Theory of Structures

DWE-3xxx

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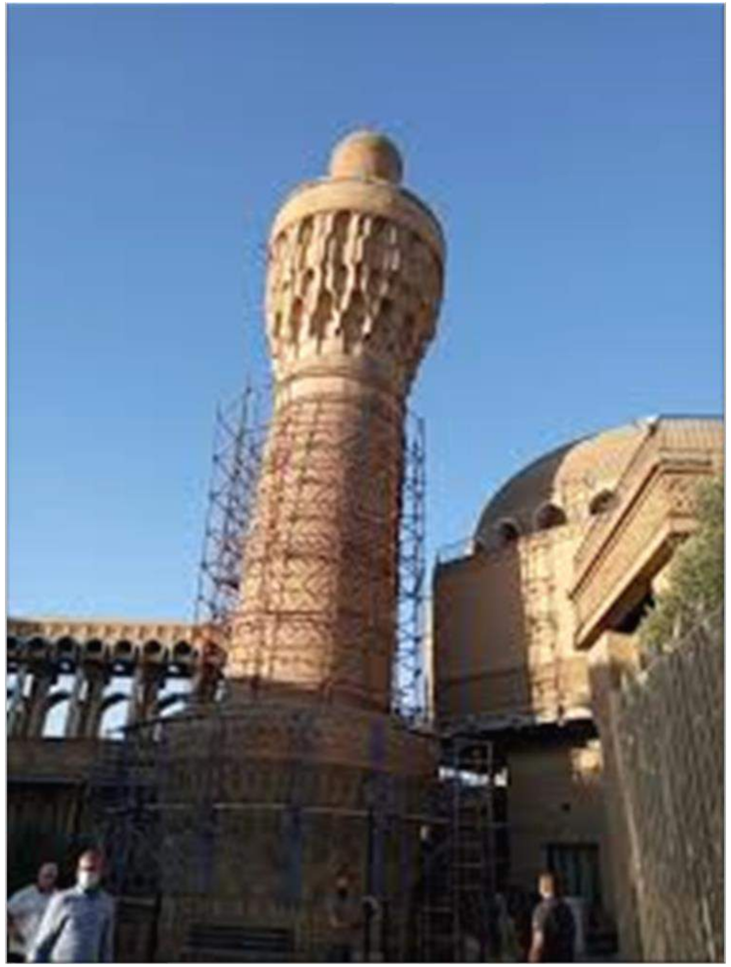



شركة توزيع الكهرباء م.ع.م
المبنى رقم (١)

Course Description:

This course covers the outlines of general principles, indeterminacy and stability, shear and moment diagrams of structures, trusses, approximate analysis, influence lines and moving concentrated loads, analysis of statically determinate structures, analysis of statically indeterminate structures.

Course Objectives:

1. To impart the principles of elastic structural analysis and behaviour of indeterminate structures.
2. Ability to idealize and analyze statically determinate and indeterminate structures.
3. To enable the student to get a feeling of how real-life structures behave.
4. Familiarity with professional and contemporary issues.

Student Outcomes:

The student after undergoing this course will be able to:

1. To understand analysis of indeterminate structures and adopt an appropriate structural analysis technique.

2. Determine response of structures by classical, iterative and matrix methods.

Text Book:

Structural Analysis by R. C. Hibbeler- 8th edition.

REFERENCES:

- Theory of Structures by S.P. Timoshenko and D. H. Young - 2nd edition.
- Theory of Structures by Yuang Yu Hsiegh.
- Structural Analysis by Aslam Kassimali, 4th edition.
- Structural and Stress Analysis by Dr. T.H.G Megson – 2nd edition, 2000.

Course Assessement:

Term Tests	Laboratory	Quizzes	Project	Final Exam
30.0%	0.0%	10.0%	----	60.0%

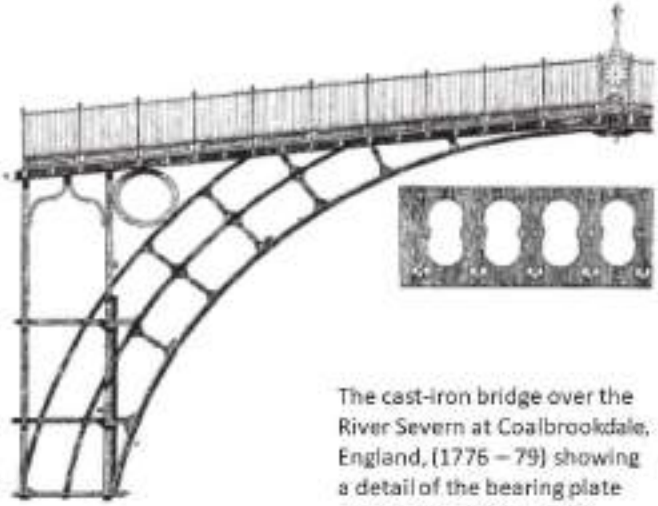
Syllabus

week	Topics Covered
1	Introduction to structural analysis
2	Determinacy and stability of structures
3	Shear and moment diagrams of structures
4	Shear and moment diagrams of structures
5	Simple Trusses and Compound Trusses
6	Complex Trusses OR Approximate Analysis of Structures
7	Influence lines and moving concentrated loads
8	Influence lines and moving concentrated loads
9	Deflection of determinate structures
10	Deflection of determinate structures
11	Analysis of indeterminate structures- Consistent deformation method.
12	Analysis of indeterminate structures- Consistent deformation method.
13	Analysis of indeterminate structures using Slope-Deflection Method
14	Analysis of indeterminate structures using Moment-Distribution Method
15	Review

Unit-1

Introduction to Structural Analysis

1.1 Types of Structural Forms



The cast-iron bridge over the River Severn at Coalbrookdale, England, (1776 – 79) showing a detail of the bearing plate [Mehrtens, 1908, p. 270]

Suspension bridge over the Menai Strait near Bangor, Wales [Dietrich, 1998, p. 115]

Röbling's Niagara Bridge [Güntheroth & Kahlow, 2005, p. 135]

The first home of the Institute of Engineers of Ways of Communication and the Russian Highways Authority – Jusupov Palace on the River Fontanka, St. Petersburg [Fedorov, 2005, p. 57]

Göltzsch Viaduct around 1850
[Conrad & Hänseroth, 1995, p. 762]

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The Garabit Viaduct shortly after completion [Eiffel, 1889]

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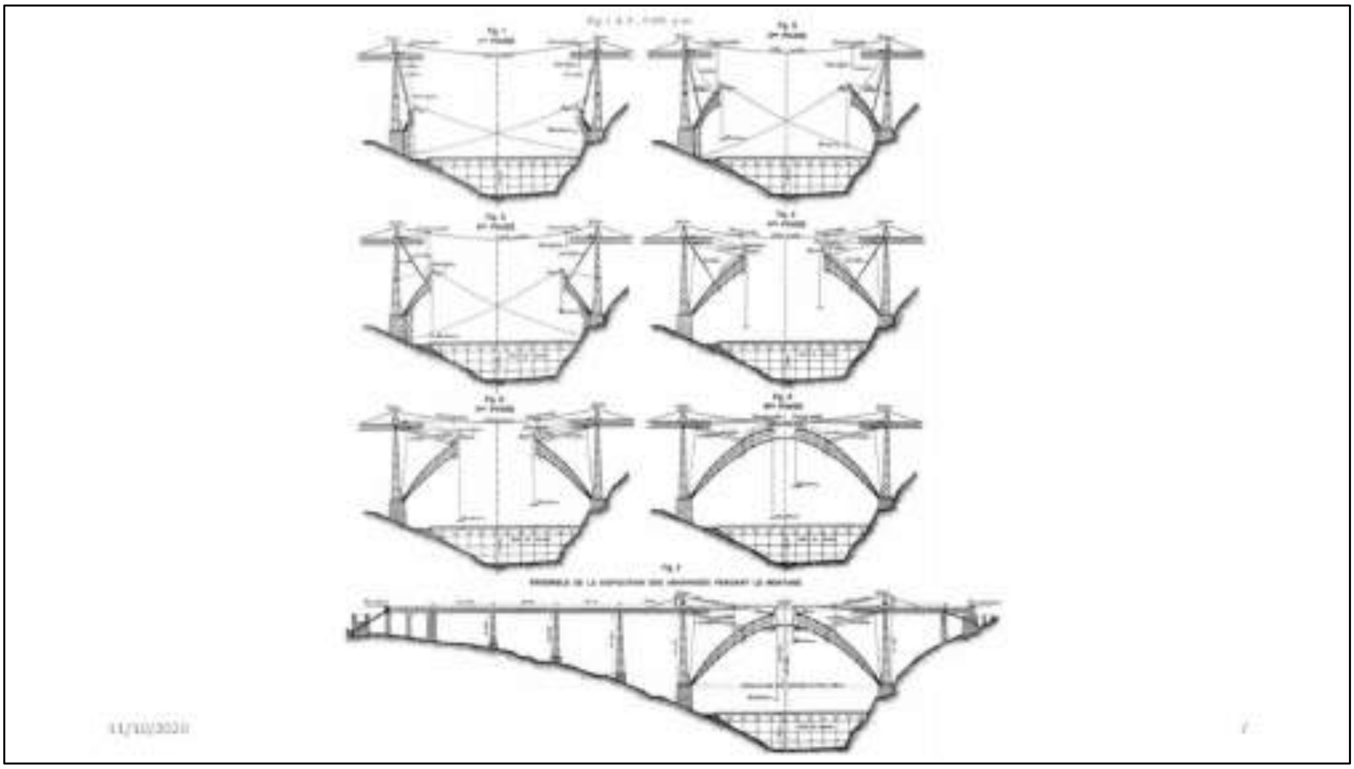
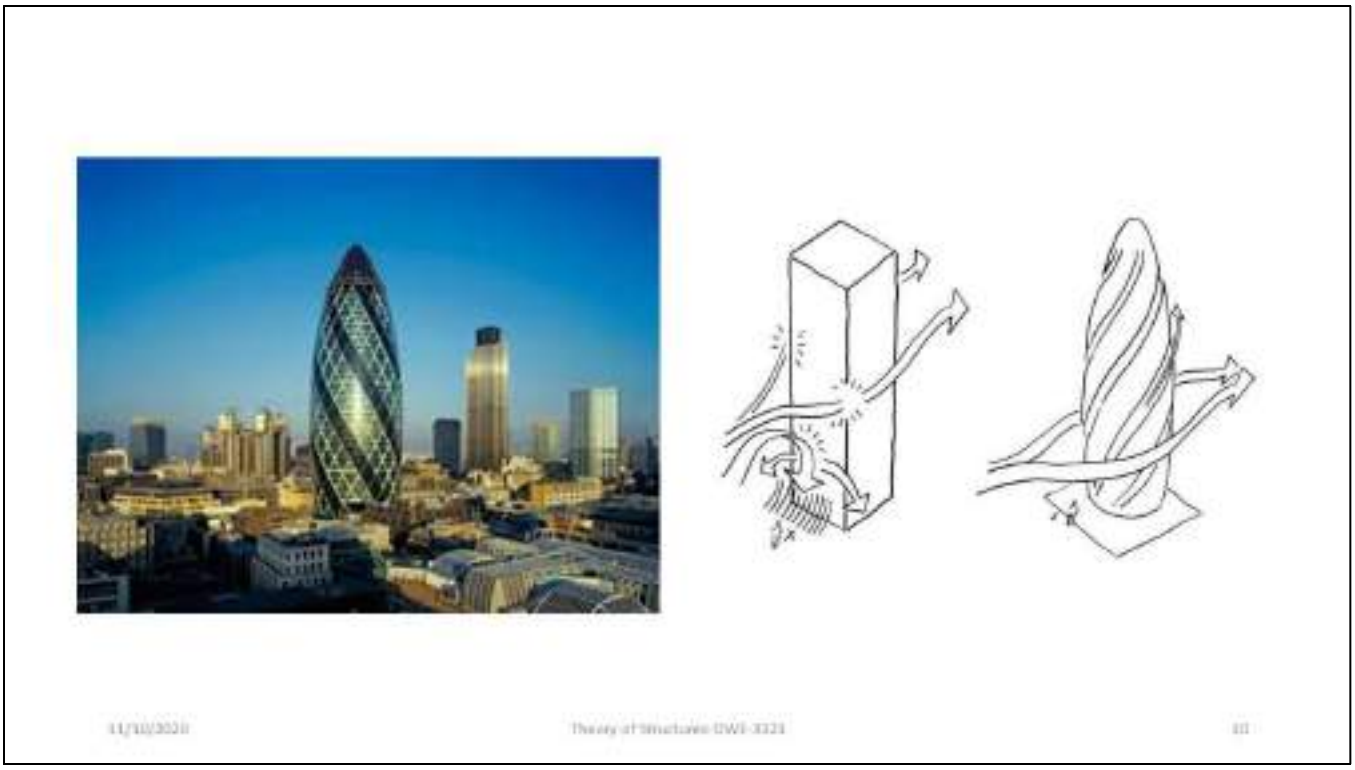
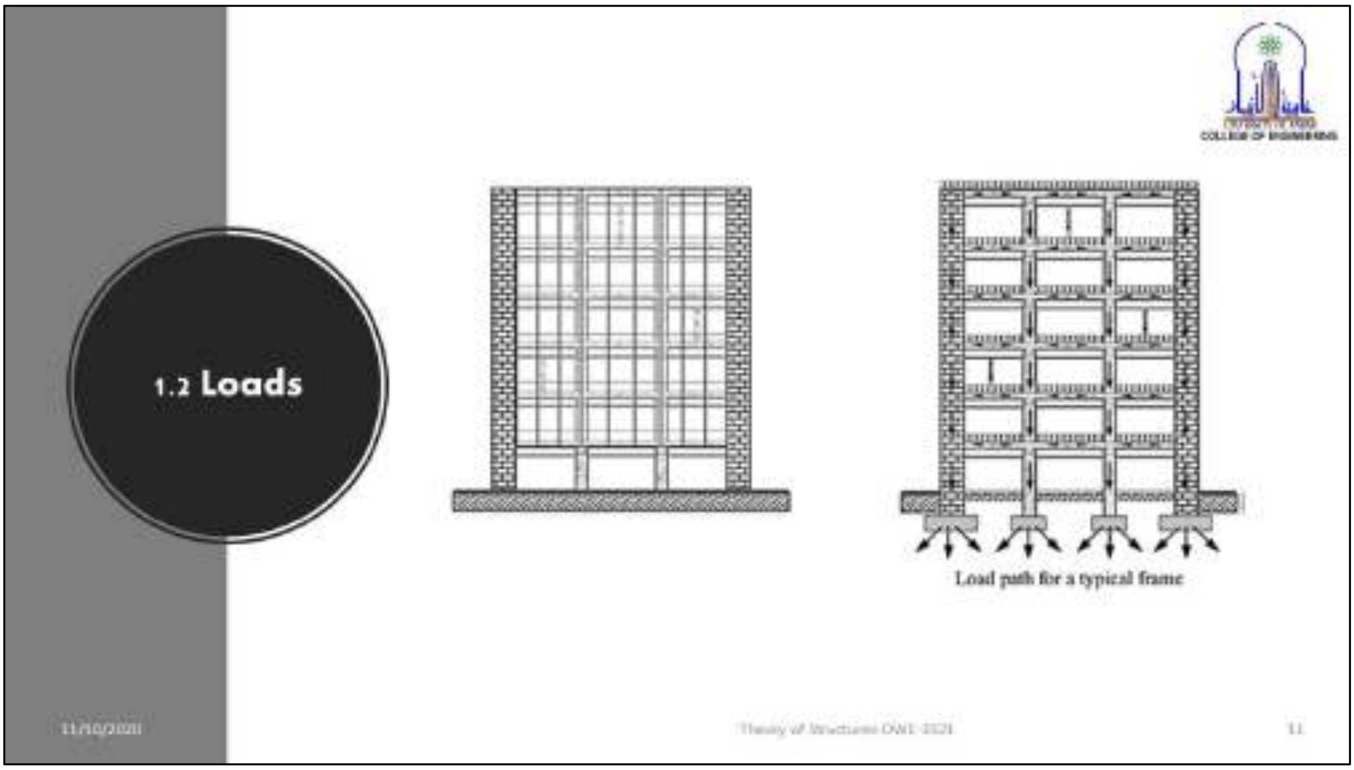
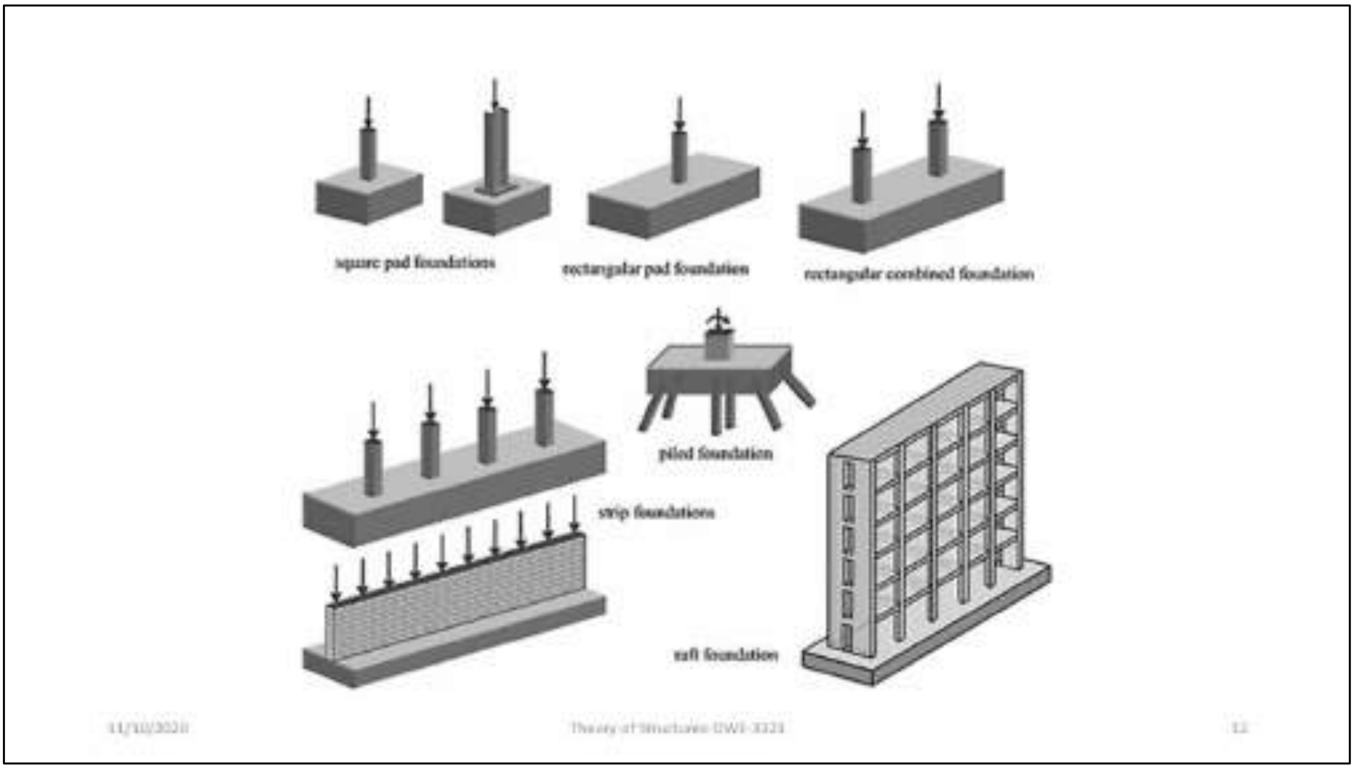
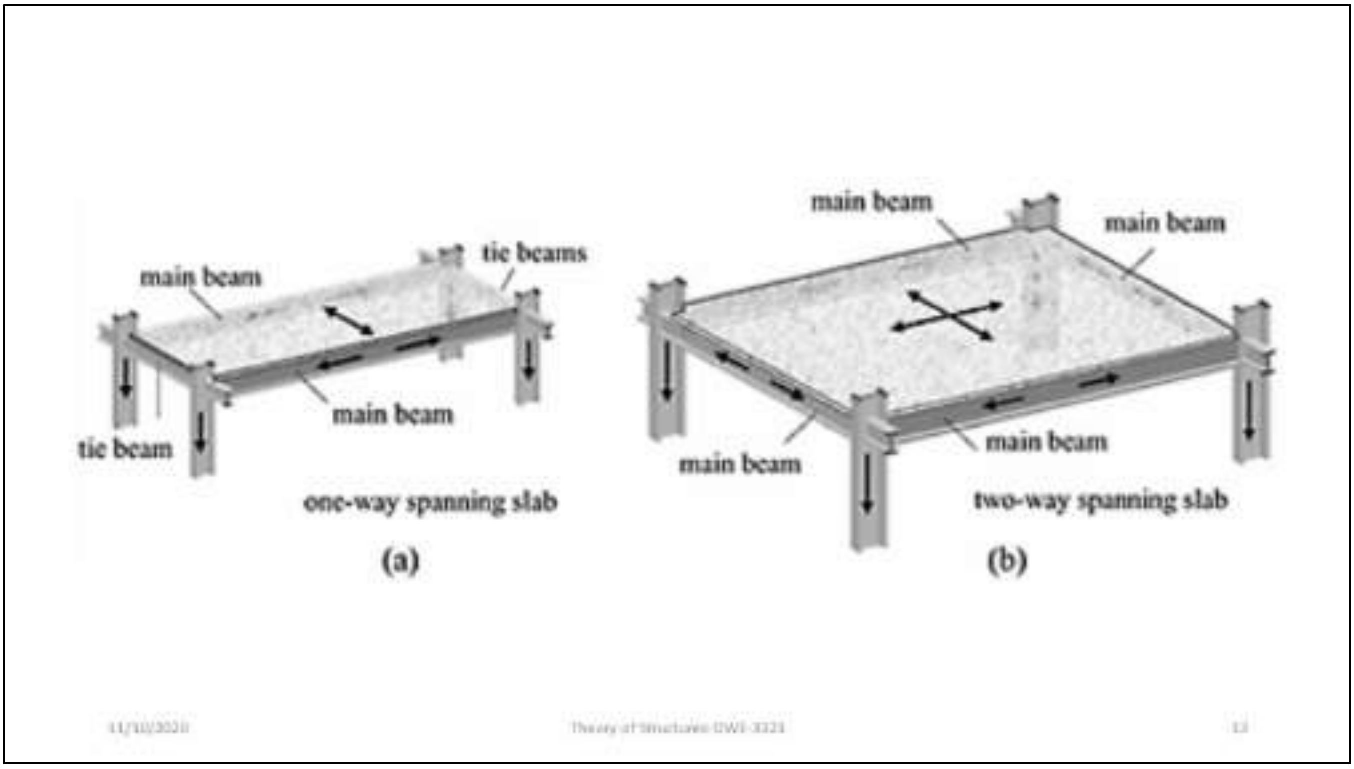
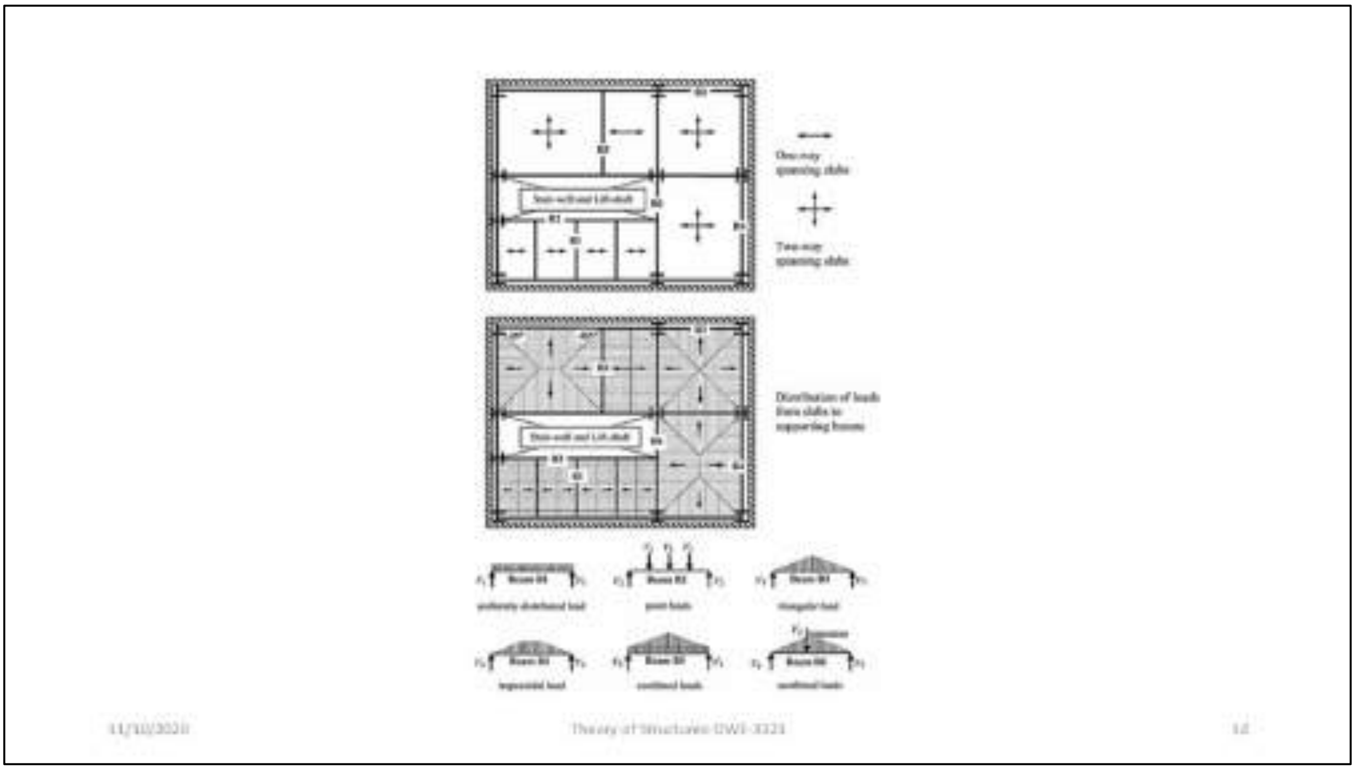





1.2 Loads
Load path for a typical frame


square pad foundations
rectangular pad foundation
rectangular combined foundation
piled foundation
strip foundations
raft foundation


main beam
tie beams
main beam
tie beam
one-way spanning slab
(a)
main beam
main beam
main beam
main beam
two-way spanning slab
(b)



- Dead Loads
- Live loads
- Moving loads
- Impact loads
- Wind loads
- Snaw loads
- Earthquake loads
- Blast loads
- Temperature load
- Soil pressure
- Hydrostatic load
- Centrifugal forces

TABLE 1–1 Codes

General Building Codes
Minimum Design Loads for Buildings and Other Structures, ASCE/SEI 7-10, American Society of Civil Engineers
International Building Code
Design Codes
Building Code Requirements for Reinforced Concrete, Am. Conc. Inst. (ACI)
Manual of Steel Construction, American Institute of Steel Construction (AISC)
Standard Specifications for Highway Bridges, American Association of State Highway and Transportation Officials (AASHTO)
National Design Specification for Wood Construction, American Forest and Paper Association (AFPA)
Manual for Railway Engineering, American Railway Engineering Association (AREA)

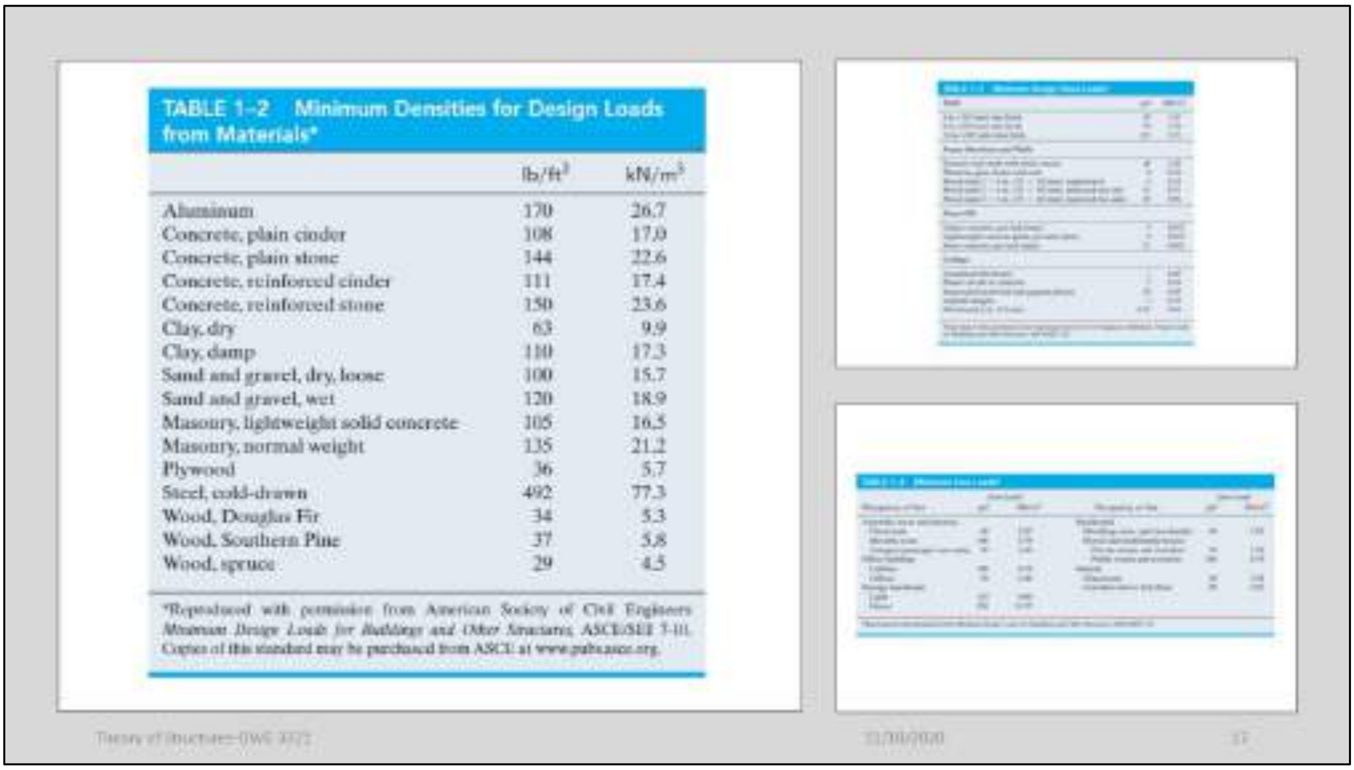
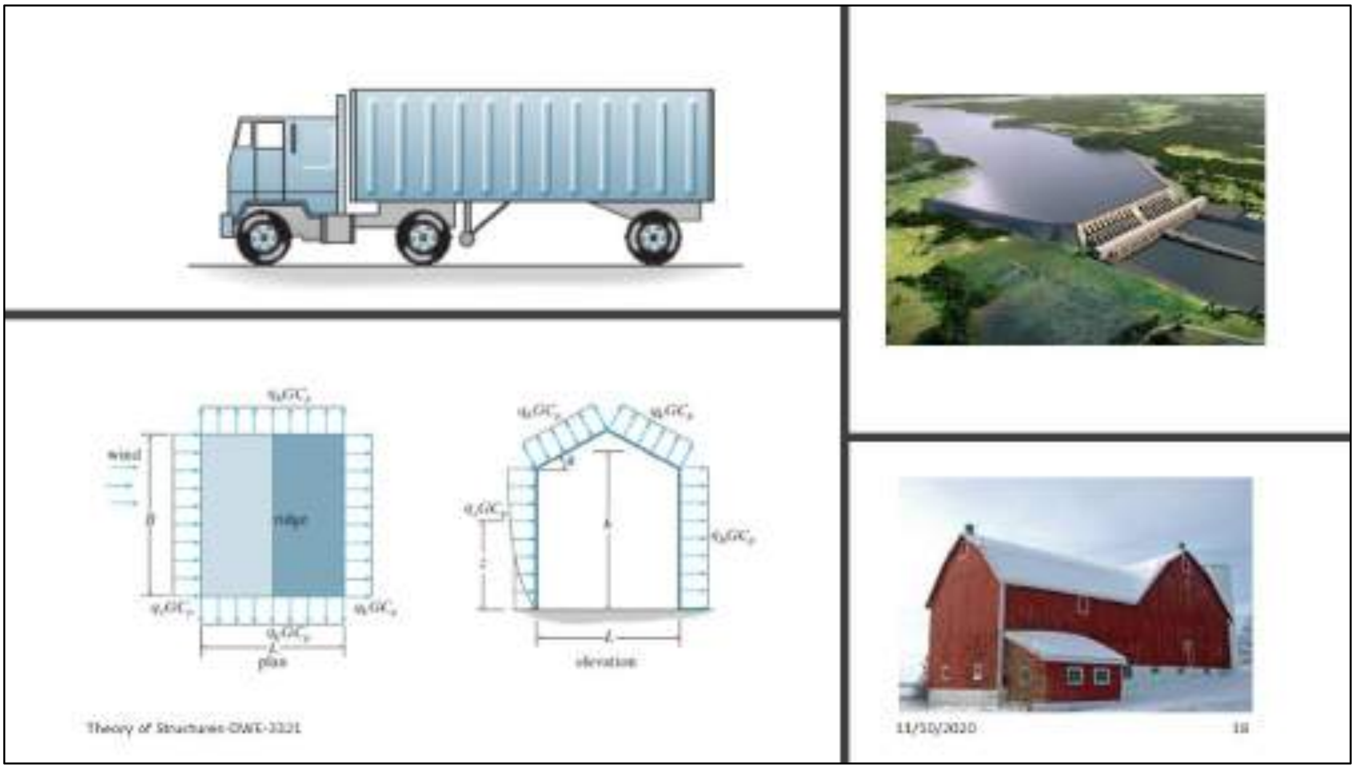


TABLE 1–2 Minimum Densities for Design Loads from Materials*

	lb/ft³	kN/m³
Aluminum	170	26.7
Concrete, plain cinder	108	17.0
Concrete, plain stone	144	22.6
Concrete, reinforced cinder	111	17.4
Concrete, reinforced stone	150	23.6
Clay, dry	63	9.9
Clay, damp	110	17.3
Sand and gravel, dry, loose	100	15.7
Sand and gravel, wet	120	18.9
Masonry, lightweight solid concrete	105	16.5
Masonry, normal weight	135	21.2
Plywood	36	5.7
Steel, cold-drawn	492	77.3
Wood, Douglas Fir	34	5.3
Wood, Southern Pine	37	5.8
Wood, spruce	29	4.5

*Reproduced with permission from American Society of Civil Engineers *Minimum Design Loads for Buildings and Other Structures*, ASCE/SEI 7-10. Copies of this standard may be purchased from ASCE at www.pubs.asce.org.



