (j = 1,2, .. . , n), define Zj == $C_0^T A_j$, the dot product of **C₀** with the j th column of **A**. The j th entry in the last row of Tableau 3-1 is **cj - Zj** (or, for a maximization program, **Zj - cj**), where cj is the cost in the second row of the tableau, immediately above **Aj** ; Once this last row has been obtained, the second row and second column of the tableau, corresponding to $\mathbf{C}^{\mathbf{T}}$ and **C₀**, respectively, become superfluous and may be eliminated.

THE SIMPLEX METHOD

STEP 1: Locate the most negative number in the bottom row of the simplex tableau, excluding the last column, and call the column in which this number appears the *work column (pivot column)*. If more than one candidate for most negative numbers exists, choose one.

***STEP* 2:** Form ratios by dividing each *positive* number in the work column, excluding the last row, into the element in the same row and last column. Designate the element in the work column that yields the *smallest* ratio as the *pivot element and the row in which pivot element is pivot row*. If more than one element yields the same smallest ratio, choose one. If no element in the work column is positive, the program has no solution.

***STEP* 3:** Use elementary row operations to convert the pivot element to 1 and then to reduce all *other* elements in the work column to 0.



Gauss-Jordan row operations.

1. Pivot row
   - a. Replace the leaving variable in the *Basic* column with the entering variable.
   - b. New pivot row = Current pivot row ÷ Pivot element
2. All other rows, including z
   New row = (Current row) − (pivot column coefficient) × (New pivot row)

***STEP* 4:** Repeat Steps 1 through 4 until there are no negative numbers in the last row, excluding the last column.

***STEP* 5:** The optimal solution is obtained by assigning to each variable in the first column that value in the corresponding row and last column. All other variables are assigned the value zero.

The associated z^*, the optimal value of the objective function, is the number in the last row and last column for a maximization program, but the *negative* of this number for a minimization program.

THE SIMPLEX METHOD طريقة السمبلكس

الخطوة ١ : حدد أعلى قيمة سالبة فى الصف السفلى من جدول السمبلكس ، باستثناء العمود الأخير ، وأطلق على العمود الذى تظهر فيه هذه القيمة « عمود العمل » . إذا وجد أكثر من رقم متساوٍ ، اختر أحدهما .

الخطوة ٢ : كون نسباً بقسمة كل رقم موجب فى عمود العمل ، باستثناء الصف الأخير ، على العنصر (الرقم) فى نفس الصف فى العمود الأخير . وحدد العنصر فى عمود العمل الذى يؤدى إلى أصغر نسبة ، وأطلق عليه « العنصر المحورى » . إذا أدى أكثر من رقم إلى نفس النسبة ، فاختر أحدهما . وإذا لم يوجد فى عمود العمل أى رقم موجب ، يكون البرنامج ليس له حل .

الخطوة ٣ : استخدم العمليات الأولية فى تحويل العنصر المحورى إلى واحد ، واختصار كل العناصر الأخرى فى عمود المحور إلى صفر .

الخطوة ٤ : استبدل المتغير x فى صف المحور والعمود الأول بالمتغير x فى الصف الأول وعمود المحور . وهذا العمود الأول هو فئة المتغيرات الأساسية الحالية (انظر فصل ٣)

الخطوة ٥ : كرر الخطوات من ١ حتى ٤ ، حتى لا تبقى هناك أعداد سالبة فى الصف الأخير ، باستثناء العمود الأخير .

الخطوة ٦ : نصل إلى الحل الأمثل بتخصيص لكل متغير فى العمود الأول قيمة مناظرة فى الصف الناظر والعمود الأخير . وكل المتغيرات الباقية تأخذ القيم صفر . والقيمة المثلى للهدف z^* المرتبطة بهذا هى العدد الموجود فى الصف الأخير والعمود الأخير ، فى حالة برنامج التعظيم ، والقيمة السالبة لهذا العدد فى حالة برنامج التصغير .

Example: Solve the following program using the simplex method

$$
\begin{aligned}
\text{maximize:}\quad & z = x_1 + 9x_2 + x_3 \\
\text{subject to:}\quad & x_1 + 2x_2 + 3x_3 \le 9 \\
& 3x_1 + 2x_2 + 2x_3 \le 15 \\
\text{with:}\quad & \text{all variables nonnegative}
\end{aligned}
$$

Sol:

Generating an initial feasible solution by converting inequality constraints to equation by adding slack variables.

X1+2x2+x3+x4=9

3X1+2x2+2x3+x5=15

Max. z = x1+9x2+x3+0x4+0x5 all variables non negative

This program is put into matrix standard form by first introducing slack variables x_4 and x_5 in the first and second constraint inequalities, respectively, and then defining

$$\mathbf{X} \equiv [x_1, x_2, x_3, x_4, x_5]^T \qquad \mathbf{C} \equiv [1, 9, 1, 0, 0]^T$$

$$\mathbf{A} \equiv \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 3 & 2 & 2 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} \equiv \begin{bmatrix} 9 \\ 15 \end{bmatrix} \qquad \mathbf{X}_0 = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$$

The costs associated with the components of $\mathbf{X}_0$, the slack variables, are zero; hence $\mathbf{C}_0 \equiv [0, 0]^T$. Tableau 3-1 becomes

		x_1	x_2	x_3	x_4	x_5	
		1	9	1	0	0	
x_4	0	1	2	3	1	0	9
x_5	0	3	2	2	0	1	15

To compute the last row of this tableau, we use the tableau simplification and first calculate each z_j by inspection: it is the dot product of column 2 and the jth column of **A**. We then subtract the corresponding cost c_j from it (maximization program). In this case, the second column is zero, and so $z_j - c_j = 0 - c_j = -c_j$. Hence, the bottom row of the tableau, excluding the last element, is just the negative of row 2. The last element in the bottom row is simply the dot product of column 2 and the final, **B**-column, and so it too is zero. At this point, the second row and second column of the tableau are superfluous. Eliminating them, we obtain Tableau 1 as the complete initial tableau.

	x_1	x_2	x_3	x_4	x_5	
x_4	1	2*	3	1	0	9
x_5	3	2	2	0	1	15
$(z_j - c_j)$:	-1	-9	-1	0	0	0

Tableau 1

	x_1	x_2	x_3	x_4	x_5	
x_2	1/2	1	3/2	1/2	0	9/2
x_5	2	0	-1	-1	1	6
	7/2	0	25/2	9/2	0	81/2

Tableau 2

We are now ready to apply the simplex method. The most negative element in the last row of Tableau 1 is -9, corresponding to the x_2-column; hence this column becomes the work column. Forming the ratios $9/2 = 4.5$ and $15/2 = 7.5$, we find that the element 2, marked by the asterisk in Tableau 1, is the pivot element, since it yields the smallest ratio. Then, applying Steps 3 and 4 to Tableau 1, we obtain Tableau 2. Since the last row of Tableau 2 contains no negative elements, it follows from Step 6 that the optimal solution is $x_2^* = 9/2$, $x_5^* = 6$, $x_1^* = x_3^* = x_4^* = 0$, with $z^* = 81/2$.

Water Resources Management and Economy

University of Anbar– College of Engineering

Dams & Water Resources Engineering

Department – 4th stage

Asst. Prof. Dr. Sadeq Oleiwi Sulaiman

The Simplex Method (Big-M method)

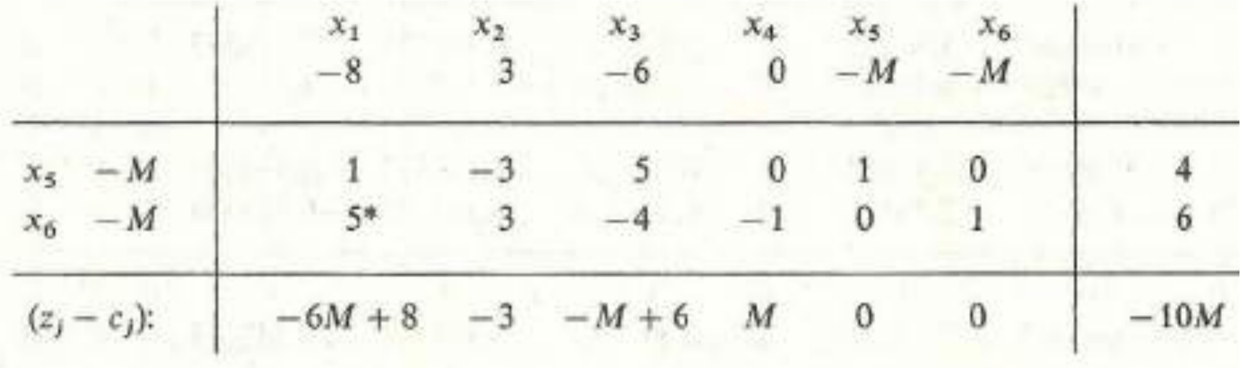
This program' is put in standard form by introducing artificial variables and substituting the appropriate coefficients into Tableau and then applying the simplex method directly.

Example 3.6: Solve the following program using the *Big* M *method*

maximize: z = -8Xl + 3x2 - 6X3
subject to: Xl - 3x2 + 5x3 = 4
5x1 + 3X2 - 4X3 ≥ 6
with: all variables nonnegative

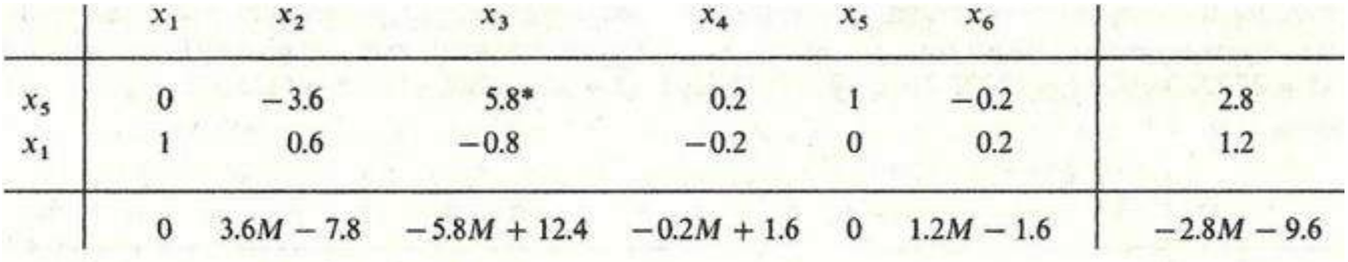
Sol:

This program' is put in standard form by introducing the surplus variable X4 in the inequality constraint and then artificial variables X5 and X6 in the two equality constraints. Substituting the appropriate coefficients into Tableau 3-1 and then applying the simplex method directly, with all calculations rounded to four significant figures and with the pivot elements designated by stars, we generate successively Tableaux 1through 4.



		x_1	x_2	x_3	x_4	x_5	x_6	
		-8	3	-6	0	$-M$	$-M$	
x_5	$-M$	1	-3	5	0	1	0	4
x_6	$-M$	5*	3	-4	-1	0	1	6
$(z_j - c_j)$:		$-6M + 8$	-3	$-M + 6$	M	0	0	$-10M$

Tableau 1



	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	0	-3.6	5.8*	0.2	1	-0.2	2.8
x_1	1	0.6	-0.8	-0.2	0	0.2	1.2
	0	$3.6M - 7.8$	$-5.8M + 12.4$	$-0.2M + 1.6$	0	$1.2M - 1.6$	$-2.8M - 9.6$

Tableau 2

	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	0	−0.6207	1	0.03448	0.1724	−0.03448	0.4828
x_1	1	0.1034*	0	−0.1724	0.1379	0.1724	1.586
	0	−0.1033	0	1.172	$M-2.138$	$M-1.172$	−15.59

Tableau 3

	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	6.003	0	1	−10.00	1.000	10.00	10.00
x_2	9.671	1	0	−1.667	1.334	1.667	15.34
	0.9990	0	0	0.9998	$M-2$	$M-0.9998$	−14.01

Tableau 4

Since M designates a large positive number, all the entries in the last row of Tableau 4, excluding the entry in the last column, are nonnegative. The optimal solution, therefore, can be read directly from it as $x_3^* = 10.00$, $x_2^* = 15.34$, and all other variables zero, with $z^* = -14.01$.

The quantity M in the previous calculations could be left as a letter only because those calculations were done by hand. Had a computer been used, a large numerical value would necessarily have been substituted for M; say, $M = 10\,000$. Then, assuming again that all numbers are rounded to *four* significant figures, the bottom row of Tableau 1 becomes

$$-60\,000 \qquad -3 \qquad -10\,000 \qquad 10\,000 \qquad 0 \qquad 0 \qquad -100\,000$$

Note that the additive constants $+8$ in the first entry and $+6$ in the third entry are lost in roundoff. The bottom row of Tableau 2 becomes

$$0 \qquad 36\,000 \qquad -58\,000 \qquad -2\,000 \qquad 12\,000 \qquad -28\,000$$

while the bottom row of Tableau 3 is

$$0 \qquad 0 \qquad 0 \qquad 0 \qquad 10\,000 \qquad 10\,000 \qquad 0$$

which signals optimality! The erroneous optimal solution would be read from Tableau 3 as $x_3^* = 0.4828$, $x_1^* = 1.586$, and all other variables zero, with $z^* = 0$.

This roundoff problem does not occur in the two-phase method since the terms that do not involve M are separated from those that do, making it impossible for the M-terms to "swamp" the others.

Modifications for Programs with Artificial Variables (two-phase method)

Whenever artificial variables are part of the initial solution Xo, the last row of Tableau 3.1 will contain the penalty cost M. To minimize roundoff error, the following modifications are incorporated into the simplex method; the resulting algorithm is the ***two-phase method.***

Change 1: The last row of Tableau 3-1 is decomposed into two rows, the first of which involves those terms not containing M, while the second involves the coefficients of M in the remaining terms.

The last row of the tableau in Example 3.1 is

-9-8M 0 -9-9M 0 M 0 - 14 - 2M

Under Change 1 it would be transformed into the two rows

- 9	**0**	**- 9**	**0**	**0**	**0**	**- 14**
- 8	**0**	**- 9**	**0**	**1**	**0**	**-2**

Change 2: Step 1 of the simplex method is applied to the last row created in Change 1 (followed by Steps 2, 3, and 4), until this row contains no negative elements. Then Step 1 is applied to those elements in the next-to-last row that are positioned over zeros in the last row.

Change 3: Whenever an artificial variable ceases to be basic-i.e., is removed from the first column of the tableau as a result of Step 4-it is deleted from the top row of the tableau, as is the entire column under it. (This modification simplifies hand calculations but is not implemented in many computer programs.)

Change 4: The last row can be deleted from the tableau whenever it contains all zeros.

Change 5: If *nonzero* artificial variables are present in the final basic set, then the program has no solution. (In contrast, zero-valued artificial variables may appear as basic variables in the final solution when one or more of the original constraint equations is redundant.)

تعديل البرنامج باستخدام المتغيرات الصناعية MODIFICATIONS FOR PROGRAMS WITH ARTIFICIAL VARIABLES

حيثما تكون المتغيرات الصناعية جزءًا من الحل الأوَّليّ X_0 ، فإن الصف الأخير من الجدول ٤ ــ ١ يحتوى على التكلفة الجزائية **M**(انظر فصل ٢) . لتقليل أخطاء الاستكمال (انظر مسألة ٤ ــ ٦) تجرى التعديلات التالية على طريقة السمبلكس ، وتكون الطريقة الناتجة هى « طريقة المرحلتين » ***two-phase method*** .

التغيير الأول : يقسم الصف الأخير فى الجدول ٤ ــ ١ إلى صفين ، يحتوى الأول منهما على الحدود التى لا تحتوى على **M** ، بينما يحتوى الثانى على معاملات **M** فى الحدود الباقية .

مثال ٤ ــ ٢ الصف الأخير فى الجدول فى المثال ٤ ــ ١

$$-9-8M \quad 0 \quad -9-9M \quad 0 \quad M \quad 0 \quad -14-2M$$

بالتغيير الأول يحول الصف إلى صفين هما :

$$\begin{array}{ccccccc} -9 & 0 & -9 & 0 & 0 & 0 & -14 \\ -8 & 0 & -9 & 0 & 1 & 0 & -2 \end{array}$$

التغيير الثانى : تطبق الخطوة الأولى فى طريقة السمبلكس على الصف الأخير الناتج من التغيير الأول .

(ويتبع بالخطوات ٢ ، ٣ ، ٤) حتى لايحتوى هذا الصف على عناصر سالبة ، ثم تطبق الخطوة الأولى على العناصر التى فى الصف قبل الأخير ، والتى فوق الأصفار فى الصف الأخير .

التغيير الثالث : عندما يتحول أى متغير صناعى إلى أساسى ــ أى ينتقل من العمود الأول فى الجدول [نتيجة تطبيق الخطوة الرابعة] ــ فإنه يحذف من الصف الأعلى بالجدول ، وكذلك من كل العمود الذى تحته . (هذا التعديل يبسط الحسابات اليدوية ، ولا يستخدم فى حالة برامج الحاسبات)

التغيير الرابع : يمكن حذف الصف الأخير من الجدول عندما يحتوى كله على أصفار .

التغيير الخامس : إذا وجدت متغيرات صناعية لا صفرية فى الفئة الأساسية النهائية ، فإن البرنامج يكون ليس له حل . (وعلى النقيض ، فإن المتغيرات الصناعية الصفرية يمكن أن تظهر كمتغيرات أساسية فى الحل النهائى عندما تكون واحدة أو أكثر من معادلة القيود الأصلية زائدة عن الحاجة .

$$\begin{aligned} \text{minimize:} \quad & z = 80x_1 + 60x_2 \\ \text{subject to:} \quad & 0.20x_1 + 0.32x_2 \le 0.25 \\ & x_1 + \quad x_2 = 1 \\ \text{with:} \quad & x_1 \text{ and } x_2 \text{ nonnegative} \end{aligned}$$

Adding a slack variable x_3 and an artificial variable x_4 to the first and second constraints, respectively, we convert the program to standard matrix form, with

$$\mathbf{X} \equiv [x_1, x_2, x_3, x_4]^T \qquad \mathbf{C} \equiv [80, 60, 0, M]^T$$

$$\mathbf{A} \equiv \begin{bmatrix} 0.20 & 0.32 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} \equiv \begin{bmatrix} 0.25 \\ 1 \end{bmatrix} \qquad \mathbf{X}_0 \equiv \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

Substituting these matrices, along with $\mathbf{C}_0 \equiv [0, M]^T$, into Tableau 3-1, we obtain Tableau 0. Since the bottom row involves M, we apply Change 1; the resulting Tableau 1 is the initial tableau for the two-phase method.

		x_1	x_2	x_3	x_4	
		80	60	0	M	
x_3	0	0.20	0.32	1	0	0.25
x_4	M	1	1	0	1	1
		$80 - M$	$60 - M$	0	0	$-M$

Tableau 0

	x_1	x_2	x_3	x_4	
x_3	0.20	0.32	1	0	0.25
x_4	1*	1	0	1	1
$(c_j - z_j)$:	80	60	0	0	0
	−1	−1	0	0	−1

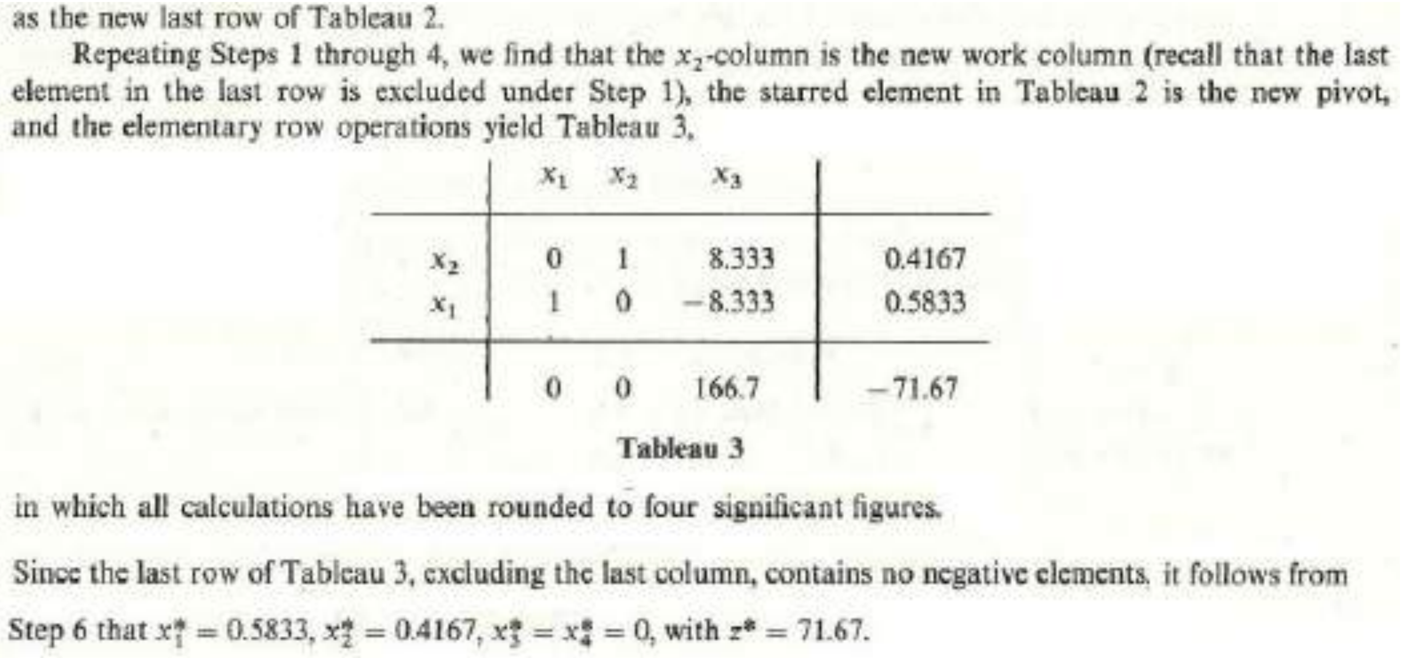
Tableau 1

	x_1	x_2	x_3	
x_3	0	0.12*	1	0.05
x_1	1	1	0	1
	0	−20	0	−80
	0	0	0	0

Tableau 2

Using both Step 1 of the simplex method and Change 2, we find that the most negative element in the last row of Tableau 1 (excluding the last column) is -1, which appears twice. Arbitrarily selecting the x_1-column as the work column, we form the ratios $0.25/0.20 = 1.25$ and $1/1 = 1$. Since the element 1, starred in Tableau 1, yields the smallest ratio, it becomes the pivot. Then, applying Steps 3 and 4 and Change 3 to Tableau 1, we generate Tableau 2. Observe that x_1 replaces the artificial variable x_4 in the first column of Tableau 2, so that the entire x_4-column is absent from Tableau 2. Now, with no artificial variables in the first column and with Change 3 implemented, the last row of the tableau should be all zeros. It is; and by Change 4 this row may be deleted, giving

0 −20 0 −80



as the new last row of Tableau 2.

Repeating Steps 1 through 4, we find that the x_2-column is the new work column (recall that the last element in the last row is excluded under Step 1), the starred element in Tableau 2 is the new pivot, and the elementary row operations yield Tableau 3,

	x_1	x_2	x_3	
x_2	0	1	8.333	0.4167
x_1	1	0	−8.333	0.5833
	0	0	166.7	−71.67

Tableau 3

in which all calculations have been rounded to four significant figures.

Since the last row of Tableau 3, excluding the last column, contains no negative elements, it follows from Step 6 that $x_1^* = 0.5833$, $x_2^* = 0.4167$, $x_3^* = x_4^* = 0$, with $z^* = 71.67$.

Water Resources Management and Economy

University of Anbar– College of Engineering

Dams & Water Resources Engineering

Department – 4th stage

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THE DUAL SIMPLEX METHOD

The (regular) simplex method moves the initial feasible but nonoptimal solution to an optimal solution while maintaining feasibility through an iterative procedure. On the other hand, the dual simplex method moves the initial optimal but infeasible solution to a feasible solution while maintaining optimality through an iterative procedure.

Iterative procedure of the Dual Simplex Method:

STEP 1: Rewrite the linear programming problem by expressing all the constraints in ≤ form and transforming them into equations through slack variables.

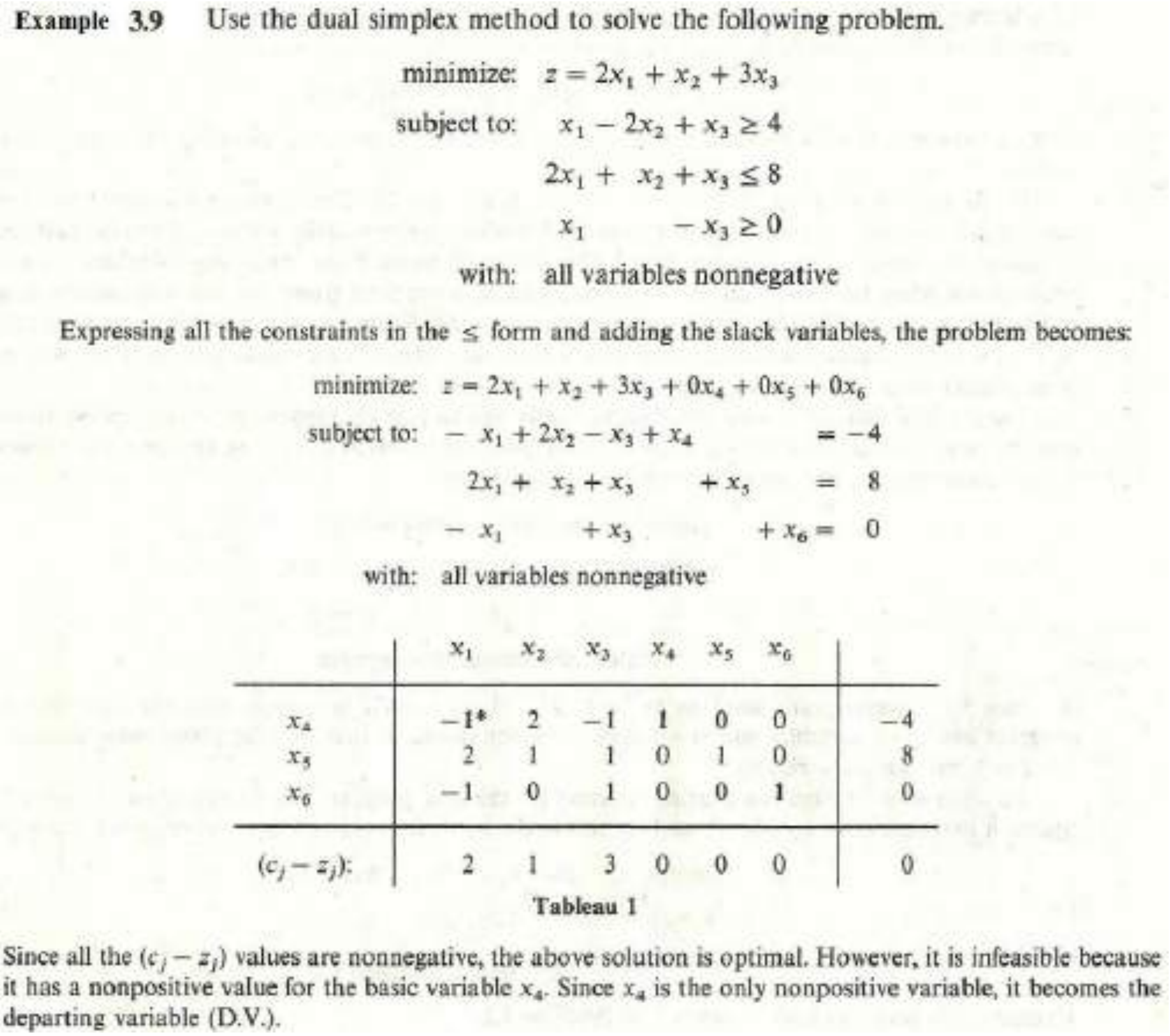
***STEP* 2:** Exhibit the above problem in the form of a simplex tableau. If the optimality condition is satisfied *and* one or more basic variables have negative values, the dual simplex method is applicable.

***STEP* 3:** Feasibility Condition: The basic variable with the most negative value becomes the departing variable (D. V.). Call the row in which this value appears the work row. If more than one· candidate for D.V. exists, choose one.

***STEP* 4:** Optimality Condition: Form ratios by dividing all but the last element of the last row of *cj* - *Zj* values (minimization problem) or the *Zj* - *Cj* values (maximization problem) by the corresponding negative coefficients of the work row. The non basic variable with the smallest absolute ratio becomes the entering variable (E.V.). Designate this element in the work row as the pivot element and the corresponding column the work column. If more than one candidate for E.V. exists, choose one. If no element in the work row is negative, the problem has no feasible solution.

***STEP* 5**: Use elementary row operations to convert the pivot element to 1 and then to reduce all the other elements in the work column to zero.