Simple Example:

EXAMPLE 1.1

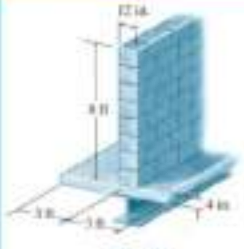


Fig. 1–8

The floor beam in Fig. 1–8 is used to support the 6-ft width of a lightweight plain concrete slab having a thickness of 4 in. The slab serves as a portion of the ceiling for the floor below, and therefore its bottom is coated with plaster. Furthermore, an 8-ft-high, 12-in.-thick lightweight solid concrete block wall is directly over the top flange of the beam. Determine the loading on the beam measured per foot of length of the beam.

SOLUTION

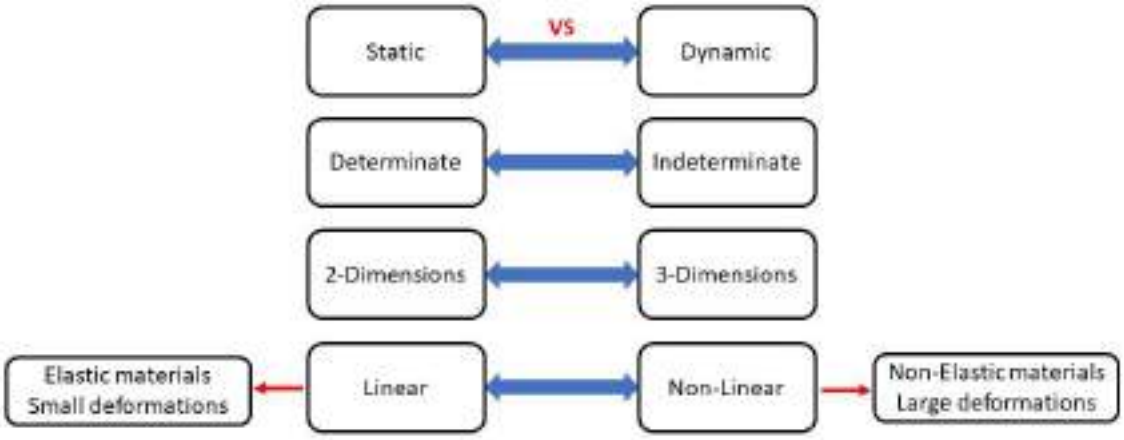
Using the data in Tables 1–2 and 1–3, we have

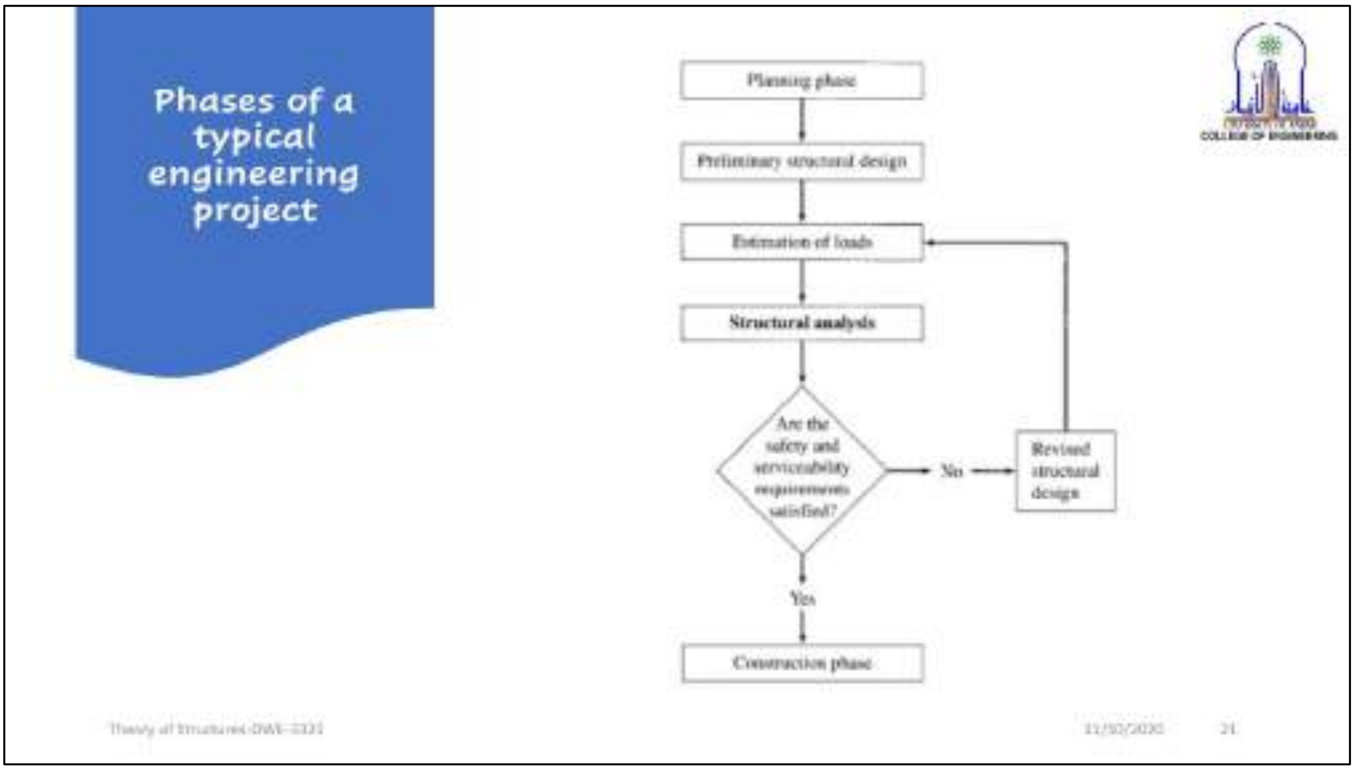
Concrete slab:	$[8\ \text{lb}/(\text{ft}^2\cdot\text{in.})](4\ \text{in.})(6\ \text{ft}) =$	192 lb/ft
Plaster ceiling:	$(5\ \text{lb}/\text{ft}^2)(6\ \text{ft}) =$	30 lb/ft
Block wall:	$(105\ \text{lb}/\text{ft}^3)(8\ \text{ft})(1\ \text{ft}) =$	840 lb/ft
Total load		1062 lb/ft = 1.06 k/ft

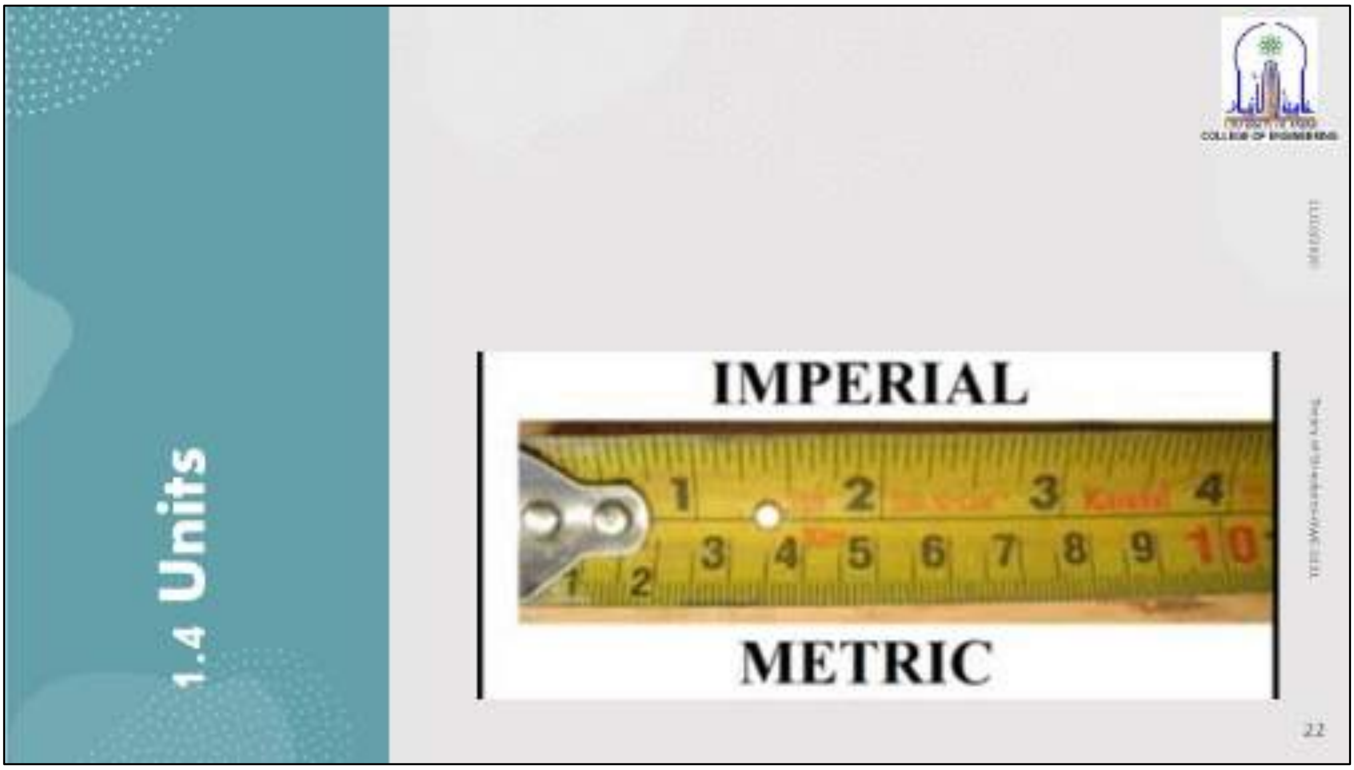
Ans.

Here the unit k stands for "kip," which symbolizes kilopounds. Hence, 1 k = 1000 lb.

1.3 Theory of Structural Analysis Classification




Phases of a typical engineering project
Planning phase
Preliminary structural design
Estimation of loads
Structural analysis
Are the safety and serviceability requirements satisfied?
No
Revised structural design
Yes
Construction phase
21


1.4 Units
IMPERIAL
METRIC
22

S.I (System International) Units: m, N, kg, sec

Imperial System Unites: ft, lb, slug, sec

MPa = 10^6 Pa = 10^6 N/mm^2
= 10^6 N/10^6 mm^2 = N/mm^2

Example:

N/mm^2 → psi (lb/in^2):

$$\frac{N}{mm^2} = \frac{N \times \frac{1}{2.24} \times \frac{lb}{N}}{mm^2 \times \left(\frac{1}{25.4}\right)^2 \times \frac{in^2}{mm^2}} = \frac{(25.4)^2}{2.24}\,\frac{lb}{in^2}$$

$$= 145\frac{lb}{in^2} = 145 \text{ psi}$$

Conversion Factors
in = 25.4 mm
m = 3.28 ft
lb = 2.24 N
Kg = 9.81 N

Example:

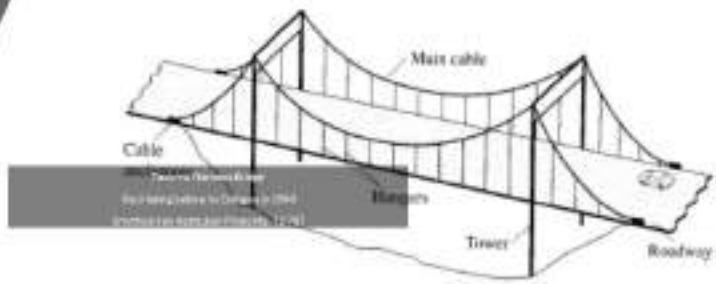
Pcf (lb/ft^3) → kN/m^3:

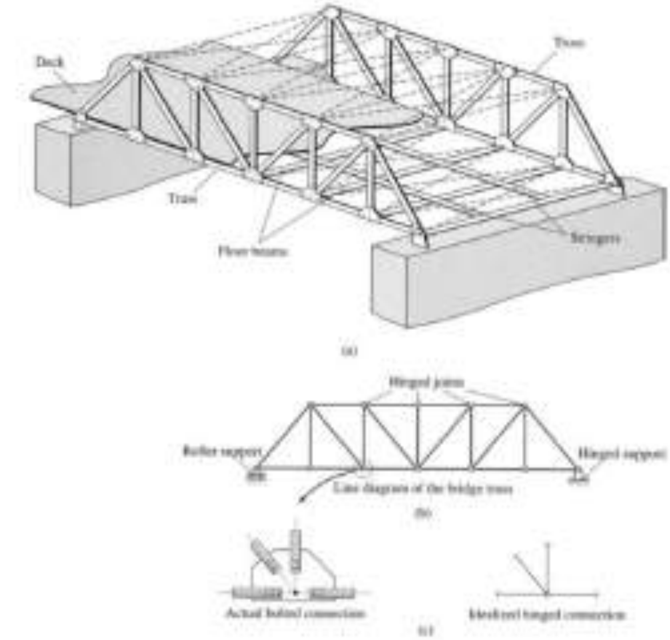
$$\frac{lb}{ft^3} = \frac{\frac{2.24}{1000}N}{\left(\frac{1}{3.28}\right)^3} = 0.079\frac{kN}{m^3}$$

1.5 Multiplication Factors

10^{3} = kilo
10^{6} = mega
10^{9} = giga
10^{12} = tetra
10^{-3} = milli
10^{-6} = micro
10^{-9} = nano

1.6 Idealization of a Structure and Loading





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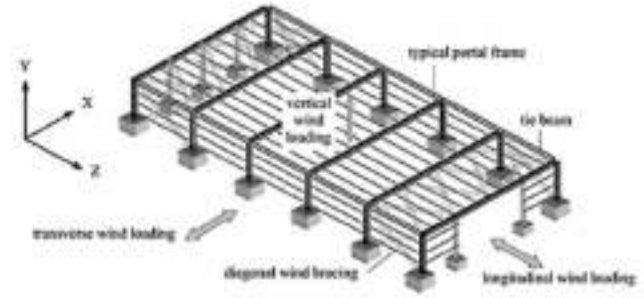
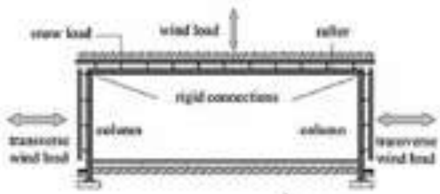
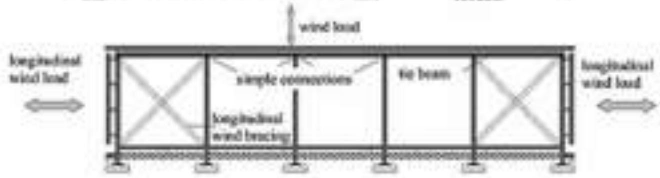




Individual frames designed as rigid-jointed in the X-Y plane for dead/imposed and transverse wind loads.
In the Y-Z plane bracing is provided (pin-jointed) to transfer the longitudinal wind forces.



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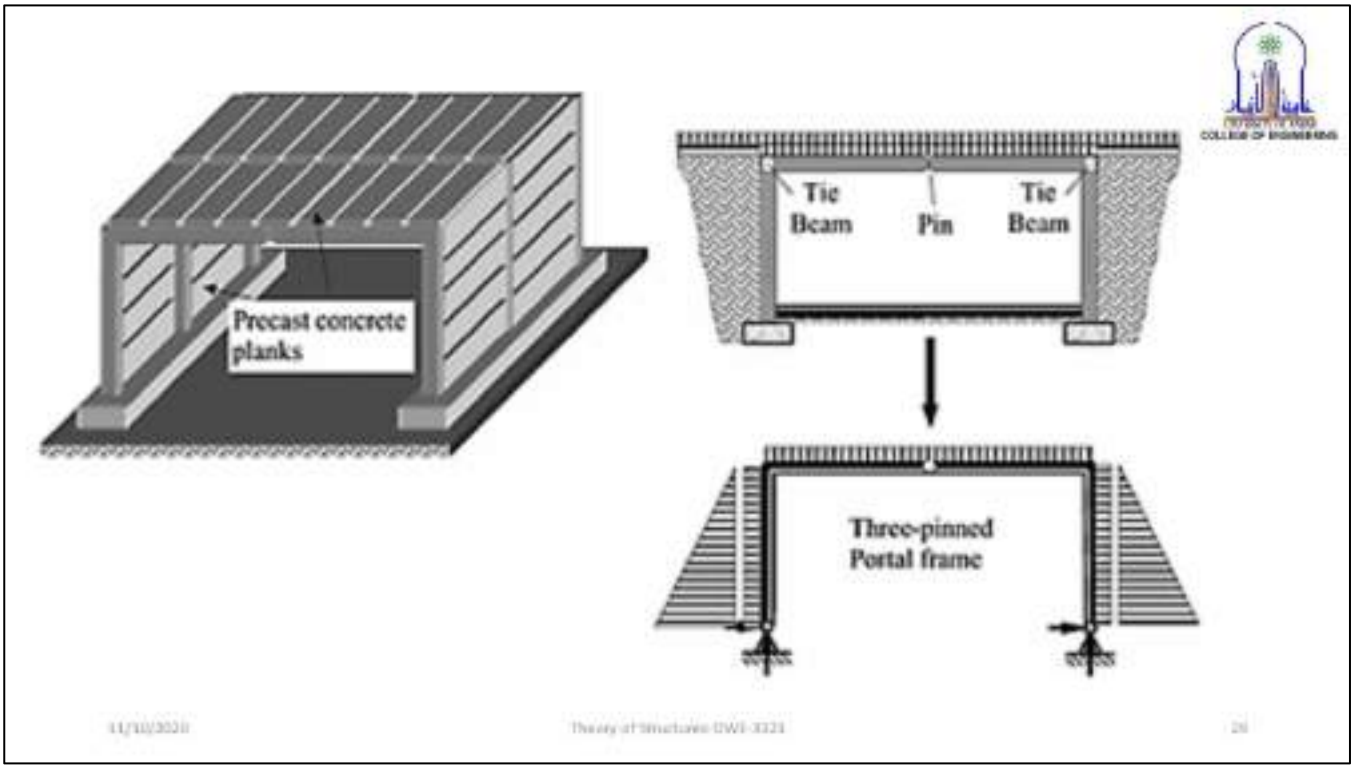
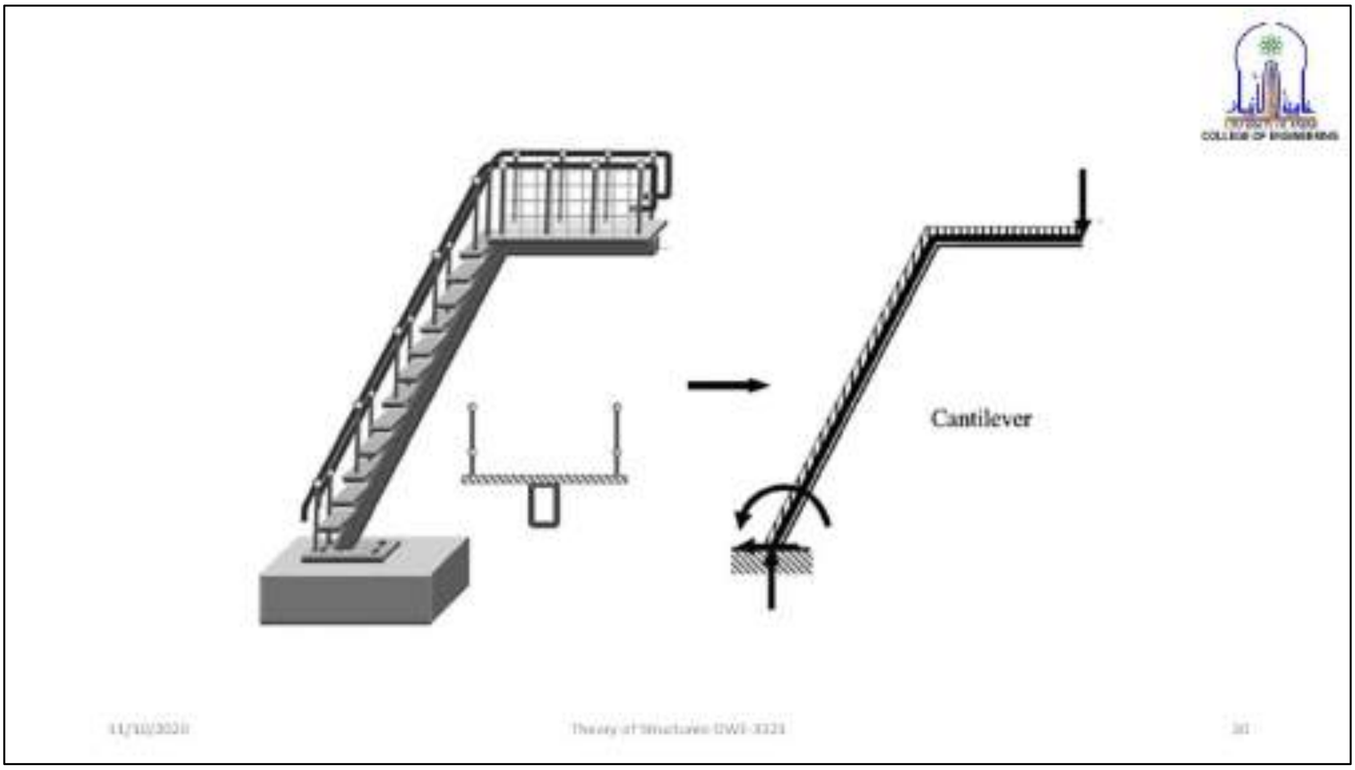
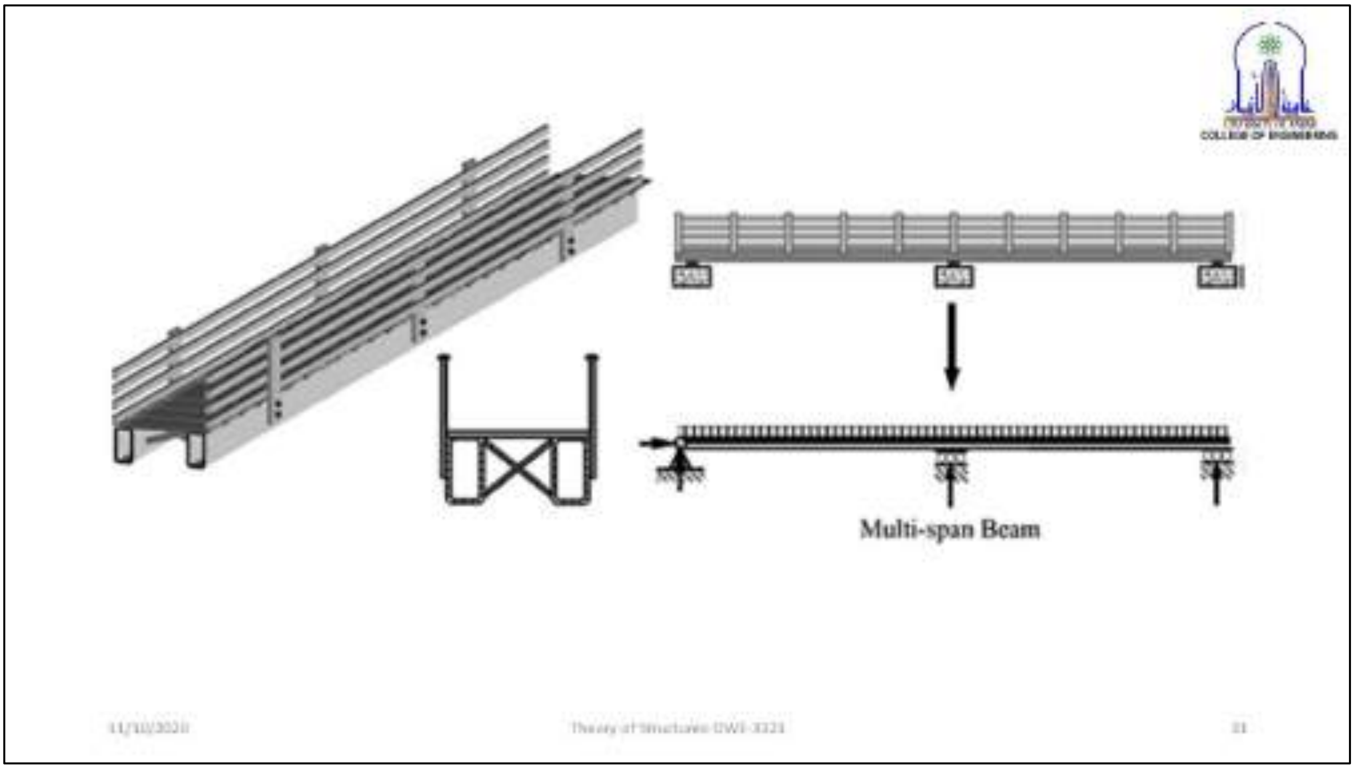
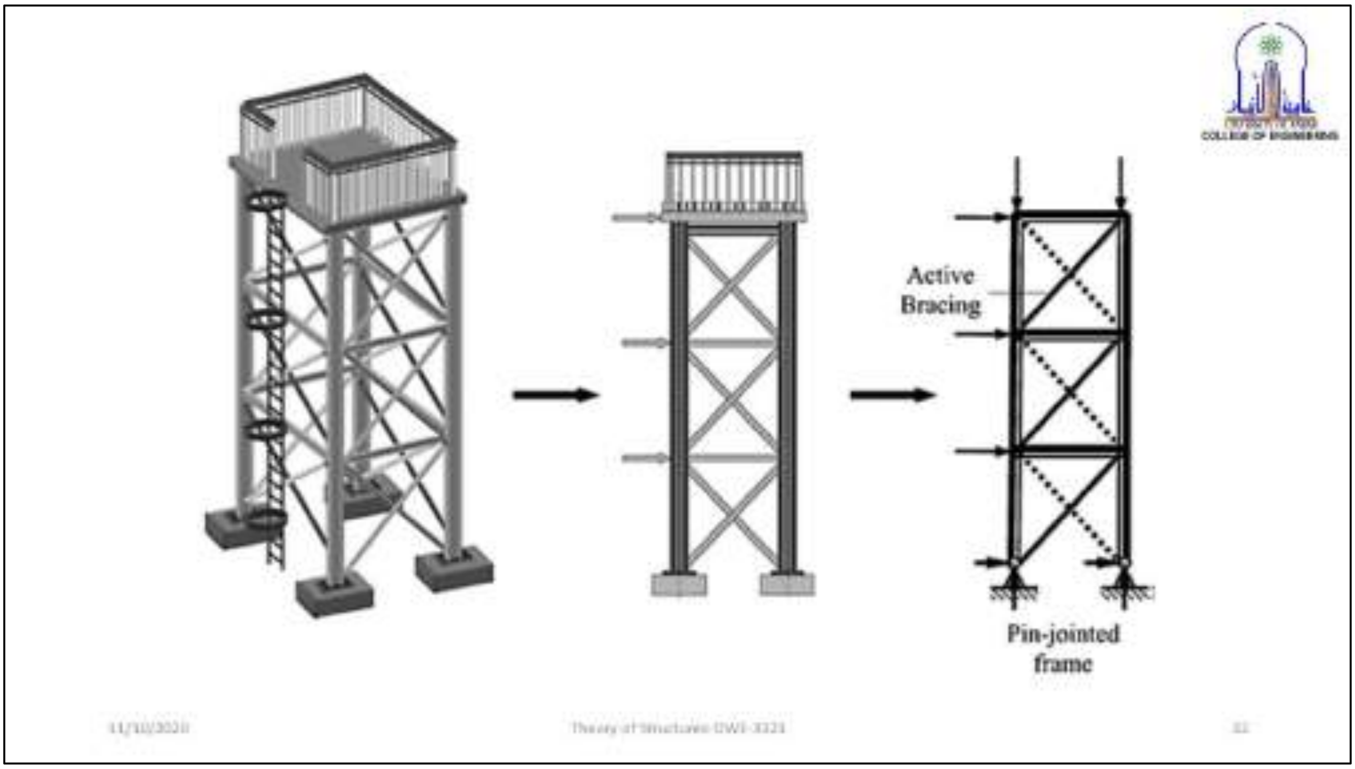
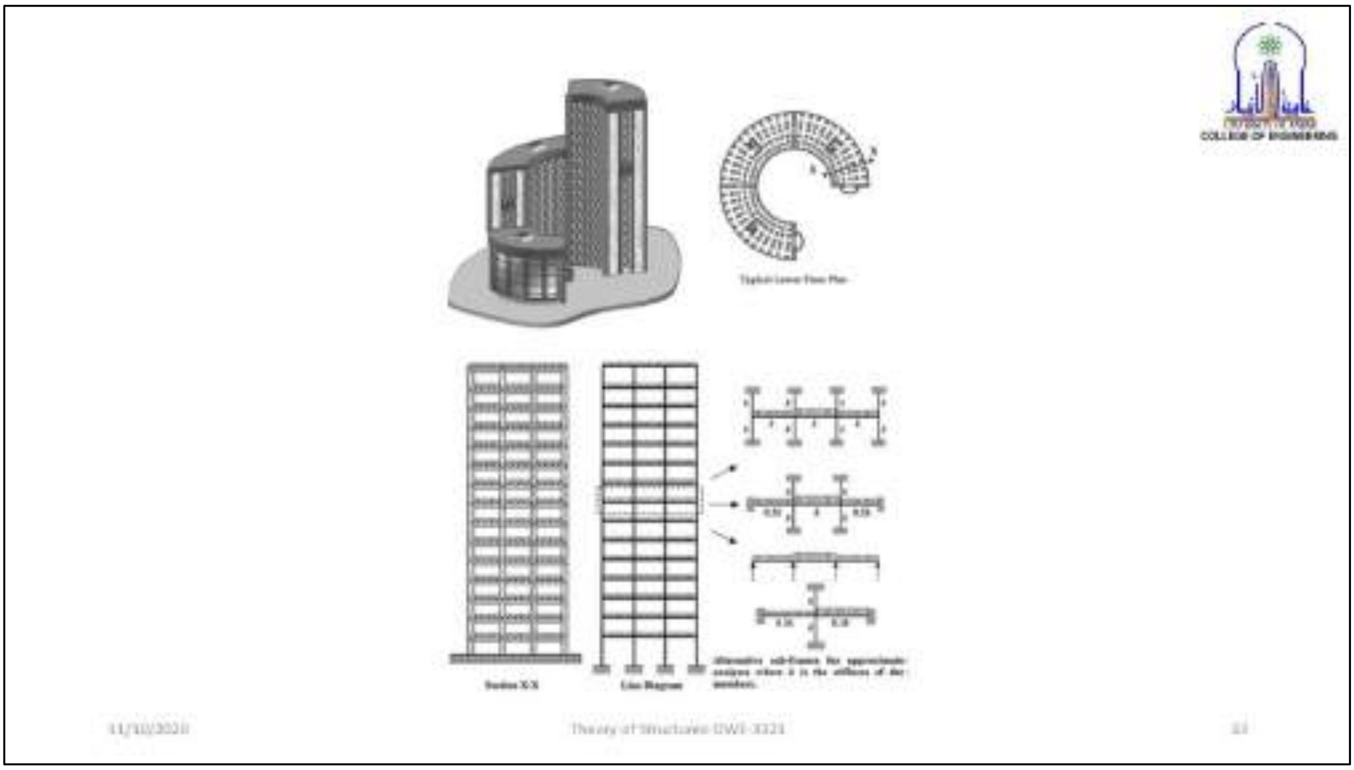
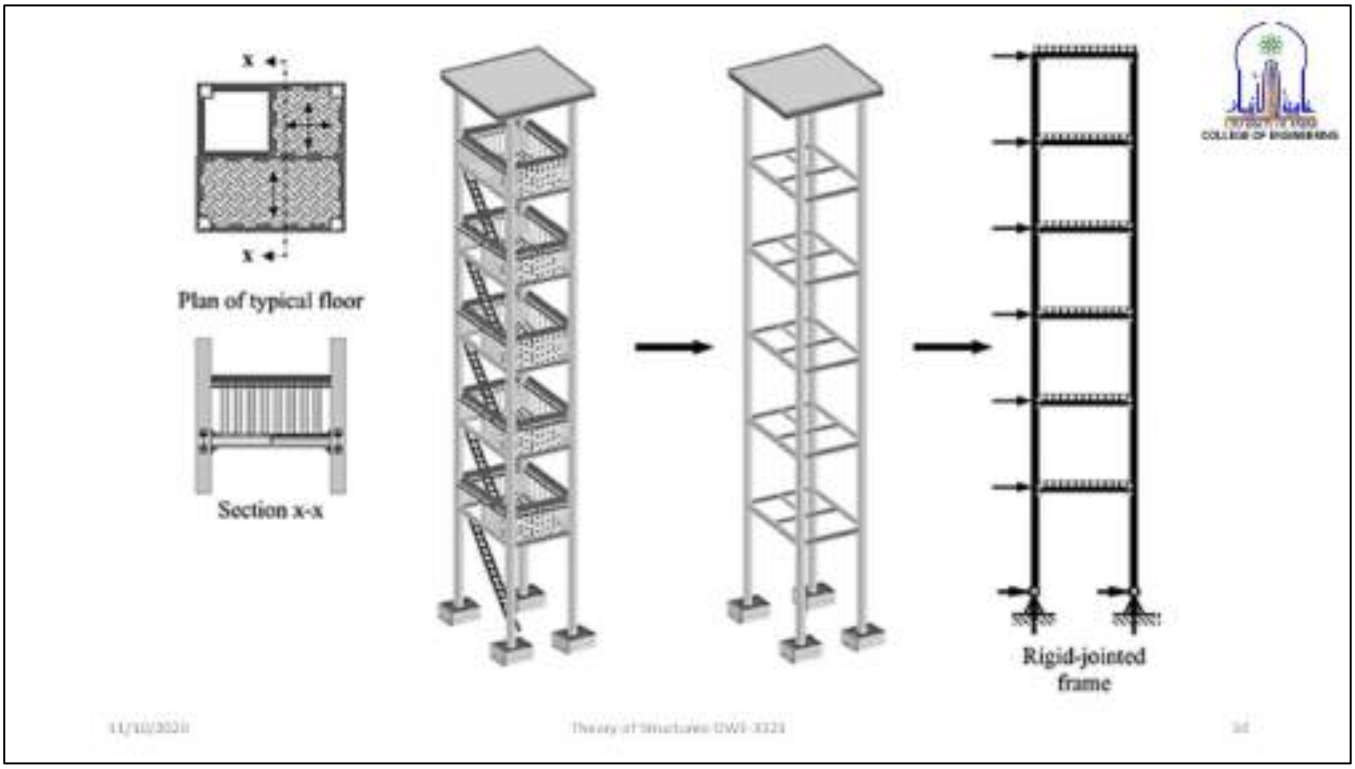