***STEP* 6:** Repeat steps 3 through 5 until there are no negative values for the basic variables.



Example 3.9 Use the dual simplex method to solve the following problem.

$$\begin{aligned} \text{minimize:}\quad & z = 2x_1 + x_2 + 3x_3 \\ \text{subject to:}\quad & x_1 - 2x_2 + x_3 \geq 4 \\ & 2x_1 + x_2 + x_3 \leq 8 \\ & x_1 - x_3 \geq 0 \\ \text{with:}\quad & \text{all variables nonnegative} \end{aligned}$$

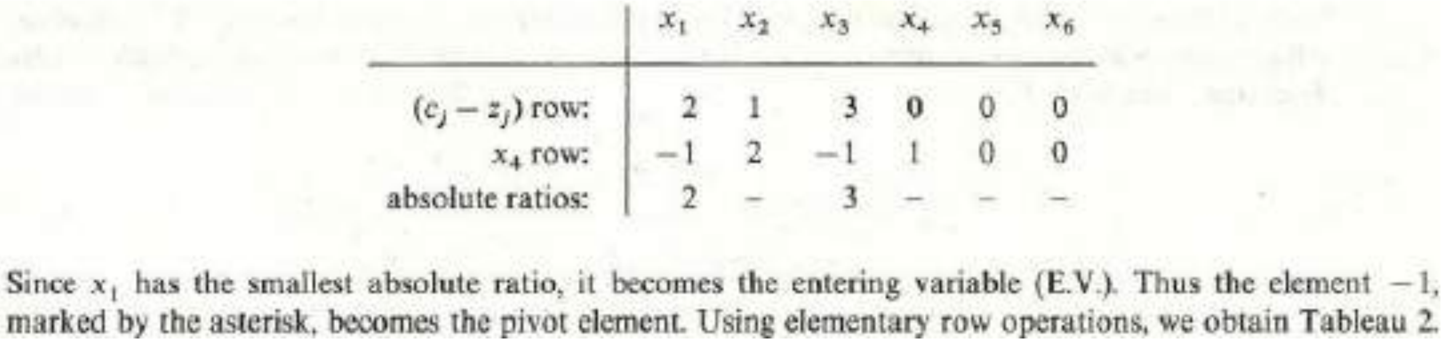
Expressing all the constraints in the ≤ form and adding the slack variables, the problem becomes:

$$\begin{aligned} \text{minimize:}\quad & z = 2x_1 + x_2 + 3x_3 + 0x_4 + 0x_5 + 0x_6 \\ \text{subject to:}\quad & -x_1 + 2x_2 - x_3 + x_4 = -4 \\ & 2x_1 + x_2 + x_3 + x_5 = 8 \\ & -x_1 + x_3 + x_6 = 0 \\ \text{with:}\quad & \text{all variables nonnegative} \end{aligned}$$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	-1^*	2	-1	1	0	0	-4
x_5	2	1	1	0	1	0	8
x_6	-1	0	1	0	0	1	0
$(c_j - z_j)$:	2	1	3	0	0	0	0

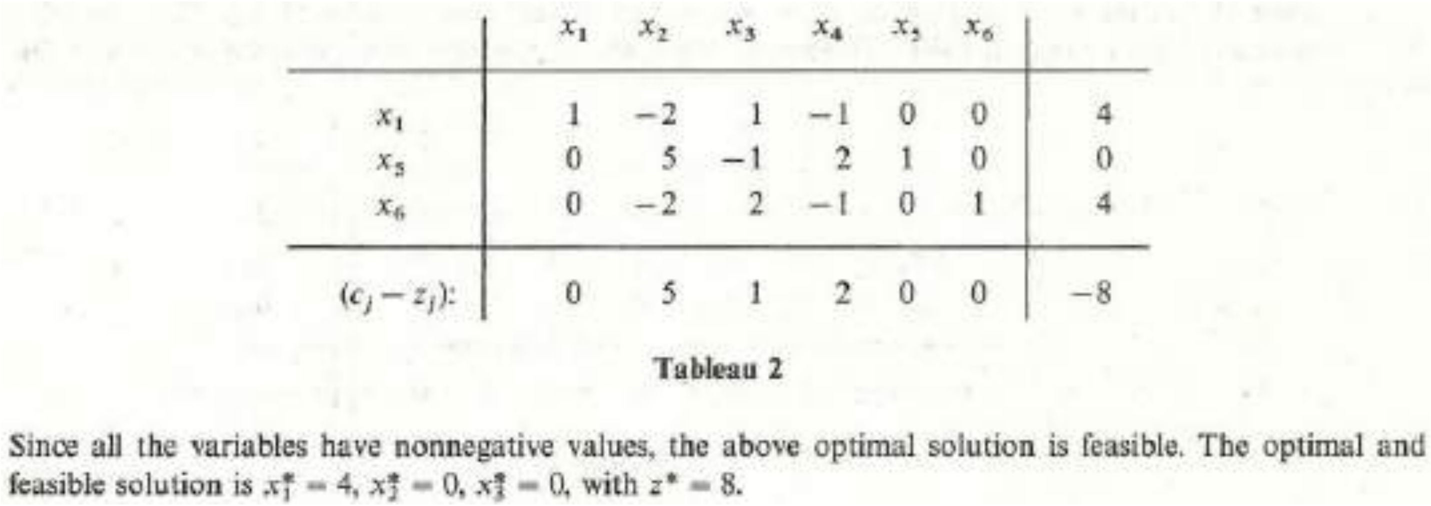
Tableau 1

Since all the $(c_j - z_j)$ values are nonnegative, the above solution is optimal. However, it is infeasible because it has a nonpositive value for the basic variable x_4. Since x_4 is the only nonpositive variable, it becomes the departing variable (D.V.).



	x_1	x_2	x_3	x_4	x_5	x_6
$(c_j - z_j)$ row:	2	1	3	0	0	0
x_4 row:	-1	2	-1	1	0	0
absolute ratios:	2	–	3	–	–	–

Since x_1 has the smallest absolute ratio, it becomes the entering variable (E.V.). Thus the element -1, marked by the asterisk, becomes the pivot element. Using elementary row operations, we obtain Tableau 2.



	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	−2	1	−1	0	0	4
x_5	0	5	−1	2	1	0	0
x_6	0	−2	2	−1	0	1	4
$(c_j - z_j)$:	0	5	1	2	0	0	−8

Tableau 2

Since all the variables have nonnegative values, the above optimal solution is feasible. The optimal and feasible solution is $x_1^* = 4$, $x_2^* = 0$, $x_3^* = 0$, with $z^* = 8$.

Solved Problems

Q1: The electrical company need to built a project of electric generator across a river. It have three types of turbines with the following properties:

Turbine type	Flow needed (ft^3/s)	Width from river needed (ft)	Net revenue .year ($)
Turbine (1)	280	35	11500
Turbine (2)	350	65	15000
Turbine (3)	500	75	19000

If the total flow of river = 12000 ft^3/s, and the net width of river = 2000 ft, find the optimum combination of turbine for maximum revenue.

Sol:

Let X1 = Number of turbines (1) in project
X2 = Number of turbines (2) in project
X3 = Number of turbines (3) in project

Objective function:

Maximize Z = 11500 X1 +15000 X2 + 19000 X3

Subjected to:

$280\ x1 + 350\ x2 + 500\ x3 \leq 12000$

$35\ x1 + 65\ x2 + 75\ x3 \leq 2000$

$X1\ ,\ x2\ ,\ x3 \geq 0$

$280\ x1 + 350\ x2 + 500\ x3 + x4 = 12000$

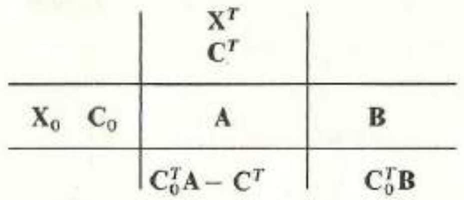
$35\ x1 + 65\ x2 + 75\ x3 + x5 = 2000$

Max. $Z = 11500\ X1 + 15000\ X2 + 19000\ X3 + 0\ x4 + 0\ x5$

$X1\ ,\ x2\ ,\ x3,\ x4\ ,\ x5 \geq 0$

$X = [\ X1\ \ X2\ \ X3\ \ X4\ \ X5]^T \qquad C = [11500\ \ 15000\ \ 19000\ \ 0\ \ 0\ ]^T$

$$A = \begin{bmatrix} 280 & 350 & 500 & 1 & 0 \\ 35 & 65 & 75 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 12000 \\ 2000 \end{bmatrix} \quad X0 = \begin{bmatrix} X4 \\ X5 \end{bmatrix} \quad C0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



	X^T C^T	
X_0 C_0	A	B
	$C_0^T A - C^T$	$C_0^T B$

Tableau 3-1 For maximization programs

		X1	X2	X3	X4	X5		ratio
		11500	15000	19000	0	0		
X4	0	280	350	500	1	0	12000	12000/500
X5	0	35	65	75	0	1	2000	2000/75
		- 11500	-15000	-19000	0	0	0	

	X1	X2	X3	X4	X5		ratio
X3	0.65	0.7	1	0.002	0	24	24/0.7
X5	-7	12.5	0	-0.15	1	200	200/12.5
	- 860	-1700	0	38	0	456000	

	X1	X2	X3	X4	X5		ratio
X3	0.95	0	1	0.01	-0.06	12.8	
X2	-0.56	1	0	-0.01	0.08	16	
	-1812	0	0	17.6	136	483200	

	X1	X2	X3	X4	X5		ratio
X1	1	0	1.05	0.01	-0.06	13.45	
X2	0	1	0.59	-0.01	0.05	23.53	
	0	0	1903.36	37.39	29.41	507563	

Number of turbines (1) used in project = 13

Number of turbines (2) used in project = 23

Number of turbines (3) used in project = 0

Total revenue = 13* 11500 + 23 * 15000 + 0 * 19000 = 494500$

Overall width of the river occupied by the turbines = 13*35 + 23*65 = 1950 ft

Overall flow of the river that passes through the turbine = 13*280 + 23*350 = 11690 cfs

Q2: An industrial water treatment plant receives 300,000 m^3 of water per day. The water must be softened and chlorinated before use. The treatment process requires at least 100 units of a chlorination chemical and 150 units of a particular softening agent. Two alternative types of water additive package contain the chlorination chemical and softening agent. Additive **A** contain 3 units of chlorination chemical and 8 units of the softening agent per package. Additive **B** contain 9 units of chlorination chemical and 4 units of the softening agent per package. Costs of water additive are **A** and **B** are 8$ and 10$ per package, respectively. Determine the combination of treatment additive **A** and **B** that will minimize cost. (Solve the linear programming model by use Big-M method).

Sol:

	Chlorination (unit)	Softening (unit)	Cost $
Additive **A** package	3	8	8
Additive **B** package	9	4	10

Let X1 = No. of **A** package needed

X2 = No. of **B** package needed

Objective function:

Minimize $Z = 8\ X1 + 10\ X2$

Subjected to:

$3\ x1 + 9\ x2 \geq 100$

$8\ x1 + 4\ x2 \geq 150$

$X1\ ,\ x2 \geq 0$

$3\ x1 + 9\ x2 - x3 \quad = 100$

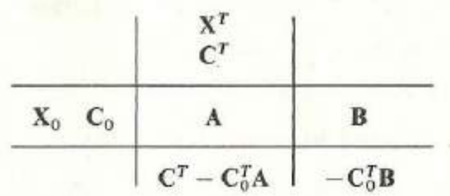
$8\ x1 + 4\ x2 \quad - x4 = 150$

$3\ x1 + 9\ x2 - x3 \quad + x5 \quad = 100$

$8\ x1 + 4\ x2 \quad - x4 \quad + x6 = 150$

Minimize $Z = 8\ x1 + 10\ x2 + 0\ x3 + 0\ x4 + M\ x5 + M\ x6$

$X1\ ,\ x2\ ,\ x3,\ x4\ ,\ x5\ ,\ x6\ \geq 0$



	X^T C^T	
X_0 C_0	A	B
	$C^T - C_0^T A$	$-C_0^T B$

Tableau 3-1 For minimization programs

		X1	X2	X3	X4	X5	X6		ratio
		8	10	0	0	M	M		
X5	M	3	9	-1	0	1	0	100	100/9
X6	M	8	4	0	-1	0	1	150	150/4
		8-11M	**10-13M**	**M**	**M**	**0**	**0**	**-250M**	

	X1	X2	X3	X4	X5	X6		ratio
X2	0.33	1	-0.11	0	0.11	0	11.11	11.11/0.33
X6	6.67	0	0.44	-1	-0.44	1	105.56	105.56/6.67
	4.6-6.67M	0	1.11-0.44M	M	-1.1+1.4M	0	-105.56M-111.1	

	X1	X2	X3	X4	X5	X6		ratio
X2	0	1	-0.13	0.05	0.13	-0.05	5.83	
X1	1	0	0.07	-0.15	-0.07	0.15	15.83	
	0	0	1.08+0.02M	0.69	0.94M-0.78	0	-183.9	

No. of **A** package needed = 16
No. of **B** package needed = 6
Number of chlorination chemical units provide for project = 3*16 + 9*6 = 102 unit
Number of softening agent units provide for project = 8*16 + 4*6 = 152 unit
Total cost = 8* 16 + 10 * 6 = 188 $

Q3: A farm need at least 800 m^3 of water daily. The water is provided by two nearby wells, and have the following properties:

	TDS (ppm)	Nitrate (ppm)	Cost $/ m^3
Well 1	980	125	0.3
Well 2	300	20	0.9

The special requirements of the crop in the farm are at most 600 ppm for TDS, and at least 50 ppm for Nitrate. The farm directorate wishes to determine the daily mixture of water from the two wells to obtain daily minimum cost. (use two-phase method)

Sol:

Let X1 = volume of water (m^3) from well 1

X2 = volume of water (m^3) from well 2

Objective function: Minimize $Z = 0.3\,X1 + 0.9\,X2$

Subjected to:

$x1 + x2 \geq 800$

$980\,x1 + 300\,x2 \leq 600\,(x1 + x2)$

$125\,x1 + 20\,x2 \geq 50\,(x1 + x2)$

$X1\,,\ x2 \geq 0$

$x1 + x2 \geq 800$

$380\,x1 - 300\,x2 \leq 0$

$75\,x1 - 30\,x2 \geq 0$

$X1\,,\ x2 \geq 0$

$x1 + x2 - x3 + x6 = 800$

$380\,x1 - 300\,x2 + x4 = 0$

$75\,x1 - 30\,x2 + x5 + x7 = 0$

Min. $Z = 0.3\,x1 + 0.9\,x2 + 0\,x3 + 0\,x4 + 0x5 + Mx6 + Mx7$

$X1\,,\ x2,\ x3\,,\ x4,\ x5,\ x6,\ x7 \geq 0$

$X = [\,X1\ \ X2\ \ X3\ \ X4\ \ X5\ \ X6\ X7\,]^T \qquad C = [0.3\ \ 0.9\ \ 0\ \ 0\ \ 0\ \ M\ \ M\,]^T$

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 380 & -300 & 0 & 1 & 0 & 0 & 0 \\ 75 & -30 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 800 \\ 0 \\ 0 \end{bmatrix} \qquad X0 = \begin{bmatrix} x6 \\ x4 \\ x7 \end{bmatrix} \quad C0 = \begin{bmatrix} 0 \\ M \\ M \end{bmatrix}$$

		X1	X2	X3	X4	X5	X6	X7		ratio
		0.3	0.9	0	0	0	M	M		
X6	0	1	1	-1	0	0	1	0	800	
X4	M	380	-300	0	1	0	0	0	0	
X7	M	75	-30	0	0	1	0	1	0	
c-z		**0.3-455M**	**0.9-270M**	**0**	**-M**	**-M**	**M**	**0**	**0**	
Separate to two phase		0.3	0.9	0	0	0	0	0	0	
		-455	-270	0	-1	-1	1	0	0	

	X1	X2	X3	X4	X5	X6	X7		ratio
X6	0	1.79	-1	0	1	0	0	800	
X1	1	-0.79	0	0.002	0	0	0	0	
X7	0	29.21	0	-1	0	-0.2	1	0	
	0	1.137	0	0	0	0	0	0	
	0	-629.45	0	0.197	-1	1	0	0	

	X1	X2	X3	X4	X5	X6		ratio
X6	0	0	-1	0.06	1	0.01	800	
X1	1	0	0	-0.025	0	0	0	
X2	0	1	0	-0.034	0	-0.007	0	
	0	0	0	0.039	0	0.008	0	
	0	0	0	-21.35	-1	-3.4	0	

	X1	X2	X3	X4	X5		ratio
X4	0	0	-16.66	1	16.66	13333.33	
X1	1	1	-0.44	0	0.44	333.33	
X2	0	1	-0.56	0	0.56	453.33	
	0	0	0.65	0	-0.65	-520	
	0	0	-355.8	0	354.8	284666.6	

بما ان كل عناصر العمود المحوري في الجدول اعلاه سلبية لذا يتوقف الحل عند هذا الحد وتكون قيم المتغيرات كما
يلي: $x1= 333.33\ m^{3,}$ $x2 = 453.33\ m^3$, $z = 520$ $

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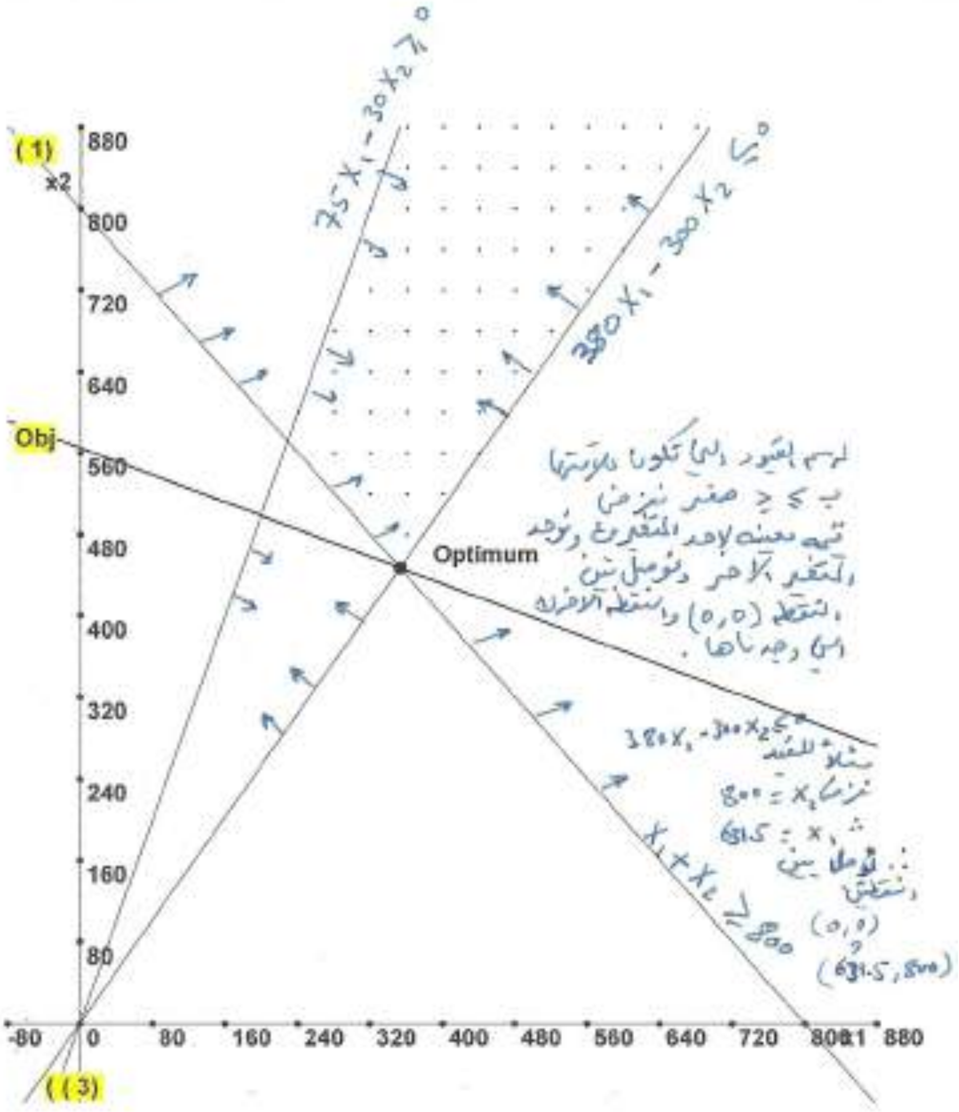
Q4: Solve Q3 Graphically

```
Summary of Optimal Solution:
   Objective Value = 508.24
   x1 = 352.94
   x2 = 447.06
```



Q5: Solve Q3 by use Big – M method

Q6:

Use the dual simplex method to solve the following problem.

$$\begin{aligned} \text{maximize:} \quad & z = -2x_1 - 3x_2 \\ \text{subject to:} \quad & x_1 + x_2 \geq 2 \\ & 2x_1 + x_2 \leq 10 \\ & x_1 + x_2 \leq 8 \\ \text{with:} \quad & x_1 \text{ and } x_2 \text{ nonnegative} \end{aligned}$$

Expressing all the constraints in the ≤ form and adding the slack variables, the problem becomes:

$$\begin{aligned} \text{maximize:} \quad & z = -2x_1 - 3x_2 + 0x_3 + 0x_4 + 0x_5 \\ \text{subject to:} \quad & -x_1 - x_2 + x_3 = -2 \\ & 2x_1 + x_2 + x_4 = 10 \\ & x_1 + x_2 + x_5 = 8 \\ \text{with:} \quad & \text{all variables nonnegative} \end{aligned}$$

	x_1	x_2	x_3	x_4	x_5	
x_3	-1^*	-1	1	0	0	-2
x_4	2	1	0	1	0	10
x_5	1	1	0	0	1	8
$(z_j - c_j)$:	2	3	0	0	0	0

Tableau 1

Since all the $(z_j - c_j)$ values are nonnegative, the above solution is optimal. However, it is infeasible because it has a nonpositive value for the basic variable x_3. Since x_3 is the only nonpositive variable, it becomes the departing variable (D.V.).

	x_1	x_2	x_3	x_4	x_5
$(z_j - c_j)$ row:	2	3	0	0	0
x_3 row:	-1	-1	1	0	0
absolute ratios:	2	3	–	–	–

Since x_1 has the smallest absolute ratio, it becomes the entering variable (E.V.). Thus the element -1, marked by the asterisk, is the pivot element. Using elementary row operations, we obtain Tableau 2.

	x_1	x_2	x_3	x_4	x_5	
x_1	1	1	-1	0	0	2
x_4	0	-1	2	1	0	6
x_5	0	0	1	0	1	6
$(z_j - c_j)$:	0	1	2	0	0	-4

Tableau 2

Since all the variables have nonnegative values, the above optimal solution is feasible. The optimal and feasible solution is $x_1^* = 2$, $x_2^* = 0$, with $z^* = -4$.

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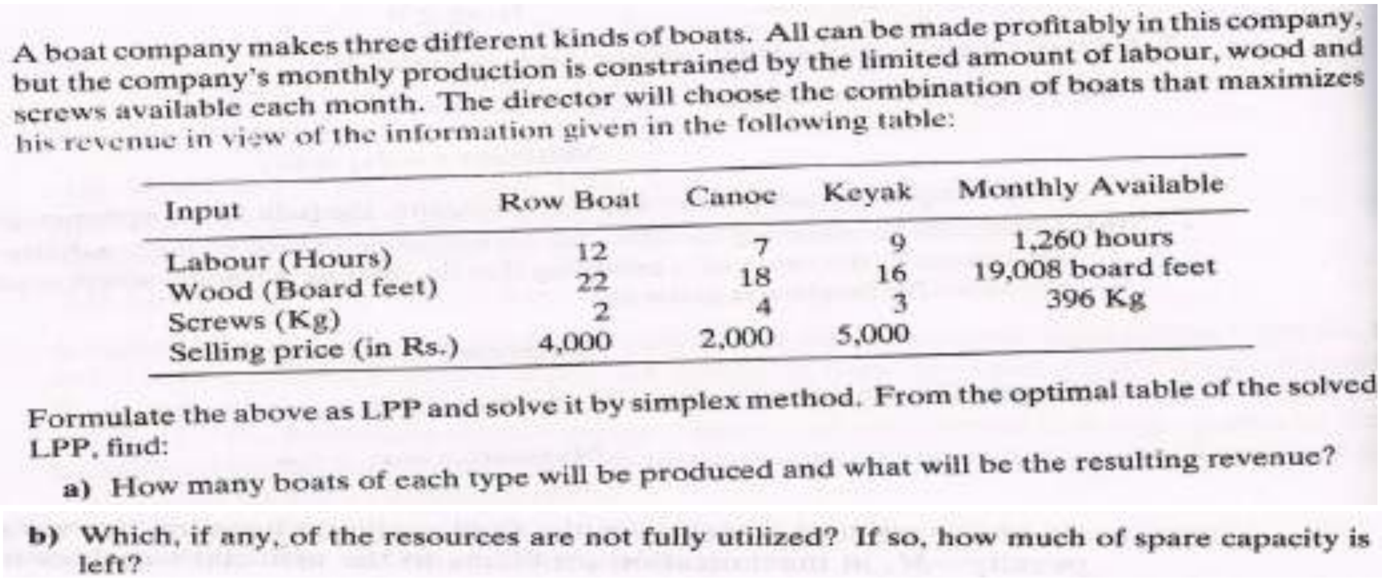
Self Test

Q1:



A boat company makes three different kinds of boats. All can be made profitably in this company, but the company's monthly production is constrained by the limited amount of labour, wood and screws available each month. The director will choose the combination of boats that maximizes his revenue in view of the information given in the following table:

Input	Row Boat	Canoe	Keyak	Monthly Available
Labour (Hours)	12	7	9	1,260 hours
Wood (Board feet)	22	18	16	19,008 board feet
Screws (Kg)	2	4	3	396 Kg
Selling price (in Rs.)	4,000	2,000	5,000	

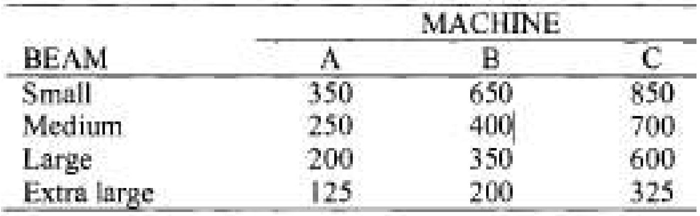
Formulate the above as LPP and solve it by simplex method. From the optimal table of the solved LPP, find:

a) How many boats of each type will be produced and what will be the resulting revenue?

b) Which, if any, of the resources are not fully utilized? If so, how much of spare capacity is left?

Q2: The combination of three technologies is used to remove a certain pollutant from wastewater. The three technologies remove 1, 2, and 3 g/m3 of the pollutants, respectively. The third technology variant seems to be the best, but it cannot be applied to more than 50% of the wastewater being treated. The costs of applying the technology variants are \$5, \$3, and \$2 per cubic meter. If 1000 m3 must be treated in a day, and at least 1500 g of pollutant has to be removed, then formulate a simple linear programming to model this optimization problem.

Q3: A steel manufacturer produces four sizes of I beams: small, medium, large, and extra-large. These beams can be produced on any one of three machine types: A, B, and C. The lengths in feet of the I beam that can be produced on the machines per hour are summarized below:



	MACHINE		
BEAM	A	B	C
Small	350	650	850
Medium	250	400	700
Large	200	350	600
Extra large	125	200	325

Assume that each machine can be used up to 50 hours per week and that the hourly operating costs of these machines are respectively \$30.00, \$50.00, and \$80.00. Further suppose that 12,000, 6000, 5000, and 7000 feet of the different size I beams are required weekly. Formulate the machine scheduling problem as a linear program.

Quiz: 20-8-2015

The combination of three technologies was used to remove a certain pollutant from wastewater. The three technologies remove 1, 2, and 3 g/m^3 of the pollutants from wastewater, respectively. The third technology seems to be the best, but it cannot be applied to more than 50% of the wastewater being treated. The costs of applying the technology variants are \$5, \$3, and \$2 per cubic meter. If at least 1000 m^3 must be treated in a day, and at least 1500 g of pollutant has to be removed. Formulate and solve a linear programming to model this optimization problem.

Sol:

Let X1 = Amount of waste water treated by using 1st technology in m^3
X2 = Amount of waste water treated by using 2nd technology in m^3
X3 = Amount of waste water treated by using 3rd technology in m^3

Objective function:

Minimize $Z = 5\ X1 + 3\ X2 + 2\ X3$

Subjected to:

$X1 + x2 + x3 \geq 1000$

$x1 + 2\ x2 + 3\ x3 \geq 1500$

$x3 \leq 0.5\ (x1+x2+x3)$ … $-\ 0.5\ x1 - 0.5\ x2 + 0.5\ x3 \leq 0$ … $0.5x1 + 0.5\ x2 - 0.5\ x3 \geq 0$

$X1\ ,\ x2,\ x3\ \geq 0$

$X1 + x2 + x3 - x4 \qquad + x7 \qquad = 1000$

$x1 + 2\ x2 + 3\ x3 \quad - x5 \qquad + x8 \qquad = 1500$

$0.5x1 + 0.5\ x2 - 0.5\ x3 - x6 \qquad + x9 = 0$

Minimize $Z = 5\ x1 + 3\ x2 + 2\ x3 + 0\ x4 + 0\ x5 + 0\ x6 + M\ x7 + M\ x8 + M\ x9$

$X1\ ,\ x2\ ,\ x3,\ x4\ ,\ x5\ ,\ x6\ ,x7\ ,\ x8,\ x9 \geq 0$

by use big-M method for solve a linear program

		X1	X2	X3	X4	X5	X6	X7	X8	X9		
		5	3	2	0	0	0	M	M	M		
X7	M	1	1	1	-1	0	0	1	0	0	1000	1000/1
X8	M	1	2	3	0	-1	0	0	1	0	1500	1500/3
X9	M	0.5	0.5	-0.5	0	0	-1	0	0	1	0	
		5-2.5M	**3-3.5M**	**2-3.5M**	**M**	**M**	**M**	**0**	**0**	**0**	**-2500M**	

	X1	X2	X3	X4	X5	X6	X7	X8	X9	solution	ratio
X7	0.66	0.33	0	-1	0.33	0	1	-0.33	0	500	500/0.66
X3	0.33	0.66	1	0	-0.33	0	0	0.33	0	500	500/0.33
X9	0.66	0.83	0	0	-0.16	-1	0	0.16	1	250	250/0.66
	4.33-1.33M	**1.66-1.16M**	**0**	**M**	**0.66-0.16M**	**M**	**0**	**-0.66+1.16M**	**0**	**-1000-750M**	

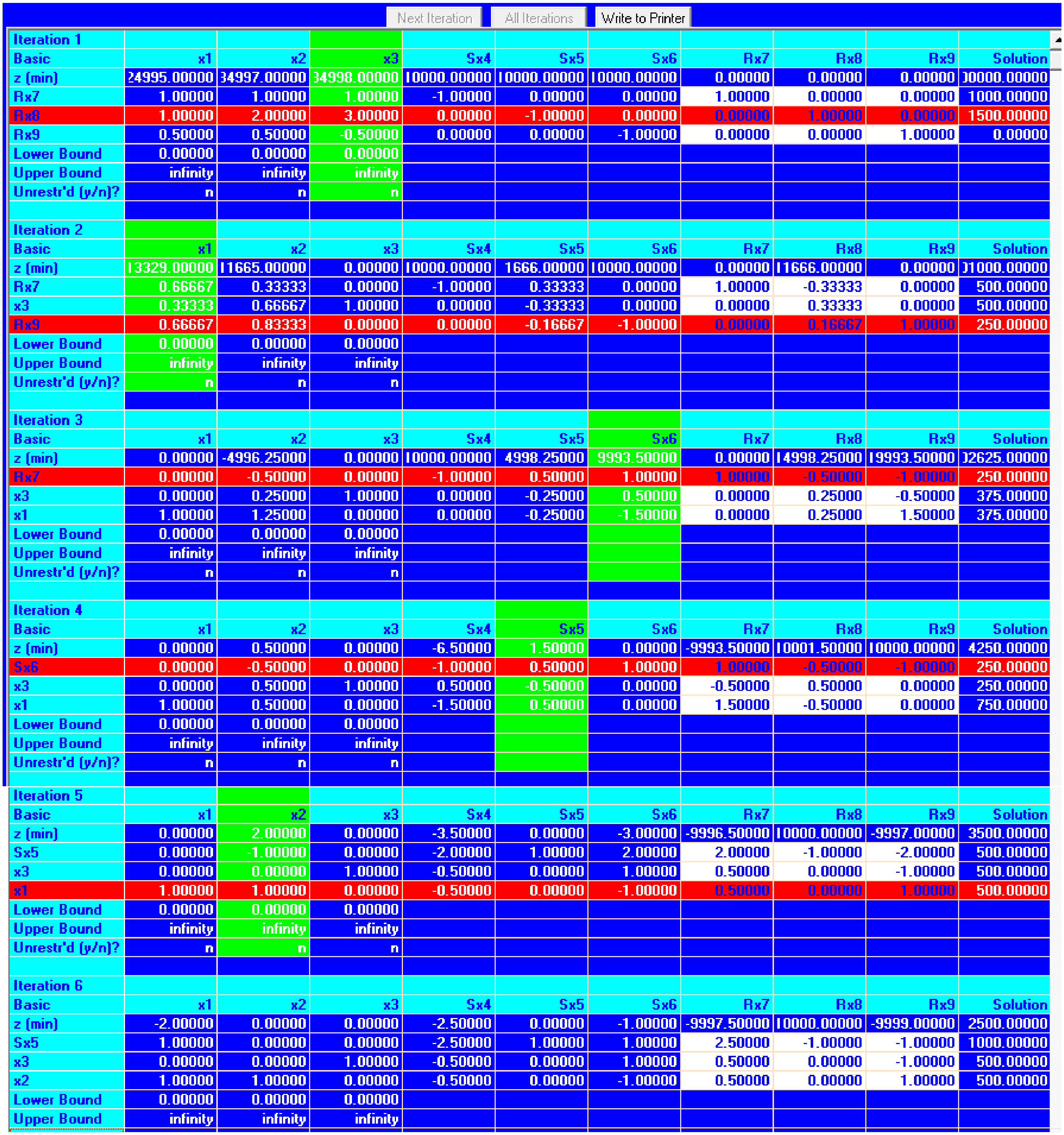
	X1	X2	X3	X4	X5	X6	X7	X8	X9	solution	ratio
X7	0	-0.5	0	-1	0.5	1	1	-0.5	-1	250	250/1
X3	0	0.25	1	0	-0.25	0.5	0	0.25	-0.5	375	375/0.5
X1	1	1.25	0	0	-0.25	-1.5	0	0.25	1.5	375	
	0	**-3.75+0.5M**	**0**	**M**	**1.75-0.5M**	**6.5-M**	**0**	**-1.75+1.5M**	**-6.5+2M**	**-2625-250M**	

	X1	X2	X3	X4	X5	X6	X7	X8	X9	solution	ratio
X6	0	-0.5	0	-1	0.5	1	1	-0.5	-1	250	250/0.5
X3	0	0.5	1	0.5	-0.5	0	-0.5	0.5	0	250	
X1	1	0.5	0	-1.5	0.5	0	1.5	-0.5	0	750	750/0.5
	0	**-0.5**	**0**	**6.5**	**-1.5**	**0**	**-6.5+M**	**1.5+M**	**M**	**-4250**	

	X1	X2	X3	X4	X5	X6	X7	X8	X9	solution	ratio
X5	0	-1	0	-2	1	2	2	-1	-2	500	
X3	0	0	1	-0.5	0	1	0.5	0	-1	500	
X1	1	1	0	-0.5	0	-1	0.5	0	1	500	
	0	**-2**	**0**	**3.5**	**0**	**3**	**-3.5+M**	**M**	**M-3**	**-3500**	

	X1	X2	X3	X4	X5	X6	X7	X8	X9	solution	ratio
X5	1	0	0								
X3	0	0	1	-0.5	0	1	0.5	0	-1	500	
X2	1	1	0	-0.5	0	-1	0.5	0	1	500	
	2	**0**	**0**	**2.5**	**0**	**1**	**-2.5+M**	**M**	**M-1**	**2500**	

The optimal solution is:
X1 = 0 m^3 , x2= 500 m^3 , x3 = 500 m^3 ,
The minimum cost for treated wastewater per day = 2500$



Iteration 1										
Basic	x1	x2	x3	Sx4	Sx5	Sx6	Rx7	Rx8	Rx9	Solution
z (min)	24995.00000	34997.00000	34998.00000	10000.00000	10000.00000	10000.00000	0.00000	0.00000	0.00000	)0000.00000
Rx7	1.00000	1.00000	1.00000	-1.00000	0.00000	0.00000	1.00000	0.00000	0.00000	1000.00000
Rx8	1.00000	2.00000	3.00000	0.00000	-1.00000	0.00000	0.00000	1.00000	0.00000	1500.00000
Rx9	0.50000	0.50000	-0.50000	0.00000	0.00000	-1.00000	0.00000	0.00000	1.00000	0.00000
Lower Bound	0.00000	0.00000	0.00000							
Upper Bound	infinity	infinity	infinity							
Unrestr'd (y/n)?	n	n	n							
Iteration 2										
Basic	x1	x2	x3	Sx4	Sx5	Sx6	Rx7	Rx8	Rx9	Solution
z (min)	13329.00000	11665.00000	0.00000	10000.00000	1666.00000	10000.00000	0.00000	11666.00000	0.00000	)1000.00000
Rx7	0.66667	0.33333	0.00000	-1.00000	0.33333	0.00000	1.00000	-0.33333	0.00000	500.00000
x3	0.33333	0.66667	1.00000	0.00000	-0.33333	0.00000	0.00000	0.33333	0.00000	500.00000
Rx9	0.66667	0.83333	0.00000	0.00000	-0.16667	-1.00000	0.00000	0.16667	1.00000	250.00000
Lower Bound	0.00000	0.00000	0.00000							
Upper Bound	infinity	infinity	infinity							
Unrestr'd (y/n)?	n	n	n							
Iteration 3										
Basic	x1	x2	x3	Sx4	Sx5	Sx6	Rx7	Rx8	Rx9	Solution
z (min)	0.00000	-4996.25000	0.00000	10000.00000	4998.25000	9993.50000	0.00000	14998.25000	19993.50000	)2625.00000
Rx7	0.00000	-0.50000	0.00000	-1.00000	0.50000	1.00000	1.00000	-0.50000	-1.00000	250.00000
x3	0.00000	0.25000	1.00000	0.00000	-0.25000	0.50000	0.00000	0.25000	-0.50000	375.00000
x1	1.00000	1.25000	0.00000	0.00000	-0.25000	-1.50000	0.00000	0.25000	1.50000	375.00000
Lower Bound	0.00000	0.00000	0.00000							
Upper Bound	infinity	infinity	infinity							
Unrestr'd (y/n)?	n	n	n							
Iteration 4										
Basic	x1	x2	x3	Sx4	Sx5	Sx6	Rx7	Rx8	Rx9	Solution
z (min)	0.00000	0.50000	0.00000	-6.50000	1.50000	0.00000	-9993.50000	10001.50000	10000.00000	4250.00000
Sx6	0.00000	-0.50000	0.00000	-1.00000	0.50000	1.00000	1.00000	-0.50000	-1.00000	250.00000
x3	0.00000	0.50000	1.00000	0.50000	-0.50000	0.00000	-0.50000	0.50000	0.00000	250.00000
x1	1.00000	0.50000	0.00000	-1.50000	0.50000	0.00000	1.50000	-0.50000	0.00000	750.00000
Lower Bound	0.00000	0.00000	0.00000							
Upper Bound	infinity	infinity	infinity							
Unrestr'd (y/n)?	n	n	n							
Iteration 5										
Basic	x1	x2	x3	Sx4	Sx5	Sx6	Rx7	Rx8	Rx9	Solution
z (min)	0.00000	2.00000	0.00000	-3.50000	0.00000	-3.00000	-9996.50000	10000.00000	-9997.00000	3500.00000
Sx5	0.00000	-1.00000	0.00000	-2.00000	1.00000	2.00000	2.00000	-1.00000	-2.00000	500.00000
x3	0.00000	0.00000	1.00000	-0.50000	0.00000	1.00000	0.50000	0.00000	-1.00000	500.00000
x1	1.00000	1.00000	0.00000	-0.50000	0.00000	-1.00000	0.50000	0.00000	1.00000	500.00000
Lower Bound	0.00000	0.00000	0.00000							
Upper Bound	infinity	infinity	infinity							
Unrestr'd (y/n)?	n	n	n							
Iteration 6										
Basic	x1	x2	x3	Sx4	Sx5	Sx6	Rx7	Rx8	Rx9	Solution
z (min)	-2.00000	0.00000	0.00000	-2.50000	0.00000	-1.00000	-9997.50000	10000.00000	-9999.00000	2500.00000
Sx5	1.00000	0.00000	0.00000	-2.50000	1.00000	1.00000	2.50000	-1.00000	-1.00000	1000.00000
x3	0.00000	0.00000	1.00000	-0.50000	0.00000	1.00000	0.50000	0.00000	-1.00000	500.00000
x2	1.00000	1.00000	0.00000	-0.50000	0.00000	-1.00000	0.50000	0.00000	1.00000	500.00000
Lower Bound	0.00000	0.00000	0.00000							
Upper Bound	infinity	infinity	infinity							

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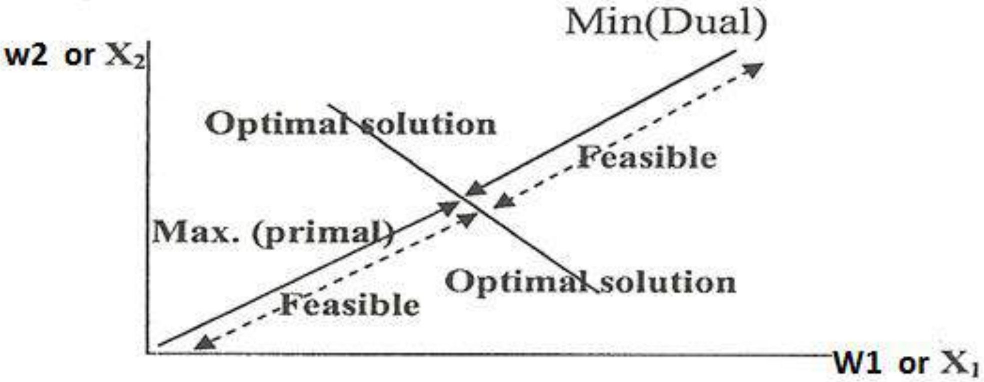
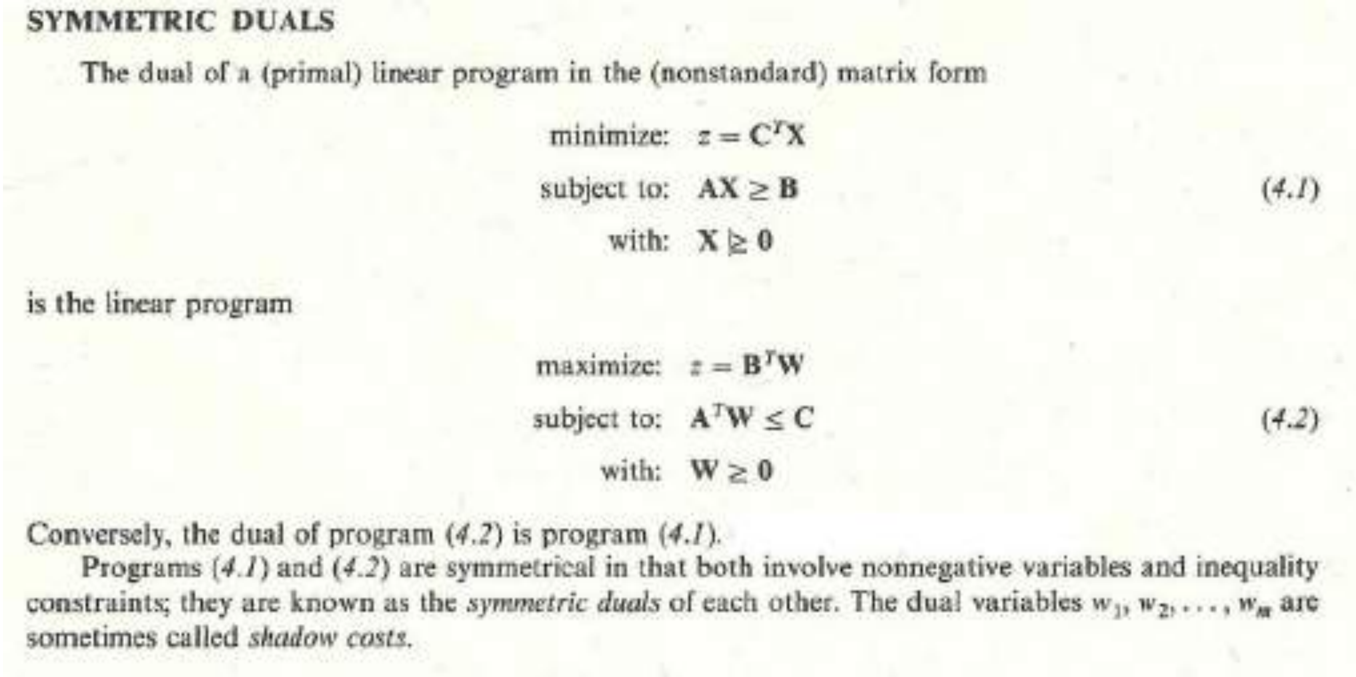
Linear Programming: Duality and Sensitivity Analysis

1- Duality

Every linear program in the variables X1, X 2 , .. . ,Xn has associated with it another linear program in the variables W1, W2, . .. , Wm (where *m* is the number of constraints in the original program), known as its *dual*. The original program, called the *primal*, completely determines the form of its dual.





SYMMETRIC DUALS

The dual of a (primal) linear program in the (nonstandard) matrix form

$$\begin{aligned} \text{minimize:} \quad & z = \mathbf{C}^T\mathbf{X} \\ \text{subject to:} \quad & \mathbf{AX} \geq \mathbf{B} \\ \text{with:} \quad & \mathbf{X} \geq \mathbf{0} \end{aligned} \tag{4.1}$$

is the linear program

$$\begin{aligned} \text{maximize:} \quad & z = \mathbf{B}^T\mathbf{W} \\ \text{subject to:} \quad & \mathbf{A}^T\mathbf{W} \leq \mathbf{C} \\ \text{with:} \quad & \mathbf{W} \geq \mathbf{0} \end{aligned} \tag{4.2}$$

Conversely, the dual of program (*4.2*) is program (*4.1*).

Programs (*4.1*) and (*4.2*) are symmetrical in that both involve nonnegative variables and inequality constraints; they are known as the *symmetric duals* of each other. The dual variables $w_1, w_2, \ldots, w_m$ are sometimes called *shadow costs*.

(Duality Theorem): If an optimal solution exists to either the primal or symmetric dual program, then the other program also has an optimal solution and the two objective functions have the same optimal value.

3-4 الخطوات العامة لتكوين المشكلة الثنائية (النموذج الثنائي المقابل) Dual

1. حدد متغير بديل غير سالب لكل قيد من قيود المشكلة الاولية Primal .
2. معاملات دالة الهدف في المشكلة الاولية تصبح ثوابت الطرف الايمن لقيود المسالة الثنائية.
3. ثوابت الطرف الايمن في المشكلة الاولية تصبح معاملات دالة الهدف في المشكلة الثنائية.
4. المبدلة (Transpose) لمصفوفة المعاملات الاولية تصبح مصفوفة المعاملات الثنائية.
5. تعكس اتجاه القيود في المشكلة الثنائية الى الاتجاه الاخر عندما كانت عليه القيود في المشكلة الاولية، فاذا كانت القيود مثلا من نوع اكبر او يساوي في المشكلة الاولية، فانها تعكس في المسالة الثنائية الى اقل او يساوي، و العكس بالعكس صحيح.
6. يعكس اتجاه دالة الهدف فاذا كان تعظيم .Max دالة الهدف في احد النموذجين فيقلب الى تصغير في النموذج الآخر او بالعكس.

ويمكن تشبيه النموذج المقابل (المسالة المقابلة، الثنائية) بأنه مقلوب النموذج الاولي (المسالة الاولية Primal) او بالعكس.

فاذا كانت الصيغة العامة للنموذج الاولي هي :

Primal model

Max. $Z = C1X1 + C2X2 + \ldots\ldots + CnXn$

Sub. To:

$a11\ x1 + a12\ x2 + a13\ x3 + \ldots\ldots. + a1n\ xn \leq b1$

$a21\ x1 + a22\ x2 + a23\ x3 + \ldots\ldots. + a2n\ xn \leq b2$

$.\quad .\quad .\quad .\quad .$

$.\quad .\quad .\quad .\quad .$

$am1\ x1 + am2\ x2 + am3\ x3 + \ldots\ldots. + amn\ xn \leq bm$

$x1, x2, x3, \ldots.., xn \geq 0$

Dual model

Min. $Z^* = b1w1 + b2w2 + \ldots\ldots\ldots + bm\ wm$,

Sub. To:

$a11\ w1 + a21\ w2 + a31\ w3 + \ldots\ldots. + am1\ wm \geq C1$

$a12\ w1 + a22\ w2 + a32\ w3 + \ldots\ldots. + am2\ wm \geq C2$

$.\quad .\quad .\quad .\quad .$

$.\quad .\quad .\quad .\quad ..$

$a1n\ w1 + a2n\ w2 + a3n\ w3 + \ldots\ldots. + amn\ wm \geq Cn$

$w1, w2, w3, \ldots.., wn \geq 0$

ملاحظة : لمعالجة القيود عندما تكون في حالة المساواة (بالقيود المكافئة لها)، اي يعبر عن كل قيد مساواة بقيدين احدهما اكبر او يساوي والأخر اقل او يساوي الطرف الايمن لقيد المساواة، وعد ذلك يصار الى تعديل جميع القيود ان تكون من نوع واحد (اي اما اكبر من او يساوي او اقل من او يساوي بضرب المختلف * -1).

Example : Determine the dual of the program

Max. $z = 5x1 + 10\ x2$

Sub. To:

$3\ x1 - 7\ x2 \leq 20$

$x1 + x2 \geq 2$

$4\ x1 + 8\ x2 = 30$

$X1, x2, x3 \geq 0$

Sol:

Primal	Dual
Max. $z = 5x1 + 10\ x2$ **Sub. To:** $3\ x1 - 7\ x2 \leq 20$ $- x1 - x2 \leq -2$ $4\ x1 + 8\ x2 \leq 30$ $- 4\ x1 - 8\ x2 \leq - 30$ $X1, x2, x3 \geq 0$	Min. $Z^* = 20\ w1 - 2\ w2 + 30\ w3 -30\ w4$ **Sub. To:** $3w1 - w2 +4\ w3 - 4w4 \geq 5$ $- 7w1 - w2 + 8w3 - 8\ w4 \geq 10$ $W1, w2, w3, w4 \geq 0$

2-3 لماذا يتم التحويل للنموذج الثنائي(المقابل) Dual:

من فوائد التحويل من النموذج الاولي Primal الى النموذج الثنائي Dual:

1. الحصول على نموذج يحتوي على عدد أقل من القيود، وبذلك سوف يختصر العمل الحسابي لجداول السمبلكس والوصول الى الحل الامثل، والحصول على نفس الحل الامثل سواء كان الحل للنموذج الاولي او الحل للنموذج الثنائي Dual.
2. للتخلص من الاشارة السالبة في الجانب الايمن (ان وجدت) اي عندما تكون المصادر ذات كميات سالبة وهو أهم ما يمكن الحصول عليه في حالة التحويل الى النموذج الثنائي.
3. لغرض التعرف على ابعاد المشكلة الاخرى (المشكلة الثنائية، البديلة) فاذا كان النموذج الاولي Primal وبصيغة الـ .Max اي المشكلة بالصيغة الربحية، فبامكاننا التعرف على النموذج الثنائي ويكون بصيغة الـ .Min وتمثيله للجانب الكلفوي (في نفس المشكلة)، ولنفس المشكلة المعبر عنها اولا بالصيغة الاولية Primal.

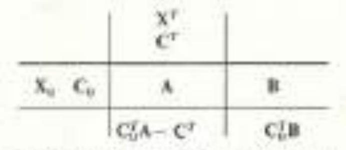
Example : Show that both primal and dual program in below problem have the same optimal value for Z, and that the solution of each is embedded in the final simplex tableau of the other.

Max. Z= 12x1 + 48 x2
Sub. To:
X1 +x2 ≤ 10
3x1 + 3x2 ≤ 24
4 x1 ≥ 8
X1, x2 ≥ 0

Sol:

1- To solve this program directly we need to add slack, surplus and artificial variables, and the application o two-phase or big-M method:

Max. Z= 12x1 + 48 x2 + 0x3 + 0x4 + 0 x5 – M x6
Sub. To:
X1 +x2 + x3 = 10
3x1 + 3x2 +x4 = 24
4 x1 -x5 + x6 = 8
X1, x2, x3, x4, x5, x6 ≥ 0



	X^T C^T	
X_0 C_0	A	B
	$C_0^T A - C^T$	$C_0^T B$

Tableau 3-1 For maximization programs

		X1	X2	X3	X4	X5	X6	solution	ratio
		12	48	0	0	0	-M		
X3	0	1	1	1	0	0	0	10	10/1
X4	0	3	3	0	1	0	0	24	24/1
X6	-M	4	0	0	0	-1	1	8	8/4
		-4M-12	**-48**	**0**	**0**	**M**	**0**	**-8M**	

	X1	X2	X3	X4	X5	X6	solution	ratio
X3	0	1	1	0	0.25	-0.25	8	8/1
X4	0	3	0	1	0.75	-0.75	18	18/3
X1	1	0	0	0	-0.25	0.25	2	

	0	**-48**	**0**	**0**	**-3**	**M+3**	**24**	
	X1	X2	X3	X4	X5	X6	solution	ratio
X3	0	0	1	-0.333	0	0	2	
X2	0	1	0	0.333	0.25	-0.25	6	
X1	1	0	0	0	0.25	-0.25	2	
Z	**0**	**0**	**0**	**16**	**9**	**M-9**	**312**	

Optimal solution: x1 = 2 , x2 = 6   , Z max = 312

ملاحظة : تكون قيمة الحل المقابل (dual) في السطر الاخير وتحت قيم المتغيرات الراكدة و الفائضة حيث ان:

W1 =0 , w2 = 16, w3 = 9

To solve the program by use dual program:

Primal	Dual
Max. Z= 12x1 + 48 x2	Min. Z* = 10 w1 + 24 w2 – 8 w3
Sub. To:	**Sub. To:**
X1 +x2 ≤ 10	w1 + 3 w2 - 4 w3 ≥ 12
3x1 + 3x ≤ 24	w1 + 3 w2 ≥ 48
- 4 x1 ≤ - 8	

To solve dual program we need to use either two phase or big M method,

Min. Z* = 10 w1 + 24 w2 – 8 w3

Sub. To:

w1 + 3 w2 - 4 w3 - w4 + w6 = 12

w1 + 3 w2 -w5 +w7 = 48

Min. Z* = 10 w1 + 24 w2 – 8 w3 +0 w4 + 0 w5 +M w6 + M w7

		W1	W2	W3	W4	W5	W6	W7	solution	ratio
		10	24	-8	0	0	M	M		
W6	M	1	3	-4	-1	0	1	0	12	12/3
W7	M	1	3	0	0	-1	0	1	48	48/3
		10-2M	**24-6M**	**-8+4M**	**M**	**M**	**0**	**0**	**-60M**	

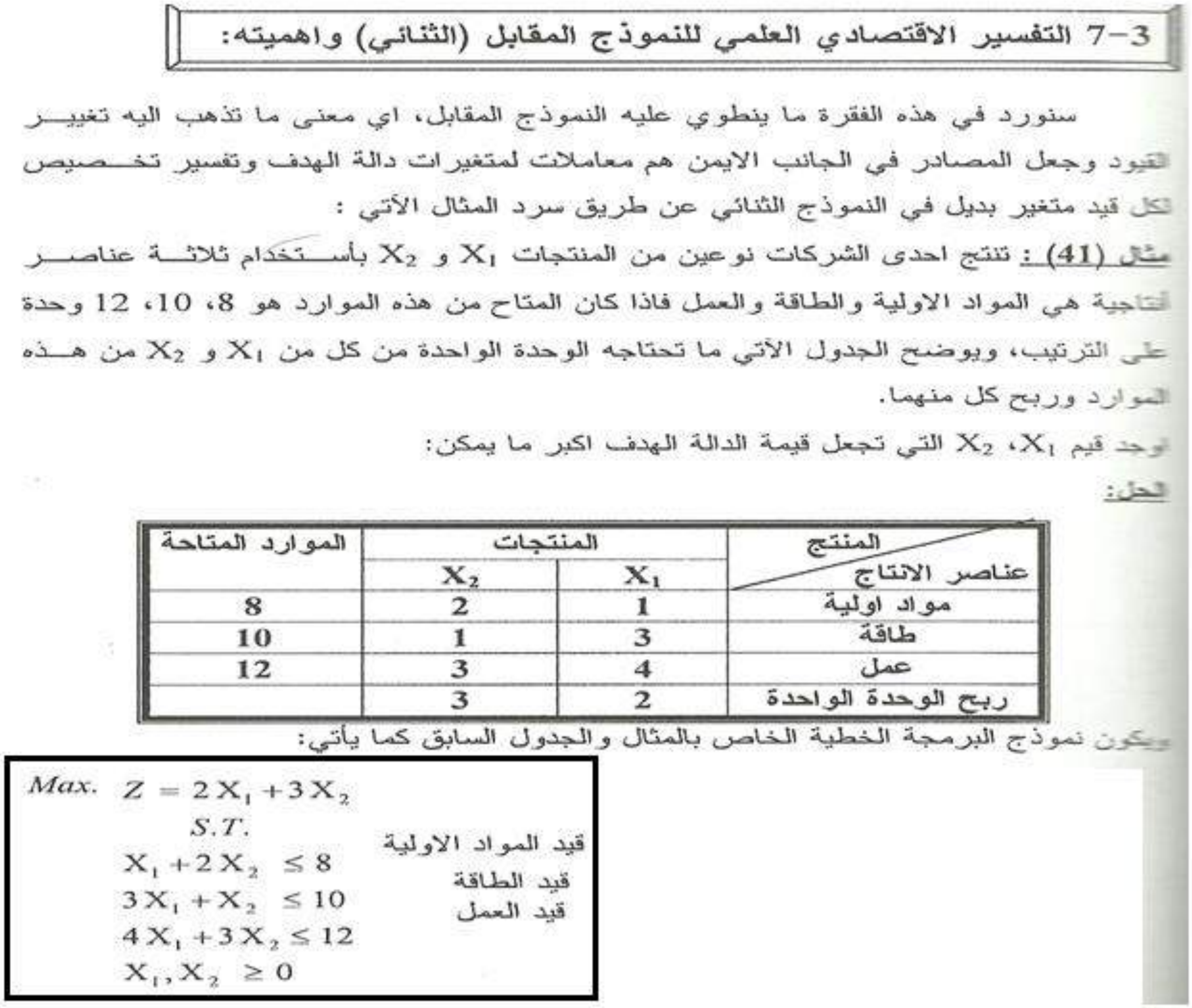
	W1	W2	W3	W4	W5	W6	W7	solution	ratio
W2	0.33	1	-1.33	-0.33	0	0.33	0	4	
W7	0	0	4	1	-1	-1	1	36	
	2	**0**	**24-4M**	**8-M**	**M**	**-8+2M**	**0**	**-96-36M**	

	W1	W2	W3	W4	W5	W6	W7	solution	ratio
W2	0	1	0	0	-0.33	0.33	0.33	16	
W3	0	0	1	0.25	-0.25	-0.25	0.25	9	
Z	**2**	**0**	**0**	**2**	**6**	**-2+M**	**-6+M**	**-312**	

Optimal solution: w1 = 0 , w2 = 16, w3 = 9 , Z *min. = - 312

ملاحظة : تكون قيمة الحل الاولي (primal) في السطر الاخير وتحت قيم المتغيرات الراكدة والفائضة حيث ان:

X1 =2 , X2 = 6, w3 = 9, Z max. = - (Z min.) = -(-312) = 312



7-3 التفسير الاقتصادي العلمي للنموذج المقابل (الثنائي) واهميته:

سنورد في هذه الفقرة ما ينطوي عليه النموذج المقابل، اي معنى ما تذهب اليه تغيير القيود وجعل المصادر في الجانب الايمن هم معاملات لمتغيرات دالة الهدف وتفسير تخصيص لكل قيد متغير بديل في النموذج الثنائي عن طريق سرد المثال الآتي :

مثال (41) : تنتج احدى الشركات نوعين من المنتجات X_1 و X_2 بأستخدام ثلاثة عناصر انتاجية هي المواد الاولية والطاقة والعمل فاذا كان المتاح من هذه الموارد هو 8، 10، 12 وحدة على الترتيب، ويوضح الجدول الآتي ما تحتاجه الوحدة الواحدة من كل من X_1 و X_2 من هذه الموارد وربح كل منهما.

وجد قيم X_1، X_2 التي تجعل قيمة الدالة الهدف اكبر ما يمكن:

الحل:

المنتج / عناصر الانتاج	المنتجات		الموارد المتاحة
	X_1	X_2	
مواد اولية	1	2	8
طاقة	3	1	10
عمل	4	3	12
ربح الوحدة الواحدة	2	3	

ويكون نموذج البرمجة الخطية الخاص بالمثال والجدول السابق كما يأتي:

$$Max.\ Z = 2X_1 + 3X_2$$

$$S.T.$$

$X_1 + 2X_2 \le 8$ قيد المواد الاولية

$3X_1 + X_2 \le 10$ قيد الطاقة

$4X_1 + 3X_2 \le 12$ قيد العمل

$X_1, X_2 \ge 0$

وحل هذا النموذج (الاولي Primal) يعطينا:

1. قيم X_1 و X_2 .
2. قيم Z المثلى التي تجعل قيمة الربح اكبر ما يمكن .
3. الوحدات غير المستغلة من المواد **X5, X4, X3**

ولكن هذا النموذج لايحدد لنا الآتي:

1. كلفة الوحدة الواحدة من X_1 و X_2.
2. الكلفة الكلية للانتاج.

ولذلك سوف نلجأ الى استخراج النموذج المقابل (الثنائي) ولذلك نفرض:

W1 : سعر الوحدة الواحدة من المواد الاولية

W2: سعر الوحدة الواحدة من الطاقة

W3: سعر الوحدة الواحدة من العمل

ويكون النموذج المقابل (الثنائي) كما يلي:

$$\text{Min. } Z^* = 8W1 + 10\,W2 + 12\,W3$$

S.T.

$$w1 + 3w2 + 4w3 \geq 2$$
$$2w1 + w2 + 3w3 \geq 3$$
$$w1, w2, w3 \geq 0$$

ويمكن التفسير لحدود دالة الهدف كما ياتي:

8w1 : كلفة المواد الاولية (اي حاصل ضرب سعر الوحدة الواحدة من المواد الاولية في كمية المواد الاولية المتوفرة).

10w2 : كلفة الطاقة (اي حاصل ضرب سعر الوحدة الواحدة من الطاقة في الكمية المتوفرة من الطاقة).

12w3 : كلفة العمل (اي حاصل ضرب سعر الوحدة الواحدة من العمل في كمية العمالة المتوفرة).

والمجموع لهذه الحدود الثلاثة يمثل الكلفة الكلية:

ولهذا نسعى الى تحقيق أقل كلفة للعملية الانتاجية (دالة الهدف للنموذج المقابل من نوع تدنية Min) مع تحقيق الارباح السابقة والتي حددت على وفق النموذج الاولي ودالة هدفه.

اما التفسير الاقتصادي لقيود النموذج الثنائي المقابل:

اولا: القيد الاول

w1 : تمثل كلفة المواد الاولية اللازمة لتصنيع الوحدة الواحدة من المنتوج X_1.

3w2 : تمثل كلفة الطاقة اللازمة لتصنيع وحدة واحدة من المنتوج X_1 .

4w3 : تمثل كلفة العمل اللازمة لتصنيع وحدة واحدة من المنتوج X_1.