Precast concrete planks
Tie Beam
Pin
Tie Beam
Three-pinned Portal frame


Cantilever

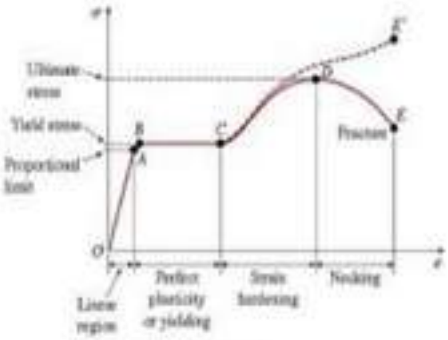

Multi-span Beam


Active
Bracing
Pin-jointed
frame

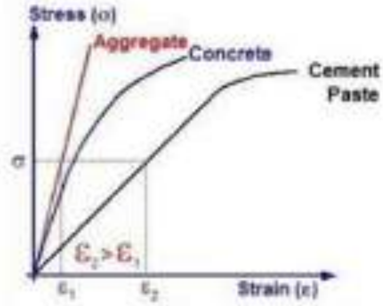



X
X
Plan of typical floor
Section x-x
Rigid-jointed
frame

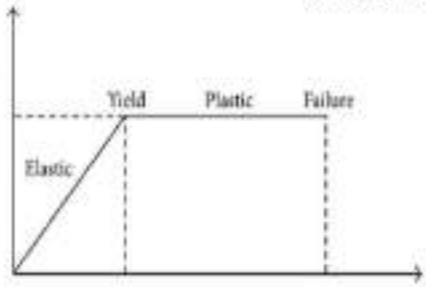
1.7 Principles of Elastic Structural Analysis



Steel



Concrete



Idealized

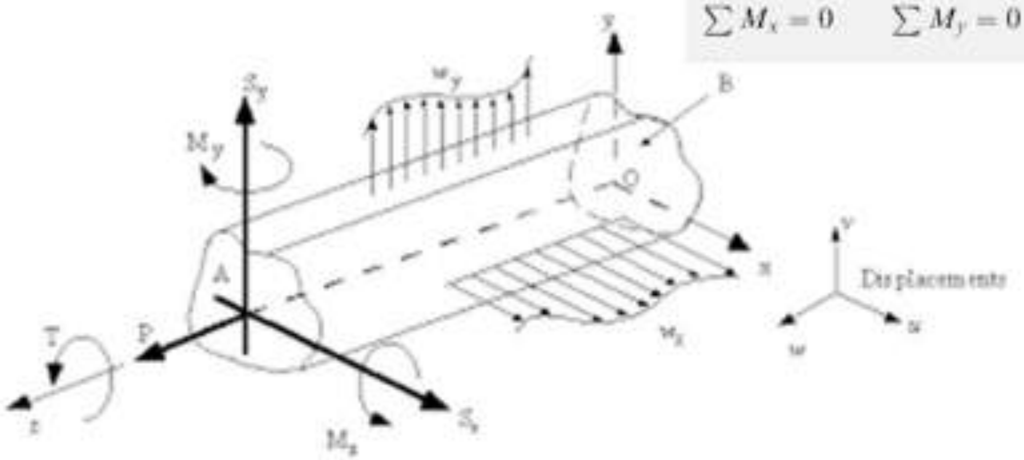
Principles:

1. Linear & Elastic
2. Small displacement principle
3. Superposition
4. Equilibrium

1.8 Equilibrium and Force Systems

A- Three-dimensional equilibrium equations:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0$$
$$\sum M_x = 0 \qquad \sum M_y = 0 \qquad \sum M_z = 0$$



1.8 Equilibrium and Force Systems

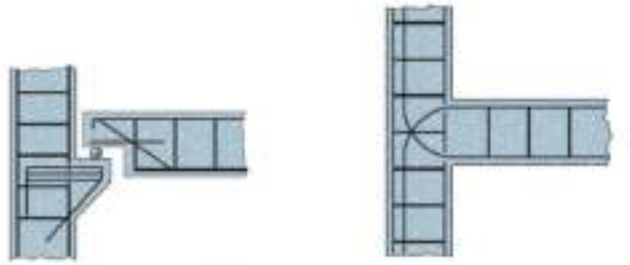
B- Two-dimensional equilibrium equations:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = 0$$

C- Real-Life Supports:

typical "pin-supported" connection (metal) (a)

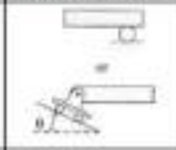
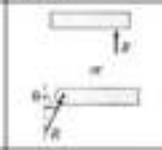
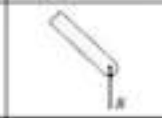
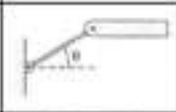
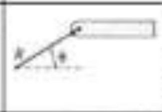
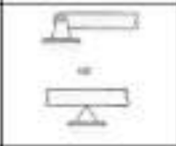
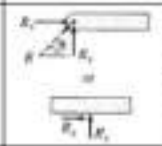
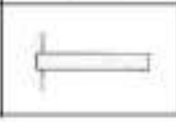
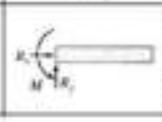
typical "fixed-supported" connection (metal) (b)



typical "roller-supported" connection (concrete) (a)

typical "fixed-supported" connection (concr (b)

D- Idealized Supports:

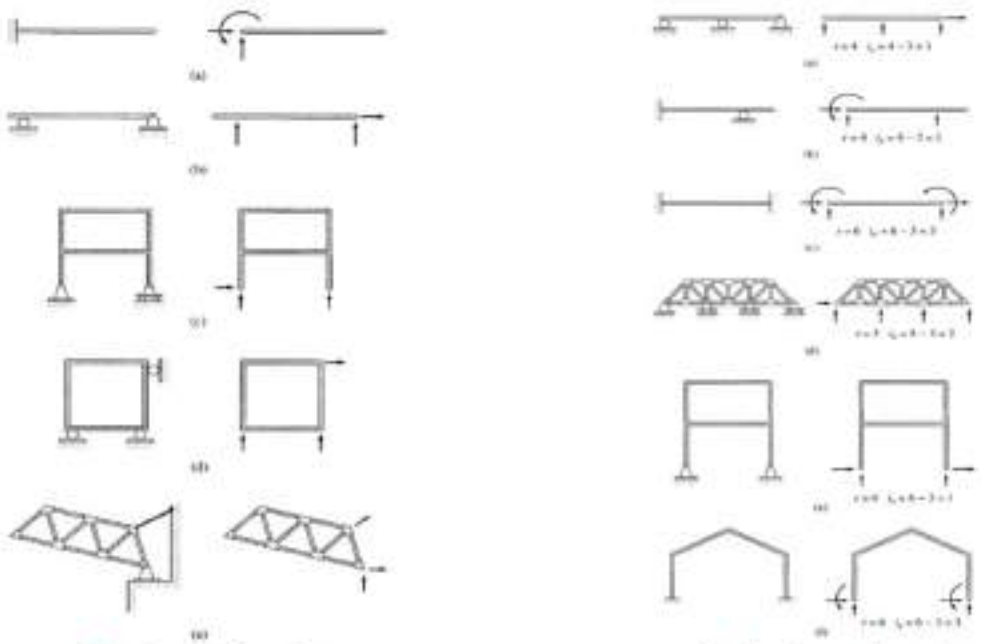
Category	Type of support	Symbolic representation	Reactions	Number of unknowns
I	Roller			1 The reaction force R acts perpendicular to the supporting surface and may be directed either into or away from the structure. The magnitude of R is the unknown.
	Rocker			
	Link			1 The reaction force R acts in the direction of the link and may be directed either into or away from the structure. The magnitude of R is the unknown.
II	Hinge			2 The reaction force R may act in any direction. It is usually convenient to represent R by its rectangular components, R_x and R_y. The magnitudes of R_x and R_y are the two unknowns.
III	Fixed			3 The reactions consist of two force components R_x and R_y and a couple of moment M. The magnitudes of R_x, R_y, and M are the three unknowns.

1.9 Stability and Indeterminacy of Structures

1. Statically determinate structures: Structures that can be analysed using equilibrium equations only.

2. Statically indeterminate structures: Structures can not be analysed using equilibrium equations only.

3. Redundant forces: The extra reactions that exceeds and can not be found by equilibrium equations.



Determinate | Indeterminate

Degree of Indeterminacy:

I.D = No. of Unknowns – No. of Equations

$$I.D = NUK - NEQ$$

(a) Beams:

NUK = Reactions (R)
NEQ = 3+C

> $r = 3n$, statically determinate
> $r > 3n$, statically indeterminate

C = No. of Conditional Equations

$$I.D = NUK - NEQ = R-(3+C)$$

Example:

(a)

$r = 3, n = 1, 3 = 3(1)$ — Statically determinate — I.D = R-(3+C) = 3-(3+0) = 0 → Determinate

(b)

$r = 5, n = 1, 5 > 3(1)$ — Statically indeterminate to the second degree — I.D = R-(3+C) = 5-(3+0) = 2 → Indeterminate 2nd Degree

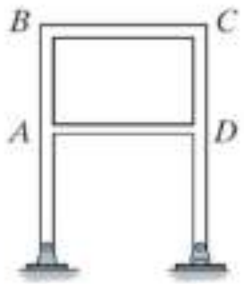
(c)

$r = 6, n = 2, 6 = 3(2)$ — Statically determinate — I.D = R-(3+C) = 4-(3+1) = 0 → Determinate

(d)

$r = 10, n = 3, 10 > 3(3)$ — Statically indeterminate to the first degree — I.D = R-(3+C) = 6-(3+2) = 1 → Indeterminate 1st Degree

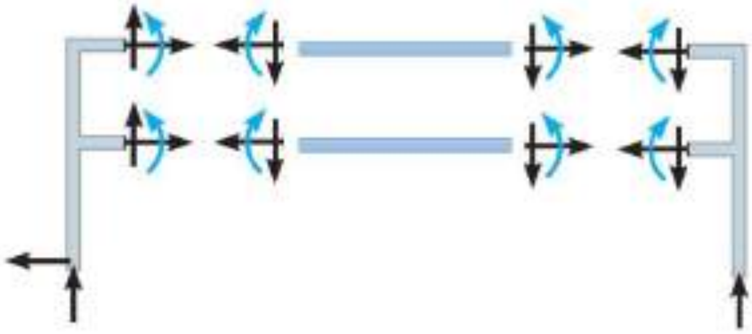
(b) Frames: Method-1 and 2



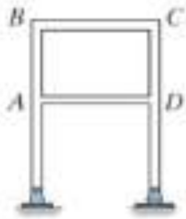
NUK = 6m+R
NEQ = 3m+3j+C

C = No. of Conditional Equations

I.D = NUK – NEQ
= 3m+R-(3j+C)



Example:



(a)

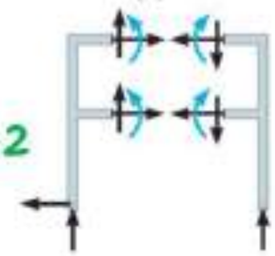
Method-1

$I.D = 3m+R-(3j+C)$

$I.D = [3(6)+3]-[3(6)+0]$

$I.D = 21-18 = 3$

Method-2



$r = 9$, $n = 2$, $9 > 6$,
Statically indeterminate to the third degree *Ans.*

(a)

Example:

Method-1

I.D = 3m+R-(3j+C)

I.D = [3(7)+9]-[3(8)+0]

I.D = 30-24 = 6

Method-2

$r = 9$, $n = 1$, $9 > 3$.
Statically indeterminate to the sixth degree *Ans.*

Example:

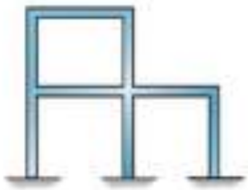
Method-1

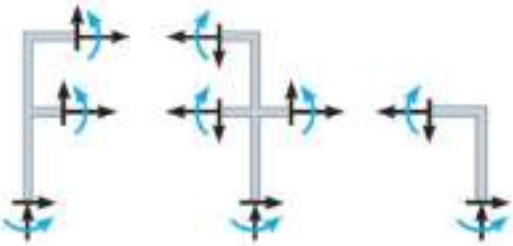
I.D = 3m+R-(3j+C)

I.D = [3(8)+9]-[3(8)+0]

I.D = 33-24 = 9

Method-2





$r = 18$, $n = 3$, $18 > 9$.
Statically indeterminate to the ninth degree *Ans.*

(b) Trusses:

NUK = m + R
NEQ = 2j

I.D = NUK − NEQ
= m+R−2j

Examples:

$I.D = m+R-2j$
$I.D = 19 + 3 - 2(11)$
$I.D = 22-22 = 0$ → **Determinate**

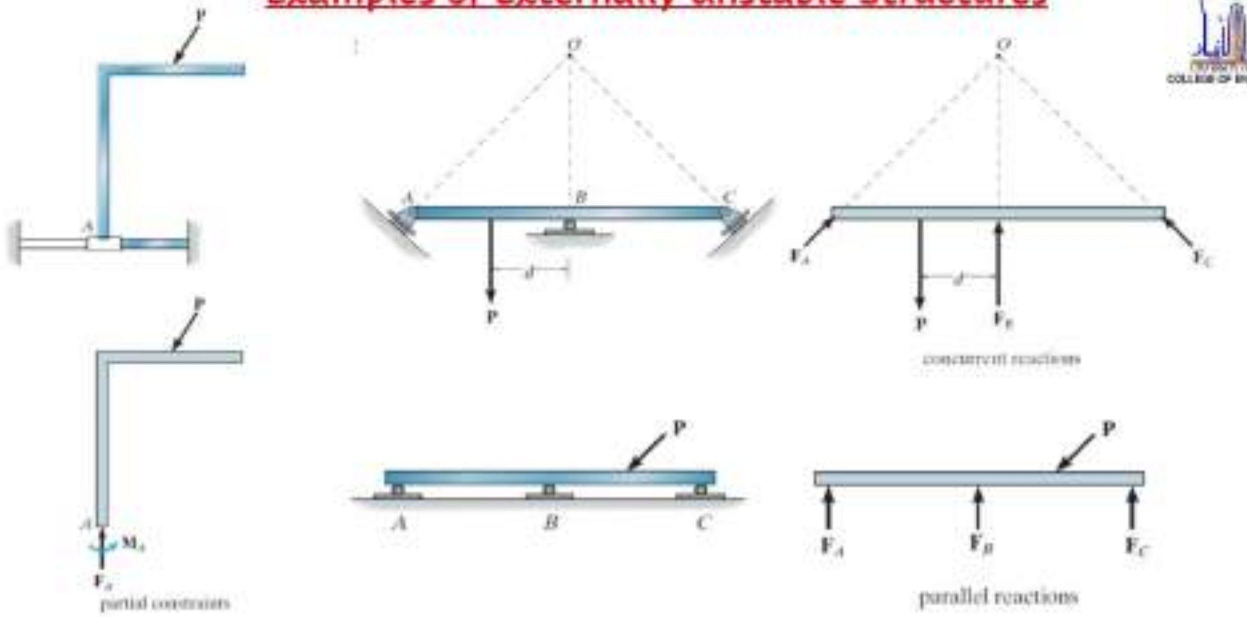
$I.D = m+R-2j$
$I.D = 9 + 3 - 2(6)$
$I.D = 12-12 = 0$ → **Determinate**

Stability

In general, when the equations of static equilibrium are satisfied, the structure is at rest and would say to be a **STABLE** structure. When the structure, or any part of it, cannot satisfy the equilibrium equations, it is said to be **UNSTABLE**!

11/10/2020

Examples of Externally Unstable Structures



Summary

For Beams:

R<C+3 → Unstable

R>C+3 → Stable Indeterminate

R=C+3 → Stable determinate

For Frames:

3M+R<3j+C → Unstable

3M+R>3j+C → Stable Indeterminate

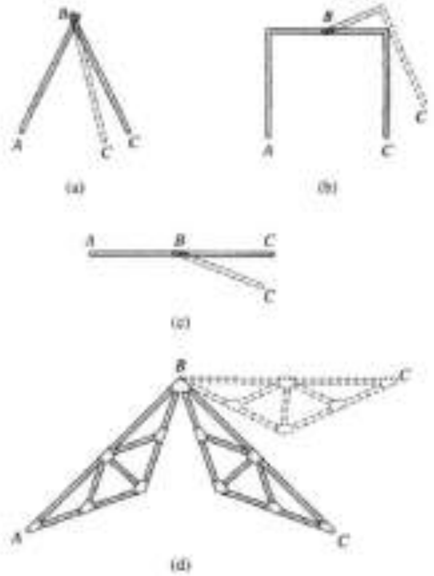
3M+R=3j+C → Stable determinate

For Trusses:

M+R<2j → Unstable

M+R>2j → Stable Indeterminate

M+R=2j → Stable determinate



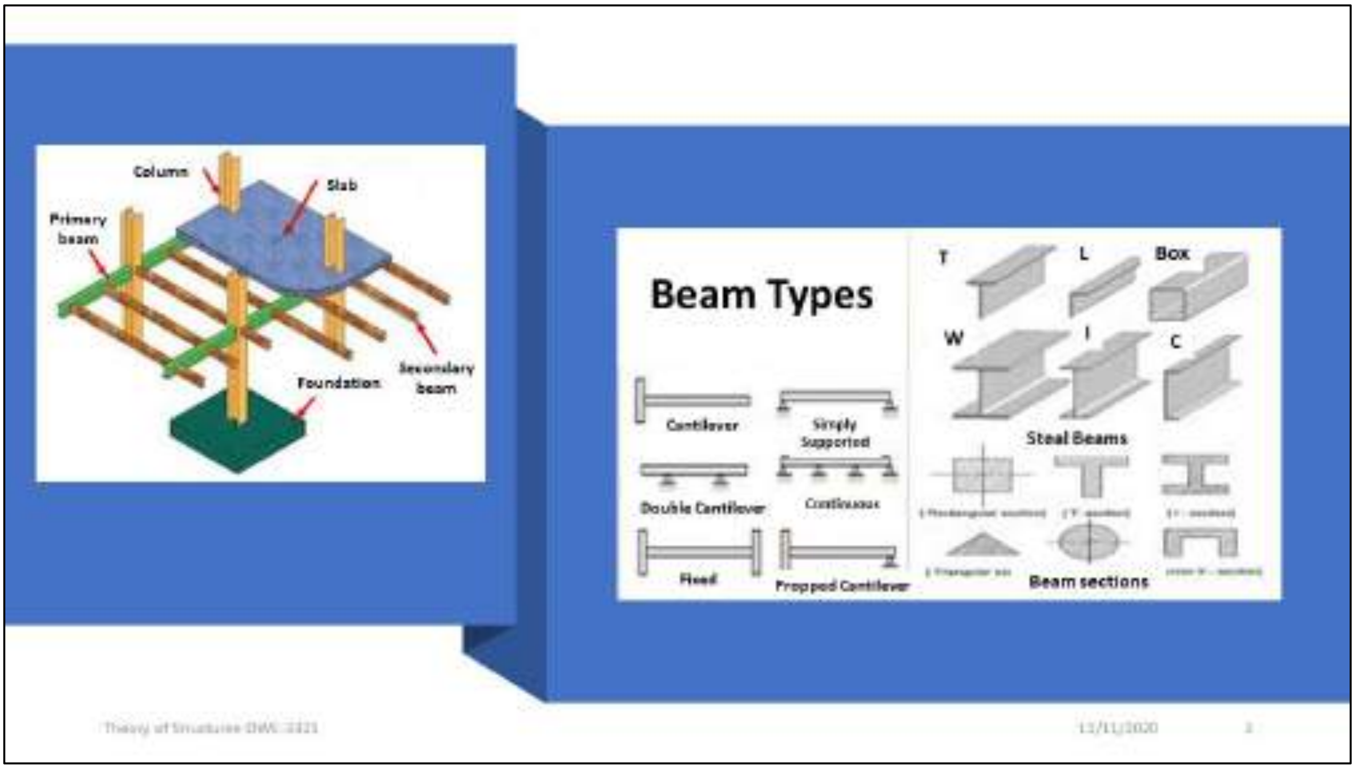
Unit-2

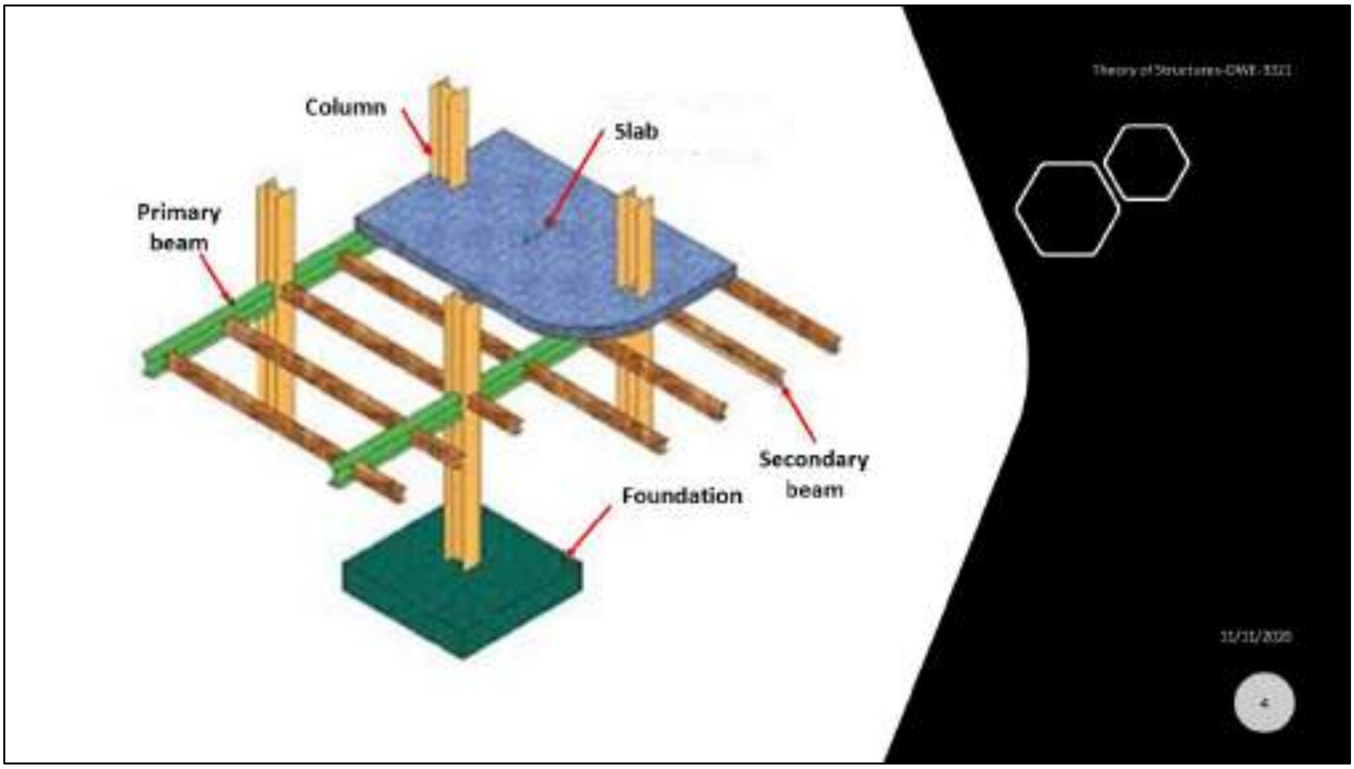
Statically Determinate Beams and Frames

1

2.1 Beams

2

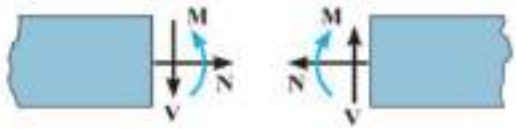

Column
Slab
Primary beam
Secondary beam
Foundation
Beam Types
Cantilever
Simply Supported
Double Cantilever
Continuous
Fixed
Propped Cantilever
T
L
Box
W
I
C
Steel Beams
Beam sections


Column
Slab
Primary beam
Secondary beam
Foundation

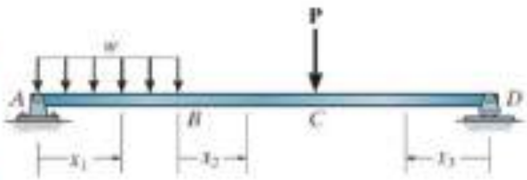
Internal Loadings Developed in Structural Members

Structural members subjected to planar loads support an internal normal force **N**, shear force **V**, and bending moment **M.** To find these values at a specific point in a member, the method of sections must be used. This requires drawing a free-body diagram of a segments of the member, and then applying the three equations of equilibrium.

Always show the three internal loadings on the section in their positive directions.



The internal shear and moment can be expressed as a function of x along the member by establishing the origin at a fixed point (normally at the left end of the member, and then using the method of sections, where the section is made a distance x from the origin). For members subjected to several loads, different x coordinates must extend between the loads.