والمجموع لهذه الحدود الثلاثة يمثل الكلفة الكلية اللازمة لتصنيع وحدة واحدة من المنتوج X_1 .

∴ القيد الاول هو كلفة انتاج وحدة واحدة من X_1 وواضح من القيد، ان الكلفة الكلية لتصنيع وحدة واحدة من X_1 يجب ان تساوي او بالحد الادتى لربح الوحدة الواحدة من المنتوج X_1 ومقداره (2).

ثانيا : القيد الثاني

2w1 : تمثل كلفة المواد الاولية اللازمة لتصنيع الوحدة الواحدة من المنتج X_2.

w2 : تمثل كلفة الطاقة اللازمة لتصنيع وحدة واحدة من المنتوج. X_2

3w3 : تمثل كلفة العمل اللازمة لتصنيع وحدة واحدة من المنتوج X_2 .

المجموع : هو الكلفة الكلية اللازمة لتصنيع وحدة واحدة من المنتوج X_2.

∴ القيد الثاني هو كلفة انتاج وحدة واحدة من X_2، وواضح من القيد ان الكلفة الكلية لتصنيع وحدة واحدة من X_2 يجب ان تساوي او بالحد الادنى لربح الوحدة الواحدة من المنتوج X_2 ومقداره (3).

Example :

$$\begin{aligned}
\text{minimize:}\quad & z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
\text{subject to:}\quad & x_1 + x_6 \geq 7 \\
& x_1 + x_2 \geq 20 \\
& x_2 + x_3 \geq 14 \\
& x_3 + x_4 \geq 20 \\
& x_4 + x_5 \geq 10 \\
& x_5 + x_6 \geq 5 \\
\text{with:}\quad & \text{all variables nonnegative}
\end{aligned}$$

To solve this program directly would require the introduction of 12 new variables, six surplus and six artificial, and the application of the two-phase method. A simpler approach is to consider the dual program:

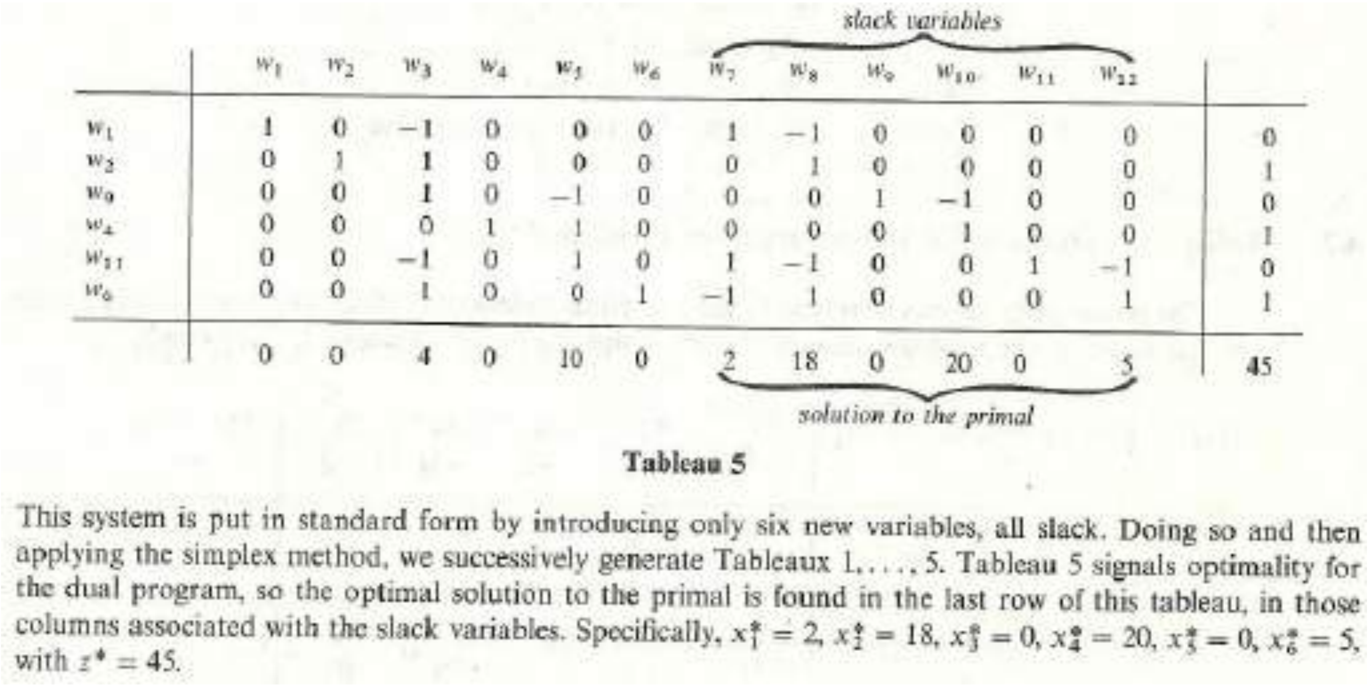
$$\begin{aligned}
\text{maximize:}\quad & z = 7w_1 + 20w_2 + 14w_3 + 20w_4 + 10w_5 + 5w_6 \\
\text{subject to:}\quad & w_1 + w_2 \leq 1 \\
& w_2 + w_3 \leq 1 \\
& w_3 + w_4 \leq 1 \\
& w_4 + w_5 \leq 1 \\
& w_5 + w_6 \leq 1 \\
& w_1 + w_6 \leq 1 \\
\text{with:}\quad & \text{all variables nonnegative}
\end{aligned}$$

		w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}	
		7	20	14	20	10	5	0	0	0	0	0	0	
w_7	0	1	1	0	0	0	0	1	0	0	0	0	0	1
w_8	0	0	1*	1	0	0	0	0	1	0	0	0	0	1
w_9	0	0	0	1	1	0	0	0	0	1	0	0	0	1
w_{10}	0	0	0	0	1	1	0	0	0	0	1	0	0	1
w_{11}	0	0	0	0	0	1	1	0	0	0	0	1	0	1
w_{12}	0	1	0	0	0	0	1	0	0	0	0	0	1	1
$(z_j - c_j)$:		−7	−20	−14	−20	−10	−5	0	0	0	0	0	0	0

Tableau 1

……………………………………………………………………………………………

…………………………………………………………………………………………..



slack variables (w_7 – w_{22})

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{22}	
w_1	1	0	−1	0	0	0	1	−1	0	0	0	0	0
w_2	0	1	1	0	0	0	0	1	0	0	0	0	1
w_9	0	0	1	0	−1	0	0	0	1	−1	0	0	0
w_4	0	0	0	1	1	0	0	0	0	1	0	0	1
w_{11}	0	0	−1	0	1	0	1	−1	0	0	1	−1	0
w_6	0	0	1	0	0	1	−1	1	0	0	0	1	1
	0	0	4	0	10	0	2	18	0	20	0	5	45

solution to the primal (w_7 – w_{22})

Tableau 5

This system is put in standard form by introducing only six new variables, all slack. Doing so and then applying the simplex method, we successively generate Tableaux 1, . . . , 5. Tableau 5 signals optimality for the dual program, so the optimal solution to the primal is found in the last row of this tableau, in those columns associated with the slack variables. Specifically, $x_1^* = 2$, $x_2^* = 18$, $x_3^* = 0$, $x_4^* = 20$, $x_5^* = 0$, $x_6^* = 5$, with $z^* = 45$.

Water Resources Management and Economy

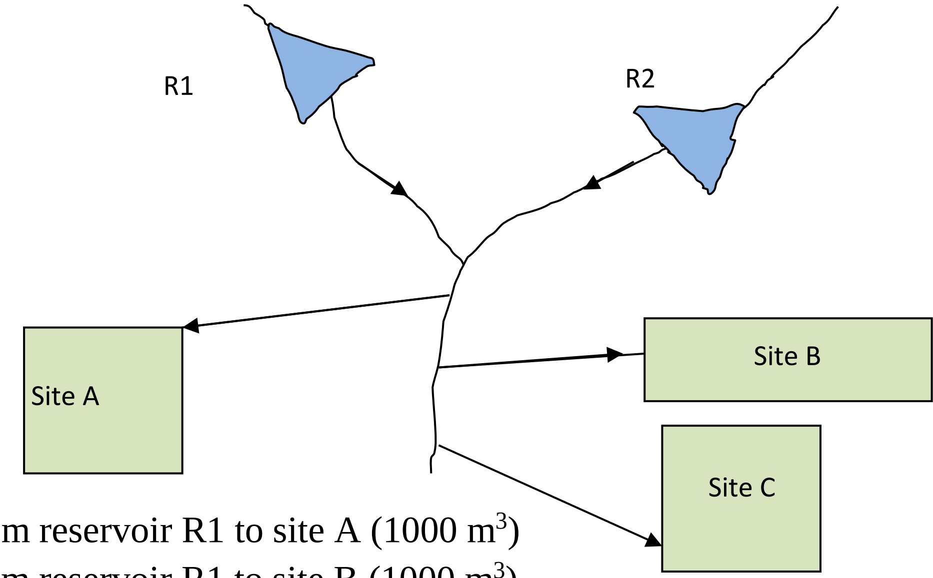
University of Anbar– College of Engineering

Dams & Water Resources Engineering

Department – 4th stage

Asst. Prof. Dr. Sadeq Oleiwi Sulaiman

Example : For a small irrigation project, two reservoirs (R1) and (R2) are available as shown in figure. The total volume of water per year that can be available from reservoir (R1) is (7,000,000 m^3) and from reservoir (R2) is (5,000,000 m^3). It is desire to convey at least (4,000,000 m^3) of water per year to sites (A), (2,000,000 m^3) of water per year to sites (B), and (6,000,000 m^3) of water per year to sites (C). If the cost for conveying each 1000 m^3 of water from reservoir (R1) to sites (A), (B) and (C) are 10$, 13$ and 15$ respectively, and from reservoir (R2) to sites (A), (B) and (C) are 14$, 11$ and 8$ respectively. Find the amount of water to be conveyed from each reservoir to minimize the total cost of conveyance of water. (use concept of duality method)



Sol:

Let :

X1= amount of water that convey from reservoir R1 to site A (1000 m^3)

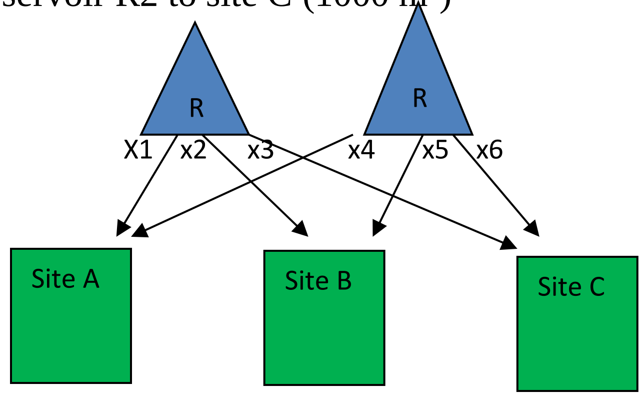
X2= amount of water that convey from reservoir R1 to site B (1000 m^3)

X3= amount of water that convey from reservoir R1 to site C (1000 m^3)

X4= amount of water that convey from reservoir R2 to site A (1000 m^3)

X5= amount of water that convey from reservoir R2 to site B (1000 m^3)

X6= amount of water that convey from reservoir R2 to site C (1000 m^3)



Obj. Fun.:

Min. $Z = 10x1 + 13x2 + 15x3 + 14x4 + 11x5 + 8x6$

Subj. to:

$X1 + x4 \geq 4000$

$X2 + x5 \geq 2000$

$X3 + x6 \geq 6000$

$X1 + x2 + x3 \leq 7000$ ………. ………. $-X1 - x2 - x3 \geq -7000$

$X4 + x5 + x6 \leq 5000$ ………. ……… $-X4 - x5 - x6 \geq -5000$

$X1, x2, x3, x4, x5, x6 \geq 0$

Convert to dual program

Max. $Z^* = 4000w1 + 2000w2 + 6000w3 - 7000w4 - 5000w5$

Sub. To:

$W1 - w4 \leq 10$

$W2 - w4 \leq 13$

$W3 - w4 \leq 15$

$W1 - w5 \leq 14$

$W2 - w5 \leq 11$

$W3 - w5 \leq 8$ $\qquad W1, w2, w3, w4, w5, w6 \geq 0$

Solve the dual program by using simplex method:

Maximize:

$Z^* = 4000w1 + 2000w2 + 6000w3 - 7000w4 - 5000w5 + 0w7 + 0w8 + 0w9 + 0w10 + 0w11 + 0w12$

Sub. To:

$W1 - w4 + w6 = 10$

$W2 - w4 + w7 = 13$

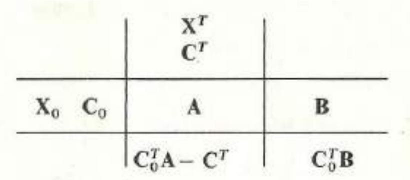
$W3 - w4 + w8 = 15$

$W1 - w5 + w9 = 14$

$W2 - w5 + w10 = 11$

$W3 - w5 + w11 = 8$

$W1, w2, w3, w4, w5, w6, W7, w8, w9, w10, w11 \geq 0$



	X^T C^T	
X_0 C_0	A	B
	$C_0^TA - C^T$	C_0^TB

Tableau 3-1 For maximization programs

---

		w1	w 2	w3	w4	w5	w6	w7	w8	w9	w10	w11		ratio
		4000	2000	6000	-7000	-5000	0	0	0	0	0	0		
W6	0	1	0	0	-1	0	1	0	0	0	0	0	**10**	
w7	0	0	1	0	-1	0	0	1	0	0	0	0	**13**	
w8	0	0	0	1	-1	0	0	0	1	0	0	0	**15**	15/1
w9	0	1	0	0	0	-1	0	0	0	1	0	0	**14**	
w10	0	0	1	0	0	-1	0	0	0	0	1	0	**11**	
w11	0	0	0	1	0	-1	0	0	0	0	0	1	**8**	8/1
		-4000	-2000	-6000	7000	5000	0	0	0	0	0	0	**0**	

	w1	w 2	w3	w4	w5	w6	w7	w8	w9	w10	w11		
W6	1	0	0	-1	0	1	0	0	0	0	0	**10**	
w7	0	1	0	-1	0	0	1	0	0	0	0	**13**	
w8	0	0	0	-1	1	0	0	1	0	0	-1	7	
w9	1	0	0	0	-1	0	0	0	1	0	0	**14**	
w10	0	1	0	0	-1	0	0	0	0	1	0	**11**	
W3	0	0	1	0	-1	0	0	0	0	0	1	**8**	
	-4000	-2000	0	7000	-1000	0	0	0	0	0	6000	**48000**	

	w1	w 2	w3	w4	w5	w6	w7	w8	w9	w10	w11		
W1	1	0	0	-1	0	1	0	0	0	0	0	10	
w7	0	1	0	-1	0	0	1	0	0	0	0	13	
w8	0	0	0	-1	1	0	0	1	0	0	-1	7	
w9	0	0	0	1	-1	-1	0	0	1	0	0	4	
w10	0	1	0	0	-1	0	0	0	0	1	0	11	
W3	0	0	1	0	-1	0	0	0	0	0	1	8	
	0	-2000	0	3000	-1000	4000	0	0	0	0	6000	88000	

	w1	w 2	w3	w4	w5	w6	w7	w8	w9	w10	w11		
W1	1	0	0	-1	0	1	0	0	0	0	0	10	
w7	0	0	0	-1	1	0	1	0	0	-1	0	2	
w8	0	0	0	-1	1	0	0	1	0	0	-1	7	
w9	0	0	0	1	-1	-1	0	0	1	0	0	4	
W2	0	1	0	0	-1	0	0	0	0	1	0	11	
W3	0	0	1	0	-1	0	0	0	0	0	1	8	
	0	0	0	3000	-3000	4000	0	0	0	2000	6000	**110000**	

	w1	w 2	w3	w4	w5	w6	w7	w8	w9	w10	w11		
W1	1	0	0	-1	0	1	0	0	0	0	0	10	
W5	0	0	0	-1	1	0	1	0	0	-1	0	2	
w8	0	0	0	0	0	0	-1	1	0	1	-1	5	
w9	0	0	0	0	0	-1	1	0	1	-1	0	6	
W2	0	1	0	-1	0	0	1	0	0	0	0	13	
W3	0	0	1	-1	0	0	1	0	0	-1	1	10	
	0	0	0	0	0	4000	3000	0	0	-1000	6000	**116000**	

	w1	w 2	w3	w4	w5	w6	w7	w8	w9	w10	w11		
W1	1	0	0	-1	0	1	0	0	0	0	0	10	
W5	0	0	0	-1	1	0	0	1	0	0	-1	7	
W10	0	0	0	0	0	0	-1	1	0	1	-1	5	
w9	0	0	0	0	0	-1	0	1	1	0	-1	11	
W2	0	1	0	-1	0	0	1	0	0	0	0	13	
W3	0	0	1	-1	0	0	0	1	0	0	0	15	
	0	0	0	0	0	4000	2000	1000	0	0	5000	**121000**	
						X1	X2	X3	X4	X5	X6	**Z**	

Amount of water that convey from reservoir R1 to site A = 4,000,000 m^3

Amount of water that convey from reservoir R1 to site B = 2,000,000 m^3

Amount of water that convey from reservoir R1 to site C = 1,000,000 m^3

Amount of water that convey from reservoir R2 to site A = 0 m^3

Amount of water that convey from reservoir R2 to site B = 0 m^3

Amount of water that convey from reservoir R2 to site C = 5,000,000 m^3

Total cost of conveyance of water per year = 121,000 $

Water Resources Management and Economy

University of Anbar– College of Engineering

Dams & Water Resources Engineering

Department – 4th stage

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Water Resources Management & Economy

4th stage – Dams & Water Resources Engineering Department – College of Engineering – University of Anbar - Iraq
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1-SENSITIVITY ANALYSIS

The scope of linear programming does not end at finding the optimal solution to the linear model of a real-life problem.
Sensitivity analysis of linear programming continues with the optimal solution to provide additional practical insight of the model.
Since this analysis examines how sensitive the optimal solution is to changes in the coefficients of the LP model, it is called sensitivity analysis. This process is also known as postoptimality analysis because it starts after the optimal solution is found.
Since we live in a dynamic world where changes occur constantly, this study of the effects on the solution due to changes in the data of a problem is very useful.

In general, we are interested in finding the effects of the following changes on the optimal LP solution:

(i) Changes in profit/unit or cost/unit (coefficients) of the objective function.

(ii) Changes in the availability of resources or capacities of production/service centers or limits on demands (requirements vector or RHS of constraints).

(iii) Changes in resource requirements/units of products or activities (technological coefficients of variables) in constraints.

(iv) Addition of a new product or activity (variable).

(v) Addition of a new constraint.

The sensitivity analysis will be discussed for linear programs of the form:
maximize: $z = C^T X$
subject to: $AX \leq B$
with: $X \geq 0$
where **X** is the column vector of unknowns; $\boldsymbol{C^T}$ is the row vector of the corresponding costs (cost vector);
A is the coefficient matrix of the constraints (matrix of technological coefficients); and
B is the column vector of the right-hand sides of the constraints (requirements vector).

To fix our ideas, the sensitivity analysis concepts will be exemplified through the following numerical problem: -

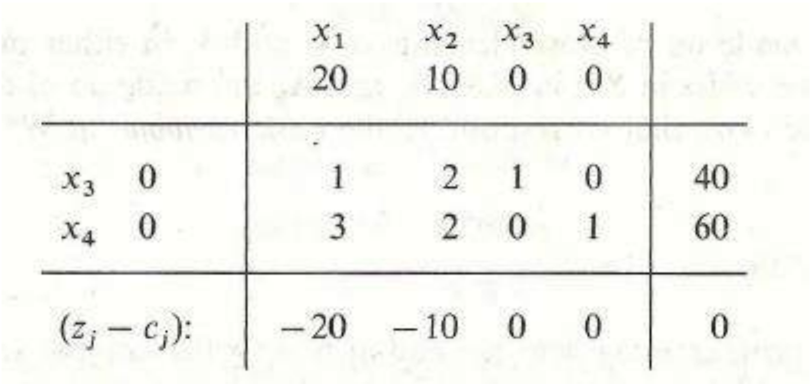
maximize: *Z* = *20* X1 + l0 X2
subject to: Xl + *2X2* ≤ 40
3xl + *2X2* ≤ 60
with: Xl and X2 nonnegative

This program is put into the following standard form by introducing the slack variables X3 and *X4 :*

maximize: Z = 20x l + l0x2 + 0x3 + 0x4
subject to: Xl + *2X2* + X3 = 40
3x 1 + *2X2* + X 4 = 60
with: all variables nonnegative

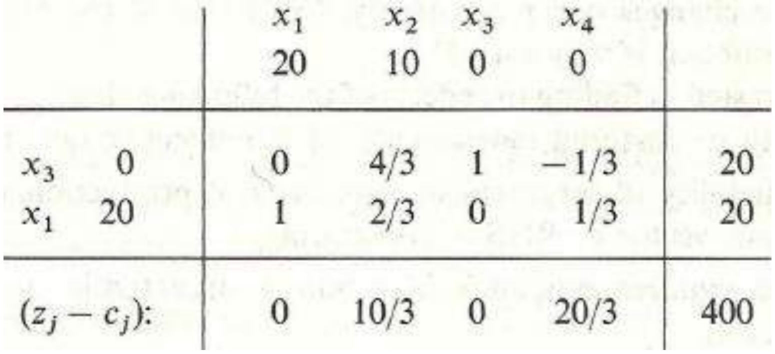
The solution for this problem is summarized as follows:

Initial Simplex Tableau:



		x_1	x_2	x_3	x_4	
		20	10	0	0	
x_3	0	1	2	1	0	40
x_4	0	3	2	0	1	60
$(z_j - c_j)$:		−20	−10	0	0	0

Final Simplex Tableau:



		x_1	x_2	x_3	x_4	
		20	10	0	0	
x_3	0	0	4/3	1	−1/3	20
x_1	20	1	2/3	0	1/3	20
$(z_j - c_j)$:		0	10/3	0	20/3	400

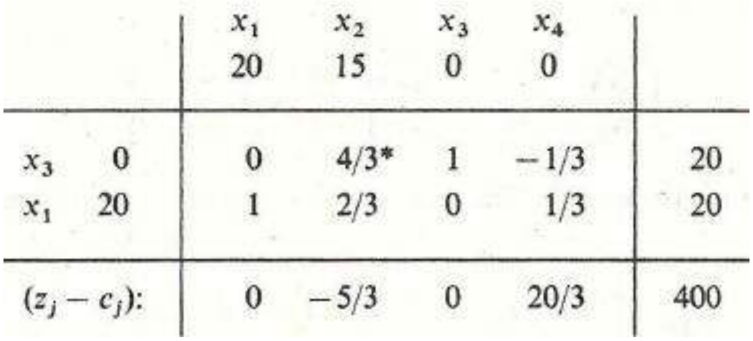
Since the last row of the above tableau contains no negative elements, the optional solution is x1 = 20, x2 = 0, with z* = 400.

For clarity of exposition, the five types of modifications are illustrated case by case below:

Example 4.1 Modification of the **cost vector** C^T

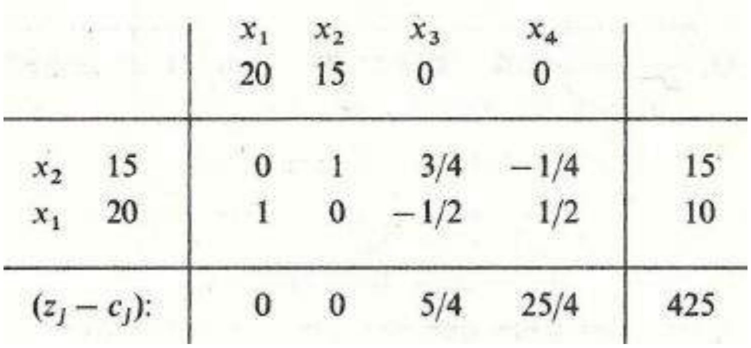
(a) Coefficients of the non basic variables
Let *the new* value of the cost coefficient corresponding to the nonbasic variable *X2* be 15 instead of 10.

The corresponding simplex tableau is



		x_1	x_2	x_3	x_4	
		20	15	0	0	
x_3	0	0	4/3*	1	−1/3	20
x_1	20	1	2/3	0	1/3	20
$(z_j - c_j)$:		0	−5/3	0	20/3	400

Since (Z2 - *C2)* < 0, the new solution is not optimal. The regular simplex method is used to reoptimize the problem, starting with *X 2* as the entering variable.
The new optimal tableau is:



		x_1	x_2	x_3	x_4	
		20	15	0	0	
x_2	15	0	1	3/4	−1/4	15
x_1	20	1	0	−1/2	1/2	10
$(z_j - c_j)$:		0	0	5/4	25/4	425

The optimal solution is x1 = 10, x2 = 15, with z* = 425.

(b) Coefficients of the basic variables.

Let the cost coefficient of the basic variable *X1* be changed from 20 to 10. Then the simplex tableau becomes:

		x_1	x_2	x_3	x_4	
		10	10	0	0	
x_3	0	0	4/3*	1	$-1/3$	20
x_1	10	1	2/3	0	1/3	20
$(z_j - c_j)$:		0	$-10/3$	0	10/3	200

Since *(Z2* – C2) < 0, the new solution is not optimal. The regular simplex method is resorted to for reoptimization, first by entering *X* 2

The new optimal tableau is

		x_1	x_2	x_3	x_4	
		10	10	0	0	
x_2	10	0	1	3/4	$-1/4$	15
x_1	10	1	0	$-1/2$	1/2	10
$(z_j - c_j)$:		0	0	5/2	5/2	250

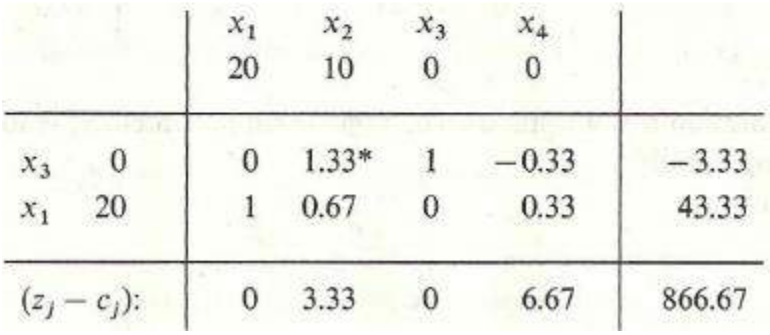
The optimal solution is x1 = 10, x2= 15, with z* = 250.

Example 4.2 Modification of the requirements **vector B**

Let the RHS of the second constraint be changed from 60 to 130.
Then $\mathbf{X_s = S^{-1} B}$ becomes

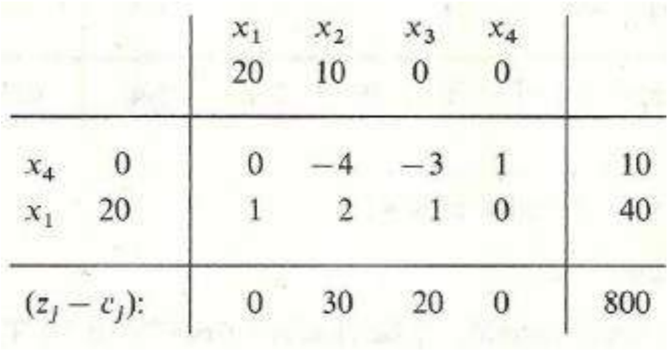
$$\begin{pmatrix} x3 \\ x1 \end{pmatrix} = S^{-1}\,\mathrm{B} = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 40 \\ 130 \end{pmatrix} = \begin{pmatrix} -3.33 \\ 43.33 \end{pmatrix}$$

Since X3 < 0, the new solution is not feasible. The **dual simplex method** used to clear the infeasibility starting with the following tableau:



		x_1	x_2	x_3	x_4	
		20	10	0	0	
x_3	0	0	1.33*	1	−0.33	−3.33
x_1	20	1	0.67	0	0.33	43.33
$(z_j - c_j)$:		0	3.33	0	6.67	866.67

The new final tableau is



		x_1	x_2	x_3	x_4	
		20	10	0	0	
x_4	0	0	−4	−3	1	10
x_1	20	1	2	1	0	40
$(z_j - c_j)$:		0	30	20	0	800

The optimal and feasible solution is x1 = 40, *x2* = 0, with z* = 800

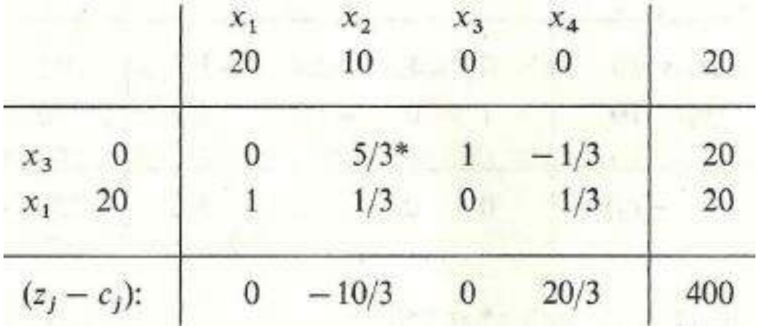
Example 4.3 Modification of the **matrix of coefficients A**

The problem becomes more complicated, when the technological coefficients of the basic variables are considered. This is because here the matrix under the starting solution changes. In this case, it may be easier to solve the new problem than resort to the sensitivity analysis approach. Therefore our analysis is limited to the case of the coefficients of nonbasic variables only.

Let the technological coefficients of X2 be changed from $(2,2)^T$ to $(2, 1)^T$.
Then the new technological coefficients of X2 in the optimal simplex tableau of the original primal problem are given by:

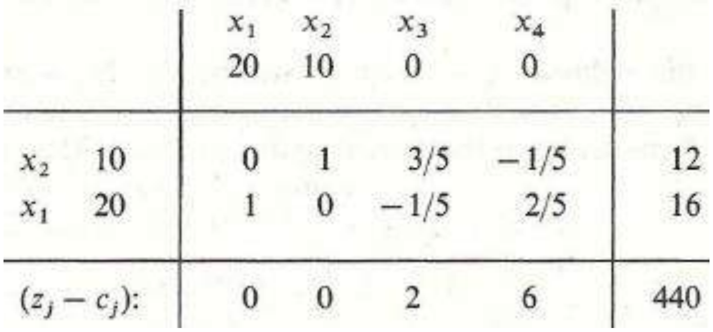
$$\boldsymbol{S^{-1}}\ \mathbf{p_2} = \begin{pmatrix} \mathbf{1} & \mathbf{-1/3} \\ \mathbf{0} & \mathbf{1/3} \end{pmatrix} \begin{pmatrix} \mathbf{2} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{5/3} \\ \mathbf{1/3} \end{pmatrix}$$

Hence the new simplex tableau becomes



		x_1	x_2	x_3	x_4	
		20	10	0	0	20
x_3	0	0	5/3*	1	−1/3	20
x_1	20	1	1/3	0	1/3	20
$(z_j - c_j)$:		0	−10/3	0	20/3	400

Since here (Z2 - C2) < 0, the new solution is not optimal. Again the regular simplex method is resorted to for reoptimization, first by entering X2.
The new optimal tableau is :



		x_1	x_2	x_3	x_4	
		20	10	0	0	
x_2	10	0	1	3/5	−1/5	12
x_1	20	1	0	−1/5	2/5	16
$(z_j - c_j)$:		0	0	2	6	440

The optimal solution is x1 = 16, x2 = 12, with z* = 440

Example 4.4 Addition of a variable

Let a new variable $\mathbf{X_k}$ be added to the original problem. This is accompanied by the addition to **A** of a column $P_k = (3, 1)^T$ and to C^T of a component $C_k = 30$. Thus the new problem becomes:

maximize: $Z = 20x1 + 10x2 + 30x_k$

subject to: $Xl + 2X2 + 3X_k \leq 40$
$3x\ 1 + 2X2 + X_k \leq 60$

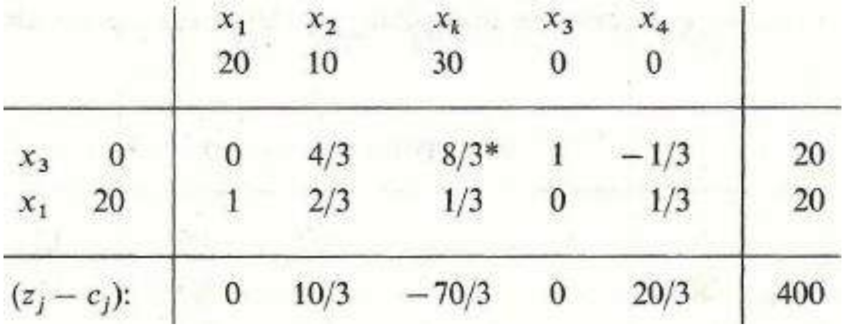
with: all variables nonnegative

Then the technological coefficients of $\mathbf{X_k}$ in the optimal tableau are:

$$S^{-1}\,\mathbf{p_K} = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8/3 \\ 1/3 \end{pmatrix}$$

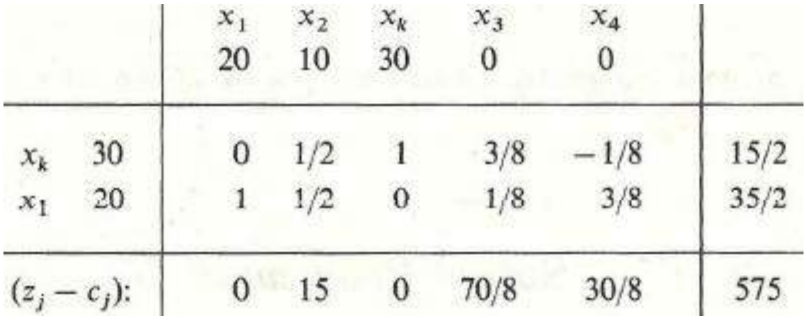
The corresponding $(Z_k - C_k) = (8/3)(0) + (1/3)(20) - 30 = 70/3$.

Thus the modified simplex tableau is:



		x_1	x_2	x_k	x_3	x_4	
		20	10	30	0	0	
x_3	0	0	4/3	8/3*	1	−1/3	20
x_1	20	1	2/3	1/3	0	1/3	20
$(z_j - c_j)$:		0	10/3	−70/3	0	20/3	400

Now entering the variable x_k, the regular simplex method is applied to obtain the following optimal tableau.



		x_1	x_2	x_k	x_3	x_4	
		20	10	30	0	0	
x_k	30	0	1/2	1	3/8	−1/8	15/2
x_1	20	1	1/2	0	1/8	3/8	35/2
$(z_j - c_j)$:		0	15	0	70/8	30/8	575

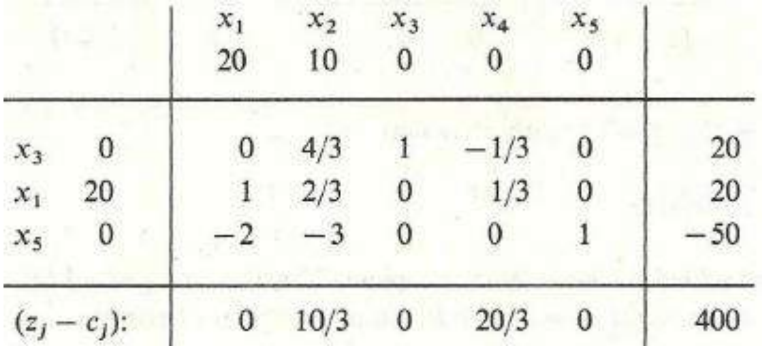
The optimal solution is $x1 = 17.5$, $x2 = 0$, $x_k = 7.5$, with $z^* = 575$.

Example 4.5 Addition of a constraint

If a new constraint added to the system is not active, it is called a secondary or redundant constraint, and the optimality of the problem remains unchanged. On the other hand, if the new constraint is active, the current optimal solution becomes infeasible.

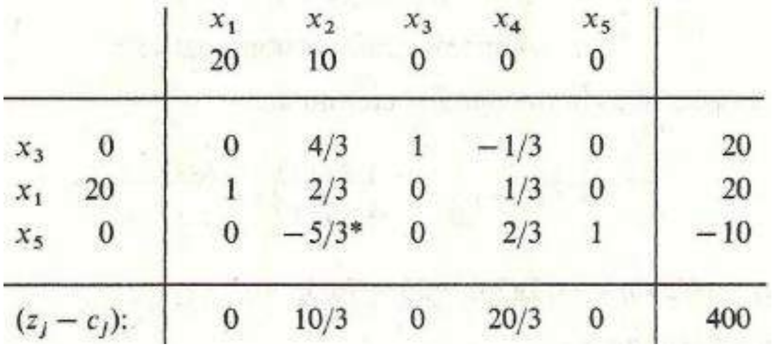
Let us consider the case of the addition of an active constraint, 2Xl + 3X2 ≥ 50 to the original problem.

The current optimal solution (x1 = 20, x2 = 0) does not satisfy the above new constraint and hence becomes infeasible. Therefore, add the new constraint to the current optimal tableau. The new slack variable is ***X5*** . The new simplex tableau is:



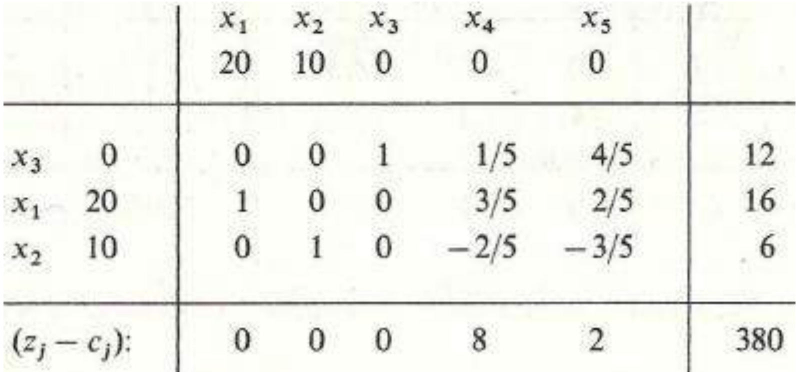
		x_1	x_2	x_3	x_4	x_5	
		20	10	0	0	0	
x_3	0	0	4/3	1	$-1/3$	0	20
x_1	20	1	2/3	0	1/3	0	20
x_5	0	-2	-3	0	0	1	-50
$(z_j - c_j)$:		0	10/3	0	20/3	0	400

By using the row operations, the coefficient of X1 in the new constraint is made zero. The modified tableau becomes



		x_1	x_2	x_3	x_4	x_5	
		20	10	0	0	0	
x_3	0	0	4/3	1	$-1/3$	0	20
x_1	20	1	2/3	0	1/3	0	20
x_5	0	0	$-5/3$*	0	2/3	1	-10
$(z_j - c_j)$:		0	10/3	0	20/3	0	400

The dual simplex method is used to overcome the infeasibility by departing the variable X5. The new tableau is:



		x_1	x_2	x_3	x_4	x_5	
		20	10	0	0	0	
x_3	0	0	0	1	1/5	4/5	12
x_1	20	1	0	0	3/5	2/5	16
x_2	10	0	1	0	$-2/5$	$-3/5$	6
$(z_j - c_j)$:		0	0	0	8	2	380

The tableau on the left gives the optimal and feasible solution as x1 = 16, x2 = 6, x3= 12, with $z^* = 380$

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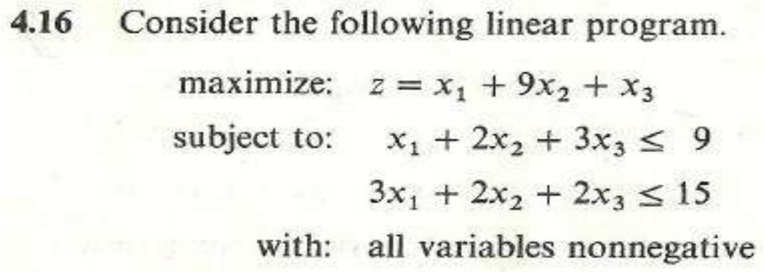
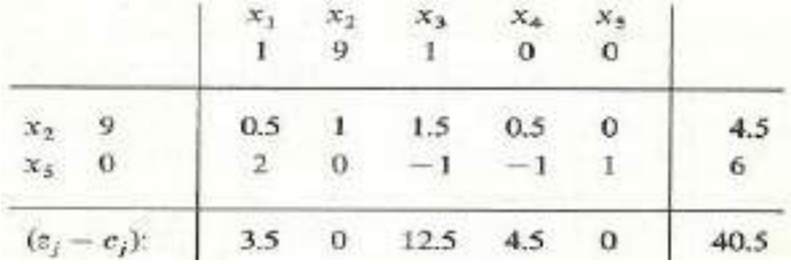
Solved Problems



4.16 Consider the following linear program.

maximize: $z = x_1 + 9x_2 + x_3$
subject to: $x_1 + 2x_2 + 3x_3 \le 9$
$3x_1 + 2x_2 + 2x_3 \le 15$
with: all variables nonnegative

The optimal simplex tableau for the standard form of the above problem (with slack variables X4 and X5 is:

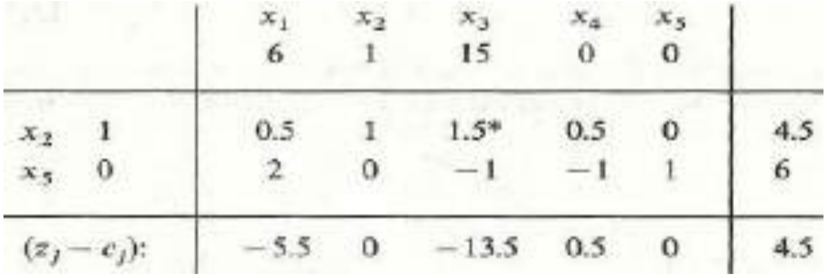


		x_1	x_2	x_3	x_4	x_5	
		1	9	1	0	0	
x_2	9	0.5	1	1.5	0.5	0	4.5
x_5	0	2	0	−1	−1	1	6
$(z_j - c_j)$:		3.5	0	12.5	4.5	0	40.5

If the new objective function is to maximize: Z = 6x1 + X2 + 15x3, find the new optimal solution by the sensitivity analysis approach.

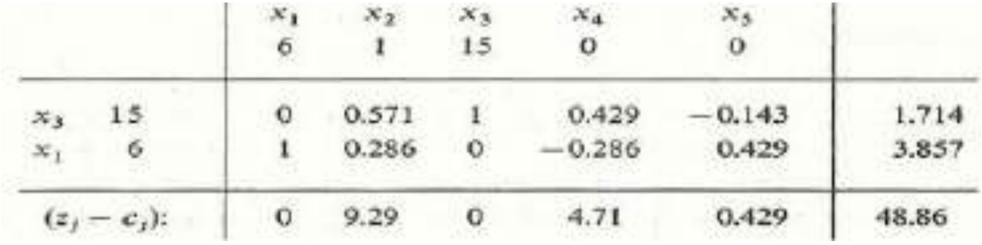
Sol:

The new simplex tableau becomes



		x_1	x_2	x_3	x_4	x_5	
		6	1	15	0	0	
x_2	1	0.5	1	1.5*	0.5	0	4.5
x_5	0	2	0	−1	−1	1	6
$(z_j - c_j)$:		−5.5	0	−13.5	0.5	0	4.5

Since not all (*Zj* - cj) values are nonnegative, the new solution is not optimal. The regular simplex method is used to reoptimize the problem, starting with X2 as the entering variable. The new optimal tableau is



		x_1	x_2	x_3	x_4	x_5	
		6	1	15	0	0	
x_3	15	0	0.571	1	0.429	−0.143	1.714
x_1	6	1	0.286	0	−0.286	0.429	3.857
$(z_j - c_j)$:		0	9.29	0	4.71	0.429	48.86

The optimal solution is x1 = 3.857, x2 = 0, x3 = 1.714, with z* = 48.86

4.18 The optimal solution to the standard form of the following LP problem

$$\begin{aligned} \text{maximize:} \quad & z = 35x_1 + 50x_2 \\ \text{subject to:} \quad & 4x_1 + 6x_2 \le 120 \\ & x_1 + x_2 \le 20 \\ & 2x_1 + 3x_2 \le 40 \\ \text{with:} \quad & x_1 \text{ and } x_2 \text{ nonnegative} \end{aligned}$$

is given below with slack variables x_3, x_4, and x_5:

		x_1	x_2	x_3	x_4	x_5	
		35	50	0	0	0	
x_3	0	0	0	1	0	−2	40
x_1	35	1	0	0	3	−1	20
x_2	50	0	1	0	−2	1	0
$(z_j - c_j)$:		0	0	0	5	15	700

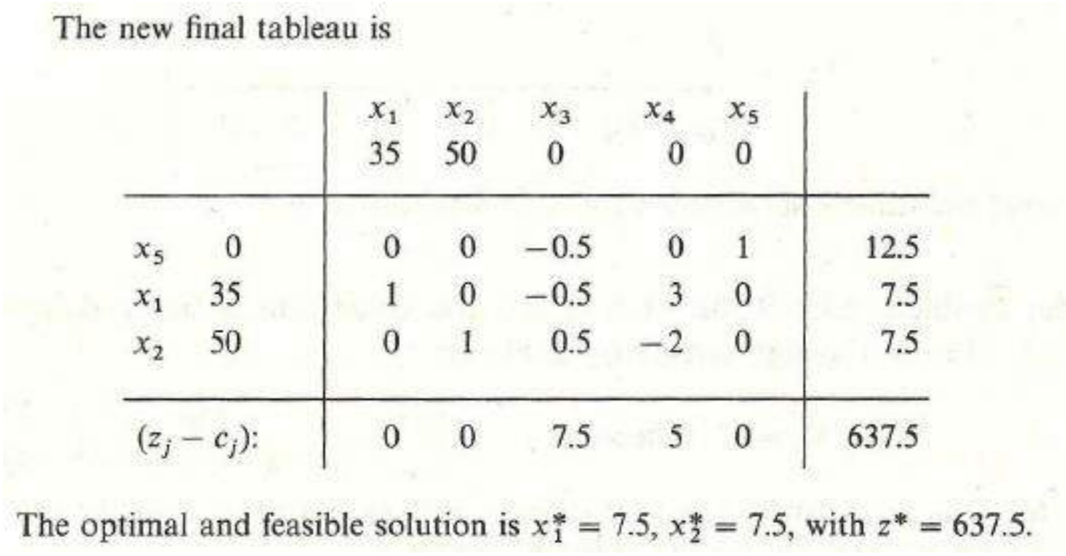
If the RHS of the constraints is changed from $(120, 20, 40)^T$ to $(75, 15, 50)^T$, find the new optimum solution by applying sensitivity analysis.

Sol:

$\mathbf{X_S} = \mathbf{S}^{-1}\mathbf{B}$ becomes
$$\begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 75 \\ 15 \\ 50 \end{pmatrix} = \begin{pmatrix} -25 \\ -10 \\ 25 \end{pmatrix}$$

Since x_1 and x_3 are negative, the new solution is not feasible. The dual simplex method is used to clear the infeasibility starting with the following tableau and departing the most negative variable x_3.

		x_1	x_2	x_3	x_4	x_5	
		35	50	0	0	0	
x_3	0	0	0	1	0	−2*	−25
x_1	35	1	0	0	3	−1	−10
x_2	50	0	1	0	−2	1	25
$(z_j - c_j)$:		0	0	0	5	15	900

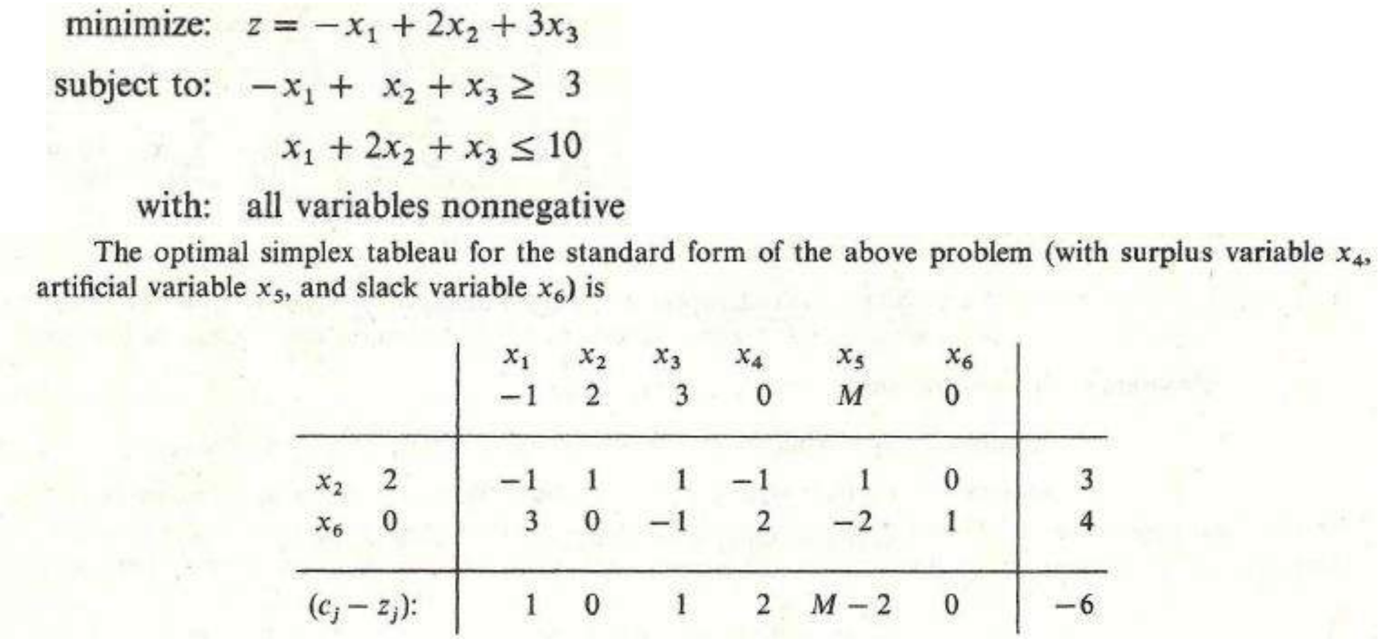


The new final tableau is

		x_1	x_2	x_3	x_4	x_5	
		35	50	0	0	0	
x_5	0	0	0	−0.5	0	1	12.5
x_1	35	1	0	−0.5	3	0	7.5
x_2	50	0	1	0.5	−2	0	7.5
$(z_j - c_j)$:		0	0	7.5	5	0	637.5

The optimal and feasible solution is $x_1^* = 7.5$, $x_2^* = 7.5$, with $z^* = 637.5$.

4.27 If a new constraint Xl + X3 ≥ 2 is added to the following linear program with following optimal solution, find the new optimum solution through sensitivity analysis.



minimize: $z = -x_1 + 2x_2 + 3x_3$

subject to: $-x_1 + x_2 + x_3 \geq 3$

$x_1 + 2x_2 + x_3 \leq 10$

with: all variables nonnegative

The optimal simplex tableau for the standard form of the above problem (with surplus variable x_4, artificial variable x_5, and slack variable x_6) is

		x_1	x_2	x_3	x_4	x_5	x_6	
		−1	2	3	0	M	0	
x_2	2	−1	1	1	−1	1	0	3
x_6	0	3	0	−1	2	−2	1	4
$(c_j - z_j)$:		1	0	1	2	$M - 2$	0	−6

Sol:

The current optimal solution (x1 = 0, x2 = 3) does not satisfy the new constraint and hence becomes infeasible. Add the new constraint to the current optimal tableau. The new slack variable is *X 7* and the new simplex tableau is:

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	
		-1	2	3	0	M	0	0	
x_2	2	-1	1	1	-1	1	0	0	3
x_6	0	3	0	-1	2	-2	1	0	4
x_7	0	-1	0	-1^*	0	0	0	1	-2
$(c_j - z_j)$:		1	0	1	2	$M-2$	0	0	-6

The dual simplex method is used to overcome the infeasibility by departing the variable x_7. The new tableau is

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	
		-1	2	3	0	M	0	0	
x_2	2	-2	1	0	-1	1	0	1	1
x_6	0	4	0	0	2	-2	1	-1	6
x_3	3	1	0	1	0	0	0	-1	2
$(c_j - z_j)$:		0	0	0	2	$M-2$	0	1	-8

The above tableau gives the optimal solution as $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 2$, with $z^* = 8$.

4.14 Consider the following linear programming (LP) problem.

$$\begin{aligned} \text{maximize:} \quad & z = 3x_1 + 2x_2 \\ \text{subject to:} \quad & 4x_1 + 3x_2 \leq 120 \\ & x_1 + 3x_2 \leq \ \ 60 \\ \text{with:} \quad & x_1 \text{ and } x_2 \text{ nonnegative} \end{aligned}$$

The optimal simplex tableau for the standard form of the above program (with slack variables x_3 and x_4) is

		x_1	x_2	x_3	x_4	
		3	2	0	0	
x_1	3	1	0.75	0.25	0	30
x_4	0	0	2.25	-0.25	1	30
$(z_j - c_j)$:		0	0.25	0.75	0	90

4.28 If a new constraint $x_1 \leq 25$ is added to Problem 4.14, find the new optimum solution through sensitivity analysis.

The current optimal solution ($x_1^* = 30$, $x_2^* = 0$) does not satisfy the new constraint and hence becomes infeasible. Add the new constraint to the current optimal tableau. The new slack variable is x_5 and the new simplex tableau is

		x_1	x_2	x_3	x_4	x_5	
		3	2	0	0	0	
x_1	3	1	0.75	0.25	0	0	30
x_4	0	0	2.25	−0.25	1	0	30
x_5	0	1	0	0	0	1	25
$(z_j - c_j)$:		0	0.25	0.75	0	0	90

By using the row operations, the coefficient of x_1 in the new constraint is made zero. The modified tableau becomes

		x_1	x_2	x_3	x_4	x_5	
		3	2	0	0	0	
x_1	3	1	0.75	0.25	0	0	30
x_4	0	0	2.25	−0.25	1	0	30
x_5	0	0	−0.75*	−2.25	0	1	−5
$(z_j - c_j)$:		0	0.25	0.75	0	0	90

The dual simplex method is used to overcome the infeasibility by departing the variable x_5.

The new tableau is

		x_1	x_2	x_3	x_4	x_5	
		3	2	0	0	0	
x_1	3	1	0	0	0	1	25
x_4	0	0	0	−1	1	3	15
x_2	2	0	1	0.33	0	−1.33	6.67
$(z_j - c_j)$:		0	0	0.67	0	0.33	88.33

The above tableau gives the optimal and feasible solution as $x_1^* = 25$, $x_2^* = 6.67$, with $z^* = 88.33$.

Water Resources Management and Economy

University of Anbar– College of Engineering

Dams & Water Resources Engineering Department – 4th stage

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Linear Programming: Extensions

THE REVISED SIMPLEX METHOD

Consider the following linear programming problem in standard matrix form:

Maximize: $Z = C^T X$

Subject to: $\mathbf{AX} = B$

With: $X \geq 0$

Where **X** is the column vector of unknowns, including all slack, surplus, and artificial variables**;** $\mathbf{C^T}$ is the row vector of corresponding costs; **A** is the coefficient matrix of the constraint equations; and **B** is the column vector of the right-hand side of the constraint equations. They are represented as follows:

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix}, \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Let Xs = the column vector of basic variables, C_S^T = the row vector of costs corresponding to Xs, and

S = the basis matrix corresponding to Xs.

STEP 1: ENTERING VECTOR P_k :

For every nonbasic vector P_j, calculate the coefficient

$Z_j - C_j = WP_j - c_j$ (maximization program), or

$C_j - Z_j = C_j - WP_j$ (minimization program), where $\mathbf{W} = \boldsymbol{C_S^T}\ \mathbf{S^{-1}}$**.**

The nonbasic vector P_j with the most negative coefficient becomes the entering vector (E.V.), P_k .

If more than one candidate for E.V. exists, choose one.

STEP 2: DEPARTING VECTOR P_r :

(a) Calculate the current basis X_s: $X_s = S^{-1} B$

(b) Corresponding to the entering vector P_k, calculate the constraint coefficients $\mathbf{t_k}$:

$\mathbf{t_k = S^{-1} P_k}$

(c) Calculate the ratio θ:

$$\theta = \min_i \left\{ \frac{(\mathbf{X}_S)_i}{t_{ik}}, t_{ik} > 0 \right\}, i = 1, 2, \ldots, m$$

The departing vector (D.V.), P_r, is the one that satisfies the above condition.

NOTE: If all $t_{ik} \leq 0$, there is no bounded solution for the problem. Stop.

STEP 3: NEW BASIS:

$$\mathbf{S}_{\text{new}}^{-1} = \mathbf{E}\mathbf{S}^{-1}, \text{ where } \mathbf{E} = (\mathbf{u}_1, \ldots, \mathbf{u}_{r-1}, \eta, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_m)$$

Note,

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{pmatrix}, \text{ where } \eta_i = \begin{Bmatrix} -\dfrac{t_{ik}}{t_{rk}}, \text{ if } i \neq r \\ \dfrac{1}{t_{rk}}, \text{ if } i = r \end{Bmatrix}$$

and $\mathbf{u}_i$ is a column vector with 1 in the ith element and 0 in the other (m - 1) elements.

Set $\mathbf{S}^{-1} = \mathbf{S}_{\text{new}}^{-1}$ and repeat steps 1 through 3, until the following optimality condition is satisfied.

$\mathbf{Zj - cj \geq 0}$ (maximization problem), or

$\mathbf{cj - Zj \geq 0}$ (minimization problem)

Then the optimal solution is as follows:

$$\mathbf{Xs = S^{-1} B}; \quad \mathbf{Z} = C_S^T \ \mathbf{Xs}$$

طريقة السمبلكس المعدلة (المحورة) Revised Simplex Method:

لطريقة السمبلكس الاعتيادية التي اكتشفها العالم الرياضي الانكليزي دانتزك Dantizge عام 1947 الفضل في ايجاد الحلول المثلى لنماذج البرمجة الخطية وعن طريق ايجاد الحل الممكن ومن ثم الانتقال من جدول للسمبلكس الى جدول آخر افضل من الجدول السابق وصولا الى الحل الامثل، ولكن الذي يؤخذ على هذه الطريقة (على سبيل النقد) هو انها تستغرق وقت طويل، وذلك لأستخراج قيم يمكن تجاوزها او الاستغناء عنها، او تكرار لقيم بعض المتغيرات التي لا يحتاجها الحل بها ومما يستدعي انه يستغرق وقتا اطول حتى لو تم استخدام الحاسبة فهناك وقت يهدر يمكن الاستفادة منه وفي حالة الاستعاضة عن طريقة السمبلكس بطريقة السمبلكس المحورة او المعدلة Revised وهنا يتم الاعتماد على استخدام المصفوفات و المتجهات.

في طريقة السمبلكس المحورة نتبع نفس العمليات التي تجري في السمبلكس الاعتيادية ولكن في كل محاولة Iteration اي اعادة جدول السمبلكس (في حالة دخول متغير وخروج متغير) لايحسب كل الجدول الذي سبق وان تم حسابه في السمبلكس الاعتيادية عند التحويل من جدول الى جدول لاحق اخر. وهنا يمكن انجاز وحساب طريقة السمبلكس المحورة من المعادلات الاصلية للنموذج اي عن طريق استخدام المصفوفات والمتجهات وكما يأتي:

①

أسلوب الحل باستخدام الطريقة المحورة Revised simplex method

1- من منطوق السؤال نوجد دالة الهدف و القيود

2- نحول المتباينات الى معادلات باضافة متغيرات زائده وناقصه وصناعيه كل حسب حاجته ثم نعيد كتابة دالة الهدف ونحدد عمود لثوابت كل متغير (P_i) في القيود مثل $P_1 = \begin{pmatrix}1\\4\end{pmatrix}$, $P_2 = \begin{pmatrix}4\\8\end{pmatrix}$ .... وهكذا

3- نقوم بعملية التهيئة initialization للحل بمعرفة (X_s) والذي يمثل صف المتغيرات صاحبة الحل الاساسي (Basic solution variables) وهي في الخطوة الاولى المتغيرات الزائده والمتغيرات الصناعيه مثل $X_s = (x_3, x_4, x_5)$ وكذلك نحدد قيمة (C_s^t) وتمثل قيم معاملات متغيرات ال X_s في دالة الهدف. ونحدد ايضا قيمة S و S^{-1} التي تمثل مصفوفة ثوابت متغيرات الحل الاساسي ومعكوسها.