Shear and moment diagrams for structural members can be drawn by plotting the shear and moment functions. They also can be plotted using the two graphical relationships.

$$\frac{dV}{dx} = w(x)$$

$$\left.\begin{matrix}\text{Slope of} \\ \text{Shear Diagram}\end{matrix}\right\} = \left\{\begin{matrix}\text{Intensity of} \\ \text{Distributed Load}\end{matrix}\right.$$

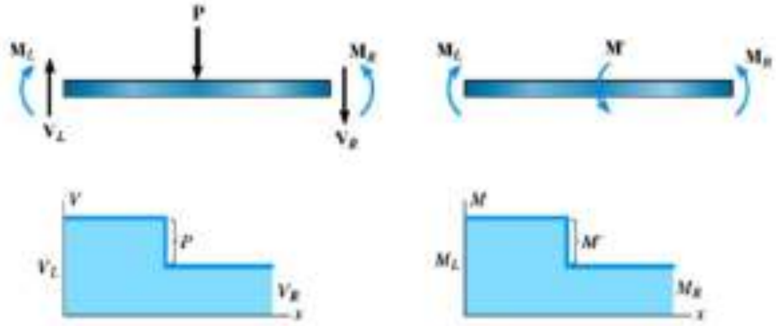
$$\frac{dM}{dx} = V$$

$$\left.\begin{matrix}\text{Slope of} \\ \text{Moment Diagram}\end{matrix}\right\} = \{\text{Shear}$$

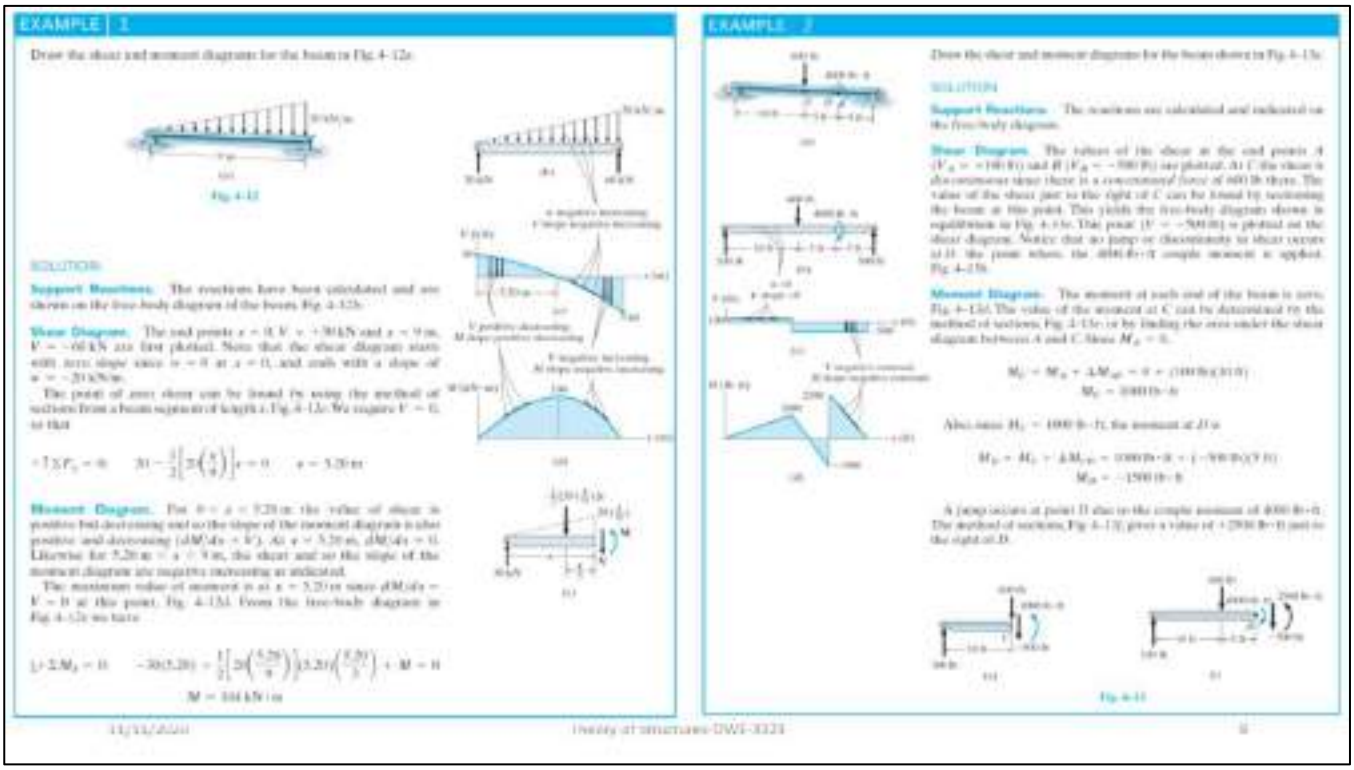
Note that a point of zero shear locates the point of maximum moment since:

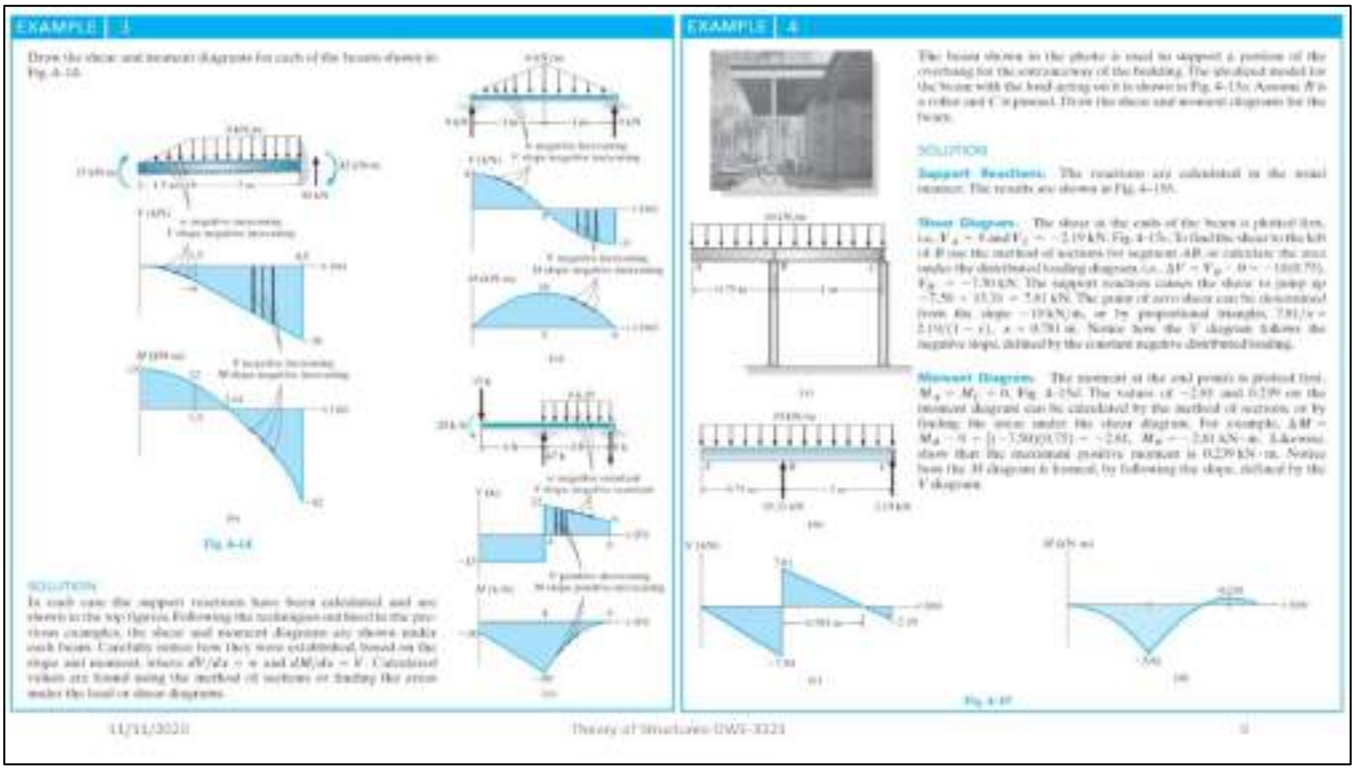
$$V = dM/dx = 0$$

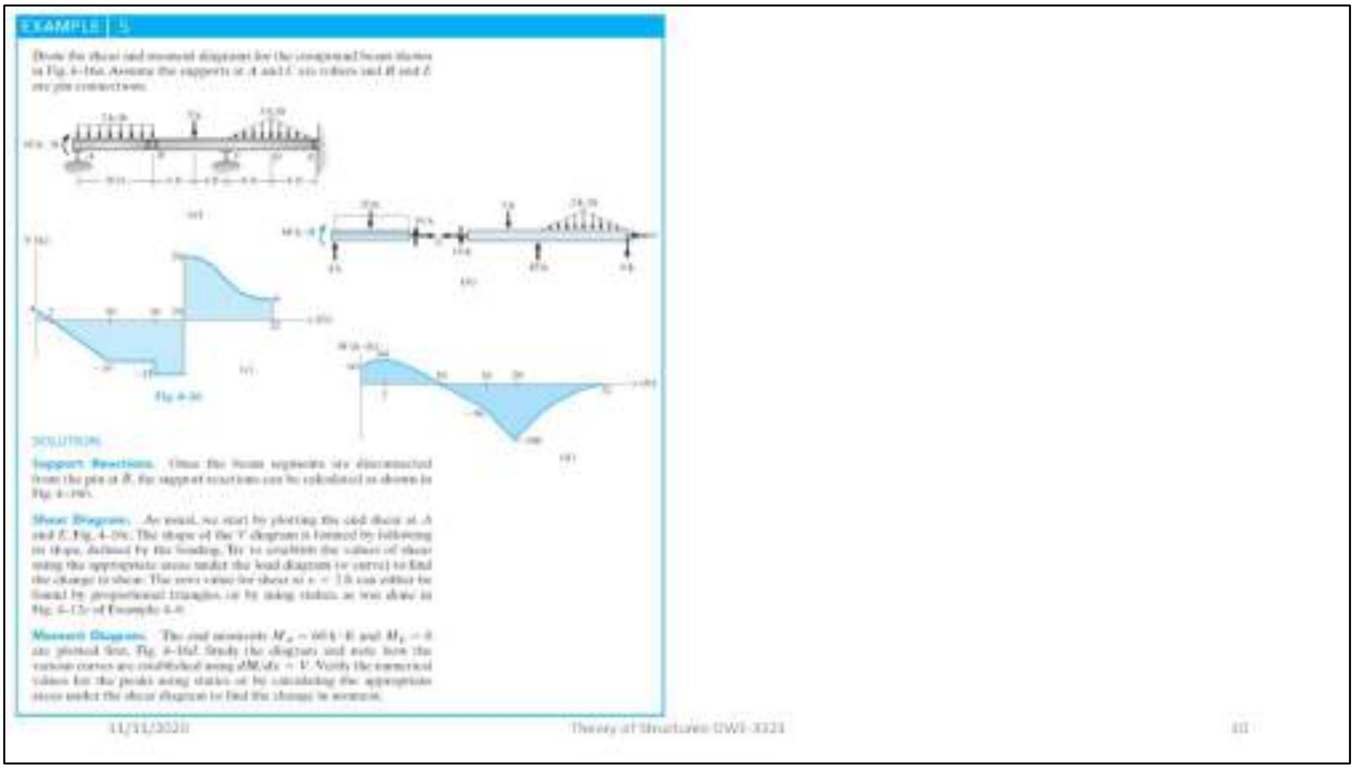
A force acting downward on the beam will cause the shear diagram to jump downwards, and a counterclockwise couple moment will cause the moment diagram to jump downwards.

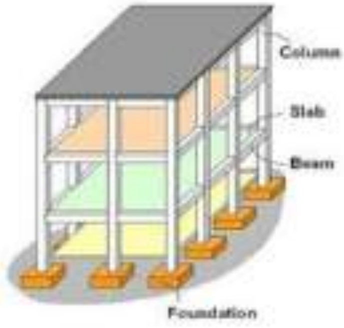


Using the method of superposition, the moment diagrams for a member can be represented by a series of simpler shapes. The shapes represent the moment diagram for each of the separate loadings. The resultant moment diagram is then the algebraic addition of the separate diagrams.




EXAMPLE 3
EXAMPLE 4
SOLUTION
11/11/2020

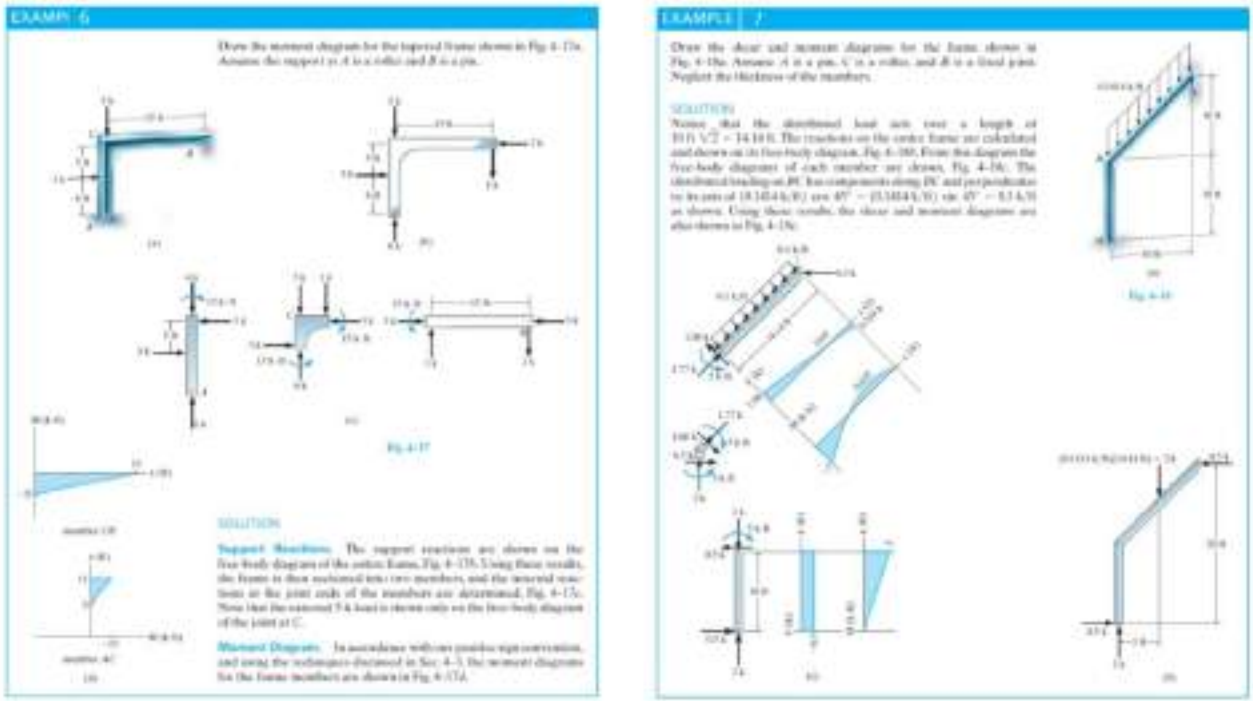

EXAMPLE 5
SOLUTION
11/11/2020



Typical RC Frame Building

2.2 Frames

11/11/2020 Theory of Structures DWE-3321 11



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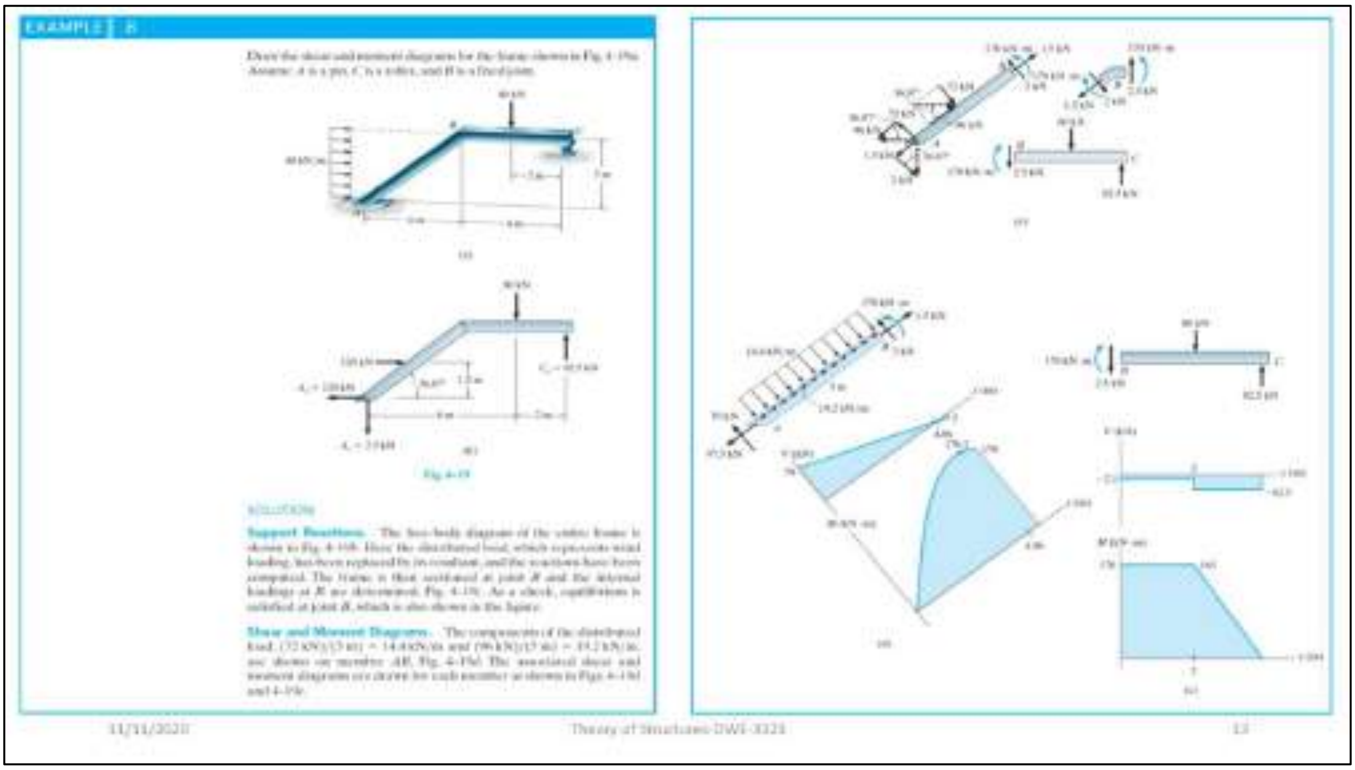
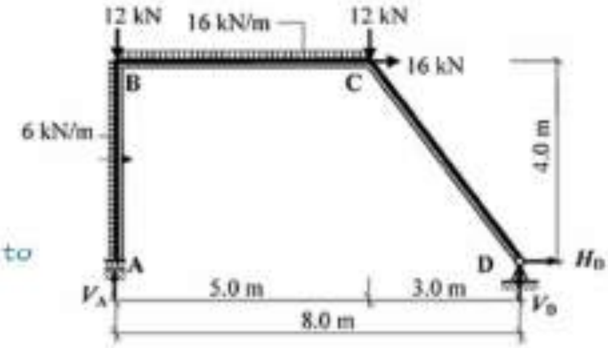


Example -9

An asymmetric portal frame is supported on a roller at A and pinned at support D as shown in Figure below. For the loading indicated:

i) determine the support reactions and,

ii) sketch the axial load, shear force and bending moment diagrams.



Solution:

Apply the three equations of static equilibrium to the force system

+ve $\uparrow \Sigma F_y = 0 \qquad V_A - 12.0 - (16.0 \times 5.0) - 12.0 + V_D = 0$

+ve $\rightarrow \Sigma F_x = 0 \qquad (6.0 \times 4.0) + 16.0 + H_D = 0$

+ve $\curvearrowright \Sigma M_A = 0 \qquad (6.0 \times 4.0)(2.0) + (16.0 \times 5.0)(2.5) + (12.0 \times 5.0) + (16.0 \times 4.0) - (V_D \times 8.0) = 0$

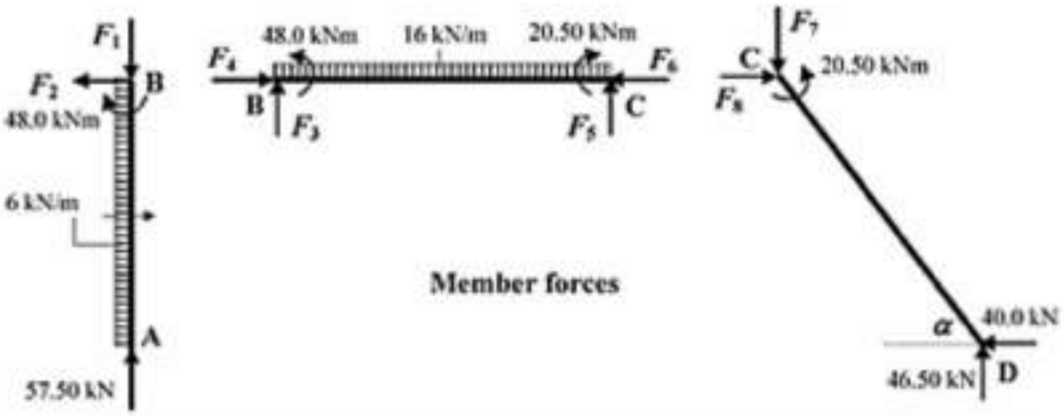
From equation (2): $40.0 + H_D = 0 \qquad \therefore H_D = -40.0$ kN $\leftarrow$

From equation (3): $372.0 - 8.0V_D = 0 \qquad \therefore V_D = +46.5$ kN $\uparrow$

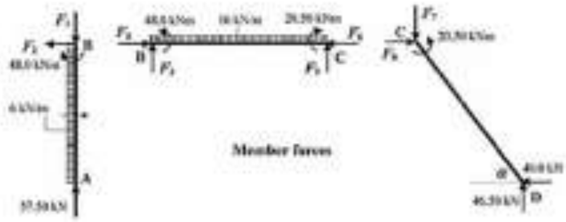
From equation (1): $V_A - 104.0 + 46.5 = 0 \qquad \therefore V_A = +57.5$ kN $\uparrow$

Assuming positive bending moments induce tension inside the frame:

MB= −(6.0×4.0)(2.0)= −48.0 kN.m
MC= +(46.5×3.0)−(40.0×4.0)= −20.50 kN.m



The values of the end-forces F1 to F8 can be determined by considering the equilibrium of each member and joint in turn.



Consider member AB:

+ve ↑ $\Sigma F_y = 0$ $+57.50 - F_1 = 0$ $\therefore F_1 = 57.50$ kN ↓

+ve → $\Sigma F_x = 0$ $+(6.0 \times 4.0) - F_2 = 0$ $\therefore F_2 = 24.0$ kN ←

Consider joint B:

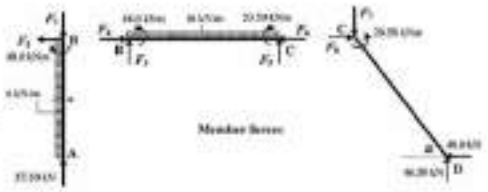
+ve ↑ $\Sigma F_y = 0$ There is an applied vertical load at joint B = 12 kN ↓

$-F_1 + F_3 = -12.0$ $\therefore F_3 = 45.50$ kN ↑

+ve → $\Sigma F_x = 0$

$-F_2 + F_4 = 0$ $\therefore F_4 = 24.0$ kN →

Consider member BC:



+ve ↑ $\Sigma F_y = 0$ $+45.5 - (16.0 \times 5.0) + F_5 = 0$ $\therefore F_5 = 34.5$ kN ↑

+ve → $\Sigma F_x = 0$ $+24.0 - F_6 = 0$ $\therefore F_6 = 24.0$ kN ←

Consider member CD:

+ve ↑ $\Sigma F_y = 0$ $+46.5 - F_7 = 0$ $\therefore F_7 = 46.5$ kN ↓

+ve → $\Sigma F_x = 0$ $-40.0 + F_8 = 0$ $\therefore F_8 = 40.0$ kN →

Check joint C:

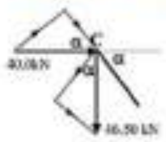
+ve ↑ ΣF_y There is an applied vertical load at joint C = 12 kN ↓

$+F_5 - F_7 = +34.5 - 46.5 = -12.0$

+ve → ΣF_x There is an applied horizontal at joint C = 16 kN →

$-F_6 + F_8 = -24.0 + 40.0 = +16.0$

Member CD:



$\alpha = \tan^{-1}(4.0/3.0) = 53.13°$

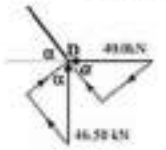
$\cos\alpha = 0.60$; $\sin\alpha = 0.80$

Assume axial compression to be positive.

At joint C

Axial force = $+(40.0 \times \cos\alpha) + (46.50 \times \sin\alpha) = +61.2$ kN

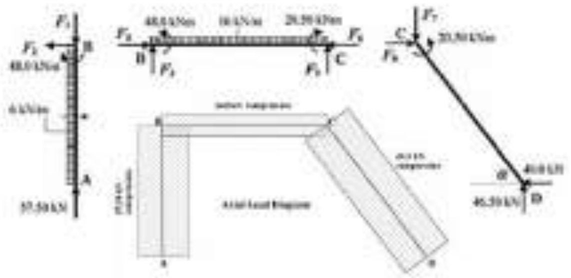
Shear force = $+(40.0 \times \sin\alpha) - (46.50 \times \cos\alpha) = +4.10$ kN

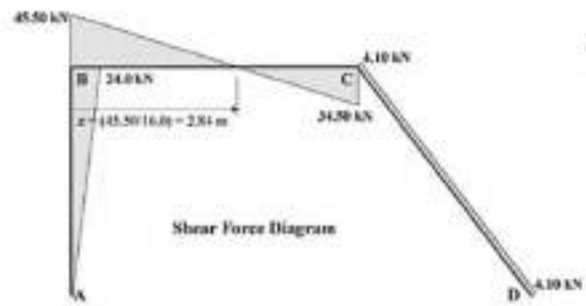


Similarly at joint D

Axial force = + 61.2 kN

Shear force = + 4.10 kN



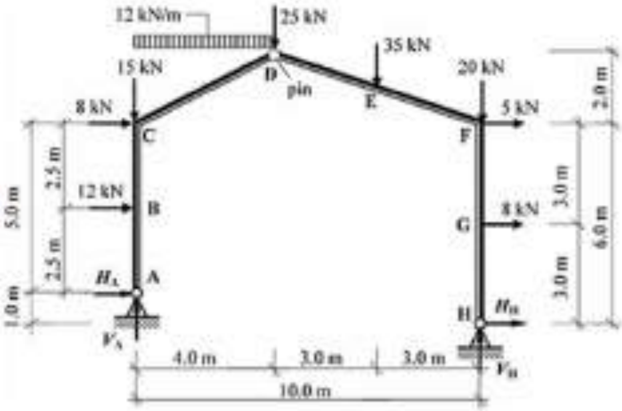


Example -10

A pitched-roof portal frame is pinned at supports A and H and members CD and DEF are pinned at the ridge as shown in Figure 5.6. For the loading indicated:

i) determine the support reactions and

ii) sketch the axial load, shear force and bending moment diagrams.



Solution:

Apply the three equations of static equilibrium to the force system in addition to the Σ moments at the pin = 0:

+ve ↑ $\Sigma F_y = 0$

$$V_A - 15.0 - (12.0 \times 4.0) - 25.0 - 35.0 - 20.0 + V_H = 0$$

+ve → $\Sigma F_x = 0$

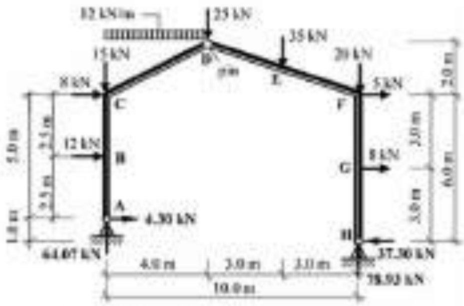
$$H_A + 12.0 + 8.0 + 5.0 + 8.0 + H_H = 0$$

+ve ↻ $\Sigma M_A = 0$

$$(12.0 \times 2.5) + (8.0 \times 5.0) + (12.0 \times 4.0)(2.0) + (25.0 \times 4.0) + (35.0 \times 7.0) + (20.0 \times 10.0) + (5.0 \times 5.0) + (8.0 \times 2.0) - (H_H \times 1.0) - (V_H \times 10.0) = 0$$

+ve ↻ $\Sigma M_{pin} = 0$ (right-hand side)

$$+ (35.0 \times 3.0) + (20.0 \times 6.0) - (5.0 \times 2.0) - (8.0 \times 5.0) - (H_H \times 8.0) - (V_H \times 6.0) = 0$$



From Equation (3): $+ 752.0 - H_H - 10.0V_H = 0$

From Equation (4): $+ 175.0 - 8.0H_H - 6.0V_H = 0$

Solve equations 3(a) and 3(b) simultaneously: $V_H = + 78.93$ kN ↑ $H_H = - 37.30$ kN ←

From Equation (2): $H_A + 33.0 + H_H = 0$ $H_A = + 4.30$ kN →

From Equation (1): $V_A - 143.0 + V_H = 0$ $V_A = + 64.07$ kN ↑

$M_B = - (4.30 \times 2.5) = - 10.75$ kNm

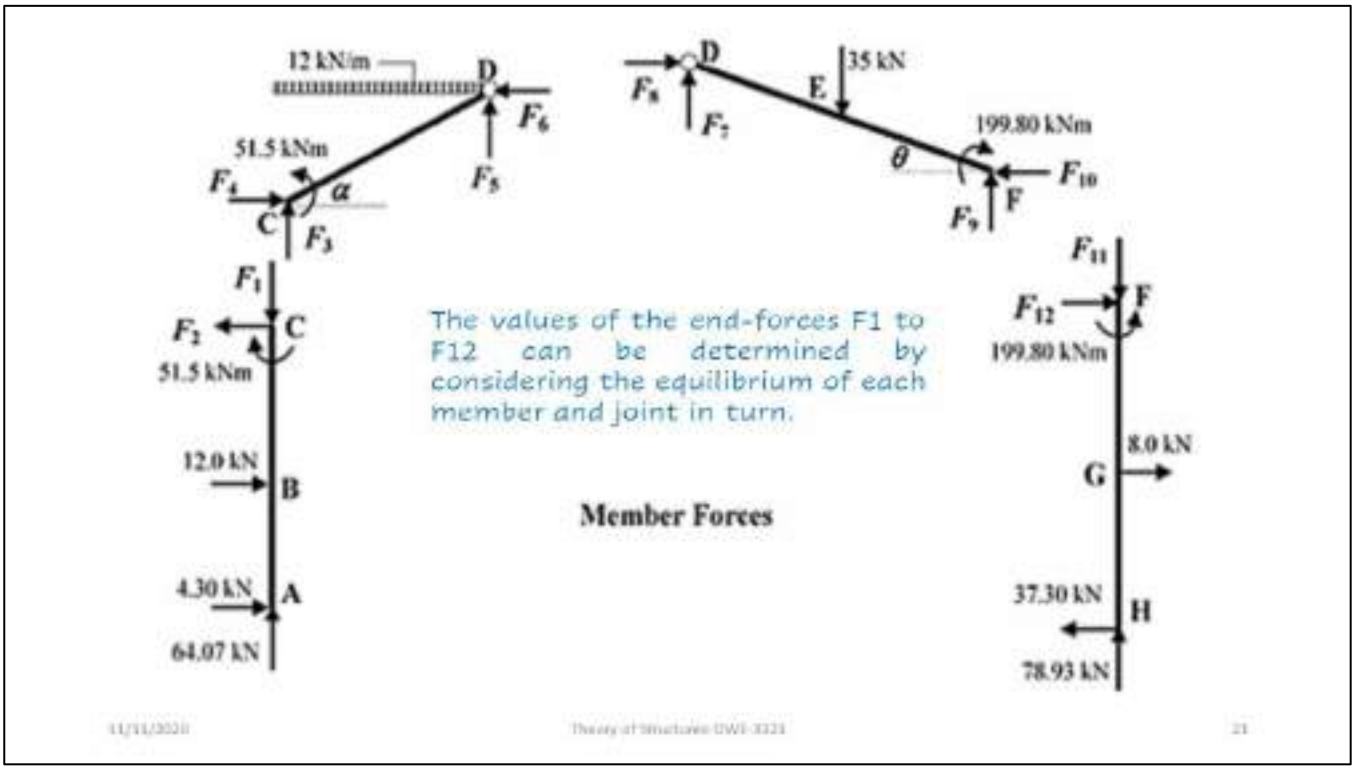
$M_C = - (4.30 \times 5.0) - (12.0 \times 2.5) = - 51.50$ kNm

$M_D =$ zero (pin)

$M_E = - (20.0 \times 3.0) + (5.0 \times 1.0) + (8.0 \times 4.0) - (37.3 \times 7.0) + (78.93 \times 3.0)$
$= - 47.31$ kNm

$M_F = + (8.0 \times 3.0) - (37.30 \times 6.0) = - 199.80$ kNm

$M_G = - (37.30 \times 3.0) = - 111.90$ kNm



Consider member ABC:

+ve ↑ $\Sigma F_y = 0$ $\quad + 64.07 - F_1 = 0$ $\quad \therefore F_1 = 64.07$ kN ↓

+ve → $\Sigma F_x = 0$ $\quad + 4.30 + 12.0 - F_2 = 0$ $\quad \therefore F_2 = 16.30$ kN ←

Consider Joint C:

+ve ↑ $\Sigma F_y = 0$ There is an applied vertical load at joint C = 15 kN ↓

$-F_1 + F_3 = -15.0$ $\quad \therefore F_3 = 49.07$ kN ↑

+ve → $\Sigma F_x = 0$ There is an applied horizontal load at joint C = 8 kN →

$-F_2 + F_4 = +8.0$ $\quad \therefore F_4 = 24.30$ kN →

Consider member CD:

+ve ↑ $\Sigma F_y = 0$ $\quad + 49.07 - (12.0 \times 4.0) + F_5 = 0$ $\quad \therefore F_5 = -1.07$ kN ↓

+ve → $\Sigma F_x = 0$ $\quad + 24.30 - F_6 = 0$ $\quad \therefore F_6 = 24.30$ kN ←

Consider member FGH:

+ve ↑ $\Sigma F_y = 0$ $\quad + 78.93 - F_{11} = 0$ $\quad \therefore F_{11} = 78.93$ kN ↓

+ve → $\Sigma F_x = 0$ $\quad - 37.30 + 8.0 + F_{12} = 0$ $\quad \therefore F_{12} = 29.30$ kN →

Consider Joint F:

+ve $\uparrow \Sigma F_y = 0$ There is an applied vertical load at joint F = 20 kN $\downarrow$

$F_{11} + F_9 = -20.0$ $\therefore F_9 = 58.93$ kN $\uparrow$

+ve $\rightarrow \Sigma F_x = 0$ There is an applied horizontal load at joint F = 5 kN $\rightarrow$

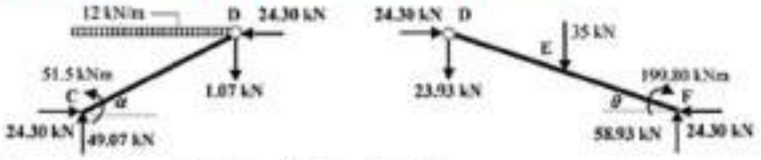
$+F_{12} - F_{10} = +5.0$ $\therefore F_{10} = 24.30$ kN $\leftarrow$

Consider member DF:

+ve $\uparrow \Sigma F_y = 0$ $+58.93 - 35.0 + F_7 = 0$ $\therefore F_7 = 23.93$ kN $\downarrow$

+ve $\rightarrow \Sigma F_x = 0$ $-24.30 + F_8 = 0$ $\therefore F_8 = 24.30$ kN $\rightarrow$

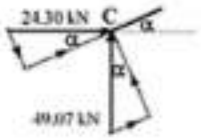
The calculated values can be checked by considering the equilibrium at joint D.