4- نقوم بالمحاولة الاولى (Iteration 1) للحل وكما يلي:

Ⓐ لتحديد العمود (المتجه Vector) الداخل من متجها = الحل غير الاساسي (Non-basic solution variables) نوجد قيمة $(Z_j - C_j)$ في حالة كون الدالة Max و $C_j - Z_j$ في حالة كونها Min وكما يلي: مثلاً لل Max

$$Z_j - C_j = C_s^t S^{-1} (P_j) - C_j = \text{مثلاً} = (C_3\ C_4\ C_5)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{bmatrix}P_1 & P_2\end{bmatrix} - \begin{bmatrix}C_1 & C_2\end{bmatrix}$$

اعمدة ثوابت متغيرات الحل غير الاساسي

صف ثوابت متغيرات الحل الاساسي في دالة الهدف

ويكون العمود الداخل Entering Vector (EV) او P_k هو عمود العنصر صاحب اعلى قيمة سالبه من العملية اعلاه حيث ان k تمثل رقم العمود الداخل

Ⓑ لتحديد الصف (المتجه) الخارج (Departing Vector) $(P_r\ ,\ DV)$ من متجها = الحل الاساسي نوجد قيمة:

$$\theta = \min\left(\frac{X_{s_i}}{t_{ik}}\right) \quad t_{ik} > 0 \quad , \quad X_{s_i} = S^{-1} B$$

$$t_i k = S^{-1} P_k$$

تمثل عمود ثوابت العمود الداخل الذي وجدناه بالخطوه السابقه

(ج) ويكون المتجه (الصف) الخارج هو صاحب أقل قيمة موجبة لـ θ مثل

الذي يمثل العنصر

$$\theta = \min \left(\begin{array}{ccc} P_3 & P_4 & P_5 \\ 4, & \frac{1}{4}, & \frac{5}{3} \end{array} \right).$$

((وإذا كانت كل قيم θ سالبة يتوقف الحل ونوجد الحل الأمثل)

(٤) لتحديد مصفوفة الحل الأساسي الجديد (S^{-1}_{new}) نوجد قيمة η حيث أن

$$\eta = \begin{pmatrix} \frac{-t_{ik}}{t_{rk}} \\ \frac{1}{t_{rk}} \end{pmatrix} \begin{array}{l} i \neq k \\ i = r \end{array}$$

حيث أن i تمثل رقم متغير الحل الأساسي
K تمثل رقم المتغير الداخل
r تمثل رقم المتغير الخارج

$$E = (u_1 + u_2 , u_3 \; \dots u_{r-1} \; \eta \; \dots u_{r+1} \dots u_m)$$

حيث أن u_i تمثل عمود متجه بقيمة 1 عند العنصر i وصفر عند بقية العناصر

مثلاً

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 1 \end{pmatrix} = (u_1 \; \eta \; u_2)$$

لو فرضنا

$$\eta = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

تكون مصفوفة الحل الأساسي الجديد:

$$S^{-1}_{new} = E S^{-1}$$

(5) بعد إيجاد مصفوفة الحل الأساسي الجديد نوجد X_s وتمثل متغيرات الحل الأساسي الجديد بعد إدخال المتغير الداخل E_v ومكان المتغير الخارج D_v ثم C_s^T وتمثل ثوابت متغيرات الحل الأساسي الجديد في دالة الهدف ونعيد الخطوة الرابعة بعمل محاولة ثانية Iteration ② وهكذا إلى أن نصل للحل الأمثل.

(6) بعد الوصول للحل الأمثل نوجد قيمة متغيرات الحل الأساسي وكما يلي:

$$X_s = \begin{pmatrix} x_4 \\ x_1 \\ \vdots \end{pmatrix} = S^{-1} B = \begin{pmatrix} \vdots \end{pmatrix}$$

$$Z = C_s^T X_s = (\cdot)$$

Solved Problems

5.1 Use the revised simplex method to solve the following problem.
Maximize : Z = l0 X1 + 11 X2
Subject to: Xl + 2X2 ≤ 150
3x l + $4X2$ ≤ 200
6x l + x2 ≤ 175
With: X1 and $X\ 2$ nonnegative

Sol:

This program is put in standard form by introducing the slack variables $X3$, $X4$, and $X5$,

$$\begin{aligned}
\text{maximize:}\quad & z = 10x_1 + 11x_2 + 0x_3 + 0x_4 + 0x_5 \\
\text{subject to:}\quad & x_1 + 2x_2 + x_3 \qquad\qquad\quad = 150 \\
& 3x_1 + 4x_2 \qquad + x_4 \qquad = 200 \\
& 6x_1 + x_2 \qquad\qquad\quad + x_5 = 175 \\
\text{with:}\quad & \text{all variables nonnegative}
\end{aligned}$$

$$\mathbf{P}_1 = \begin{pmatrix}1\\3\\6\end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix}2\\4\\1\end{pmatrix}, \mathbf{P}_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{P}_4 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{P}_5 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \mathbf{B} = \begin{pmatrix}150\\200\\175\end{pmatrix}$$

Initialization:

$$\mathbf{X}_S = (x_3, x_4, x_5)^T;\ \mathbf{C}_S^T = (0, 0, 0)$$

$$\mathbf{S} = (\mathbf{P}_3, \mathbf{P}_4, \mathbf{P}_5) = \mathbf{I} = \mathbf{S}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Iteration No.1:

The nonbasic vectors are $\mathbf{P}_1$ and $\mathbf{P}_2$.

(*a*) Entering Vector:

$$\mathbf{W} = \mathbf{C}_S^T\mathbf{S}^{-1} = (0, 0, 0)\mathbf{I} = (0, 0, 0)$$

$$(z_1 - c_1, z_2 - c_2) = \mathbf{W}(\mathbf{P}_1, \mathbf{P}_2) - (c_1, c_2) = (0, 0, 0)\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 6 & 1 \end{pmatrix} - (10, 11) = (-10, -11)$$

Since the most negative coefficient corresponds to $\mathbf{P}_2$, it becomes the entering vector (**E.V.**).

(*b*) Departing Vector:

$$\mathbf{X}_S = \mathbf{S}^{-1}\mathbf{B} = \mathbf{IB} = \mathbf{B} = \begin{pmatrix} 150 \\ 200 \\ 175 \end{pmatrix}$$

$$\mathbf{t}_2 = \mathbf{S}^{-1}\mathbf{P}_2 = \mathbf{IP}_2 = \mathbf{P}_2 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$\theta = \min\left\{\frac{150}{2}, \frac{200}{4}, \frac{175}{1}\right\} = 50$$

Since the minimum ratio corresponds to $\mathbf{P}_4$, it becomes the departing vector (**D.V.**).

(*c*) New Basis:

$$\eta = \begin{pmatrix} \dfrac{-t_{32}}{t_{42}} \\ \dfrac{1}{t_{42}} \\ \dfrac{-t_{52}}{t_{42}} \end{pmatrix} = \begin{pmatrix} -2/4 \\ 1/4 \\ -1/4 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/4 \\ -1/4 \end{pmatrix}; \qquad \mathbf{E} = (\mathbf{u}_1, \eta, \mathbf{u}_3)$$

$$\mathbf{S}_{new}^{-1} = \mathbf{ES}^{-1} = \mathbf{EI} = \mathbf{E} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1/4 & 1 \end{pmatrix}$$

Summary of Iteration No. 1:

$$\mathbf{X}_S = (x_3, x_2, x_5)^T;\quad \mathbf{C}_S^T = (0, 11, 0)$$

Iteration No. 2:
Now the nonbasic vectors are $\mathbf{P}_1$ and $\mathbf{P}_4$.

(*a*) Entering Vector:

$$\mathbf{W} = \mathbf{C}_S^T\mathbf{S}^{-1} = (0, 11, 0)\begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1/4 & 1 \end{pmatrix} = (0, 11/4, 0)$$

$$(z_1 - c_1, z_4 - c_4) = \mathbf{W}(\mathbf{P}_1, \mathbf{P}_4) - (c_1, c_4) = (0, 11/4, 0)\begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 6 & 0 \end{pmatrix} - (10, 0) = (-7/4, 11/4)$$

Since the most negative coefficient corresponds to $\mathbf{P}_1$, it becomes the entering vector (**E.V.**).

(*b*) Departing Vector:

$$\mathbf{X}_S = \mathbf{S}^{-1}\mathbf{B} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1/4 & 1 \end{pmatrix}\begin{pmatrix} 150 \\ 200 \\ 175 \end{pmatrix} = \begin{pmatrix} 50 \\ 50 \\ 125 \end{pmatrix}$$

$$t_1 = \mathbf{S}^{-1}\mathbf{P}_1 = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1/4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 3/4 \\ 21/4 \end{pmatrix}$$

$$\theta = \min\left\{-, \frac{50}{3/4}, \frac{125}{21/4}\right\} = 500/21$$

Since the minimum ratio corresponds to $\mathbf{P}_5$, it becomes the departing vector (**D.V.**).

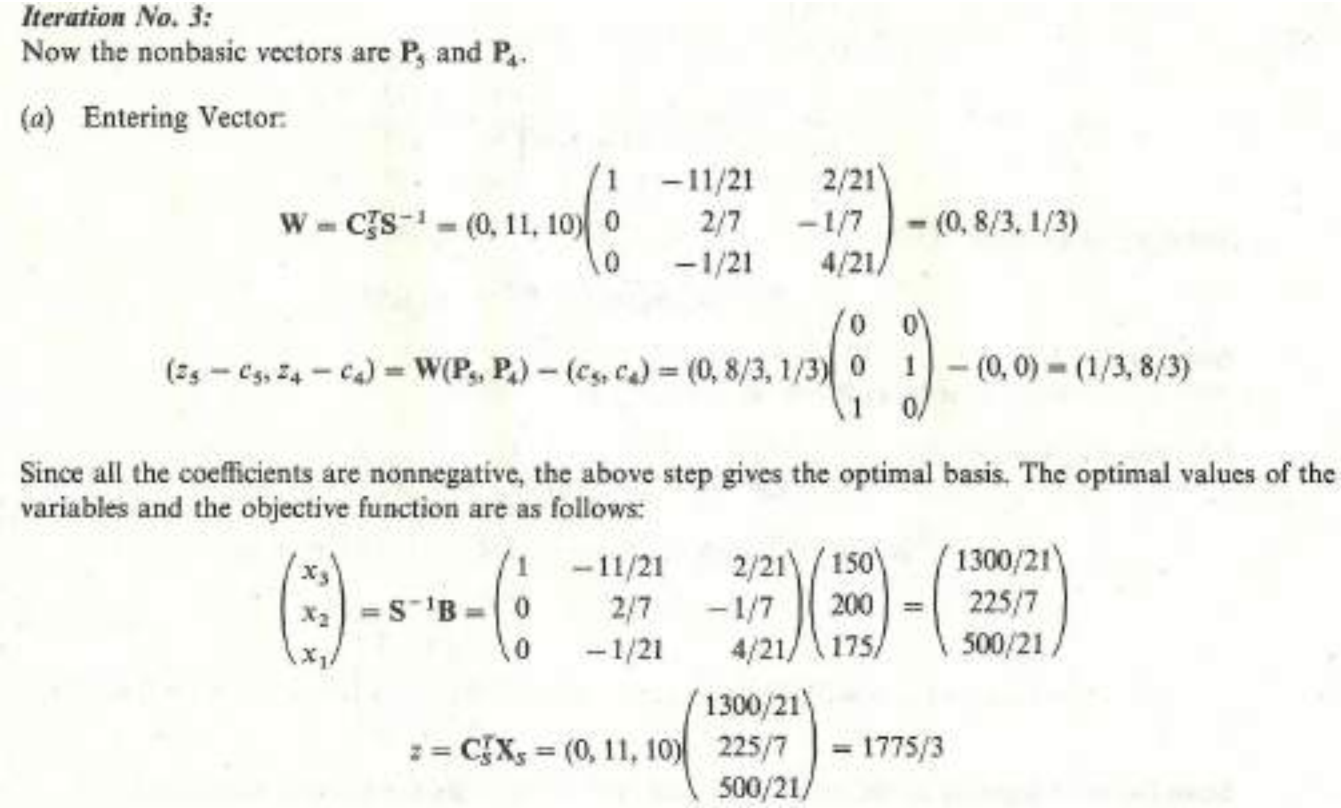
(c) New Basis:

$$\eta = \begin{pmatrix} \dfrac{-t_{31}}{t_{51}} \\ \dfrac{-t_{21}}{t_{51}} \\ \dfrac{1}{t_{51}} \end{pmatrix} = \begin{pmatrix} -\dfrac{-1/2}{21/4} \\ -\dfrac{3/4}{21/4} \\ \dfrac{1}{21/4} \end{pmatrix} = \begin{pmatrix} 2/21 \\ -1/7 \\ 4/21 \end{pmatrix}; \qquad \mathbf{E} = (\mathbf{u}_1, \mathbf{u}_2, \eta)$$

$$\mathbf{S}_{\text{new}}^{-1} = \mathbf{E}\mathbf{S}^{-1} = \begin{pmatrix} 1 & 0 & 2/21 \\ 0 & 1 & -1/7 \\ 0 & 0 & 4/21 \end{pmatrix}\begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1/4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -11/21 & 2/21 \\ 0 & 2/7 & -1/7 \\ 0 & -1/21 & 4/21 \end{pmatrix}$$

Summary of Iteration No. 2:

$$\mathbf{X}_S = (x_3, x_2, x_1)^T; \qquad \mathbf{C}_S^T = (0, 11, 10)$$



Iteration No. 3:
Now the nonbasic vectors are $\mathbf{P}_5$ and $\mathbf{P}_4$.

(*a*) Entering Vector:

$$\mathbf{W} = \mathbf{C}_S^T\mathbf{S}^{-1} = (0, 11, 10)\begin{pmatrix} 1 & -11/21 & 2/21 \\ 0 & 2/7 & -1/7 \\ 0 & -1/21 & 4/21 \end{pmatrix} = (0, 8/3, 1/3)$$

$$(z_5 - c_5, z_4 - c_4) = \mathbf{W}(\mathbf{P}_5, \mathbf{P}_4) - (c_5, c_4) = (0, 8/3, 1/3)\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} - (0, 0) = (1/3, 8/3)$$

Since all the coefficients are nonnegative, the above step gives the optimal basis. The optimal values of the variables and the objective function are as follows:

$$\begin{pmatrix} x_5 \\ x_2 \\ x_1 \end{pmatrix} = \mathbf{S}^{-1}\mathbf{B} = \begin{pmatrix} 1 & -11/21 & 2/21 \\ 0 & 2/7 & -1/7 \\ 0 & -1/21 & 4/21 \end{pmatrix}\begin{pmatrix} 150 \\ 200 \\ 175 \end{pmatrix} = \begin{pmatrix} 1300/21 \\ 225/7 \\ 500/21 \end{pmatrix}$$

$$z = \mathbf{C}_S^T\mathbf{X}_S = (0, 11, 10)\begin{pmatrix} 1300/21 \\ 225/7 \\ 500/21 \end{pmatrix} = 1775/3$$

Water Resources Management and Economy

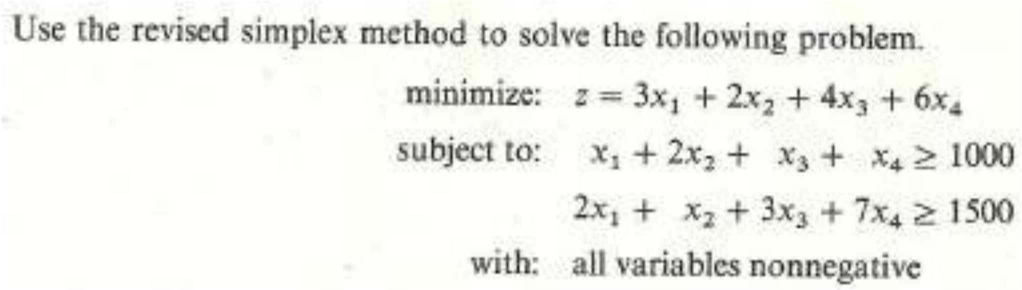
University of Anbar– College of Engineering

Dams & Water Resources Engineering

Department – 4th stage

Asst. Prof. Dr. Sadeq Oleiwi Sulaiman

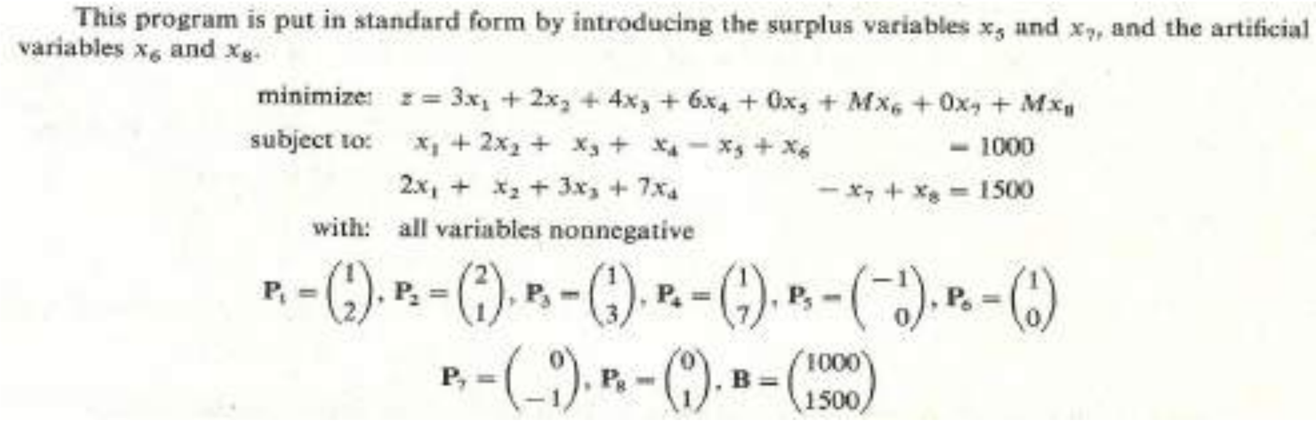
Example 2:



Use the revised simplex method to solve the following problem.

$$\begin{aligned}\text{minimize:}\quad & z = 3x_1 + 2x_2 + 4x_3 + 6x_4\\ \text{subject to:}\quad & x_1 + 2x_2 + x_3 + x_4 \geq 1000\\ & 2x_1 + x_2 + 3x_3 + 7x_4 \geq 1500\\ \text{with:}\quad & \text{all variables nonnegative}\end{aligned}$$

Sol:



This program is put in standard form by introducing the surplus variables x_5 and x_7, and the artificial variables x_6 and x_8.

$$\begin{aligned}\text{minimize:}\quad & z = 3x_1 + 2x_2 + 4x_3 + 6x_4 + 0x_5 + Mx_6 + 0x_7 + Mx_8\\ \text{subject to:}\quad & x_1 + 2x_2 + x_3 + x_4 - x_5 + x_6 = 1000\\ & 2x_1 + x_2 + 3x_3 + 7x_4 - x_7 + x_8 = 1500\\ \text{with:}\quad & \text{all variables nonnegative}\end{aligned}$$

$$\mathbf{P}_1 = \begin{pmatrix}1\\2\end{pmatrix},\ \mathbf{P}_2 = \begin{pmatrix}2\\1\end{pmatrix},\ \mathbf{P}_3 = \begin{pmatrix}1\\3\end{pmatrix},\ \mathbf{P}_4 = \begin{pmatrix}1\\7\end{pmatrix},\ \mathbf{P}_5 = \begin{pmatrix}-1\\0\end{pmatrix},\ \mathbf{P}_6 = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\mathbf{P}_7 = \begin{pmatrix}0\\-1\end{pmatrix},\ \mathbf{P}_8 = \begin{pmatrix}0\\1\end{pmatrix},\ \mathbf{B} = \begin{pmatrix}1000\\1500\end{pmatrix}$$

Initialization:

$$\mathbf{X}_S = (x_6, x_8)^T; \qquad \mathbf{C}_S^T = (M, M)$$

$$\mathbf{S} = (\mathbf{P}_6, \mathbf{P}_8) = \mathbf{I} = \mathbf{S}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Iteration No. 1:

The nonbasic vectors are $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_4$, $\mathbf{P}_5$, and $\mathbf{P}_7$.

(*a*) Entering Vector:

$$\mathbf{W} = \mathbf{C}_S^T\mathbf{S}^{-1} = (M, M)\mathbf{I} = (M, M)$$

$$\begin{aligned}(c_1 - z_1, c_2 - z_2, c_3 - z_3, c_4 - z_4, c_5 - z_5, c_7 - z_7) &= (c_1, c_2, c_3, c_4, c_5, c_7) - \mathbf{W}(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4, \mathbf{P}_5, \mathbf{P}_7) \\ &= (3, 2, 4, 6, 0, 0) - (M, M)\begin{pmatrix} 1 & 2 & 1 & 1 & -1 & 0 \\ 2 & 1 & 3 & 7 & 0 & -1 \end{pmatrix} \\ &= (-3M + 3, -3M + 2, -4M + 4, -8M + 6, M, M)\end{aligned}$$

Since the most negative coefficient corresponds to $\mathbf{P}_4$, it becomes the entering vector (**E.V.**).

(*b*) Departing Vector:

$$\mathbf{X}_S = \mathbf{S}^{-1}\mathbf{B} = \mathbf{IB} = \mathbf{B} = \begin{pmatrix} 1000 \\ 1500 \end{pmatrix}; \qquad t_4 = \mathbf{S}^{-1}\mathbf{P}_4 = \mathbf{IP}_4 = \mathbf{P}_4 = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

$$\theta = \min\{1000, 1500/7\} = 1500/7$$

Since the minimum ratio corresponds to $\mathbf{P}_8$, it becomes the departing vector (**D.V.**).

(c) New Basis:

$$\boldsymbol{\eta} = \begin{pmatrix} \dfrac{-t_{64}}{t_{84}} \\ \dfrac{1}{t_{84}} \end{pmatrix} = \begin{pmatrix} -1/7 \\ 1/7 \end{pmatrix}; \qquad \mathbf{E} = (\mathbf{u}_1, \boldsymbol{\eta})$$

$$\mathbf{S}_{\text{new}}^{-1} = \mathbf{ES}^{-1} = \mathbf{EI} = \mathbf{E} = \begin{pmatrix} 1 & -1/7 \\ 0 & 1/7 \end{pmatrix}$$

Summary of Iteration No. 1:

$$\mathbf{X}_S = (x_6, x_4)^T; \qquad \mathbf{C}_S^T = (M, 6)$$

Iteration No. 2:

Now the nonbasic vectors are $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_8$, $\mathbf{P}_5$, and $\mathbf{P}_7$.

(*a*) Entering Vector:

$$\mathbf{W} = \mathbf{C}_S^T\mathbf{S}^{-1} = (M, 6)\begin{pmatrix} 1 & -1/7 \\ 0 & 1/7 \end{pmatrix} = (M, 6/7 - M/7)$$

$$\begin{aligned}(c_1 - z_1, c_2 - z_2, c_3 - z_3, c_8 - z_8, c_5 - z_5, c_7 - z_7) &= (c_1, c_2, c_3, c_8, c_5, c_7) - \mathbf{W}(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_8, \mathbf{P}_5, \mathbf{P}_7) \\ &= (3, 2, 4, M, 0, 0) - (M, 6/7 - M/7) \\ &\quad \times \begin{pmatrix} 1 & 2 & 1 & 0 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & -1 \end{pmatrix} \\ &= (-5M/7 + 9/7, -13M/7 + 8/7, -4M/7 + 10/7, \\ &\quad 8M/7 - 6/7, M, -M/7 + 6/7)\end{aligned}$$

Since the most negative coefficient corresponds to $\mathbf{P}_2$, it becomes the entering vector (**E.V.**).

(*b*) Departing Vector:

$$\mathbf{X}_S = \mathbf{S}^{-1}\mathbf{B} = \begin{pmatrix} 1 & -1/7 \\ 0 & 1/7 \end{pmatrix}\begin{pmatrix} 1000 \\ 1500 \end{pmatrix} = (1000 - 1500/7, 1500/7) = (5500/7, 1500/7)$$

$$\mathbf{t}_2 = \mathbf{S}^{-1}\mathbf{P}_2 = \begin{pmatrix} 1 & -1/7 \\ 0 & 1/7 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 13/7 \\ 1/7 \end{pmatrix}$$

$$\theta = \min\left\{\frac{5500/7}{13/7}, \frac{1500/7}{1/7}\right\} = \min\left\{\frac{5500}{3}, 1500\right\} = \frac{5500}{3}$$

Since the minimum ratio corresponds to $\mathbf{P}_6$, it becomes the departing vector (**D.V.**).

(*c*) New Basis:

$$\boldsymbol{\eta} = \begin{pmatrix} \dfrac{1}{t_{62}} \\ \dfrac{-t_{42}}{t_{62}} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{13/7} \\ -\dfrac{1/7}{13/7} \end{pmatrix} = \begin{pmatrix} 7/13 \\ -1/13 \end{pmatrix}; \qquad \mathbf{E} = (\boldsymbol{\eta}, \mathbf{u}_2)$$

$$\mathbf{S}_{\text{new}}^{-1} = \mathbf{E}\mathbf{S}^{-1} = \begin{pmatrix} 7/13 & 0 \\ -1/13 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1/7 \\ 0 & 1/7 \end{pmatrix} = \begin{pmatrix} 7/13 & -1/13 \\ -1/13 & 2/13 \end{pmatrix}$$

Summary of Iteration No. 2:

$$\mathbf{X}_S = (x_2, x_4)^T; \qquad \mathbf{C}_S^T = (2, 6)$$

Iteration No. 3:

Now the nonbasic vectors are $\mathbf{P}_1$, $\mathbf{P}_6$, $\mathbf{P}_3$, $\mathbf{P}_8$, $\mathbf{P}_5$, and $\mathbf{P}_7$.

(*a*) Entering Vector:

$$\mathbf{W} = \mathbf{C}_S^T\mathbf{S}^{-1} = (2, 6)\begin{pmatrix} 7/13 & -1/13 \\ -1/13 & 2/13 \end{pmatrix} = (8/13, 10/13)$$

$$\begin{aligned}(c_1 - z_1, c_6 - z_6, c_3 - z_3, c_8 - z_8, c_5 - z_5, c_7 - z_7) &= (c_1, c_6, c_3, c_8, c_5, c_7) - \mathbf{W}(\mathbf{P}_1, \mathbf{P}_6, \mathbf{P}_3, \mathbf{P}_8, \mathbf{P}_5, \mathbf{P}_7) \\ &= (3, M, 4, M, 0, 0) - (8/13, 10/13) \\ &\quad \times \begin{pmatrix} 1 & 1 & 1 & 0 & -1 & 0 \\ 2 & 0 & 3 & 1 & 0 & -1 \end{pmatrix} \\ &= 11/13, -8/13 + M, 14/13, -10/13 + M, \\ &\quad 8/13, 10/13)\end{aligned}$$

Since all the coefficients are nonnegative, the above step gives the optimal basis. The optimal values of the variables and the objective function are as follows:

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \mathbf{S}^{-1}\mathbf{B} = \begin{pmatrix} 7/13 & -1/13 \\ -1/13 & 2/13 \end{pmatrix}\begin{pmatrix} 1000 \\ 1500 \end{pmatrix} = \begin{pmatrix} 5500/13 \\ 2000/13 \end{pmatrix}$$

$$z = \mathbf{C}_S^T\mathbf{X}_S = (2, 6)\begin{pmatrix} 5500/13 \\ 2000/13 \end{pmatrix} = 23000/13$$

Example:

An industry must have a water supply of at least $4\text{x}10^6$ liters/day of a quality such that total dissolved solids (TDS) is kept below 100 mg/l. The water can be obtained from two sources: (1) purchase from the city system at $100 per million liters, and (2) pump from a nearby stream at $50 per million liters. The concentration of TDS in the city source is 50 mg/l. TDS in the stream is 200 mg/l. Water from the two sources is completely mixed before it is used. The city can supply up to $3.5\text{x}10^6$ l/day, and water rights permit pumping up to $2\text{x}10^6$ l/day from the stream.

a. Formulate a linear program to optimize the amount of water used from each source. Define your decision variables and the meaning of the objective function and constraints.

b. Use the revised simplex method to determine the optimal solution.

Quiz 24-11-2015: Revised simplex

An aqueduct constructed to supply water to industrial users has an excess capacity in the months of June, July, and August of 14,000 acft, 18,000 acft, and 6,000 acft, respectively. It is proposed to develop not more than 10,000 acres of new land by utilizing the excess aqueduct capacity for irrigation water deliveries. Two crops, hay and grain, are to be grown. Their monthly water requirements and expected net returns are given in the following table:

	Monthly Water Requirement (acft/acre)			
	June	July	August	Return, $/acre
Hay	2	1	1	100
Grain	1	2	0	120

Formulate and solve a linear program to optimize the irrigation development. (use revised simplex method)

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Integer Programming: The Transportation Algorithm

The transportation algorithm is a special class of linear programs that deals with shipping a commodity from sources (e.g. factories, reservoirs,….etc.) to destinations (e.g. warehouses, farms, ……etc.).

The objective: is to determine the shipping schedule that minimize the total shipping cost while satisfying supply and demand limits.

The application of the transportation model can be extended to other areas of operation, including inventory control, employment scheduling, and personal assignment.

Definition of the transportation algorithm

The general problem is represented by the network below:

There are (m) sources and (n) destinations, each represented by a node. The arcs represented the routes linking the sources and the destinations. Arc (i,j) joining the source (i) to destination (j) carries two pieces of information: the transportation cost per unit (Cij), and the amount shipped(Xij). The amount of supply at source (i) is (ai), and the amount of demand at destination (j) is (bj).

The objective of the transportation model is to determine the unknowns Xij that will minimize the total transportation cost while satisfying all the supply and demand restrictions.

STANDARD FORM

It is assumed that the total supply and total demand are equal; that is,

$\sum_{i=1}^{m} ai = \sum_{j=1}^{n} bj$ ………… (1)

Equation (1) is guaranteed by creating either a fictitious (dummy) destination with a demand equal to the surplus if total demand is less than total supply or a fictitious source with a supply equal to the shortage if total demand exceeds total supply.

The standard mathematical model for this problem is:

$$\text{minimize: } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to: } \sum_{j=1}^{n} x_{ij} = a_i \quad (i = 1, \ldots, m)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad (j = 1, \ldots, n)$$

$$\text{with: all } x_{ij} \text{ nonnegative and integral}$$

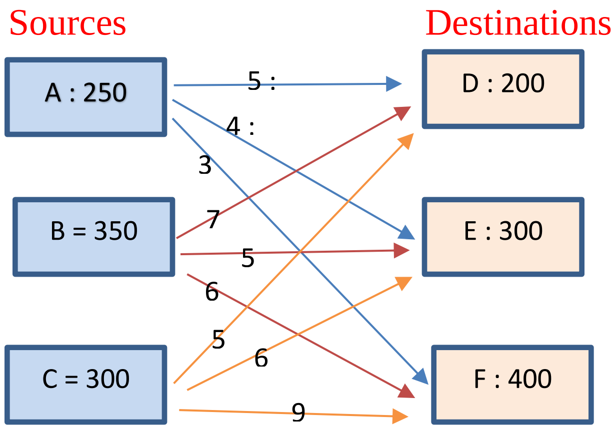
Example: a company has three factories, A, B, and C. the capacities of the three plants are 250, 350, and 300 pieces respectively. The company exported its products to three warehouses D, E, and F that have a capacity of 200, 300, and 400 pieces, respectively. The transportation costs per piece on deferent routes in dollar are given in table below:

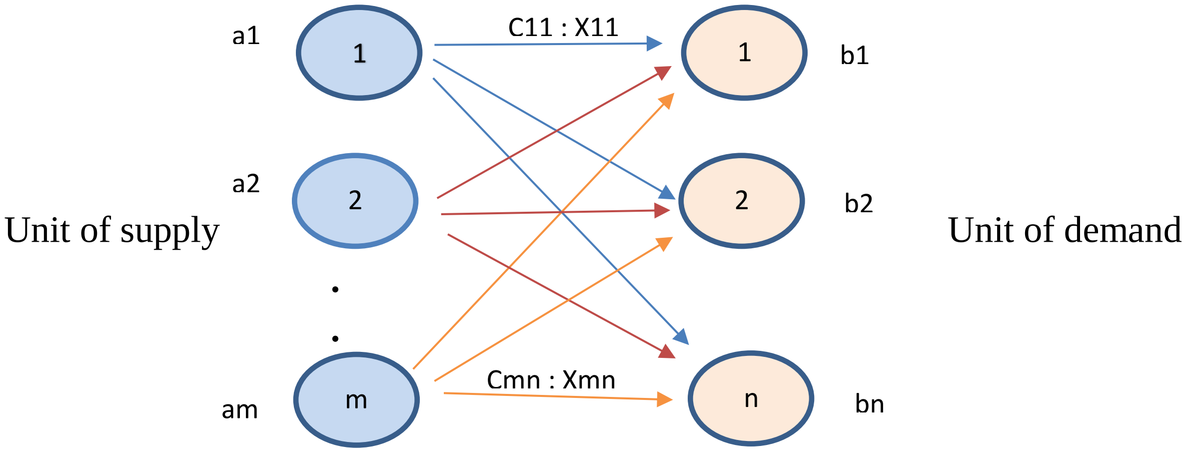
1- Representation of the transportation model with nodes and arcs.
2- Simulate the transportation problem using transportation tableau.

To From	D	E	F
A	5	4	3
B	7	5	6
C	5	6	9

Sol:

1- Representation of the transportation model with nodes and arcs.





The problems of transportation model can be solved with more conveniently using the transportation tableau as shown in table below: **(Tableau 1)**

Destination Source	1	2	……….	n	Supply	Ui
1	X11 C11	X12 C12	……….	X1n C1n	a1	U1
2	X21 C21	X22 C22	……….	X2n C2n	a2	U2
. .			……….		. .	
m	Xm1 Cm1	Xm2 Cm2	……….	Xmn Cmn	a_m	Um
(dummy)	0	0	……….	0		
Demand	b1	b2		bn		
Vj	V1	V2		Vn		

Where :
- Cij = unit transportation cost
- Xij = the amount of unit that shipped
- ai = the amount of supply at source i
- bj = the amount of demand at destination j
- m = number of sources
- n = number of destinations

Destination Source	D	E	F	Supply
A	5	4	3	250
B	7	5	6	350
C	5	6	9	300
Demand	200	300	400	

Determination of the starting solution

A general transportation model with (m) sources and (n) destinations has (m+n) constraint equations, one for each source and each destination. However, because the transportation model is always balanced (sum of the supply = sum of the demand), one of these equations is redundant. Thus, the model has m+n-1 independent constraint equations, which means that the starting basic solution consists of (m+n-1) basic variables. Thus in the above example the starting solution must have (3+3-1=5) basic variables.

The transportation algorithm is the simplex method specialized to the format of **Tableau 1**; as usual, it involves

(i) Finding an initial, basic feasible solution;
(ii) Testing the solution for optimality;
(iii) Improving the solution when it is not optimal; and
(iv) Repeating steps (ii) and (iii) until the optimal solution is obtained.

The special structure of the transportation problem allows starting basic solution using one of these methods:

1- Initial solution using **Northwest corner** method with: **A)** Modified distribution method. **B)** Stepping stone method.
2- **Vogel's** approximation method
3- The **Hungarian** method.

Northwest Corner with Modified Distribution method

Beginning with the (1,1) cell in **Tableau 1** (the northwest corner), allocate to X11 as many units as possible without violating the constraints. This will be the smaller of **a1** and **b1**. Thereafter, continue by moving one cell to the right, if some supply remains, or, if not, one cell down. At each step, allocate as much as possible to the cell (variable) under consideration without violating the constraints: the sum of the ***i*th**-row allocations cannot exceed **ai**, the sum of the ***j*th**-column allocations cannot exceed **b*j*** , and no allocation can be negative. The allocation may be zero.

Variables that are assigned values by this starting procedure become the basic variables in the initial solution. The unassigned variables are nonbasic and, therefore, zero. We adopt the convention of not entering the nonbasic variables in Tableau 1- they are understood to be zero-and of indicating basic-variable allocations in boldface type.

TEST FOR OPTIMALITY

Assign one (anyone) of the **Ui** or **Vj** in Tableau 1 the value zero and calculate the remaining Ui and Vj so that for each basic variable Ui + Vj = Cij Then, for each nonbasic variable, calculate the quantity Cij - ui – Vj . If all these latter quantities are nonnegative, the current solution is optimal; otherwise, the current solution is not optimal.

IMPROVING THE SOLUTION

Definition: A loop is a sequence of cells in Tableau 1 such that:

(i) each pair of consecutive cells lie in either the same row or the same column;
(ii) no three consecutive cells lie in the same row or column;
(iii) the first and last cells of the sequence lie in the same row or column;
(iv) no cell appears more than once in the sequence.

Example 8.1 The sequences {(1, 2), (1, 4), (2, 4), (2, 6), (4, 6), (4, 2)} and {(1, 3), (1, 6), (3, 6), (3, 1), (2, 1), (4, 2), (2, 4), (2, 3)} illustrated in Figs. 8-1 and 8-2, respectively, are loops. Note that a row or column can have more than two cells in the loop (as the second row of Fig. 8-2), but no more than two can be consecutive.

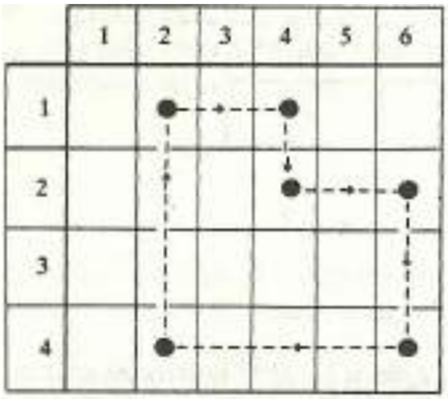


Fig. 8-1

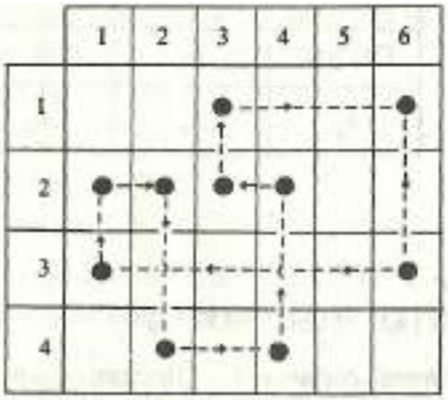


Fig. 8-2

Consider the nonbasic variable corresponding to the most negative of the quantities **Cij - *Ui* – *Vj*** calculated in the test for optimality; it is made the incoming variable. Construct a loop consisting exclusively of this incoming variable (cell) and current basic variables (cells). Then allocate to the incoming cell as many units as possible such that, after appropriate adjustments have been made to the other cells in the loop, the supply and demand constraints are not violated, all allocations remain nonnegative, and one of the old basic variables has been reduced to zero (whereupon it ceases to be basic).

DEGENERACY

In view of condition (8.1), only n + m - 1 of the constraint equations in system (8.2) are independent. Then, by Problems 2.19 and 2.20 a non-degenerate basic feasible solution will be characterized by positive values for exactly n + m - 1 basic variables. If the process of improving the current basic solution results in two or more current basic variables being reduced to zero simultaneously, only one is allowed to become nonbasic (solver's choice, although the variable with the largest unit shipping cost is preferred).

The other variable(s) remains (remain) basic, but with a zero allocation, thereby rendering the new basic solution degenerate.

The northwest corner rule always generates an initial basic solution; but it may fail to provide n + m - 1 positive values, thus yielding a degenerate solution.

If Vogel's method is used, and does not yield that same number of positive values, additional variables with zero allocations must be designated as basic (see Problem 8.6). The choice is arbitrary, to a point: basic variables cannot form loops, and preference is usually given to variables with the lowest associated shipping costs.

Improving a degenerate solution may result in replacing one basic variable having a zero value by another such. (This occurs at the first improvement in Problem 8.4.) Although the two degenerate solutions are effectively the same-only the designation of the basic variables has changed, not their values-the additional iteration is necessary for the transportation algorithm to proceed.

Example 2:

Solve the previous example using Northwest modified distribution method.

Sol:

Destination Source	D	E	F	Supply	Ui
A	200 $\boxed{5}$	50 $\boxed{4}$	$\boxed{3}$	250	
B	$\boxed{7}$	250 $\boxed{5}$	100 $\boxed{6}$	350	
C	$\boxed{5}$	$\boxed{6}$	300 $\boxed{9}$	300	
Demand	200	300	400		
Vj					

m+n-1 = 3+3-1 = 5 basic variable

To determine whether the initial allocation found in Tableau above is optimal, we first calculate the terms Ui and Vj with respect to the basic-variable cells of the tableau. Arbitrarily choosing U 1 = 0 (choose any row or column contains more basic variables than any other row or column, this choice will simplify the computations, if it is same choose any one), we find:

(1, 1) cell: U1 + V1 = C11 , 0+ V1 = 5, or V1 = 5
(1, 2) cell: U1 + V2 = C12, 0+ V2 = 4, or V2 = 4
(2,2) cell: U2 + V2 = C22, U2+ 4 = 5, or U2 = 1
(2 , 3) cell: U2 + V3 = C23, 1 + V3 = 6, or V3 = 5
(3, 3) cell: U3 + V3 = C33, U3 + 5 = 9, or U3 = 4

These values are shown in Tableau below. Next, we calculate the quantities Cij - *Ui* - *Vj* for each non-basic variable cell of Tableau above.

(1, 3) cell: C13-U1 –V3 = 3- 0-5 = -2
(2, 1) cell: C21-U2 –V1 = 7 - 1- 5 = 1
(3,1) cell: C31-U3 –V1 = 5-4-5 = -4
(3 ,2) cell: C32-U3 –V2 = 6- 4 -4 = -2

These results also are recorded in Tableau below, in parentheses.

Destination Source	D	E	F	Supply	Ui
A	200 $\boxed{5}$	50 $\boxed{4}$	(-2) $\boxed{3}$	250	0
B	(1) $\boxed{7}$	250 $\boxed{5}$	100 $\boxed{6}$	350	1
C	(-4) + $\boxed{5}$	(-2) $\boxed{6}$	300 $\boxed{9}$	300	4
Demand	200	300	400		
Vj	5	4	5		

Since at least one of these *(cij - Uj* – vj)-values is negative, the current solution is not optimal, and a better solution can be obtained by increasing the allocation to the variable (cell) having the largest negative entry, here the (3, 1) cell of Tableau above. We do so by placing a boldface plus sign (signaling an increase) in the (3, 1) cell and identifying a loop containing, besides this cell, only basic-variable cells. Such a loop is shown by the heavy lines in Tableau above. We now increase the allocation to the (3,1) cell as much as possible, simultaneously adjusting the other cell allocations in the loop so as not to violate the supply, demand, or nonnegativity constraints. The new basic solution, also degenerate, is given in Tableau below.

estination Source	D	E	F	Supply	Ui
A	(4) $\boxed{5}$	250 $\boxed{4}$	(-2) + $\boxed{3}$	250	1
B	(5) $\boxed{7}$	50 $\boxed{5}$	300 $\boxed{6}$	350	0
C	200 $\boxed{5}$	(-2) $\boxed{6}$	100 $\boxed{9}$	300	3
Demand	200	300	400		
Vj	2	5	6		

For each basic variable

Let U2 = 0

(2, 2) cell: U2 + V2 = C22 , 0+ V2 = 5, or V2 = 5

(2, 3) cell: U2 + V3 = C23, 0+ V3 = 6, or V3 = 6

(1,2) cell: U1 + V2 = C12, U1+ 5 = 4, or U1 = -1

(3, 3) cell: U3 + V3 = C33, U3 + 6 = 9, or U3 = 3

(3, 1) cell: U3 + V1 = C31, 3 + V1 = 5, or V1 = 2

Next, we calculate the quantities Cij - *Ui* - *Vj* for each non-basic variable cell of Tableau above.

(1, 1) cell: C11-U1 –V1 = 5- (-1)-2 = 4

(1, 3) cell: C13-U1 –V3 = 3- (-1)-6 = -2

(2,1) cell: C21-U2 –V1 = 7-0-2 = 5

(3 ,2) cell: C32-U3 –V2 = 6- 3 -5 = -2

Destination Source	D	E	F	Supply	Ui
A	(6) 5	(2) 4	**250** 3	250	3
B	(5) 7	300 5	50 6	350	6
C	200 5	(-2) + 6	100 9	300	9
Demand	200	300	400		
Vi	-4	-1	0		

For each basic variable

Let V3 = 0

(1,3) cell: U1 + V3 = C13 , U1+ 0 = 3, or U1 = 3

(2, 3) cell: U2 + V3 = C23, U2+ 0 = 6, or U2 = 6

(3, 3) cell: U3 + V3 = C33, U3 + 0 = 9, or U3 = 9

(2,2) cell: U2 + V2 = C22, 6+ V2 = 5, or V2 = -1

(3, 1) cell: U3 + V1 = C31, 9 + V1 = 5, or V1 = -4

Next, we calculate the quantities Cij - *Ui* - *Vj* for each non-basic variable cell of Tableau above.

(1, 1) cell: C11-U1 –V1 = 5-3- (-4) = 6

(1, 2) cell: C12-U1 –V2 = 4-3- (-1) = 2

(2,1) cell: C21-U2 –V1 = 7-6- (-4) = 5

(3 ,2) cell: C32-U3 –V2 = 6-9- (-1) =-2

Destination Source	D	E	F	Supply	Ui
A	(4) $\boxed{5}$	(2) $\boxed{4}$	**250** $\boxed{3}$	250	-3
B	(3) $\boxed{7}$	200 $\boxed{5}$	150 $\boxed{6}$	350	0
C	200 $\boxed{5}$	100 $\boxed{6}$	(2) $\boxed{9}$	300	1
Demand	200	300	400		
Vi	4	5	6		

For each basic variable

Let U2 = 0

(2,2) cell: U2 + V2 = C22, 0+ V2 = 5, or V2 = 5

(2, 3) cell: U2 + V3 = C23, 0+ V3 = 6, or V3 = 6

(1,3) cell: U1 + V3 = C13 , U1+ 6 = 3, or U1 = -3

(3, 2) cell: U3 + V2 = C33, U3 + 5 = 6, or U3 = 1

(3, 1) cell: U3 + V1 = C31, 1 + V1 = 5, or V1 = 4

Next, we calculate the quantities Cij - *Ui* - *Vj* for each non-basic variable cell of Tableau above.

(1, 1) cell: C11-U1 –V1 = 5-(-3)- 4 = 4

(1, 2) cell: C12-U1 –V2 = 4-(-3)- 5 = 2

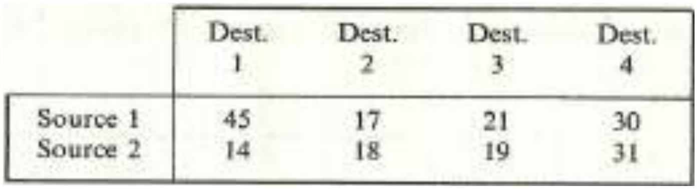
(2,1) cell: C21-U2 –V1 = 7-0- 4 = 3

(3 ,3) cell: C33-U3 –V3 = 9-1-6 = 2

It is seen that each Cij - Ui – Vj is nonnegative; hence the new solution is optimal. That is, X13 = 250, X22 = 200, X23 = 150, X31 = 200, X32 = 100, with all other variables non-basic and, therefore, zero. Furthermore,

z* = 250(3) + 200(5) + 150(6) + 200(5) + 100(6) = 4250 $

8.1 A car rental company is faced with an allocation problem resulting from rental agreements that allow cars to be returned to locations other than those at which they were originally rented. At the present time, there are two locations (sources) with 15 and 13 surplus cars, respectively, and four locations (destinations) requiring 9, 6, 7, and 9 cars, respectively. Unit transportation costs (in dollars) between the locations are as follows:



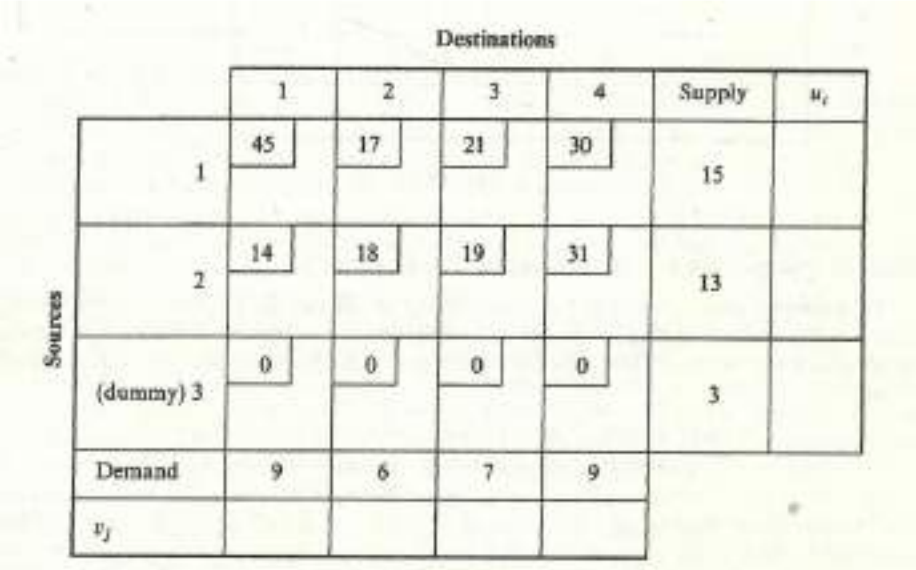
	Dest. 1	Dest. 2	Dest. 3	Dest. 4
Source 1	45	17	21	30
Source 2	14	18	19	31

Set up the initial transportation tableau (Tableau 8-1) for the minimum-cost schedule.

Sol:

Since the total demand (9 + 6 + 7 + 9 = 31) exceeds the total supply (15 + 13 = 28), a dummy source is created having a supply equal to the 3-unit shortage. In reality, shipments from this fictitious source are never made, so the associated shipping costs are taken as zero. Positive allocations from this source to a destination represent cars that cannot be delivered due to a shortage of supply; they are shortages a destination will experience under an optimal shipping schedule .

. For this problem, Tableau 8-1 becomes Tableau 1A. The *xij'* u_i, and v_j are not 'entered, since they are unknown at the moment.



		Destinations					
		1	2	3	4	Supply	u_i
Sources	1	45	17	21	30	15	
	2	14	18	19	31	13	
	(dummy) 3	0	0	0	0	3	
	Demand	9	6	7	9		
	v_j						

Tableau 1A

2 For an $m \times n$ transportation tableau, show that the northwest corner rule evaluates $n + m - 1$ of the variables.

Observe that after treating the (1, 1) cell, the rule is applied *in the same form* to a subtableau, the new northwest corner being either the original (1, 2) cell or the original (2, 1) cell. Suppose then (mathematical induction) that the result holds for the subtableau, which is either $m \times (n-1)$ or $(m-1) \times n$. In either case, $n + m - 2$ variables are evaluated in the subtableau, so that

$$(n + m - 2) + 1 = n + m - 1$$

variables are evaluated in the tableau. Since the result obviously holds when $n = m = 1$, the proof by induction is complete.

3 Use the northwest corner rule to obtain an initial allocation to Tableau 1A.

We begin with x_{11} and assign it the minimum of $a_1 = 15$ and $b_1 = 9$. Thus, $x_{11} = 9$, leaving six surplus cars at the first source. We next move one cell to the right and assign $x_{12} = 6$. These two allocations together exhaust the supply at the first source, so we move one cell down and consider x_{22}. Observe, however, that the demand at the second destination has been satisfied by the x_{12} allocation. Since we cannot deliver additional cars to it without exceeding its demand, we must assign $x_{22} = 0$ and the move one cell to the right. Continuing in this manner, we obtain the degenerate solution (fewer than $4 + 3 - 1 = 6$ *positive* entries) depicted in Tableau 1B.

	1	2	3	4	Supply	u_i
1	45 9	17 6	21	30	15	
2	14	18 0	19 7	31 6	13	
(dummy) 3	0	0	0	0 3	3	
Demand	9	6	7	9		
v_j						

Tableau 1B

4 Solve the transportation problem described in Problem 8.1.

To determine whether the initial allocation found in Tableau 1B is optimal, we first calculate the terms u_i and v_j with respect to the basic-variable cells of the tableau. Arbitrarily choosing $u_2 = 0$ (since the second row contains more basic variables than any other row or column, this choice will simplify the computations), we find:

(2, 2) cell: $u_2 + v_2 = c_{22}$, $0 + v_2 = 18$, or $v_2 = 18$
(2, 3) cell: $u_2 + v_3 = c_{23}$, $0 + v_3 = 19$, or $v_3 = 19$
(2, 4) cell: $u_2 + v_4 = c_{24}$, $0 + v_4 = 31$, or $v_4 = 31$
(1, 2) cell: $u_1 + v_2 = c_{12}$, $u_1 + 18 = 17$, or $u_1 = -1$
(1, 1) cell: $u_1 + v_1 = c_{11}$, $-1 + v_1 = 45$, or $v_1 = 46$
(3, 4) cell: $u_3 + v_4 = c_{34}$, $u_3 + 31 = 0$, or $u_3 = -31$

These values are shown in Tableau 1C. Next we calculate the quantities $c_{ij} - u_i - v_j$ for each nonbasic-variable cell of Tableau 1B.

(1, 3) cell: $c_{13} - u_1 - v_3 = 21 - (-1) - 19 = 3$
(1, 4) cell: $c_{14} - u_1 - v_4 = 30 - (-1) - 31 = 0$
(2, 1) cell: $c_{21} - u_2 - v_1 = 14 - 0 - 46 = -32$
(3, 1) cell: $c_{31} - u_3 - v_1 = 0 - (-31) - 46 = -15$
(3, 2) cell: $c_{32} - u_3 - v_2 = 0 - (-31) - 18 = 13$
(3, 3) cell: $c_{33} - u_3 - v_3 = 0 - (-31) - 19 = 12$

These results also are recorded in Tableau 1C, in parentheses.

	1	2	3	4	Supply	u_i
1	45 9	17 6	21 (3)	30 (0)	15	−1
2	14 (−32) +	18 0	19 7	31 6	13	0
(dummy) 3	0 (−15)	0 (13)	0 (12)	0 3	3	−31
Demand	9	6	7	9		
v_j	46	18	19	31		

Tableau 1C

Since at least one of these $(c_{ij} - u_i - v_j)$-values is negative, the current solution is not optimal, and a better solution can be obtained by increasing the allocation to the variable (cell) having the largest negative entry, here the (2, 1) cell of Tableau 1C. We do so by placing a boldface plus sign (signaling an increase) in the (2, 1) cell and identifying a loop containing, besides this cell, only basic-variable cells. Such a loop is shown by the heavy lines in Tableau 1C. We now increase the allocation to the (2, 1) cell as much as possible, simultaneously adjusting the other cell allocations in the loop so as not to violate the supply, demand, or nonnegativity constraints. Any positive allocation to the (2, 1) cell would force x_{22} to become negative. To avoid this, but still make x_{21} basic, we assign $x_{21} = 0$ and remove x_{22} from our set of basic variables. The new basic solution, also degenerate, is given in Tableau 1D.

We now check whether this solution is optimal. Working directly on Tableau 1D, we first calculate the new u_i and v_j with respect to the new basic variables, and then compute $c_{ij} - u_i - v_j$ for each nonbasic-variable cell. Again we arbitrarily choose $u_2 = 0$, since the second row contains more basic variables than any other row or column. These results are shown in parentheses in Tableau 1E. Since two entries are negative, the current solution is not optimal, and a better solution can be obtained by increasing the allocation to the (1, 4) cell. The loop whereby this is accomplished is indicated by heavy lines in Tableau 1E; it consists of the cells (1, 4), (2, 4), (2, 1), and (1, 1). Any amount added to cell (1, 4) must be simultaneously subtracted from cells (1, 1) and (2, 4) and then added to cell (2, 1), so as not to violate the supply–demand constraints. Therefore, no more than six cars can be added to cell (1, 4) without forcing x_{24} negative. Consequently, we reassign $x_{14} = 4$, make the appropriate adjustments in the loop, and remove x_{24} as a basic variable. The new, nondegenerate basic solution is shown in Tableau 1F.

	1	2	3	4	Supply	u_i
1	45 9	17 6	21	30	15	
2	14 0	18	19 7	31 6	13	
(dummy) 3	0	0	0	0 3	3	
Demand	9	6	7	9		
v_j						

Tableau 1D

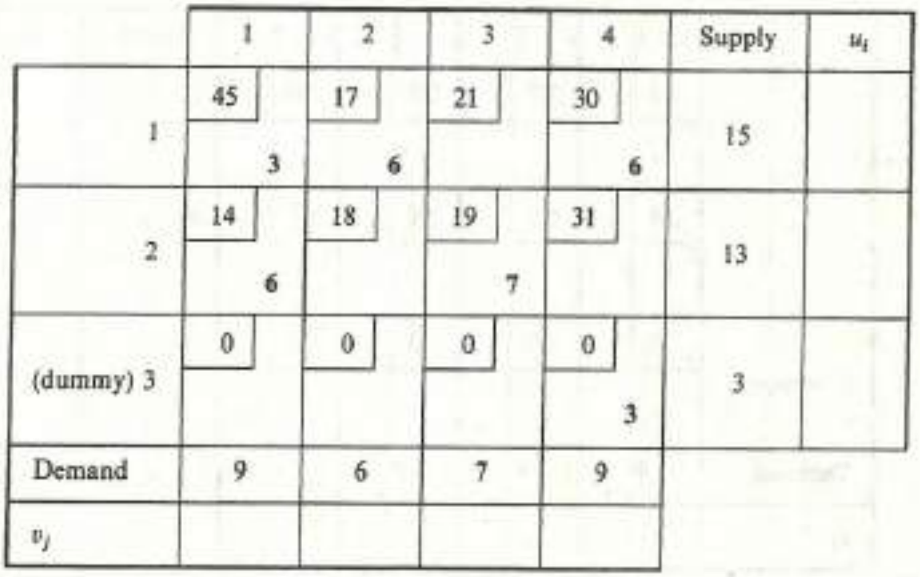
	1	2	3	4	Supply	u_i
1	45 9	17 6	21 (−29)	30 (−32) +	15	31
2	14 0	18 (32)	19 7	31 6	13	0
(dummy) 3	0 (17)	0 (45)	0 (12)	0 3	3	−31
Demand	9	6	7	9		
v_j	14	−14	19	31		

Tableau 1E

After one further optimality test (negative) and consequent change of basis, we obtain Tableau 1H, which also shows the results of the optimality test of the new basic solution. It is seen that each $c_{ij} - u_i - v_j$ is nonnegative; hence the new solution is optimal. That is, $x_{12}^* = 6$, $x_{13}^* = 3$, $x_{14}^* = 6$, $x_{21}^* = 9$, $x_{23}^* = 4$, $x_{34}^* = 3$, with all other variables nonbasic and, therefore, zero. Furthermore,

$$z^* = 6(17) + 3(21) + 6(30) + 9(14) + 4(19) + 3(0) = \$547$$

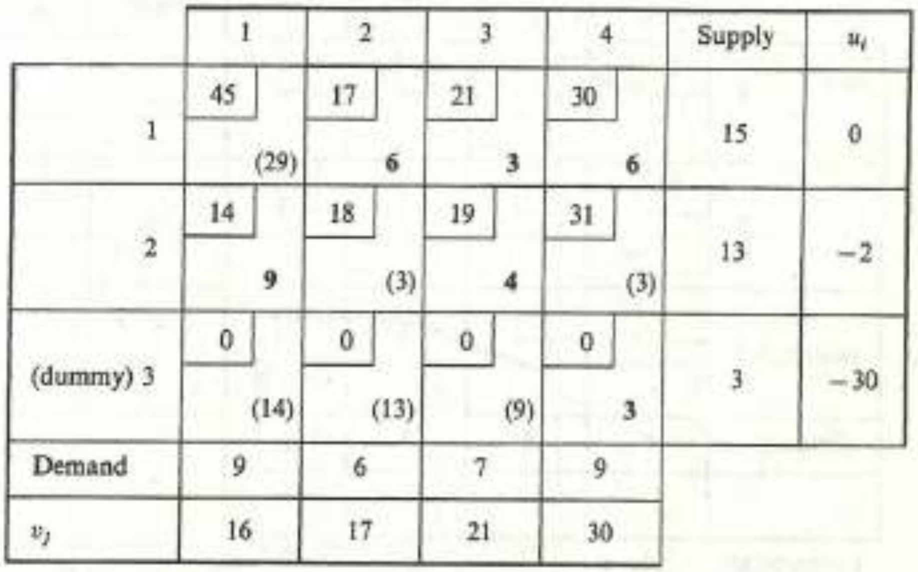
The fact that some positive allocation comes from the dummy source indicates that not all demands can be met under this optimal schedule. In particular, destination 4 will receive three fewer cars than it needs.



	1	2	3	4	Supply	u_i
1	45 3	17 6	21	30 6	15	
2	14 6	18	19 7	31	13	
(dummy) 3	0	0	0	0 3	3	
Demand	9	6	7	9		
v_j						

Tableau 1F

……



	1	2	3	4	Supply	u_i
1	45 (29)	17 6	21 3	30 6	15	0
2	14 9	18 (3)	19 4	31 (3)	13	−2
(dummy) 3	0 (14)	0 (13)	0 (9)	0 3	3	−30
Demand	9	6	7	9		
v_j	16	17	21	30		

Tableau 1H

Water Resources Management and Economy

University of Anbar– College of Engineering

Dams & Water Resources Engineering

Department – 4th stage

Asst. Prof. Dr. Sadeq Oleiwi Sulaiman

Vogel's method.

For each row and each column having some supply or some demand remaining,

1- calculate its *difference*, which is the nonnegative difference between the two smallest shipping costs cij associated with unassigned variables in that row or column.
2- Consider the row or column having the largest difference; in case of a tie, arbitrarily choose one. In this row or column, locate that unassigned variable (cell) having the smallest unit shipping cost and allocate to it as many units as possible without violating the constraints.
3- Recalculate the new differences and repeat the above procedure until all demands are satisfied. See Problems 8.5 and 8.6.

Variables that are assigned values by either one of these starting procedures become the basic variables in the initial solution. The unassigned variables are nonbasic and, therefore, zero.

We adopt the convention of not entering the nonbasic variables in Tableau 8-1- they are understood to be zero-and of indicating basic-variable allocations in boldface type.

The northwest corner rule is the simpler of the two rules to apply. However, Vogel's method, which takes into account the unit shipping costs, usually results in a closer-to-optimal starting solution (see Problem 8.5).

Ex: Determine the minimum cost-shipping for previous example using Vogel approximation method:

ا- نبدأ اولاً· نوصل بجدول طريقة او موديل النقل خالي من الارقام

		2	1	X			
		2	1	3			
		0	1	3			
Supply \ Demand		D	E	F			
A	250	X $\boxed{5}$	X $\boxed{4}$	250 $\boxed{3}$	1	X	X
B	350	X $\boxed{7}$	200 $\boxed{5}$	150 $\boxed{6}$	1	1	2
C	300	200 $\boxed{5}$	100 $\boxed{6}$	X $\boxed{9}$	1	1	1
	900 / 900	200	300	400			

2- نحسب الفرق بين اقل قيمتين لكل سطر ولكل عمود ونسجلها فوق الاعمدة وبجانب الصفوف.