+ve $\rightarrow \Sigma F_x$ $-24.30 + 24.30 = 0$

+ve $\uparrow \Sigma F_y$ $-1.07 - 23.93 = -25.0$ kN (equal to the applied vertical load at D).

Member CD:



$\alpha = \tan^{-1}(2.0/4.0) = 26.565°$

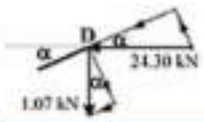
$\cos\alpha = 0.894$; $\sin\alpha = 0.447$

Assume axial compression to be positive.

At joint C

Axial force = $+(24.30 \times \cos\alpha) + (49.07 \times \sin\alpha) = +43.66$ kN

Shear force = $-(24.30 \times \sin\alpha) + (49.07 \times \cos\alpha) = +33.01$ kN

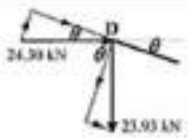


At joint D

Axial force = $+(24.30 \times \cos\alpha) + (1.07 \times \sin\alpha) = +22.20$ kN

Shear force = $-(24.30 \times \sin\alpha) + (49.07 \times \cos\alpha) = -9.91$ kN

Member DEF:



$\theta = \tan^{-1}(2.0/6.0) = 18.435°$

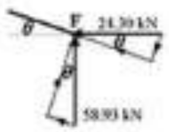
$\cos\theta = 0.947$; $\sin\theta = 0.316$

Assume axial compression to be positive.

At joint D

Axial force = $+(24.30 \times \cos\theta) + (23.93 \times \sin\theta) = +30.57$ kN

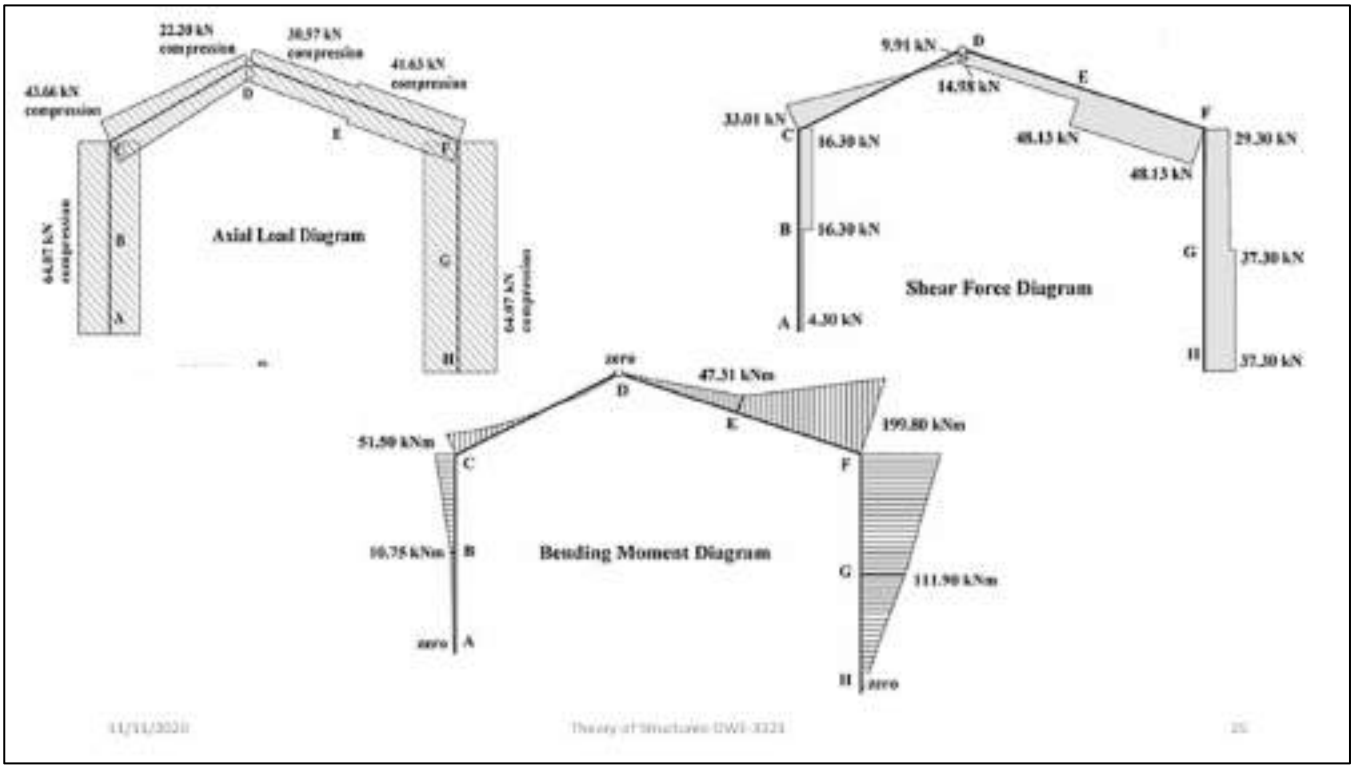
Shear force = $+(24.30 \times \sin\theta) - (23.93 \times \cos\theta) = +14.98$ kN



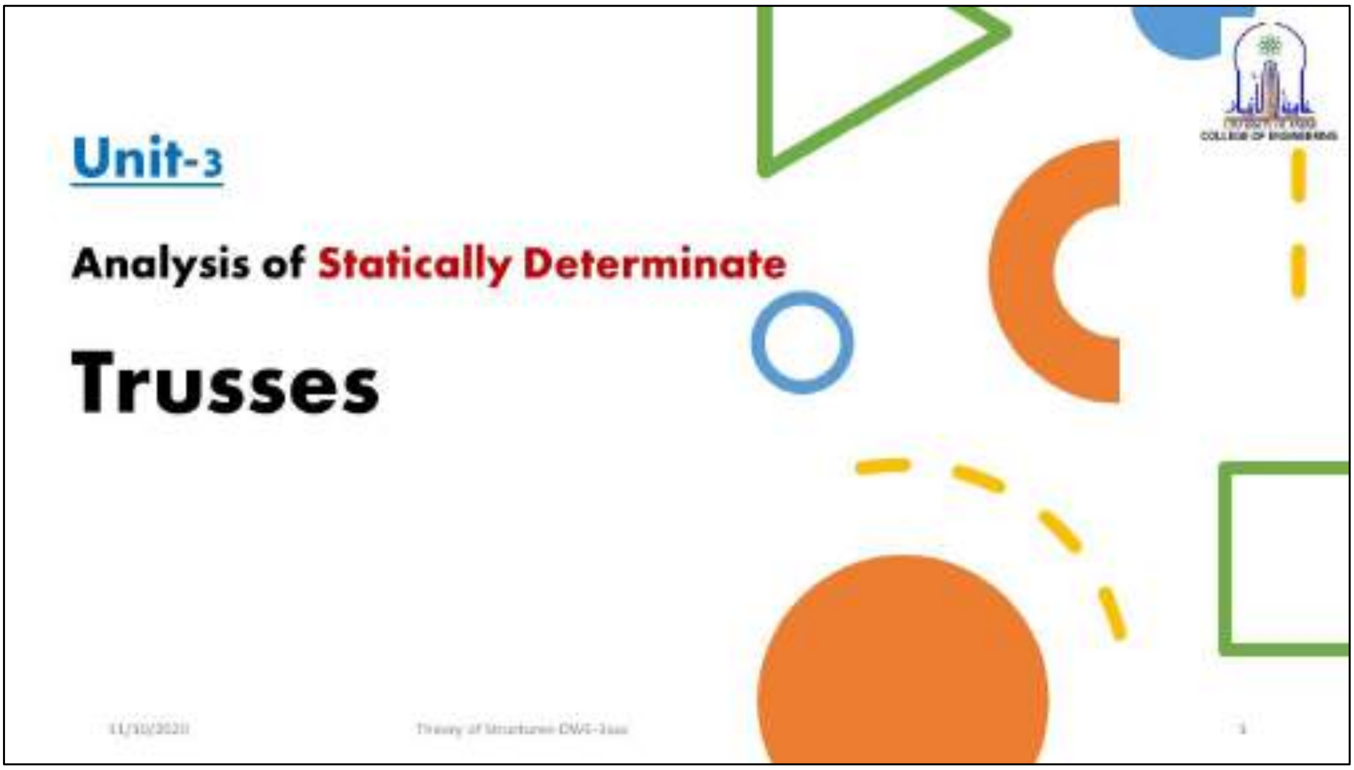
At joint F

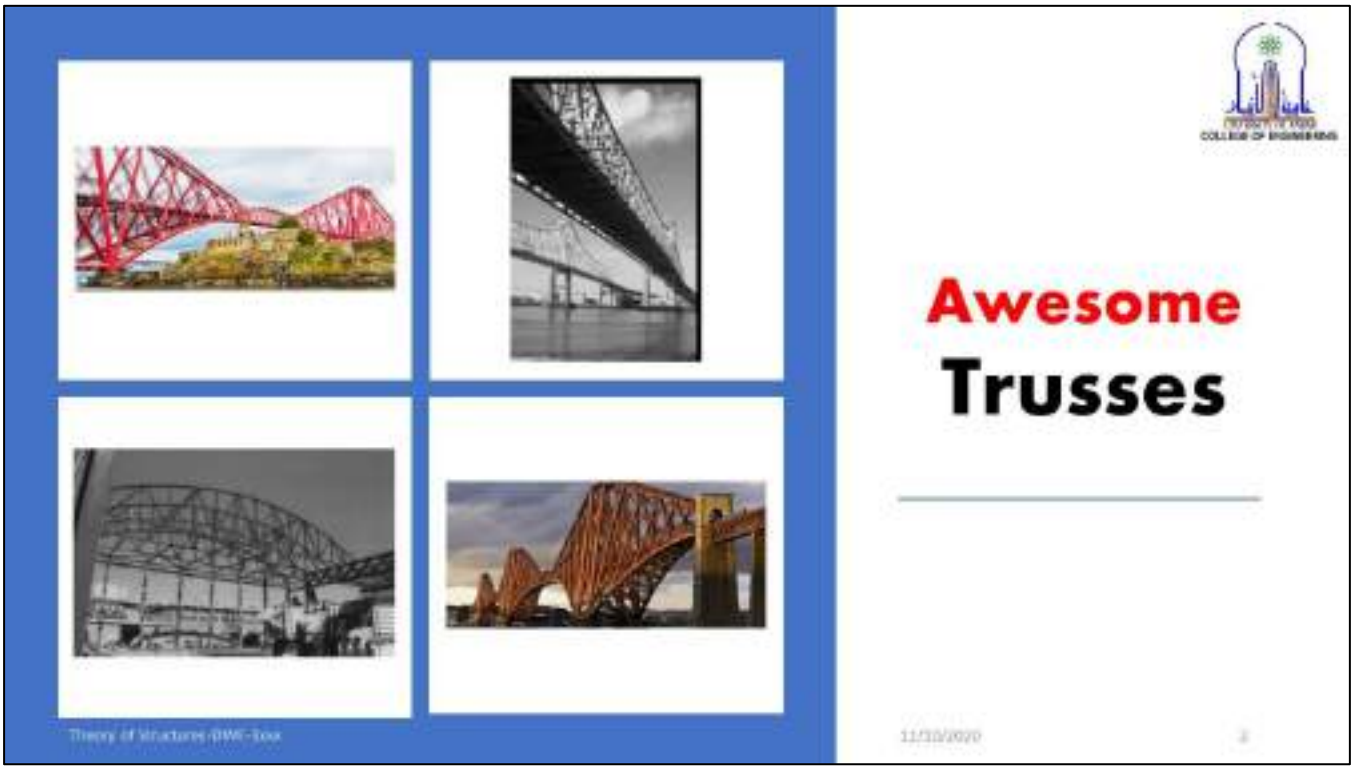
Axial force = $+(24.30 \times \cos\theta) + (58.93 \times \sin\theta) = +41.63$ kN

Shear force = $-(24.30 \times \sin\theta) + (58.93 \times \cos\theta) = +48.13$ kN



Axial Load Diagram
22.20 kN compression
30.57 kN compression
41.63 kN compression
43.66 kN compression
64.87 kN compression
64.87 kN compression
Shear Force Diagram
9.91 kN
14.58 kN
33.01 kN
16.30 kN
16.30 kN
4.30 kN
48.13 kN
48.13 kN
29.30 kN
37.30 kN
37.30 kN
Bending Moment Diagram
zero
47.31 kNm
199.80 kNm
51.50 kNm
10.75 kNm
111.90 kNm
zero
zero


Unit-3
Analysis of Statically Determinate
Trusses


Awesome
Trusses

Common Types of Trusses:

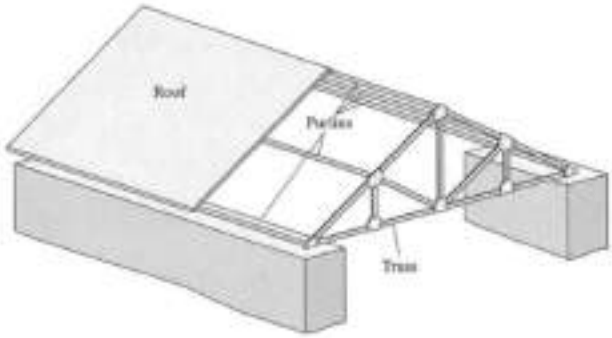
A truss is a structure composed of slender members joined together at their end points. The joint connections are usually formed by bolting or welding the ends of the members to a common plate, called a gusset plate, as shown in Fig. 3-1, or by simply passing a large bolt or pin through each of the members.

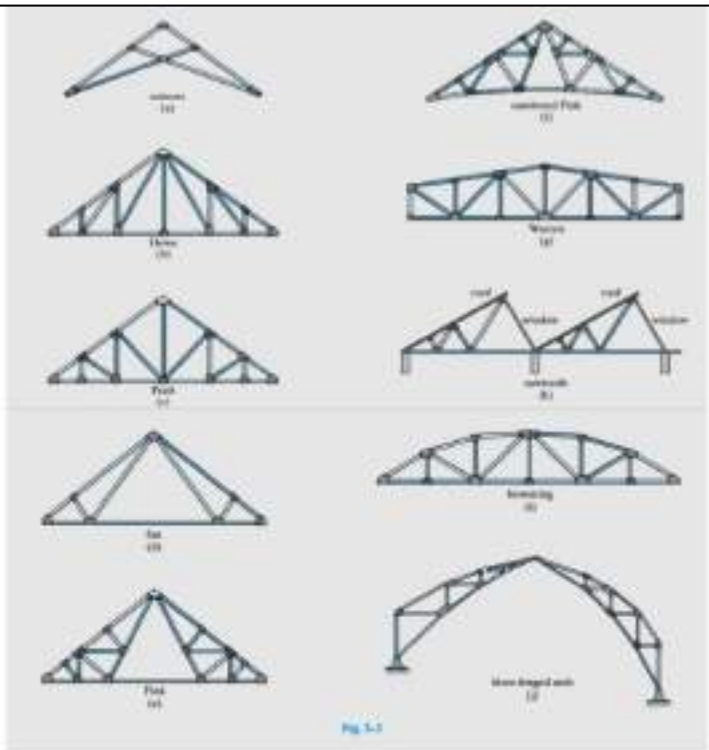
gusset plate

* Planar trusses lie in a single plane and are often used to support roofs and bridges.

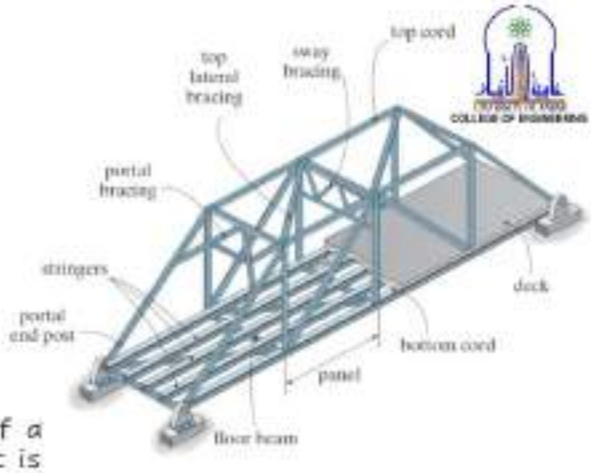
Roof Trusses:

Roof trusses are often used as part of an industrial building frame, such as the one shown in Fig. 3-2 . Trusses used to support roofs are selected on the basis of the span, the slope, and the roof material.



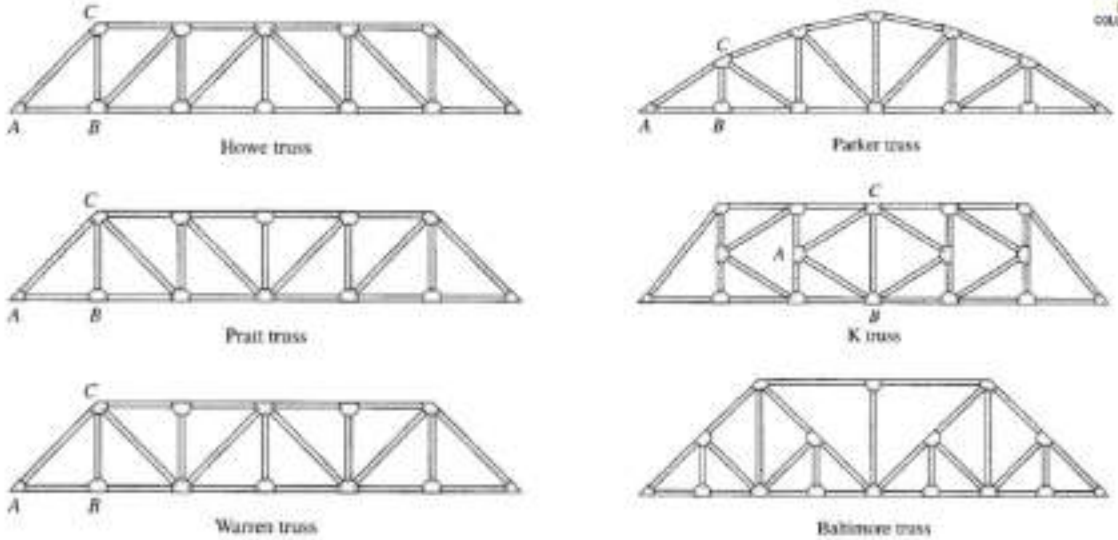


Types of Roof Trusses



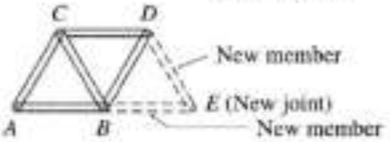
Bridge Trusses: The main structural elements of a typical bridge truss are shown in Fig. 3–4. Here it is seen that a load on the deck is first transmitted to stringers, then to floor beams, and finally to the joints of the two supporting side trusses. The top and bottom cords of these side trusses are connected by top and bottom lateral bracing, which serves to resist the lateral forces caused by wind and the sidesway caused by moving vehicles on the bridge.

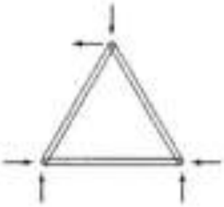
Common Types of Bridge Trusses

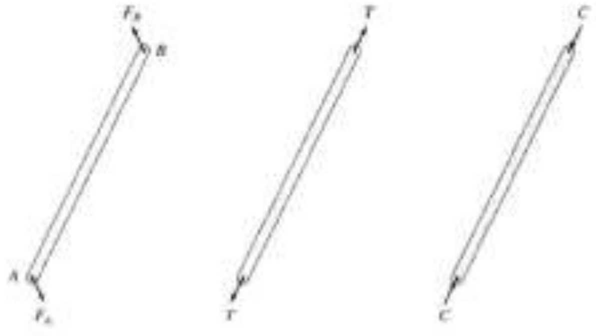


Assumptions for Design: To design both the members and the connections of a truss, it is first necessary to determine the force developed in each member when the truss is subjected to a given loading. In this regard, two important assumptions will be made in order to idealize the truss.

1. The members are joined together by smooth pins.
2. All loadings are applied at the joints.
3. Each truss member acts as an axial force member, and therefore the forces acting at the ends of the member must be directed along the axis of the member. If the force tends to elongate the member, it is a tensile force (T) ; whereas if the force tends to shorten the member, it is a compressive force (C).





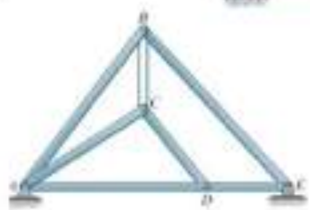


(a) (b) Axial Tension (c) Axial Compression

Classification of Coplanar Trusses:
Before beginning the force analysis of a truss, it is important to classify the truss as simple, compound, or complex, and then to be able to specify its determinacy and stability.

1) Simple Truss: The simplest framework that is rigid or stable is a trianle.

Therefore, a simple truss is constructed starting with a basic triangular element and connecting two members to form additional elements. As each additional element of two members is placed on a truss, the number of joints is increased by one.



2) Compound Truss :

This truss is formed by connecting two or more simple trusses together. This type of truss is often used for large spans.

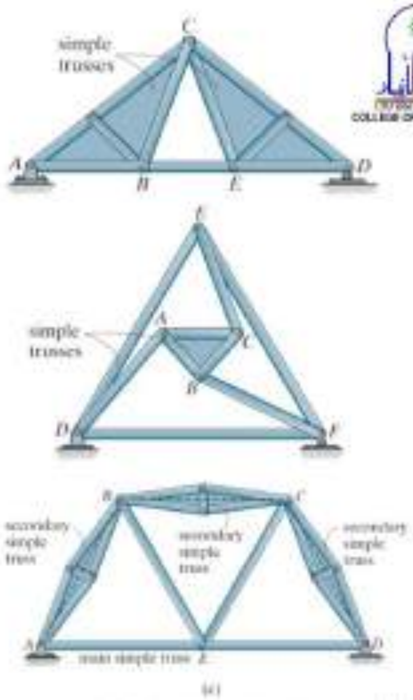
There are three ways in which simple trusses may be connected to form a compound truss:

A. Trusses may be connected by a common joint and bar.

B. Trusses may be joined by three bars.

C. Trusses may be joined where bars of a large simple truss, called the main truss, have been substituted by simple trusses, called secondary trusses.

*Compound trusses are best analysed by applying both the method of joints and the method of sections.



Various types of compound trusses

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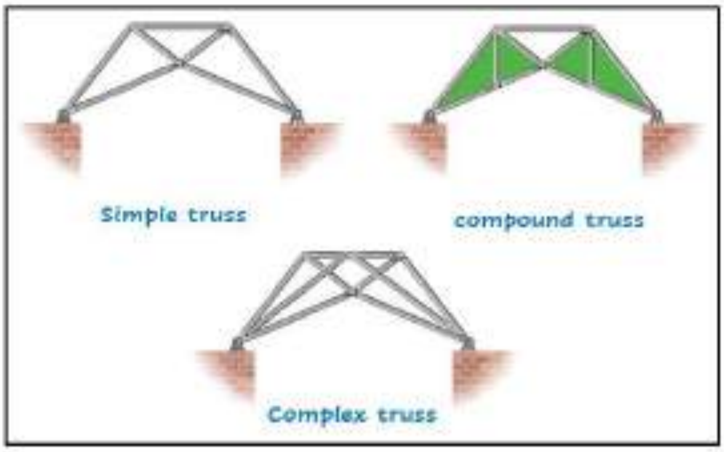
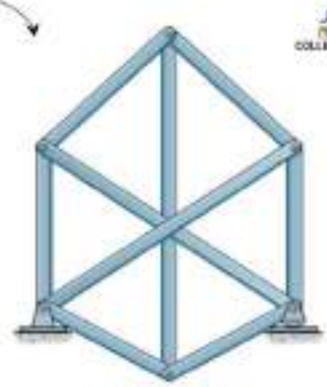
3) Complex Truss :

A complex truss is one that cannot be classified as being either simple or compound.





Complex truss

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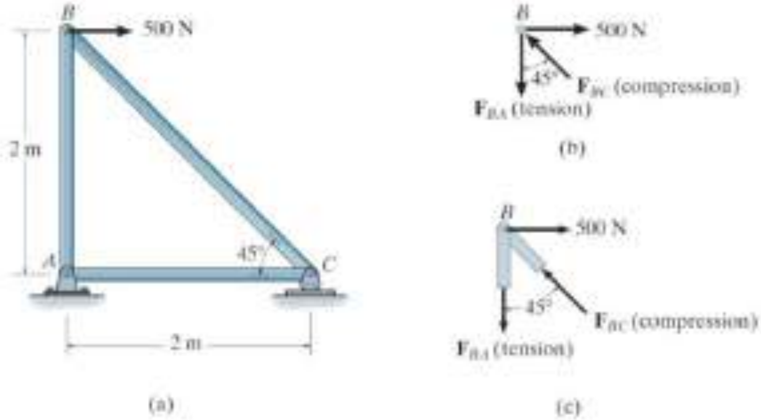
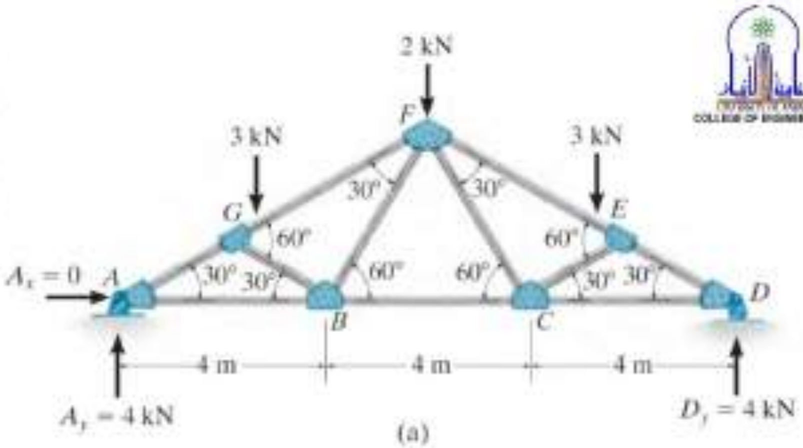
Method of Joints:

If a truss is in equilibrium, then each of its joints must also be in equilibrium. Hence, the method of joints consists of satisfying the equilibrium conditions $\sum F_x = 0$ and $\sum F_y = 0$ and for the forces exerted on the pin at each joint of the truss.



Example:

Determine the force in each member of the roof truss shown in the photo. The dimensions and loadings are shown in the figure. State whether the members are in tension or compression.



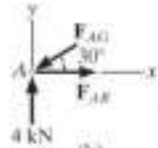
Solution:

Only the forces in half the members have to be determined, since the truss is symmetric with respect to both loading and geometry.

Joint A, We can start the analysis at joint A. **Why?**

$$+\uparrow \Sigma F_y = 0; \quad 4 - F_{AG} \sin 30^\circ = 0 \qquad F_{AG} = 8 \text{ kN (C)}$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \quad F_{AB} - 8 \cos 30^\circ = 0 \qquad F_{AB} = 6.928 \text{ kN (T)}$$



(b)

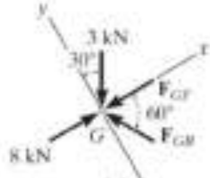
Joint G. In this case note how the orientation of the x, y axes avoids simultaneous solution of equations.

$$+\nwarrow \Sigma F_y = 0; \quad F_{GB} \sin 60^\circ - 3 \cos 30^\circ = 0$$

$$F_{GB} = 3.00 \text{ kN (C)}$$

$$+\nearrow \Sigma F_x = 0; \quad 8 - 3 \sin 30^\circ - 3.00 \cos 60^\circ - F_{GF} = 0$$

$$F_{GF} = 5.00 \text{ kN (C)}$$



(c)

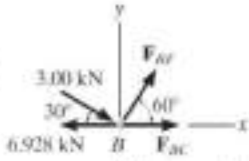
Joint B.

$$+\uparrow \Sigma F_y = 0; \quad F_{BF} \sin 60^\circ - 3.00 \sin 30^\circ = 0$$

$$F_{BF} = 1.73 \text{ kN (T)}$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \quad F_{BC} + 1.73 \cos 60^\circ + 3.00 \cos 30^\circ - 6.928 = 0$$

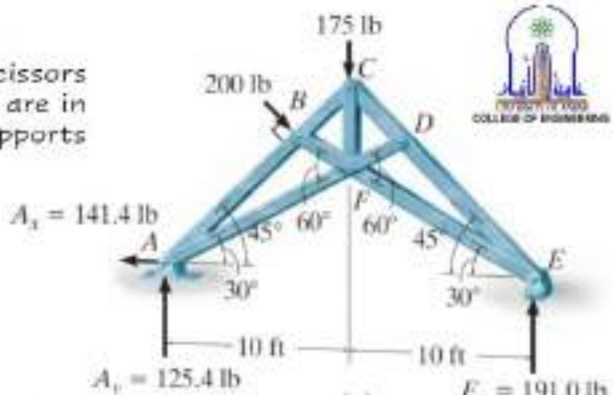
$$F_{BC} = 3.46 \text{ kN (T)}$$



(d)

Example:

Determine the force in each member of the scissors truss shown figure. State whether the members are in tension or compression. The reactions at the supports are given.



(a)

Solution:

Joint E.

$$+\nearrow \Sigma F_y = 0; \quad 191.0 \cos 30^\circ - F_{ED} \sin 15^\circ = 0$$

$$F_{ED} = 639.1 \text{ lb (C)}$$

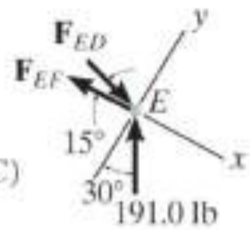
$$+\searrow \Sigma F_x = 0; \quad 639.1 \cos 15^\circ - F_{EF} - 191.0 \sin 30^\circ = 0$$

$$F_{EF} = 521.8 \text{ lb (T)}$$

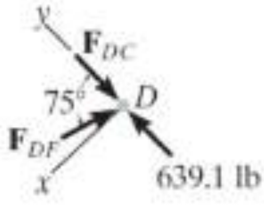
Joint D.

$$+\swarrow \Sigma F_x = 0; \quad -F_{DF} \sin 75^\circ = 0 \qquad F_{DF} = 0$$

$$+\nwarrow \Sigma F_y = 0; \quad -F_{DC} + 639.1 = 0 \qquad F_{DC} = 639.1 \text{ lb (C)}$$



(b)



(c)

Joint C.

$\xrightarrow{+}\Sigma F_x = 0; \quad F_{CB}\sin 45^\circ - 639.1\sin 45^\circ = 0$

$F_{CB} = 639.1 \text{ lb (C)}$

$+\uparrow\Sigma F_y = 0; \quad -F_{CF} - 175 + 2(639.1)\cos 45^\circ = 0$

$F_{CF} = 728.8 \text{ lb (T)}$

Joint B.

$+\nwarrow\Sigma F_y = 0; \quad F_{BF}\sin 75^\circ - 200 = 0 \quad F_{BF} = 207.1 \text{ lb (C)}$

$+\swarrow\Sigma F_x = 0; \quad 639.1 + 207.1\cos 75^\circ - F_{BA} = 0$

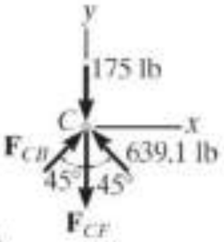
$F_{BA} = 692.7 \text{ lb (C)}$

Joint A.

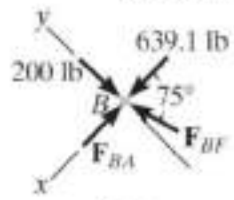
$\xrightarrow{+}\Sigma F_x = 0; \quad F_{AF}\cos 30^\circ - 692.7\cos 45^\circ - 141.4 = 0$

$F_{AF} = 728.9 \text{ lb (T)}$

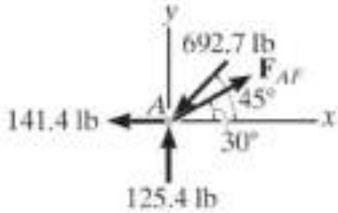
$+\uparrow\Sigma F_y = 0; \quad 125.4 - 692.7\sin 45^\circ + 728.9\sin 30^\circ = 0$ **Check**



(d)

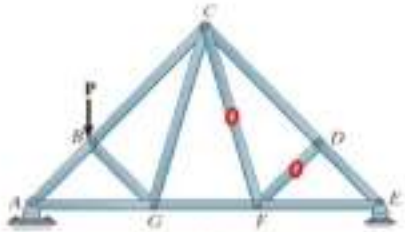


(e)

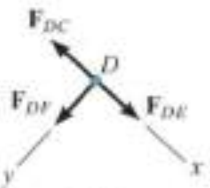


(f)

Zero – Force Members:

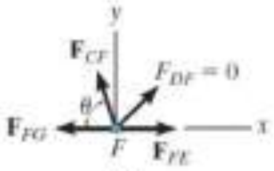


(a)



(b)

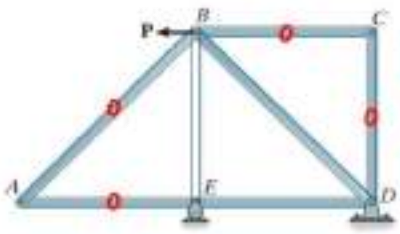
$+\swarrow\Sigma F_y = 0; F_{DF} = 0$



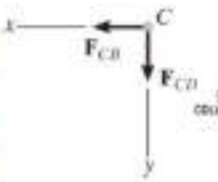
(c)

$+\uparrow\Sigma F_y = 0; F_{CF}\sin\theta + 0 = 0$

$F_{CF} = 0 \text{ (since } \sin\theta \neq 0)$



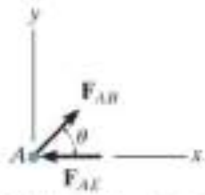
(a)



$\xleftarrow{+}\Sigma F_x = 0; F_{CB} = 0$

$+\downarrow\Sigma F_y = 0; F_{CD} = 0$

(b)



$+\uparrow\Sigma F_y = 0; F_{AB}\sin\theta = 0$

$F_{AB} = 0 \text{ (since } \sin\theta \neq 0)$

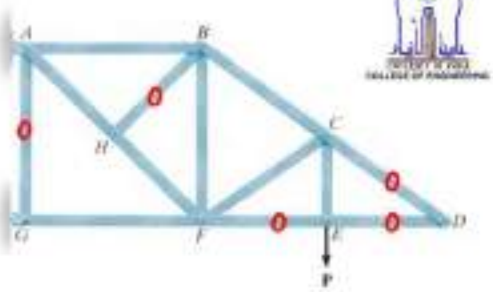
$\xrightarrow{+}\Sigma F_x = 0; -F_{AE} + 0 = 0$

$F_{AE} = 0$

(c)

Example:

Using the method of joints, indicate all the members of the truss shown in figure that have zero force.



Solution:

Joint D.

$+\uparrow\Sigma F_y = 0; \quad F_{DC}\sin\theta = 0 \quad F_{DC} = 0$

$\xrightarrow{+}\Sigma F_x = 0; \quad F_{DE} + 0 = 0 \quad F_{DE} = 0$

Joint E.

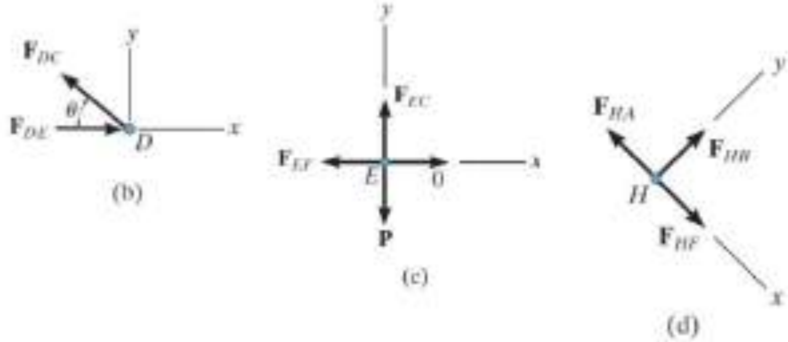
$\xrightarrow{+}\Sigma F_x = 0; \quad F_{EF} = 0$

Joint H.

$+\nearrow\Sigma F_y = 0; \quad F_{HB} = 0$

Joint G.

$+\uparrow\Sigma F_y = 0; \quad F_{GA} = 0$



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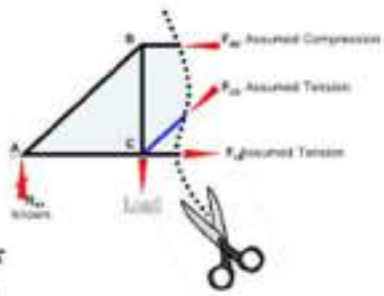
Method of Sections:

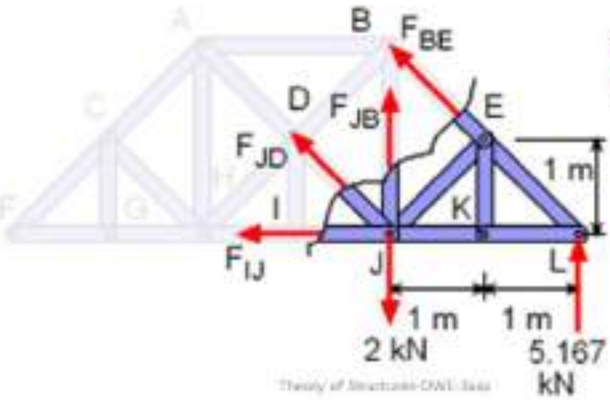
When the method of sections is used to determine the force in a particular member, a decision must be made as to how to "cut" or section the truss. Since only three independent equilibrium equations ($\sum F_x = 0$, $\sum F_y = 0$ **and** $\sum M_o = 0$) can be applied to the isolated portion of the truss, try to select a section that, in general, passes through not more than three members in which the forces are unknown.

Assume the forces on cut members act as external forces on the cut



$$\sum M_L = 0$$

$$\sum M_J = 0$$



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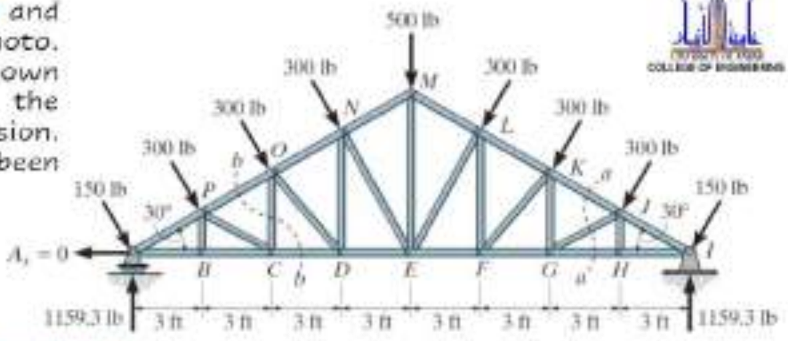
Example:

Determine the force in members **GJ** and **CO** of the roof truss shown in the photo. The dimensions and loadings are shown in the figure. State whether the members are in tension or compression. The reactions at the supports have been calculated.



(a)

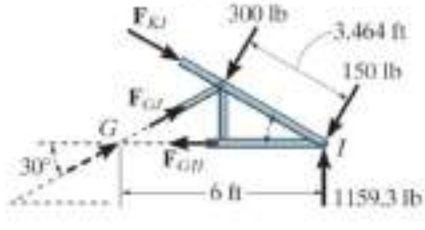
Solution:

Member GJ.

Free-Body Diagram. The force in member GJ can be obtained by considering the section aa . Taking the free-body diagram of the right part of this section:

$$\circlearrowleft + \Sigma M_I = 0; \qquad -F_{GJ}\sin 30°(6) + 300(3.464) = 0$$

$$F_{GJ} = 346 \text{ lb (C)}$$



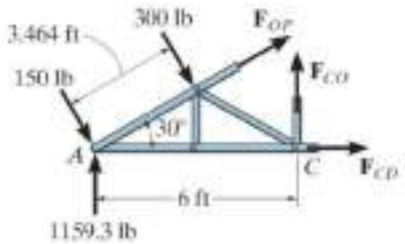
(b)

Member CO.

The force in CO can be obtained by using section bb. Taking the free-body diagram of the left portion of the section:

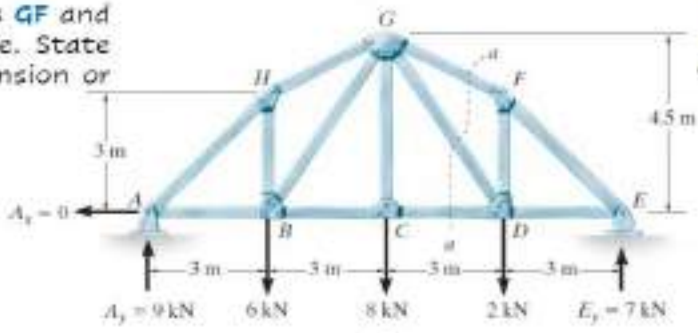
$$\circlearrowleft + \Sigma M_A = 0; \qquad -300(3.464) + F_{CO}(6) = 0$$

$$F_{CO} = 173 \text{ lb (T)}$$



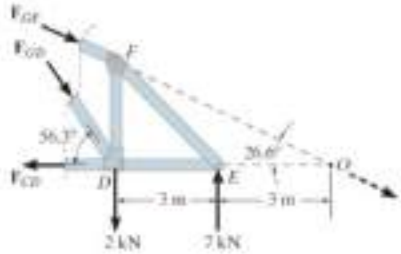
(c)

Example:
Determine the force in members **GF** and **GD** of the truss shown in figure. State whether the members are in tension or compression.



(a)

Solution:

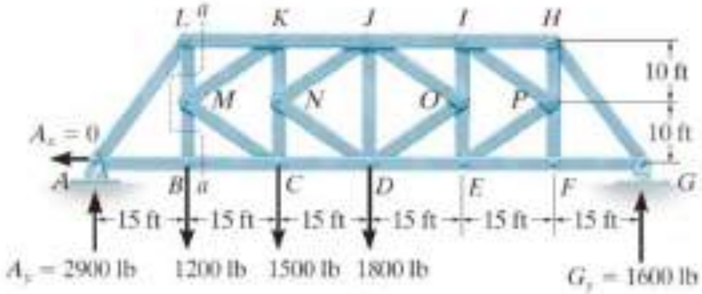


(b)

$$\circlearrowleft + \Sigma M_D = 0; \qquad -F_{GF} \sin 26.6^\circ(6) + 7(3) = 0$$
$$F_{GF} = 7.83 \text{ kN (C)}$$

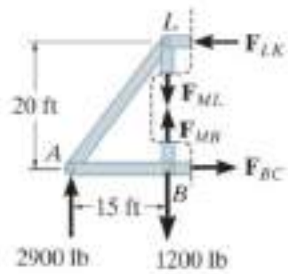
$$\circlearrowleft + \Sigma M_O = 0; \qquad -7(3) + 2(6) + F_{GD} \sin 56.3^\circ(6) = 0$$
$$F_{GD} = 1.80 \text{ kN (C)}$$

Example:
Determine the force in members **BC** and **MC** of the K-truss shown in the figure. State whether the members are in tension or compression.



(a)

Solution:



(b)

$$\circlearrowleft + \Sigma M_L = 0; \qquad -2900(15) + F_{BC}(20) = 0$$
$$F_{BC} = 2175 \text{ lb (T)}$$

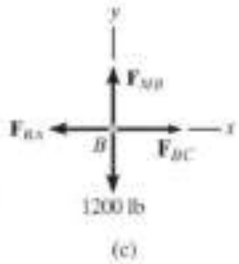
The force in **MC** can be obtained indirectly by first obtaining the force in **MB** from vertical force equilibrium of joint B, i.e., **F_{MB}=1200 lb (T)** Then:

$$+\uparrow \Sigma F_y = 0; \qquad 2900 - 1200 + 1200 - F_{ML} = 0$$
$$F_{ML} = 2900 \text{ lb (T)}$$

$$\xrightarrow{+} \Sigma F_x = 0; \quad \left(\frac{3}{\sqrt{13}}\right)F_{MC} - \left(\frac{3}{\sqrt{13}}\right)F_{MK} = 0$$

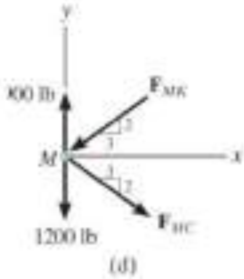
$$+\uparrow \Sigma F_y = 0; \quad 2900 - 1200 - \left(\frac{2}{\sqrt{13}}\right)F_{MC} - \left(\frac{2}{\sqrt{13}}\right)F_{MK} = 0$$

$$F_{MK} = 1532 \text{ lb (C)} \qquad F_{MC} = 1532 \text{ lb (T)} \qquad \textit{Ans.}$$



(c)

Hint:
It is also possible to solve for the force in **MC** by using the result for In this case, pass a vertical section through **LK,MK,MC**, and **BC**. Isolate the left section and apply $\sum M_K = 0$.



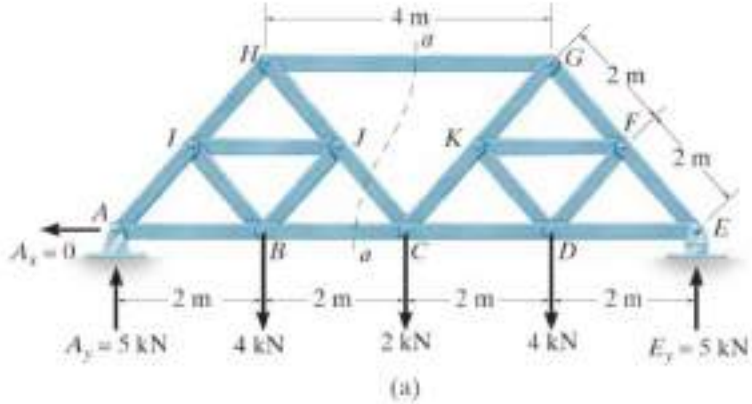
(d)

Compound Trusses:

If this type of truss is best analysed by applying both the method of joints and the method of sections. It is often convenient to first recognize the type of construction and then perform the analysis using the following procedure.

Mixed Analysis Method:

Compound trusses can be analysed using mixed method where section method can be used to find member forces that will help in solving the other ones using joint method or vice versa.

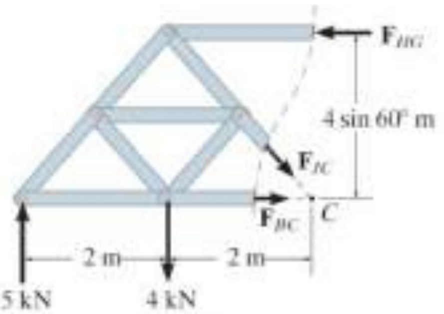


(a)

Solution:

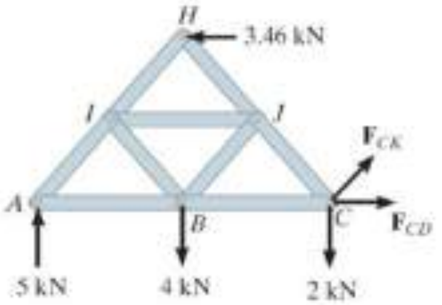
$$\circlearrowleft + \Sigma M_C = 0; \quad -5(4) + 4(2) + F_{HG}(4 \sin 60^\circ) = 0$$

$$F_{HG} = 3.46 \text{ kN (C)}$$



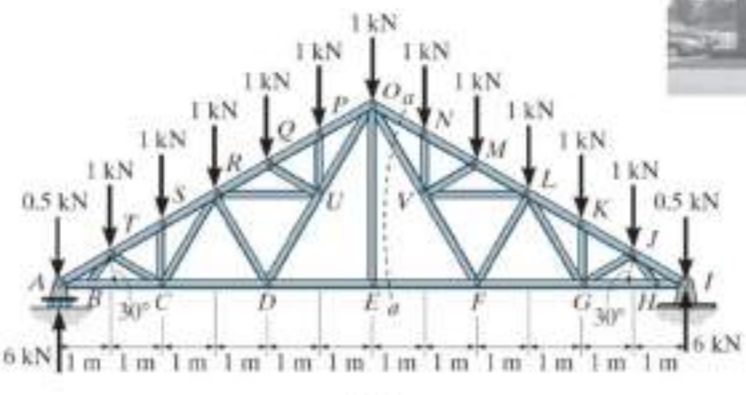
(b)

Joint A: Determine the force in *AB* and *AI*.
Joint H: Determine the force in *HI* and *HJ*.
Joint I: Determine the force in *IJ* and *IB*.
Joint B: Determine the force in *BC* and *BJ*.
Joint J: Determine the force in *JC*.



(c)

Example:
Compound roof trusses are used in a garden centre, as shown in the photo. They have the dimensions and loading shown in Fig. a. Indicate how to analyse this truss.



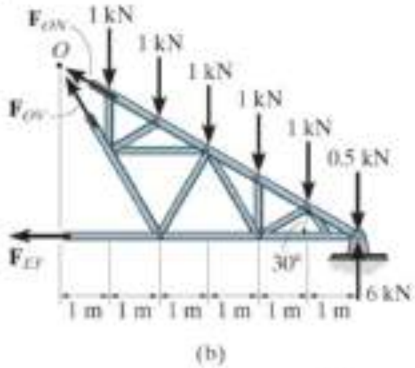
(a)

Solution:

$$\circlearrowleft +\Sigma M_O = 0; \quad -1(1) - 1(2) - 1(3) - 1(4) - 1(5) - 0.5(6) + 6(6) - F_{EF}(6 \tan 30^\circ) = 0$$

$$F_{EF} = 5.20 \text{ kN (T)} \qquad \textit{Ans.}$$

By inspection notice that BT, EO, and HJ are zero-force members since $+\uparrow \Sigma F_y = 0$ at joints B, E, and H, respectively. Also, by applying $+\nwarrow \Sigma F_y = 0$ (perpendicular to AO) at joints P, Q, S, and T, we can directly determine the force in members PU, QU, SC, and TC, respectively.



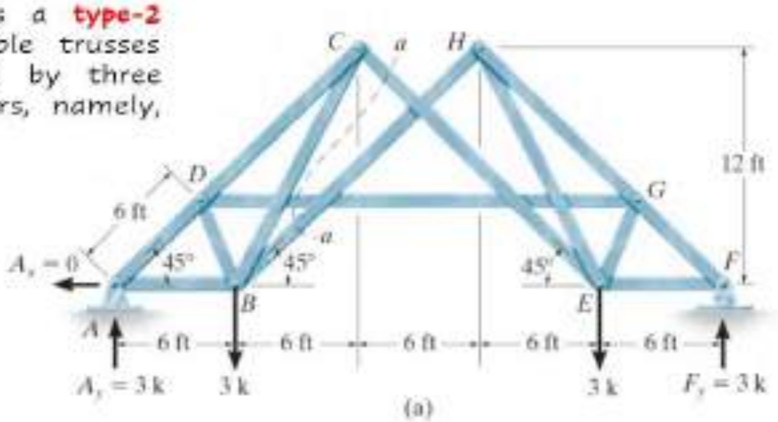
(b)

It is a good practice to try solving it yourself !

Example:
Indicate how to analyse the compound truss shown in the figure.

Solution:

The truss may be classified as a **type-2** compound truss since the simple trusses **ABCD** and **FEHG** are connected by three nonparallel or nonconcurrent bars, namely, **CE**, **BH**, and **DG**.



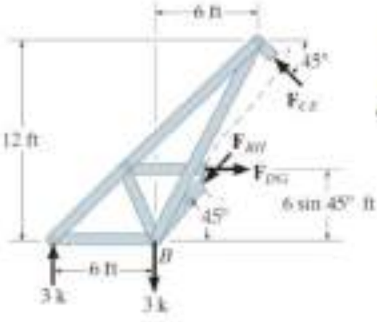
(a)

$$\circlearrowleft + \Sigma M_B = 0; \quad -3(6) - F_{DG}(6 \sin 45^\circ) + F_{CE} \cos 45^\circ (12) + F_{CE} \sin 45^\circ (6) = 0$$

$$+\uparrow \Sigma F_y = 0; \quad 3 - 3 - F_{BH} \sin 45^\circ + F_{CE} \sin 45^\circ = 0$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \quad -F_{BH} \cos 45^\circ + F_{DG} - F_{CE} \cos 45^\circ = 0$$

$$F_{BH} = F_{CE} = 2.68 \text{ k (C)} \qquad F_{DG} = 3.78 \text{ k (T)}$$

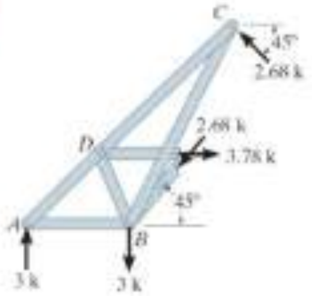


(b)

Practice, Practice, and Practice !

Hint:

Joint A: Determine the force in *AB* and *AD*.
Joint D: Determine the force in *DC* and *DB*.
Joint C: Determine the force in *CB*.

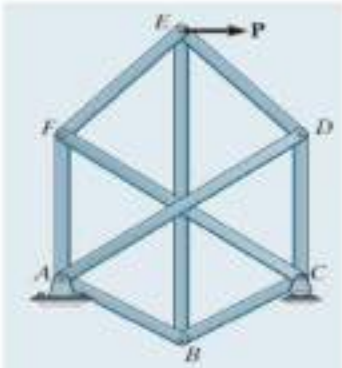


(c)

10/11/2020

Complex Trusses:

If this The member forces in a complex truss can be determined using the method of joints; however, the solution will require writing the two equilibrium equations for each of the j joints of the truss and then solving the complete set of 2j equations simultaneously. This approach may be impractical for hand calculations, especially in the case of large trusses. Therefore, a more direct method for analysing a complex truss, referred to as the method of substitute members, will be presented here.

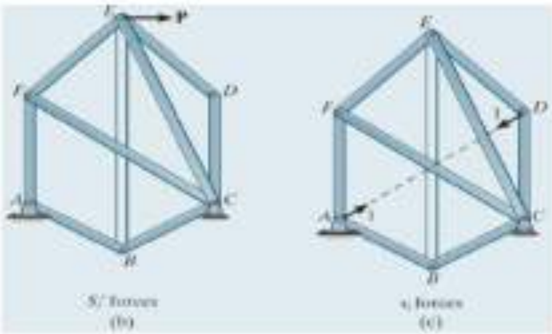


(a)

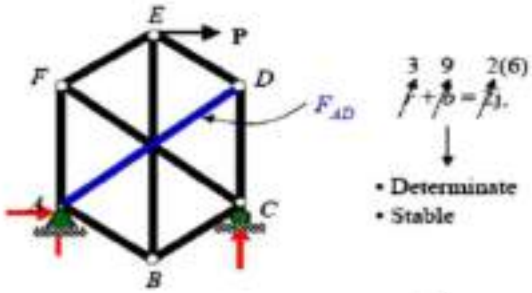
10/11/2020

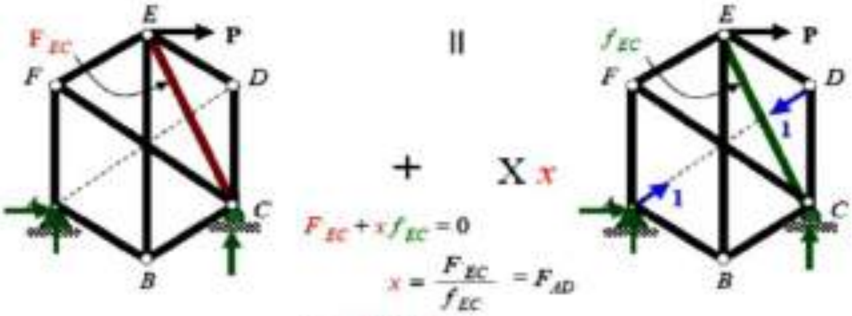
Procedure of Analysis:

1- Reduction to Stable Simple Truss
2- External Loading on Simple Truss
3- Remove External Loading from Simple Truss
4- Superposition



Complex Trusses





$F_{EC} + x f_{EC} = 0$

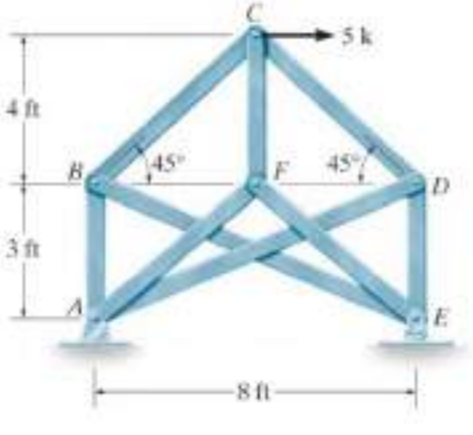
$x = \frac{F_{EC}}{f_{EC}} = F_{AD}$

$F_i = F'_i + x f_i$

Example:

Determine the force in each member of the complex truss shown in the figure. Assume joints B, F, and D are on the same horizontal line.
State whether the members are in tension or compression.