3- نحدد السطر او العمود الذي يحوي اكبر فرق ونحدد من هذا العمود او السطر المربع صاحب الكلفة الاقل ونضع فيه اكبر كمية ممكنة من وحدات البضاعة بشرط ان لا تتجاوز هذه الكمية اي من القيود الافقية او العمودية.

4- في حالة استنفاذ المربع صاحب الكلفة الاقل لكل الكمية المعروضة في السطر المعني نلغي بقية المربعات الموجودة في السطر عن طريق وضع علامة (X) في المربعات الخالية.

5- نعيد الخطوات من (1-4) على الجدول مع ملاحظة ان حساب الفروقات للاسطر والاعمدة تقتصر على المربعات الخالية. حيث يتبين بان العمود الثالث يحتوي اكبر فرق والمربع (C_{23}) صاحب اقل تكلفة في هذا العمود وبما ان المخزن (F) يحتاج الى (400) وحدة من البضاعة الموجودة به في الخطوة السابقة (250) وحدة لذلك نضع فقط (150) وحدة لنصل الى كامل طاقة خزن في المخزن (F) ثم نضع اشارة (X) في المربع (C_{33}) لان المخزن لا يتحمل المزيد من شحنات البضاعة.

6- نعيد الخطوات من (1-4) مرة اخرى على الجدول الجديد والمربعات الخالية فقط. ويتبين من ذلك ان العمود الاول والسطر الثاني حصلا على نفس الرقم (2) وسوف نعطي الاولوية للسطر (2) لكي يتم استنفاذ بقية الوحدات الموجودة في المستودع (B) واعطاءها الى المربع (C_{22}) لانه صاحب الكلفة الاقل ونضع اشارة (X) للمربع (C_{21}) لنفاذ كامل كمية مخزن B

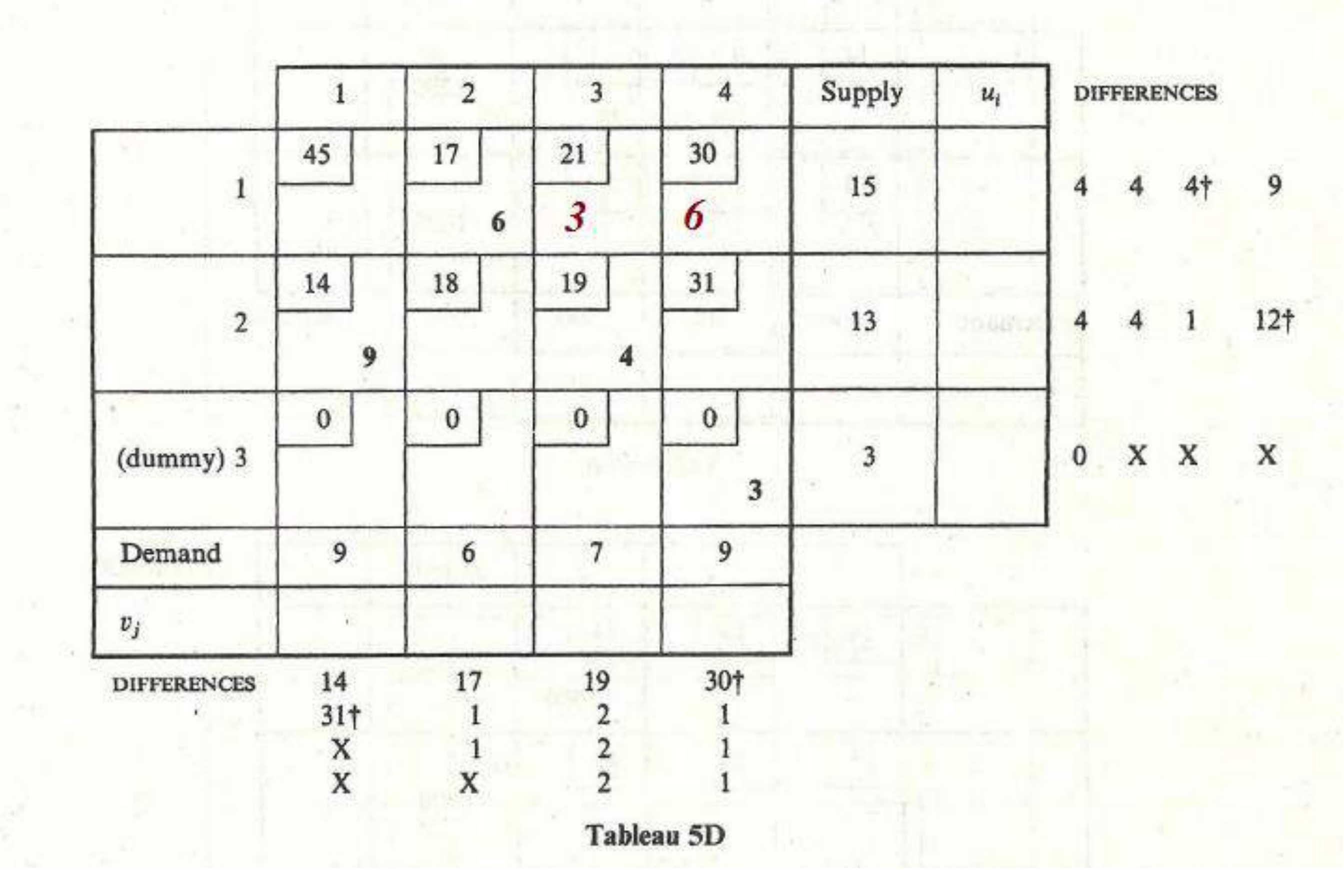
⑦ اتمام عملية التوزيع وانهاؤها بتوزيع وحدات المخزن (C) حسب امكانية مخازن المستهلكين (D) و (E).

8.5 Use Vogel's method to determine an initial basic solution to the transportation problem described in Problem 8.1.

Sol:.

The two smallest costs in row 1 of Tableau lA are 17 and 21; their difference is 4. The two smallest costs in row 2 are 14 and 18; their difference is also 4. The two smallest costs in row 3are both 0; so their difference is 0. Repeating this analysis on the columns, we generate the differences shown beside Tableau SA. Since the largest of these differences, indicated by a t, occurs in column 4, we locate the variable (cell) in this column having the lowest unit shipping cost and allocate to it as many units as possible. Thus *X 34* = 3, exhausting the supply of source 3 and *eliminating row* 3 *from further consideration.*

We now compute the differences for each row and column anew, without reference to the elements in row 3. The results are shown beside Tableau 5B, where the entry X for the second difference in row 3 means simply that this row has been eliminated. The largest difference appears in column 1, and the ·variable in this column having the smallest cost is X 21 (since row 3 is no longer under consideration). We assign X 21 = 9, thereby satisfying the demand of destination 1. Accordingly, column 1 will not be involved in the ensuing calculations.



	1	2	3	4	Supply	u_i	DIFFERENCES			
1	45	17 6	21 3	30 6	15		4	4	4†	9
2	14 9	18	19 4	31	13		4	4	1	12†
(dummy) 3	0	0	0	0 3	3		0	X	X	X
Demand	9	6	7	9						
v_j										
DIFFERENCES	14	17	19	30†						
	31†	1	2	1						
	X	1	2	1						
	X	X	2	1						

Tableau 5D

With row 3 and column 1 eliminated, the new differences are shown beside Tableau *SC*, 'where, again, an X indicates that a computation was not required. The largest difference occurs in row 1, and the variable in this row having the lowest unit cost is X 12' Note that even if C;I had been less than 17, Xll would not have been selected here, since it falls in a column that has been eliminated. We set $X 12 = 6$, thereby meeting the demand of destination 2 and removing column 2 from further calculations.

With row 3 and columns 1 and 2 no longer considered, the new differences are shown beside Tableau 5D. The largest difference occurs in row 2, and the smallest cost in that row and in columns still under consideration is 19. Consequen't1y, we assign $X23 = 4$, which with the earlier assignment X21 = 9 exhausts the supply of source 2 and removes row 2 from further consideration.

With rows 2 and 3 eliminated, we no longer can calculate differences for the remaining columns. This is a signal that the remaining allocations are uniquely determined. Here we must set Xl3 = 3 and $X 14 = 6$ if we are, to meet all demands without exceeding supplies. The result is the allocation shown in Tableau IH, which was determined in Problem 8.4 to be optimal.

Water Resources Economy Quiz 2-1-2016

Two reservoirs are available to supply the water needs of three cities. Each reservoir can supply up to 50 million gallons of water per day. Each city would like to receive 40 million gallons per day.

For each million gallons per day of unmet demand, there is a penalty. At city 1, the penalty is $20; at city 2, the penalty is $22; and at city 3, the penalty is $23. The cost of transporting 1 million gallons of water from each reservoir to each city is shown in Table below.

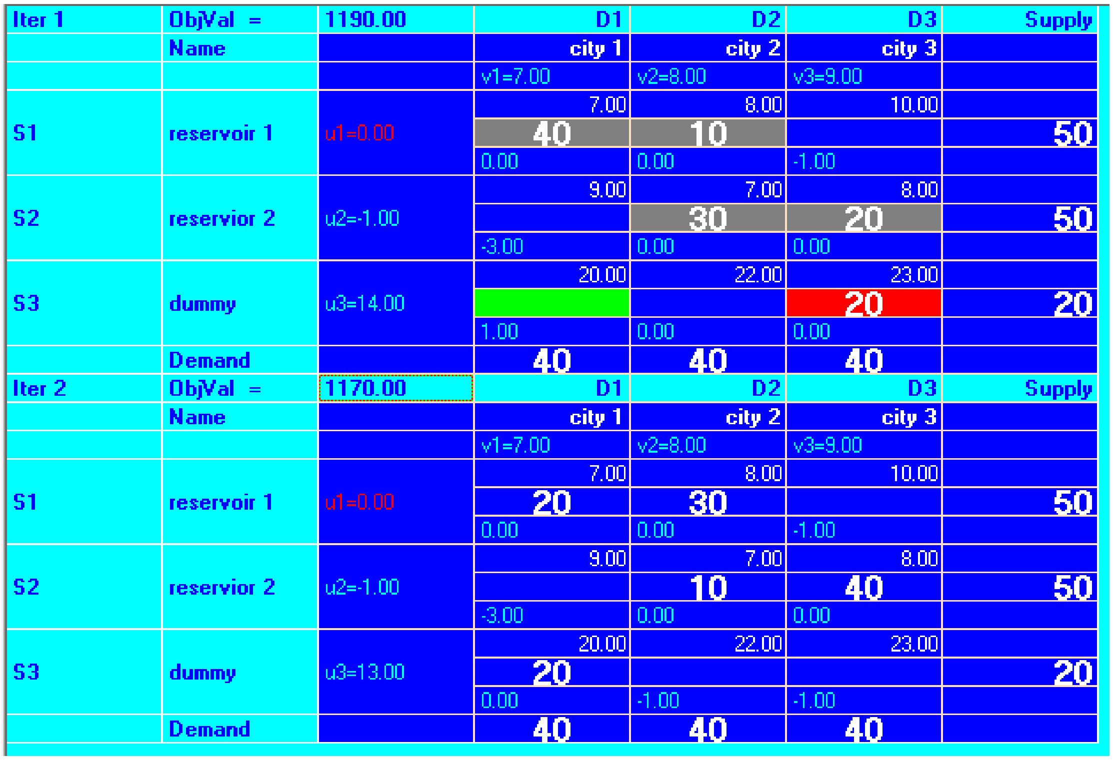
Formulate a balanced transportation problem that can be used to minimize the sum of shortage and transport costs. (use north west corner method)

Shipping Costs for Reservoir			
To From	City 1	City 2	City 3
Reservoir 1	$7	$8	$10
Reservoir 2	$9	$7	$8

Sol:

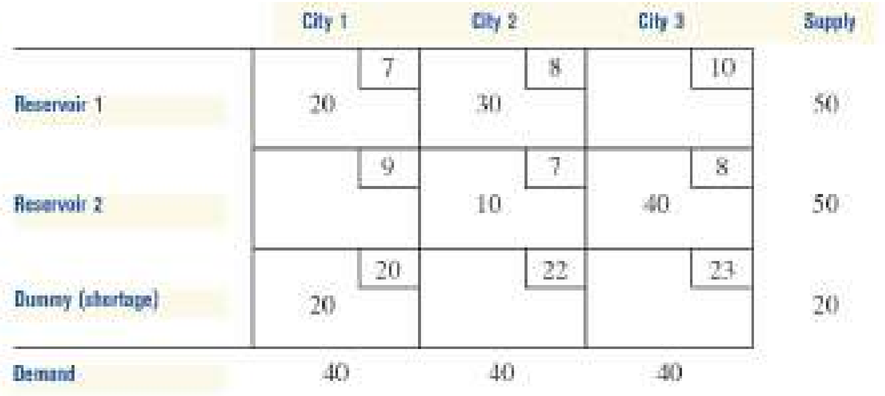
Solution In this problem, Daily supply _ 50 _ 50 _ 100 million gallons per day Daily demand _ 40 _ 40 _ 40 _ 120 million gallons per day To balance the problem, we add a dummy (or shortage) *supply point* having a supply of 120 _ 100 _ 20 million gallons per day. The cost of shipping 1 million gallons from the dummy supply point to a city is just the shortage cost per million gallons for that city. Table 5 shows the balanced transportation problem and its optimal solution. Reservoir 1 should send 20 million gallons per day to city 1 and 30 million gallons per day to city 2, whereas

Reservoir 2 should send 10 million gallons per day to city 2 and 40 million gallons per day to city 3. Twenty million gallons per day of city 1's demand will be unsatisfied.



Iter 1	ObjVal =	1190.00	D1	D2	D3	Supply
	Name		city 1	city 2	city 3	
			v1=7.00	v2=8.00	v3=9.00	
			7.00	8.00	10.00	
S1	reservoir 1	u1=0.00	40	10		50
			0.00	0.00	-1.00	
			9.00	7.00	8.00	
S2	reservior 2	u2=-1.00		30	20	50
			-3.00	0.00	0.00	
			20.00	22.00	23.00	
S3	dummy	u3=14.00			20	20
			1.00	0.00	0.00	
	Demand		40	40	40	
Iter 2	ObjVal =	1170.00	D1	D2	D3	Supply
	Name		city 1	city 2	city 3	
			v1=7.00	v2=8.00	v3=9.00	
			7.00	8.00	10.00	
S1	reservoir 1	u1=0.00	20	30		50
			0.00	0.00	-1.00	
			9.00	7.00	8.00	
S2	reservoir 2	u2=-1.00		10	40	50
			-3.00	0.00	0.00	
			20.00	22.00	23.00	
S3	dummy	u3=13.00	20			20
			0.00	-1.00	-1.00	
	Demand		40	40	40	

TABLE 5
Transportation Tableau for Reservoir



	City 1	City 2	City 3	Supply
Reservoir 1	7 20	8 30	10	50
Reservoir 2	9	7 10	8 40	50
Dummy (shortage)	20 20	22	23	20
Demand	40	40	40	

Quiz 15-8-2016

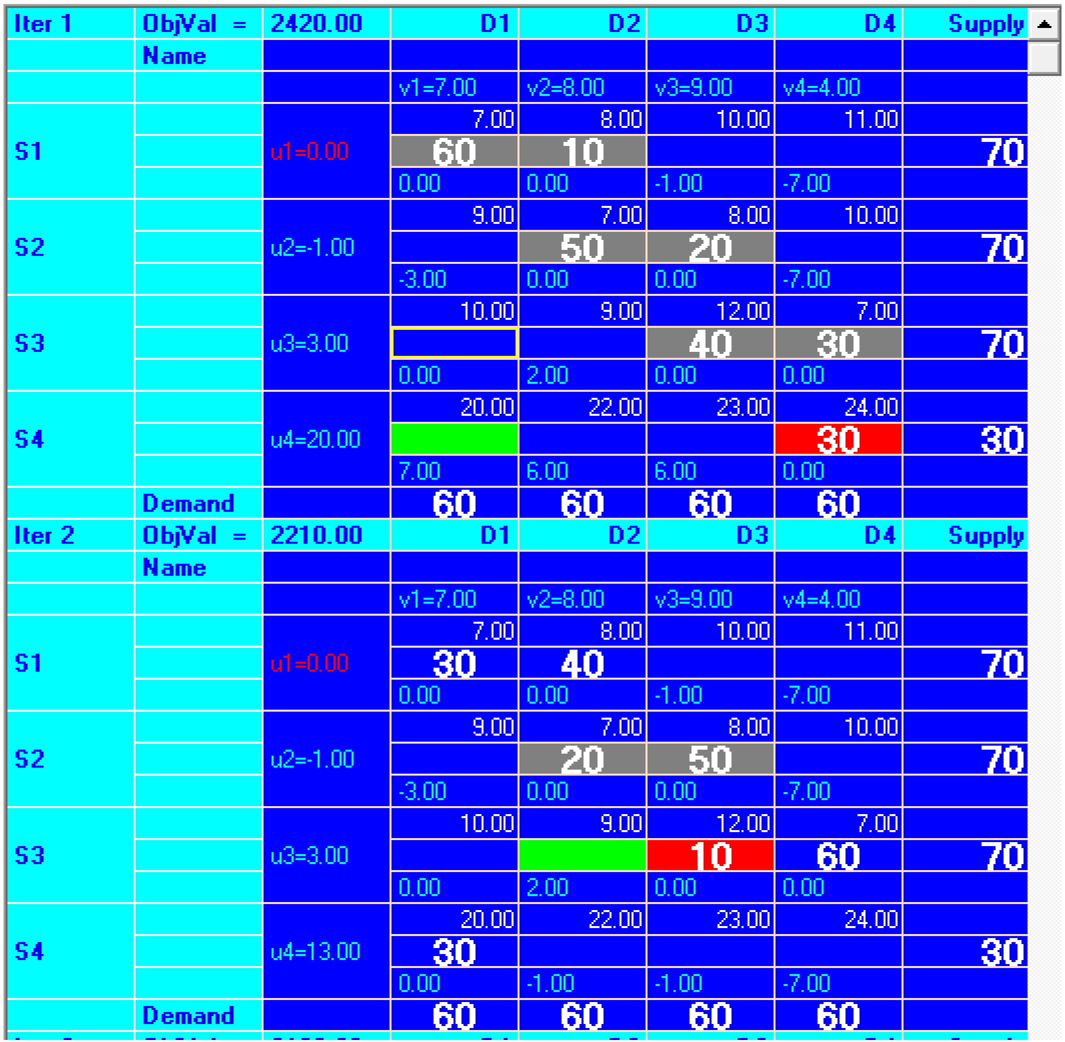
Three reservoirs are available to supply the water needs of four cities. Each reservoir can supply up to 70 million gallons of water per day. Each city would like to receive 60 million gallons per day. For each million gallons per day of unmet demand, there is a penalty. At city 1, the penalty is \$20; at city 2, the penalty is \$22; at city 3, the penalty is \$23; and at city 4, the penalty is \$24. The cost of transporting 1 million gallons of water from each reservoir to each city is shown in Table below.

Formulate a balanced transportation problem that can be used to minimize the sum of shortage and transport costs. (use north west corner method)

Shipping Costs for Reservoir				
To From	City 1	City 2	City 3	City 4
Reservoir 1	\$7	\$8	\$10	11\$
Reservoir 2	\$9	\$7	\$8	10\$
Reservoir 3	10\$	9\$	12\$	7\$

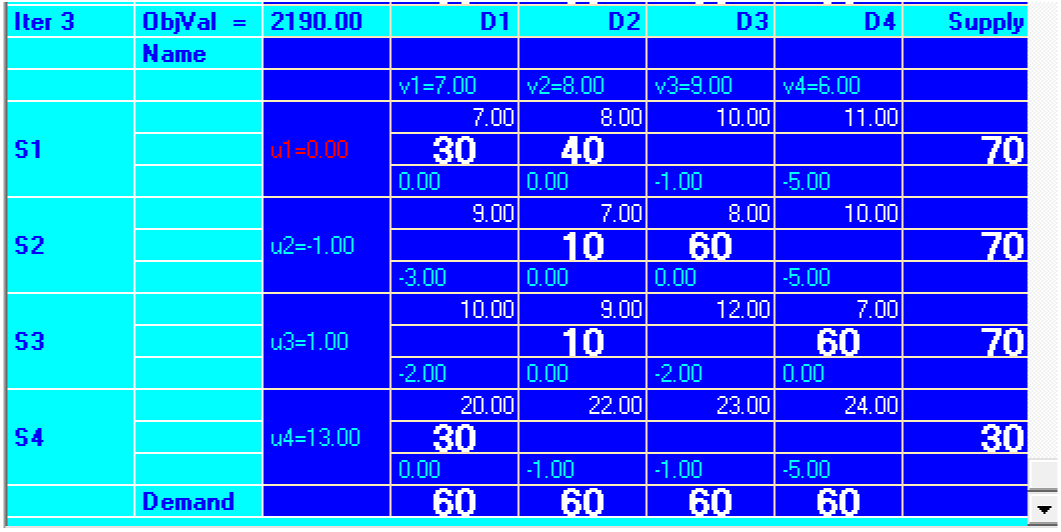
Sol:

Shipping Costs for Reservoir					
To From	City 1	City 2	City 3	City 4	Supply
Reservoir 1	\$7	\$8	\$10	11\$	70
Reservoir 2	\$9	\$7	\$8	10\$	70
Reservoir 3	10\$	9\$	12\$	7\$	70
Dummy	20\$	22\$	23\$	24\$	30
Dimand	60	60	60	60	



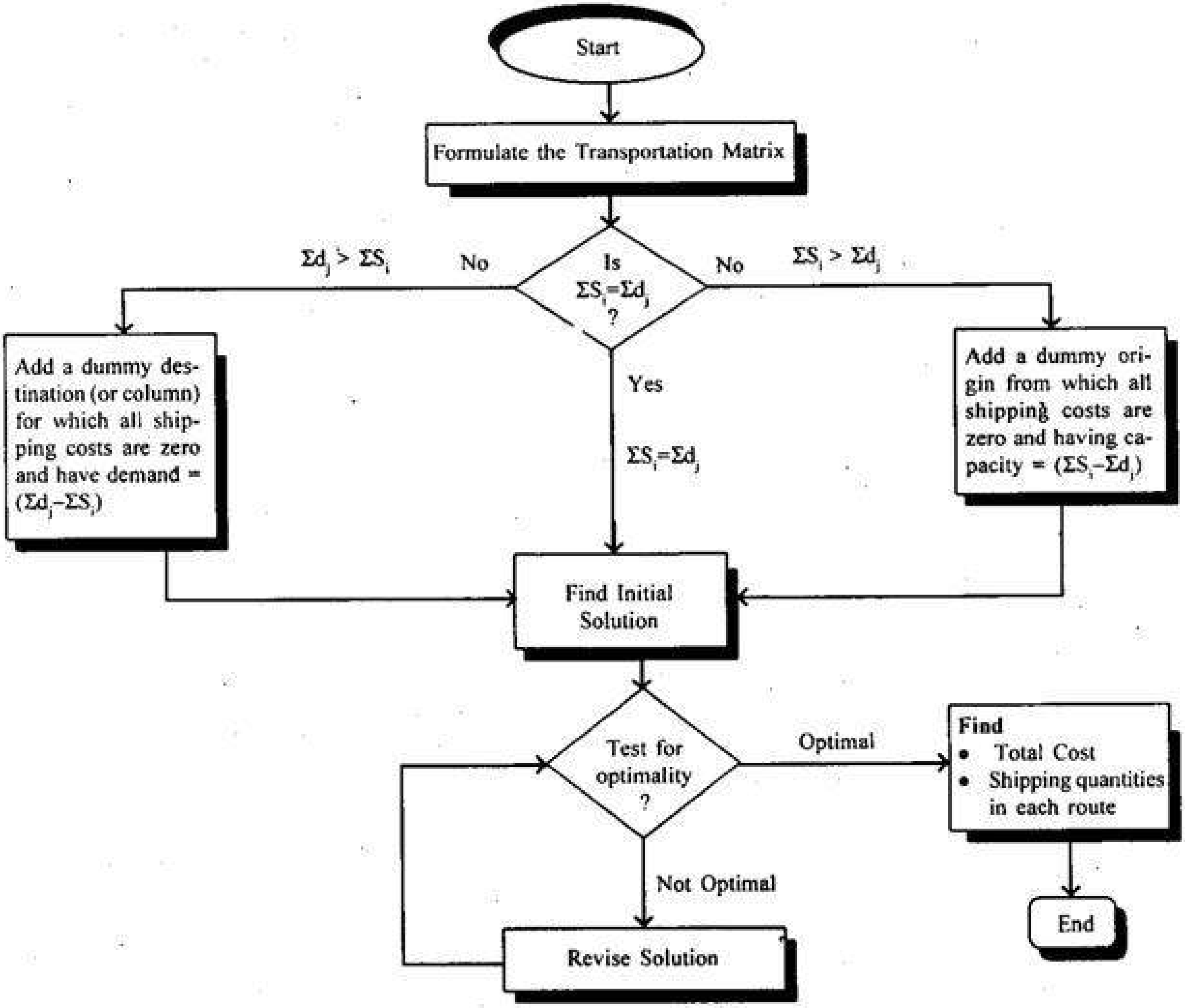
Iter 1	ObjVal =	2420.00	D1	D2	D3	D4	Supply
	Name						
			v1=7.00	v2=8.00	v3=9.00	v4=4.00	
			7.00	8.00	10.00	11.00	
S1		u1=0.00	60	10			70
			0.00	0.00	-1.00	-7.00	
			9.00	7.00	8.00	10.00	
S2		u2=-1.00		50	20		70
			-3.00	0.00	0.00	-7.00	
			10.00	9.00	12.00	7.00	
S3		u3=3.00			40	30	70
			0.00	2.00	0.00	0.00	
			20.00	22.00	23.00	24.00	
S4		u4=20.00	🟩			🟥 30	30
			7.00	6.00	6.00	0.00	
	Demand		60	60	60	60	

Iter 2	ObjVal =	2210.00	D1	D2	D3	D4	Supply
	Name						
			v1=7.00	v2=8.00	v3=9.00	v4=4.00	
			7.00	8.00	10.00	11.00	
S1		u1=0.00	30	40			70
			0.00	0.00	-1.00	-7.00	
			9.00	7.00	8.00	10.00	
S2		u2=-1.00		20	50		70
			-3.00	0.00	0.00	-7.00	
			10.00	9.00	12.00	7.00	
S3		u3=3.00		🟩	🟥 10	60	70
			0.00	2.00	0.00	0.00	
			20.00	22.00	23.00	24.00	
S4		u4=13.00	30				30
			0.00	-1.00	-1.00	-7.00	
	Demand		60	60	60	60	

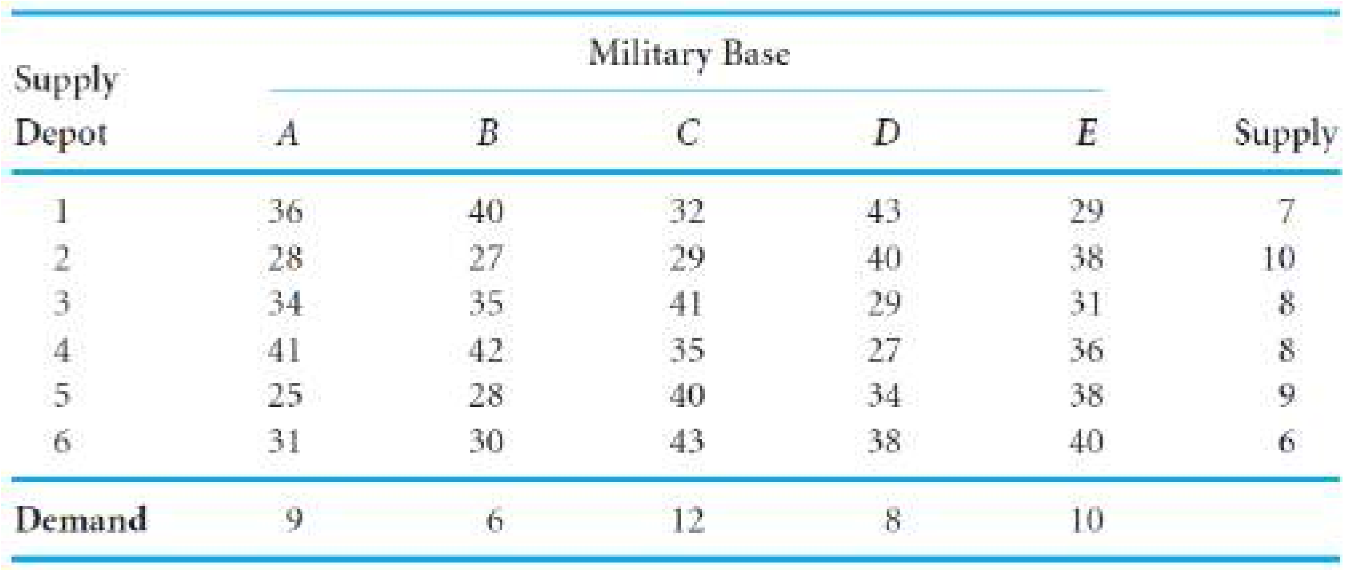


Iter 3	ObjVal =	2190.00	D1	D2	D3	D4	Supply
	Name						
			v1=7.00	v2=8.00	v3=9.00	v4=6.00	
			7.00	8.00	10.00	11.00	
S1		u1=0.00	30	40			70
			0.00	0.00	-1.00	-5.00	
			9.00	7.00	8.00	10.00	
S2		u2=-1.00		10	60		70
			-3.00	0.00	0.00	-5.00	
			10.00	9.00	12.00	7.00	
S3		u3=1.00		10		60	70
			-2.00	0.00	-2.00	0.00	
			20.00	22.00	23.00	24.00	
S4		u4=13.00	30				30
			0.00	-1.00	-1.00	-5.00	
	Demand		60	60	60	60	

Flow Chart Solution For the transportation Problem



Example: During the Gulf War, Operation Desert Storm required large amounts of military materiel and supplies to be shipped daily from supply depots in the United States to bases in the Middle East. The critical factor in the movement of these supplies was speed. The following table shows the number of planeloads of supplies available each day from each of six supply depots and the number of daily loads demanded at each of five bases. (Each planeload is approximately equal in tonnage.) Also included are the transport hours per plane, including loading and fueling, actual flight time, and unloading and refueling. Determine the optimal daily flight schedule that will minimize total transport time.



Supply	Military Base					
Depot	A	B	C	D	E	Supply
1	36	40	32	43	29	7
2	28	27	29	40	38	10
3	34	35	41	29	31	8
4	41	42	35	27	36	8
5	25	28	40	34	38	9
6	31	30	43	38	40	6
Demand	9	6	12	8	10	

Example: The Hardrock Concrete Company has plants in three locations and is currently working on three major construction projects, each located at a different site. The shipping cost per truckload of concrete, daily plant capacities, and daily project requirements are provided in the accompanying table. Formulate an initial feasible solution to Hardrock's transportation problem using Vogel Approximation Method.

To From	Project A	Project B	Project C	Plant Capacities
Plant 1	**$10**	**$ 4**	**$11**	**70**
Plant 2	**12**	**5**	**8**	**50**
Plant 3	**9**	**7**	**6**	**30**
Project Requirements	**40**	**50**	**60**	