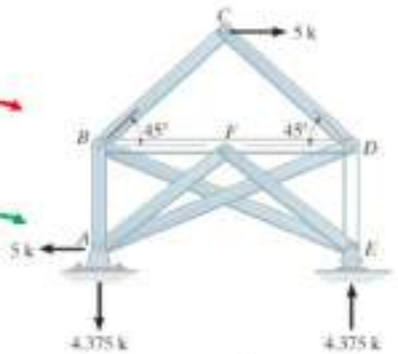
Solution:



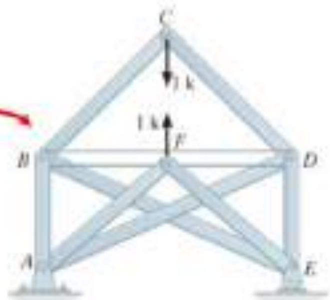
(a)

1- Reduction to Stable Simple Truss

2- External Loading on Simple Truss



3- Remove External Loading from Simple Truss



(c)

4- Superposition

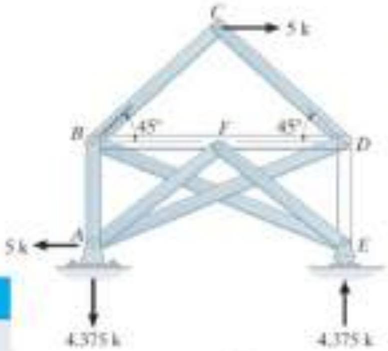
$$S_{DB} = S'_{DB} + xs_{DB} = 0$$

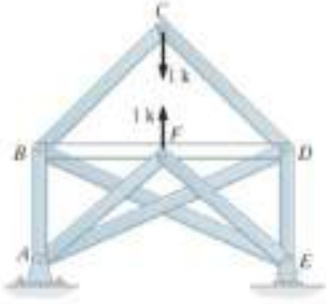
$$-2.50 + x(1.167) = 0 \qquad x = 2.143$$

$$S_i = S'_i + xs_i$$

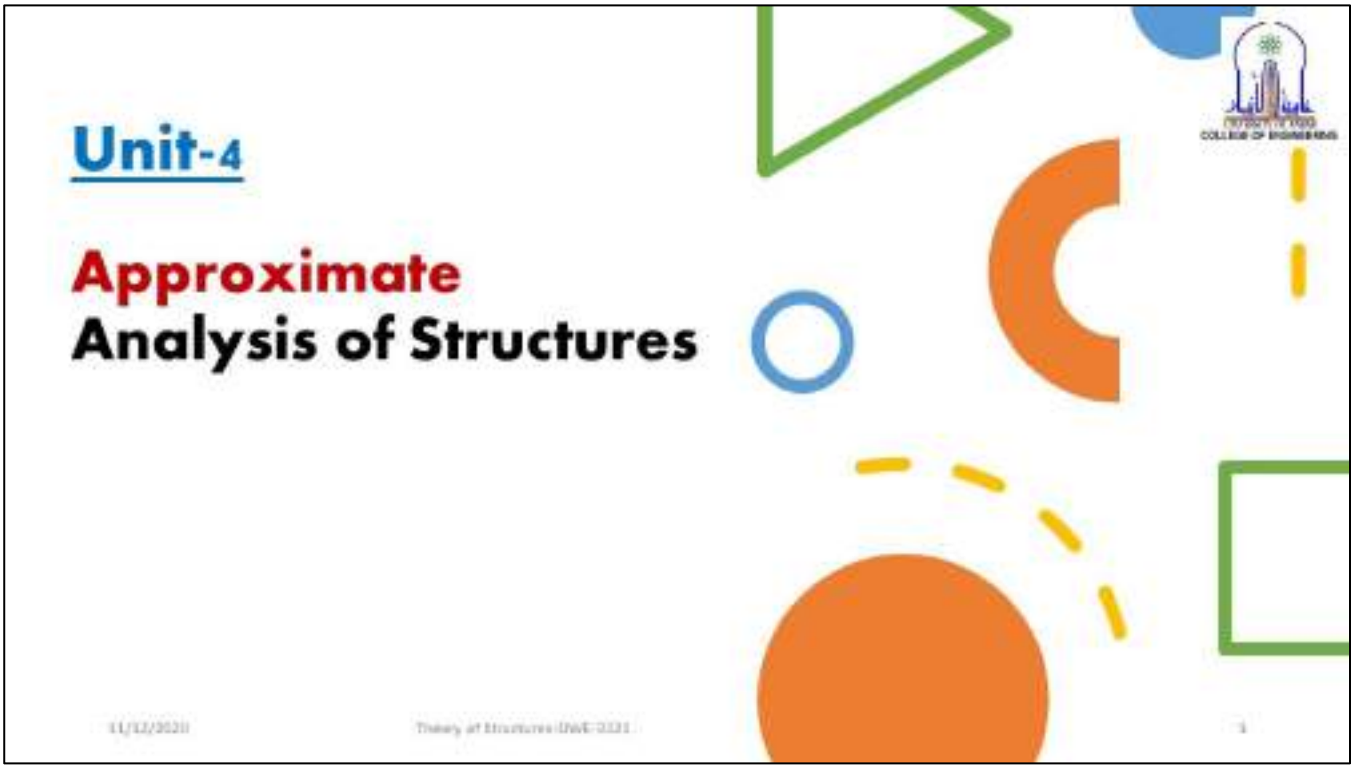
TABLE 1

Member	S'_i	s_i	xs_i	S_i
CB	3.54	−0.707	−1.52	2.02 (T)
CD	−3.54	−0.707	−1.52	5.05 (C)
FA	0	0.833	1.79	1.79 (T)
FE	0	0.833	1.79	1.79 (T)
EB	0	−0.712	−1.53	1.53 (C)
ED	−4.38	−0.250	−0.536	4.91 (C)
DA	5.34	−0.712	−1.53	3.81 (T)
DB	−2.50	1.167	2.50	0
BA	2.50	−0.250	−0.536	1.96 (T)
CB				2.14 (T)

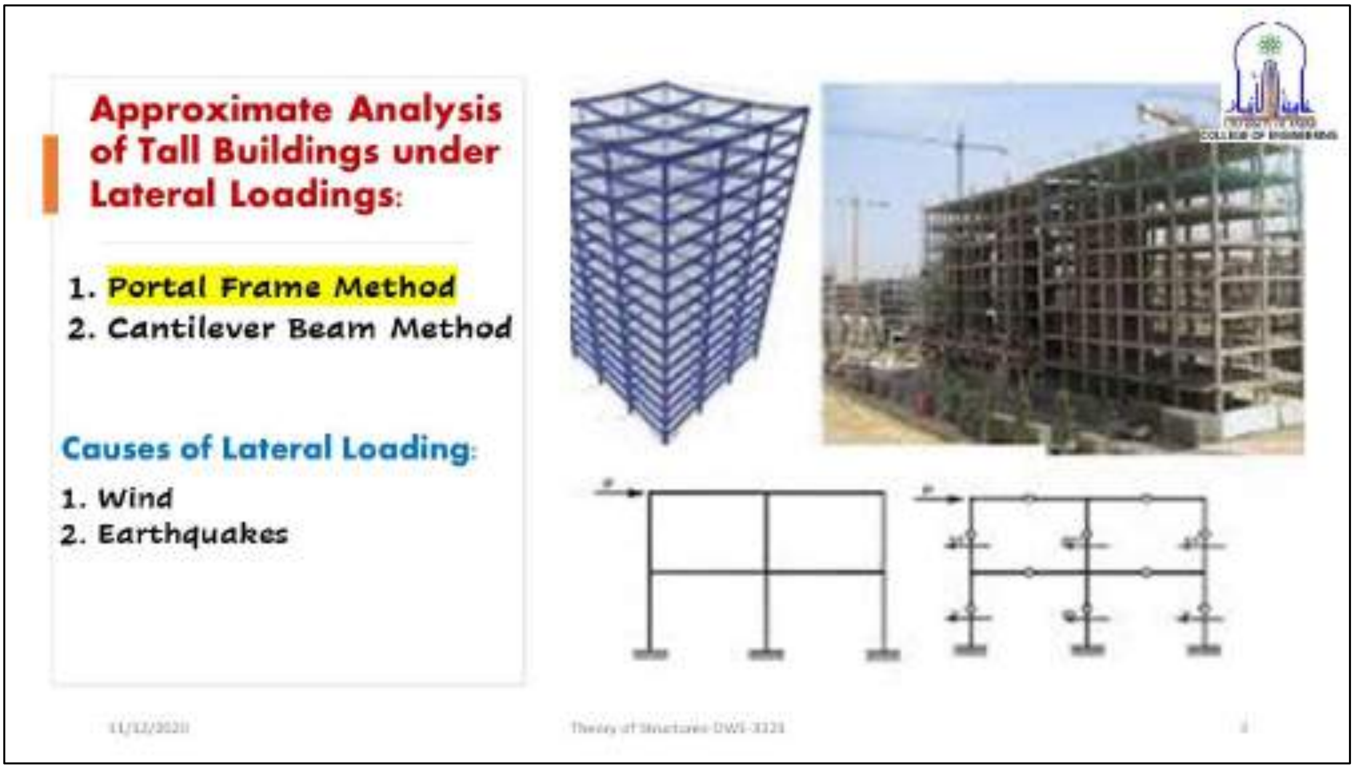


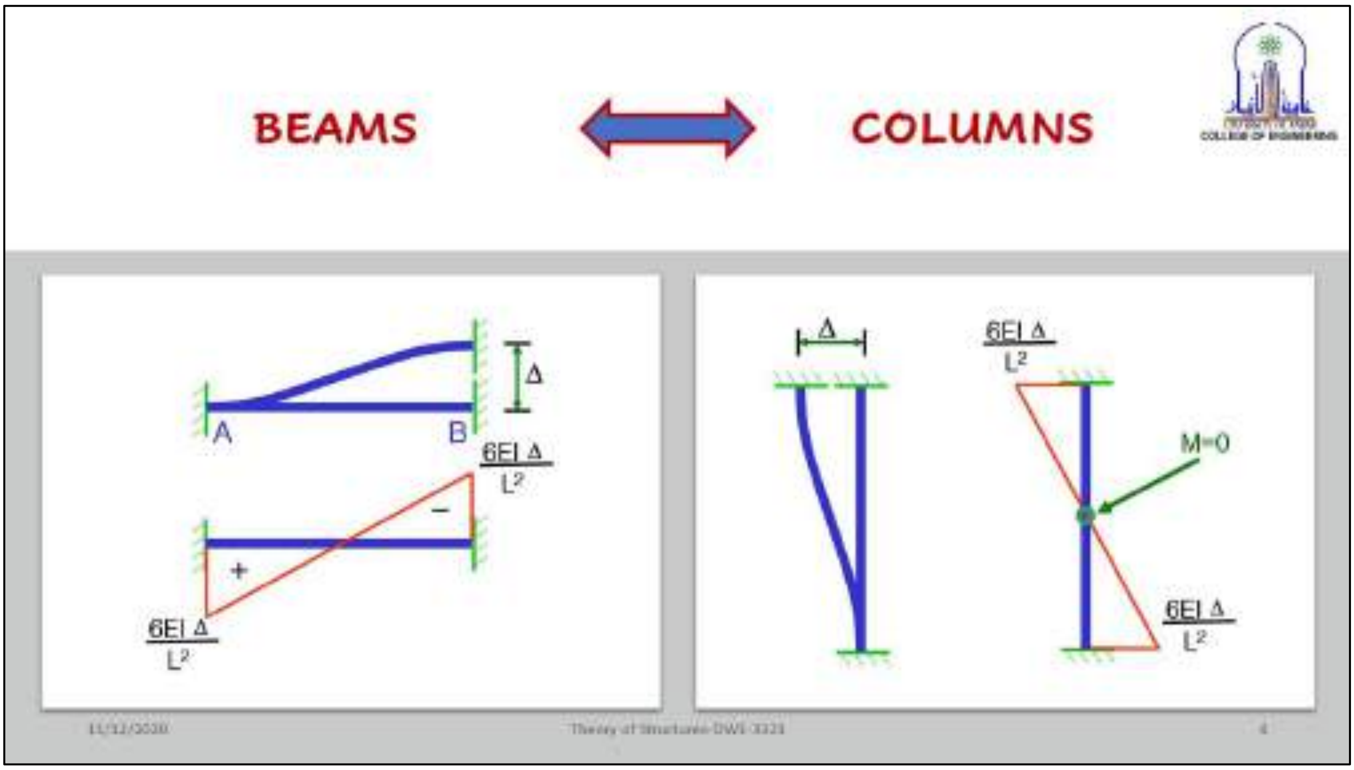


(c)

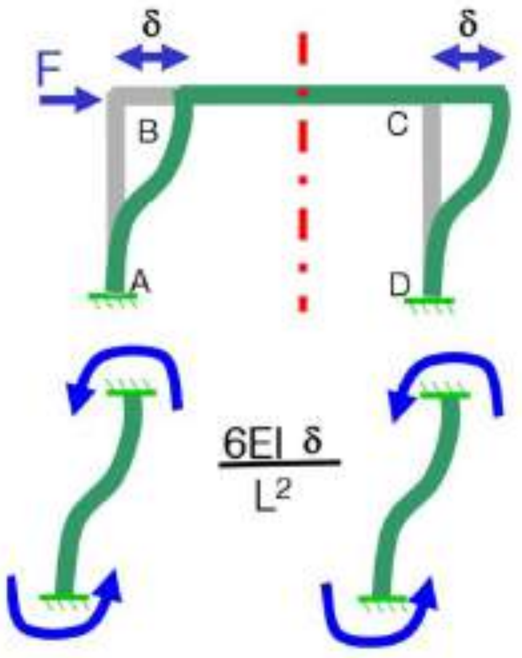

Unit-4
Approximate
Analysis of Structures


Awesome
Structures

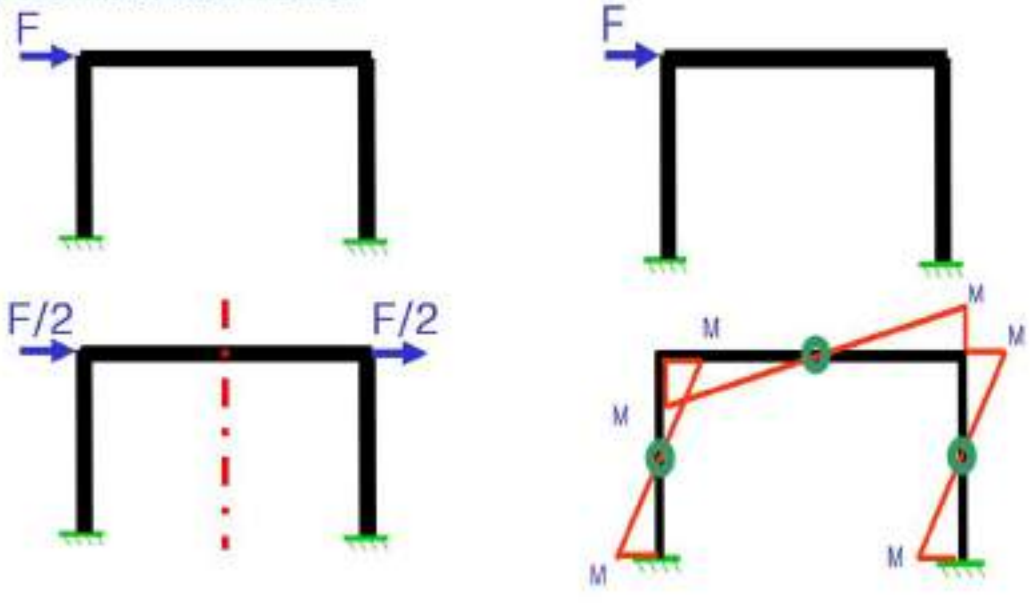

Approximate Analysis of Tall Buildings under Lateral Loadings:
1. Portal Frame Method
2. Cantilever Beam Method
Causes of Lateral Loading:
1. Wind
2. Earthquakes


BEAMS
COLUMNS
A
B
Δ
6EI Δ / L²
+
−
M=0

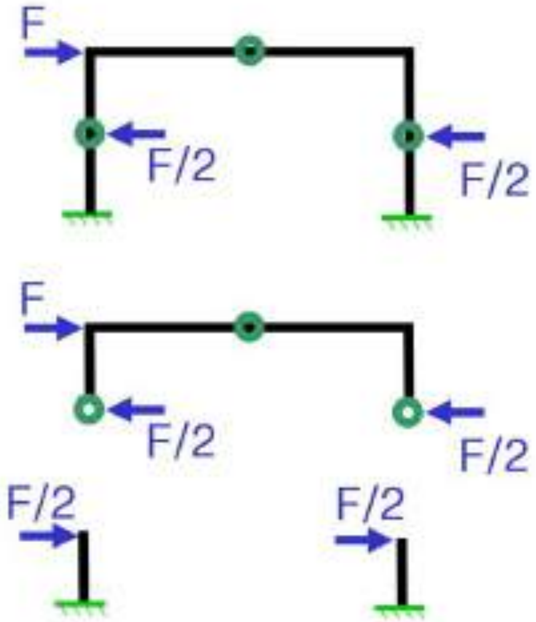
Anti-Symmetry

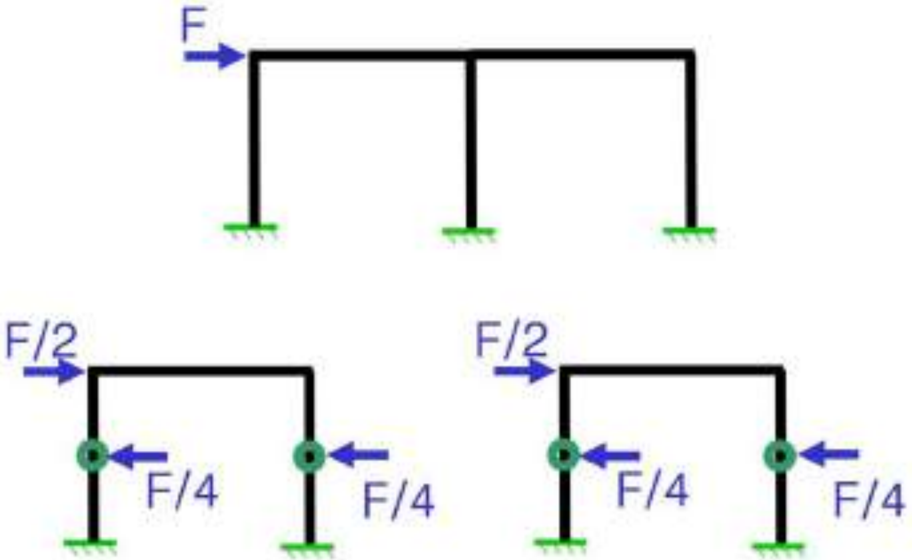


Criterion-1: when frame is subjected to lateral loads, we can put intermediate hinge in the mid of each member

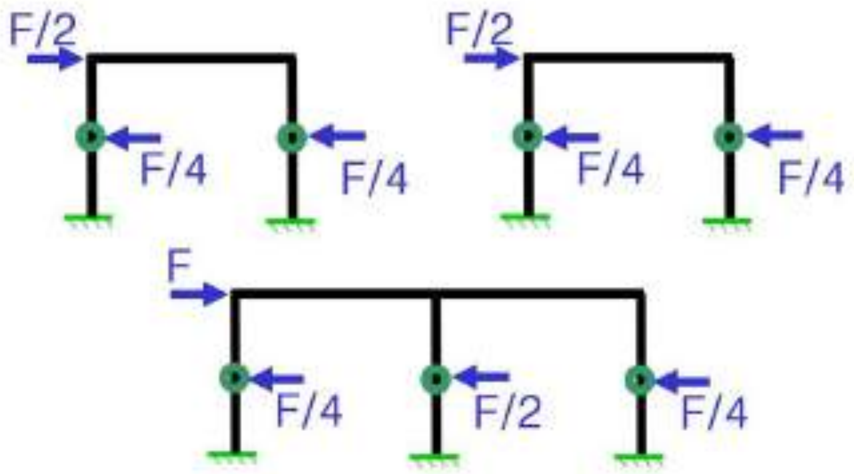


One Storey Frames:

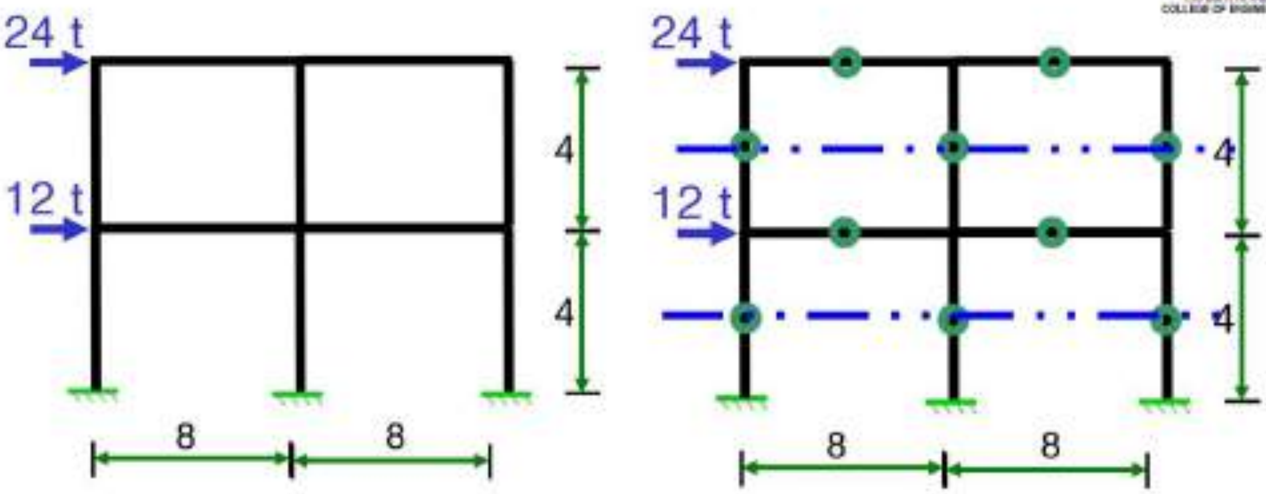


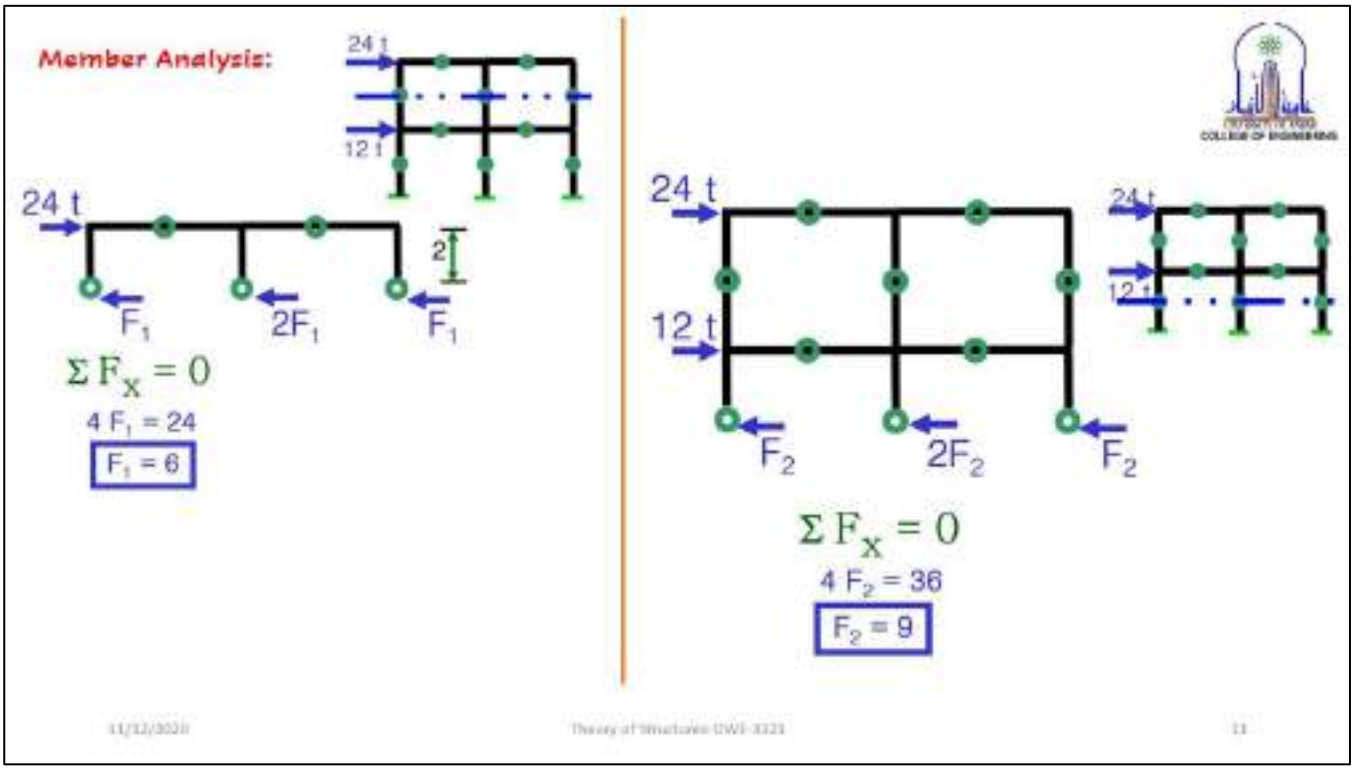


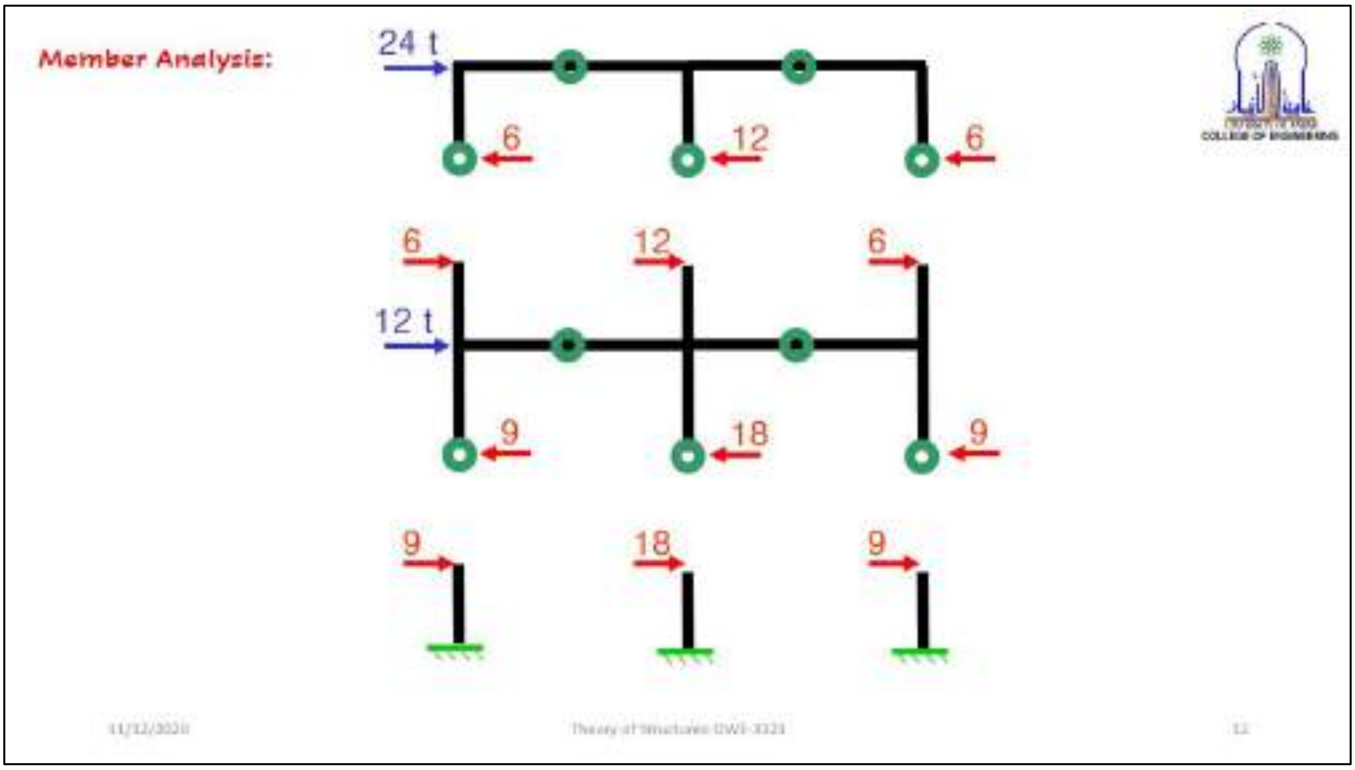
Criterion-2: When frame is subjected to lateral loads, the interior column carries the double of the exterior columns

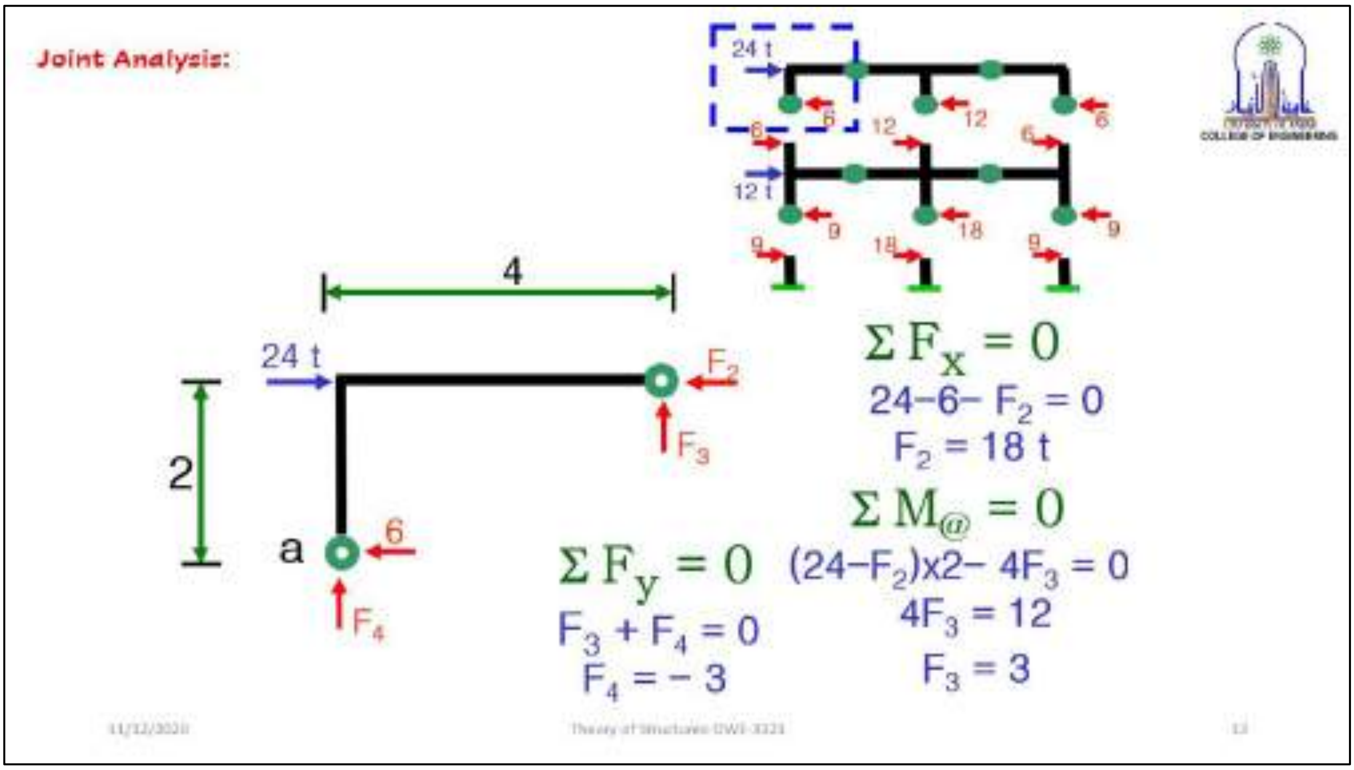


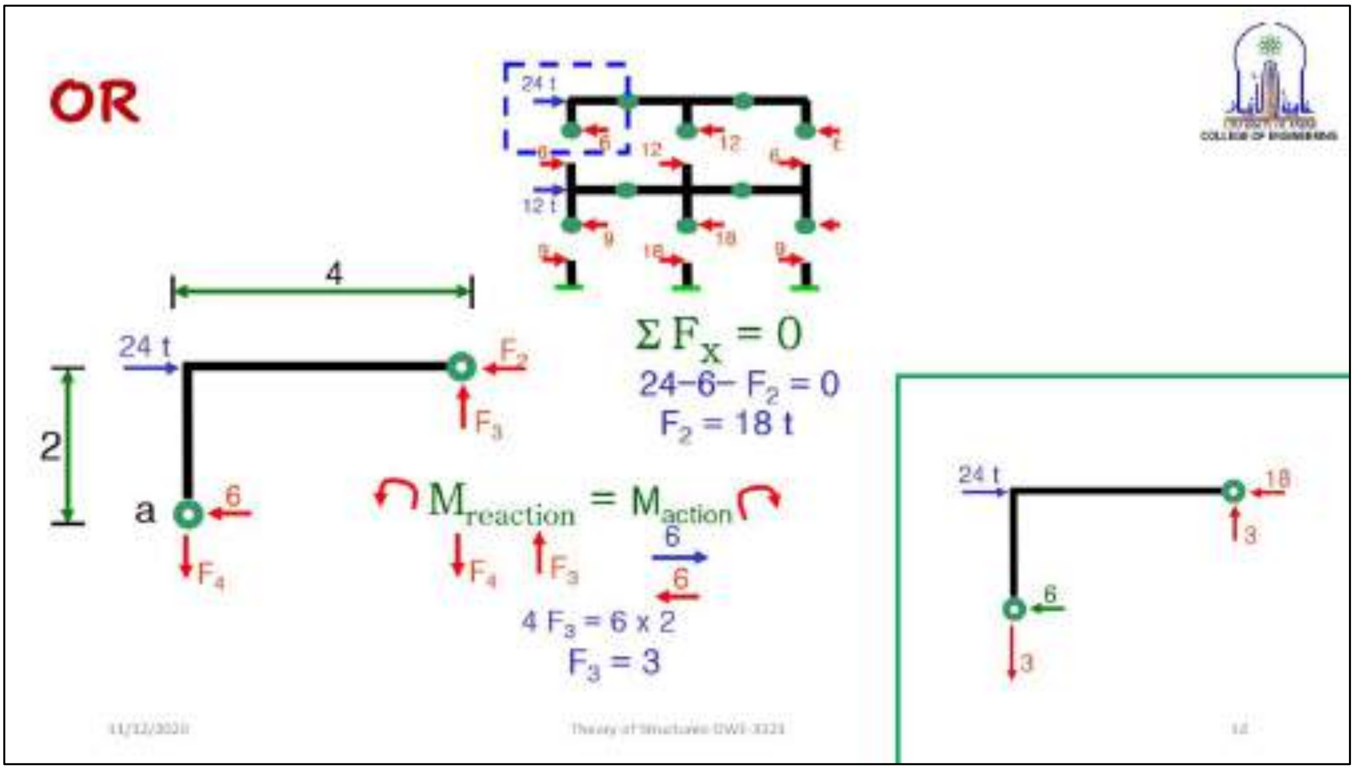
Example-1:

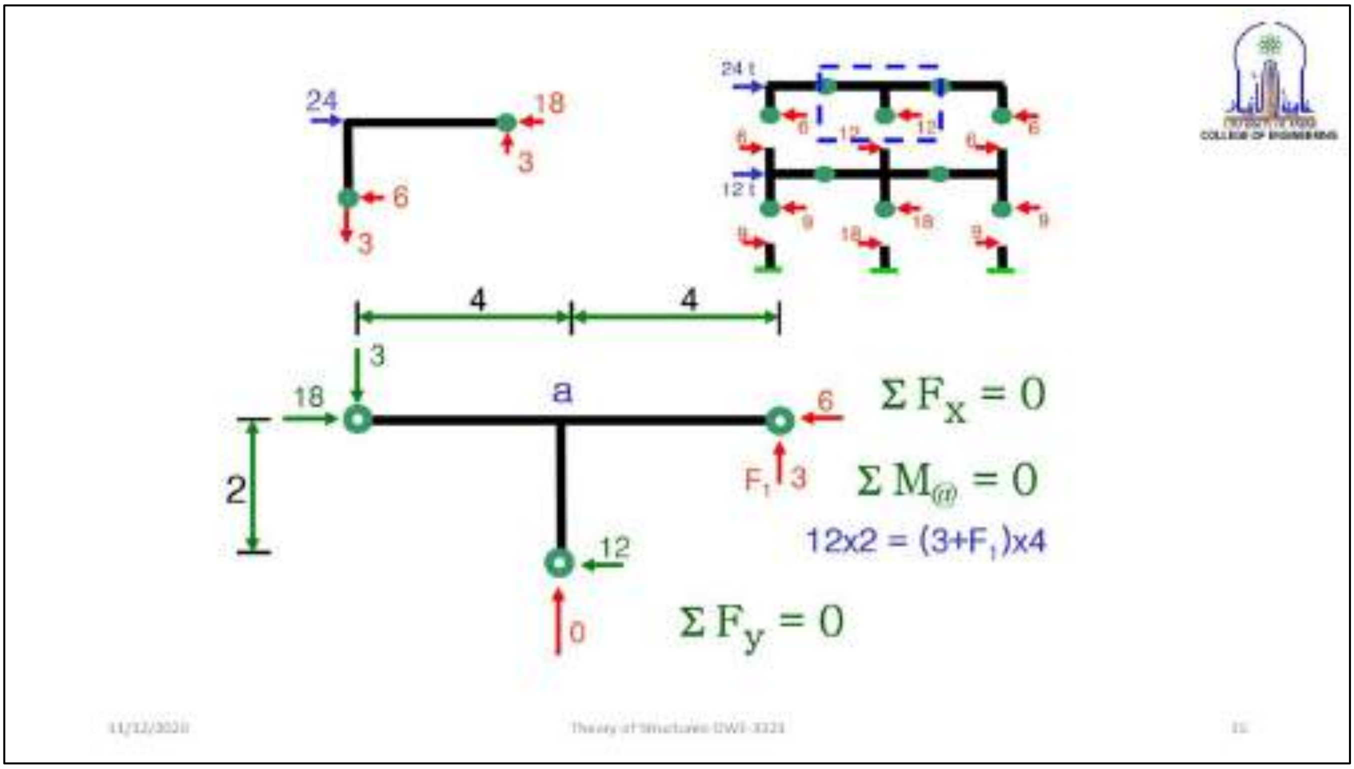



Member Analysis:
24 t
12 t
24 t
2
F1
2F1
F1
ΣFx = 0
4 F1 = 24
F1 = 6
24 t
12 t
F2
2F2
F2
ΣFx = 0
4 F2 = 36
F2 = 9
11/12/2020
Theory of Structures CIVE 3323

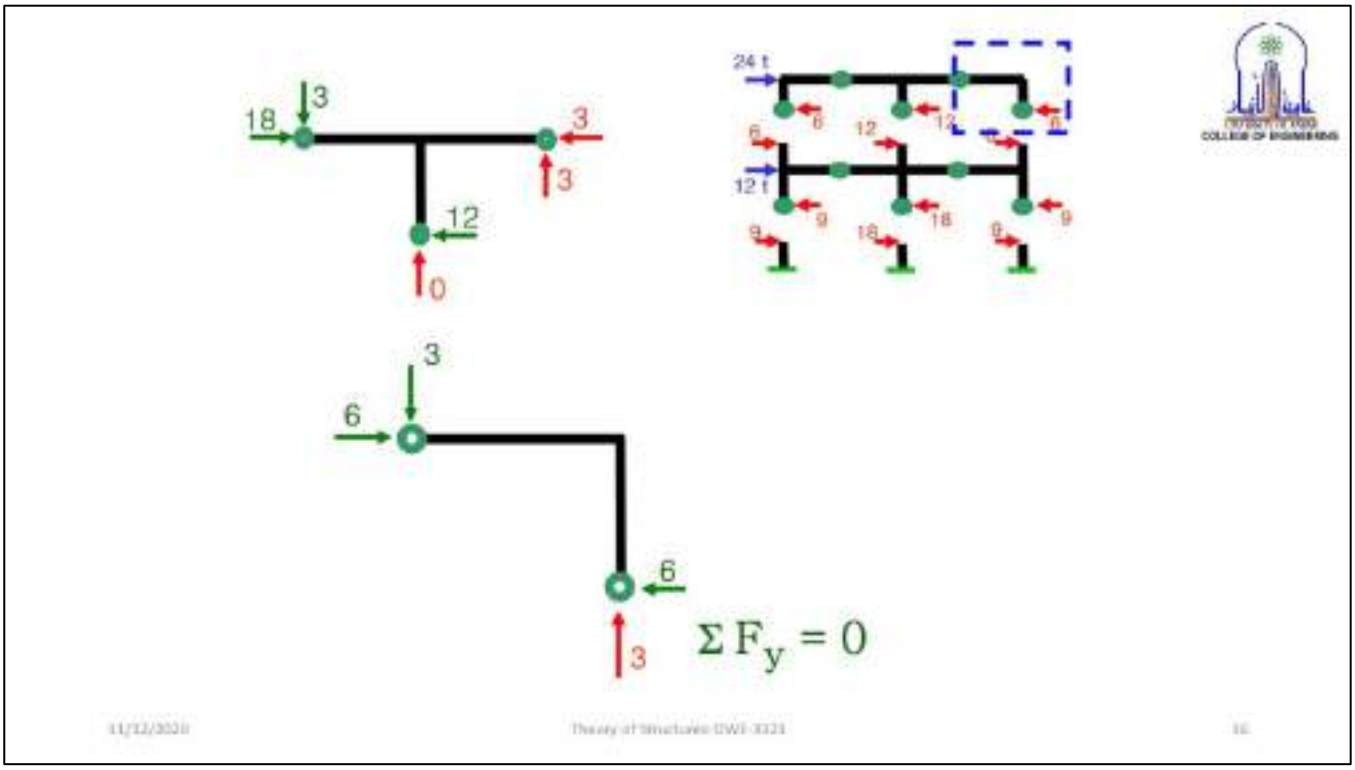

Member Analysis:
24 t
6
12
6
6
12
6
12 t
9
18
9
9
18
9
11/12/2020
Theory of Structures CIVE 3323


Joint Analysis:
24 t
4
2
a
6
F2
F3
F4
ΣFx = 0
24−6− F2 = 0
F2 = 18 t
ΣM@ = 0
(24−F2)x2− 4F3 = 0
4F3 = 12
F3 = 3
ΣFy = 0
F3 + F4 = 0
F4 = − 3

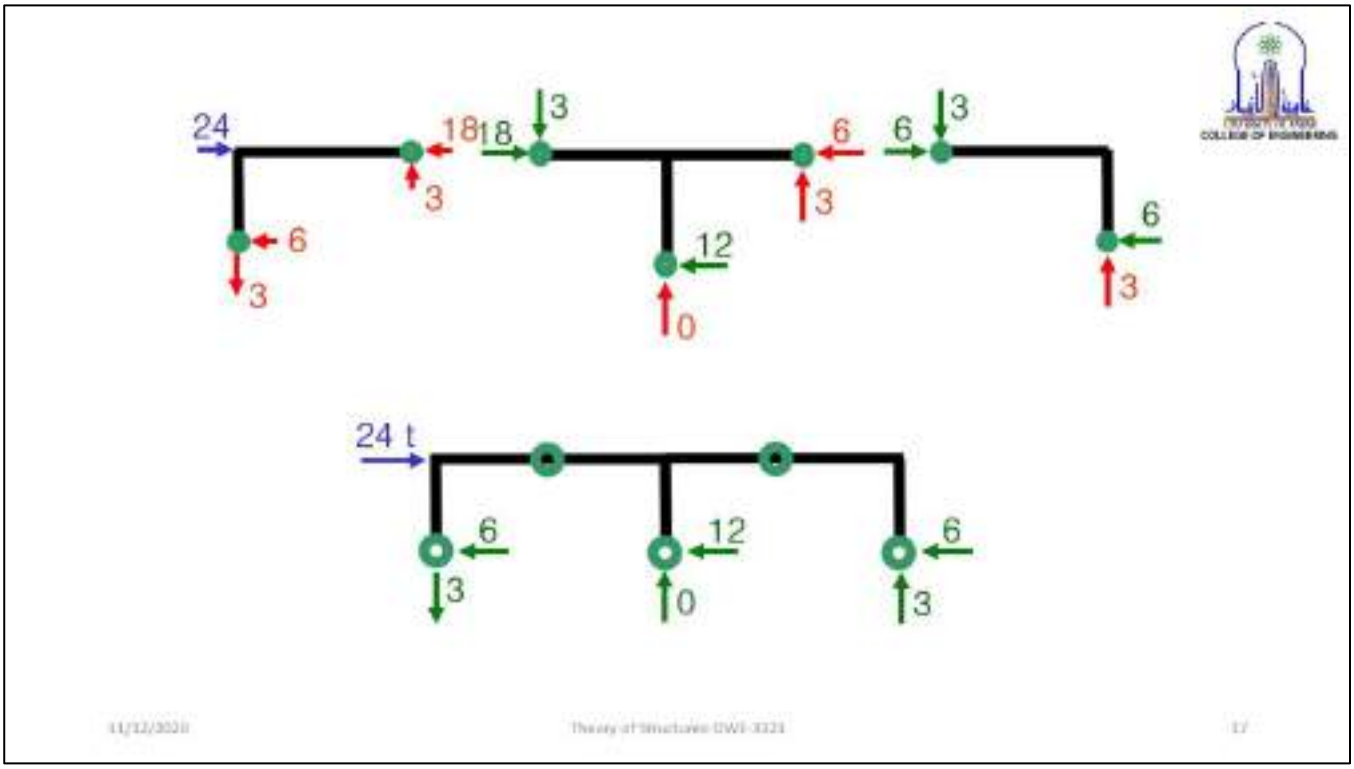

OR
24 t
4
2
a
6
F2
F3
F4
ΣFx = 0
24−6− F2 = 0
F2 = 18 t
Mreaction = Maction
4 F3 = 6 x 2
F3 = 3
18
3



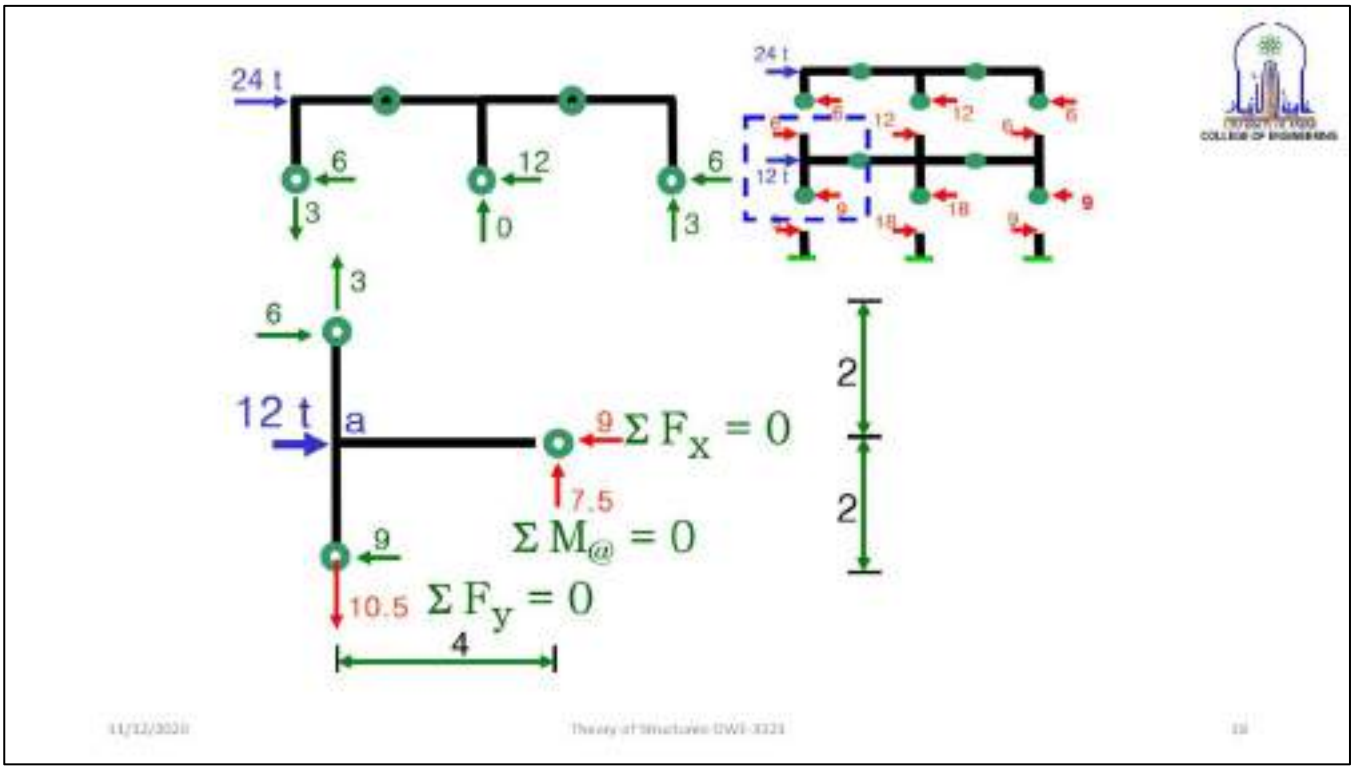
24
18
3
6
3
24 t
12 t
4
4
3
18
a
6
2
F_1 3
12
0
$\Sigma F_x = 0$
$\Sigma M_{@} = 0$
12x2 = (3+F_1)x4
$\Sigma F_y = 0$



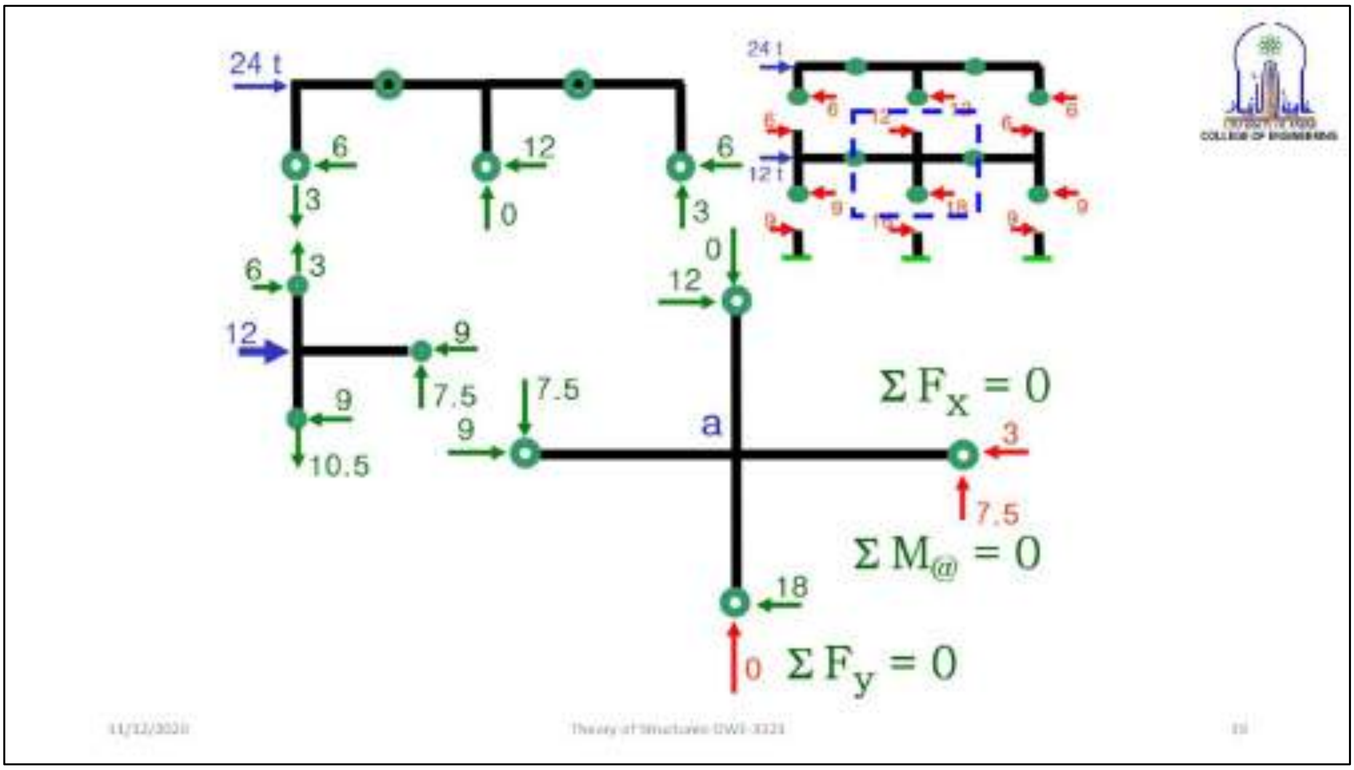
18
3
3
3
12
0
24 t
12 t
3
6
6
3
$\Sigma F_y = 0$

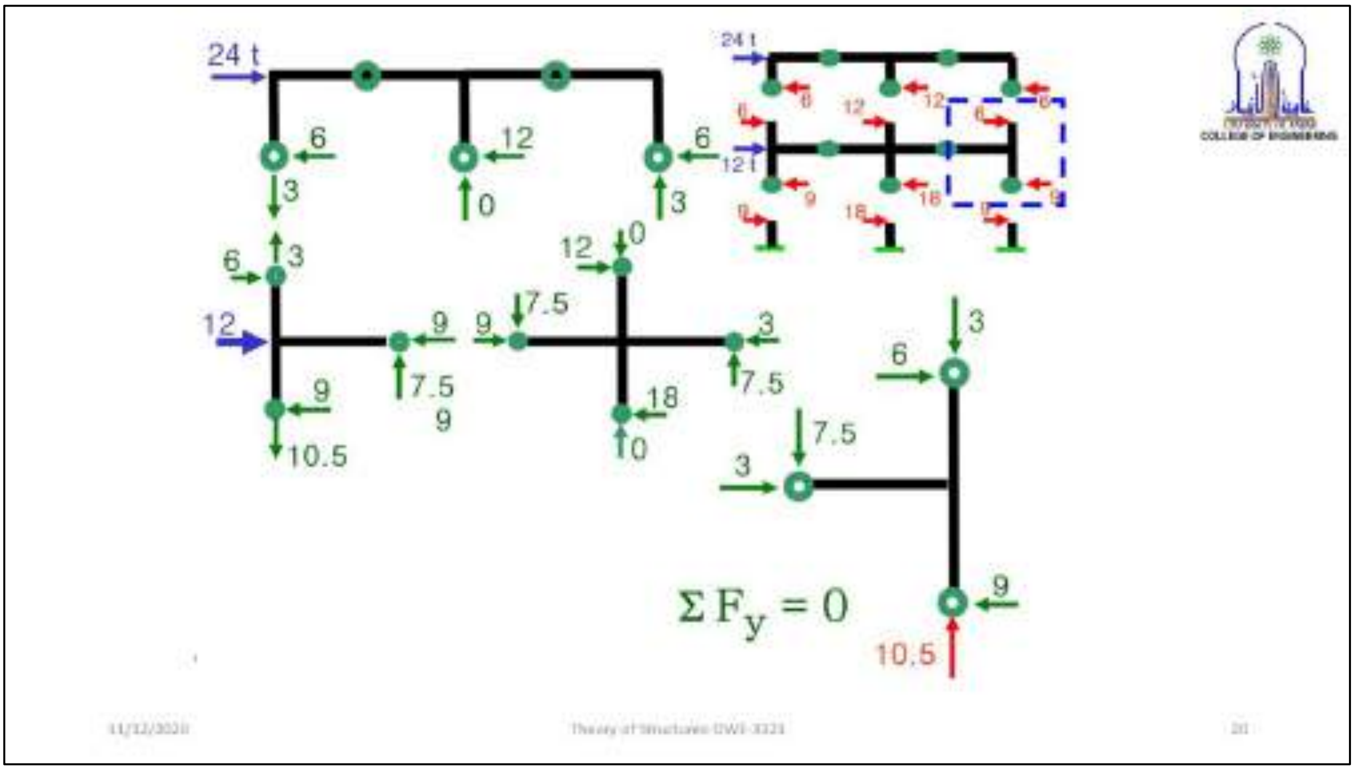


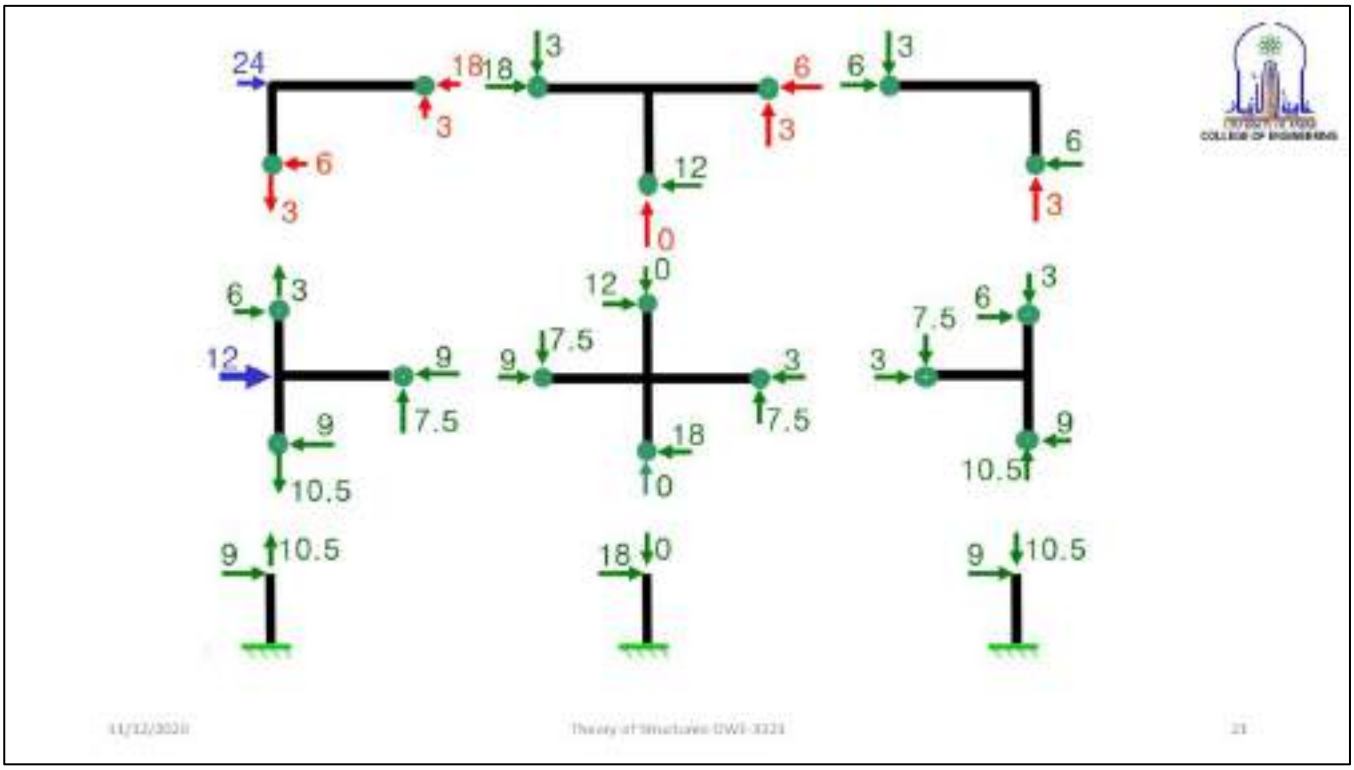
24
18
18
3
6
6
3
3
6
3
3
12
0
3
6
3
24 t
6
12
6
3
0
3

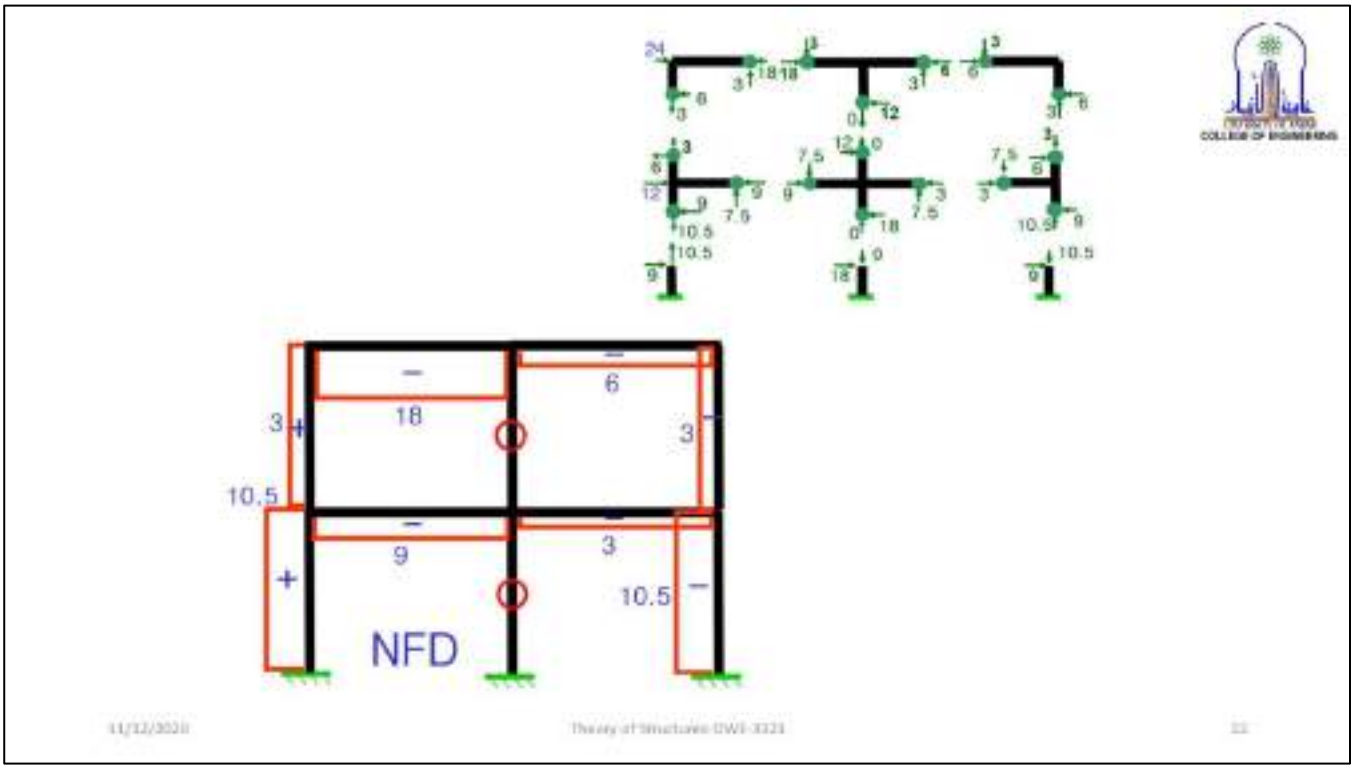


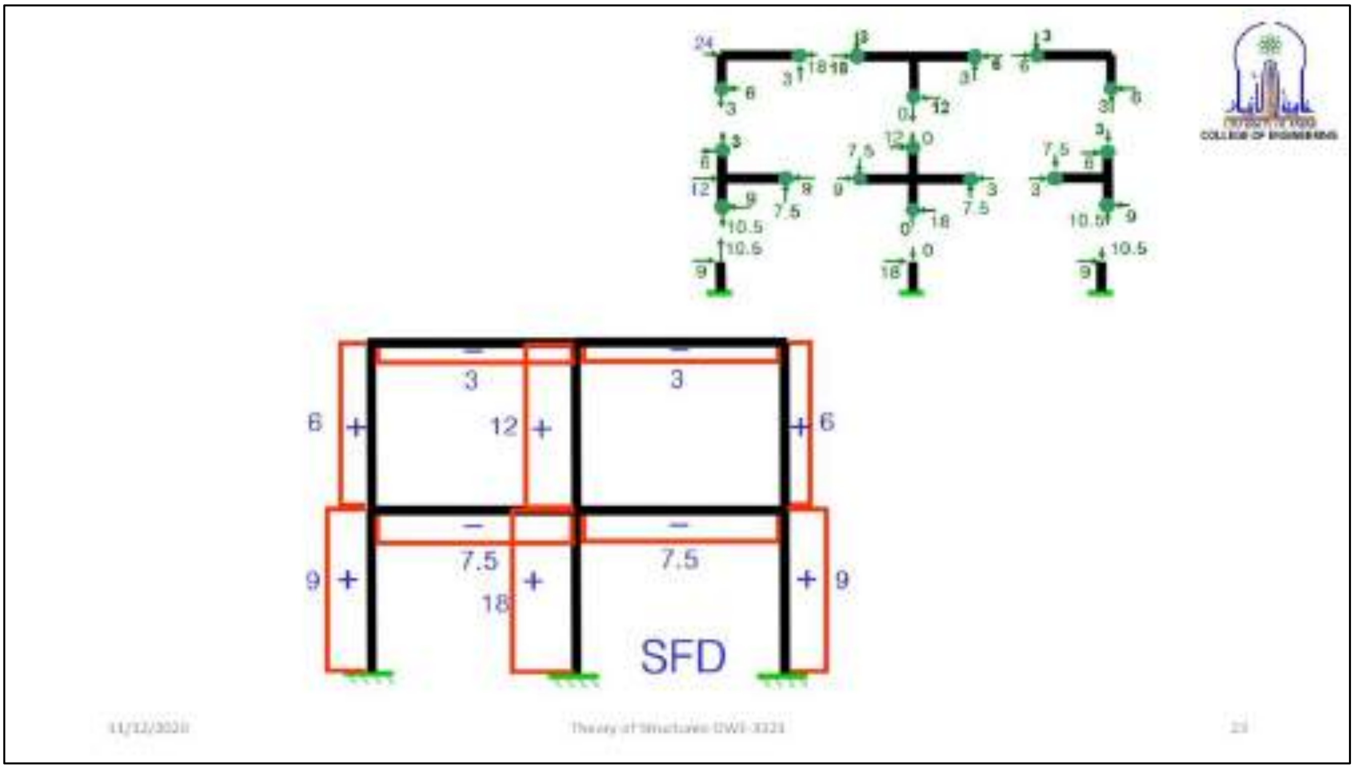
24 t
6
12
6
3
0
3
3
6
12 t
a
9
7.5
9
10.5
4
2
2
$\Sigma F_x = 0$
$\Sigma M_@ = 0$
$\Sigma F_y = 0$

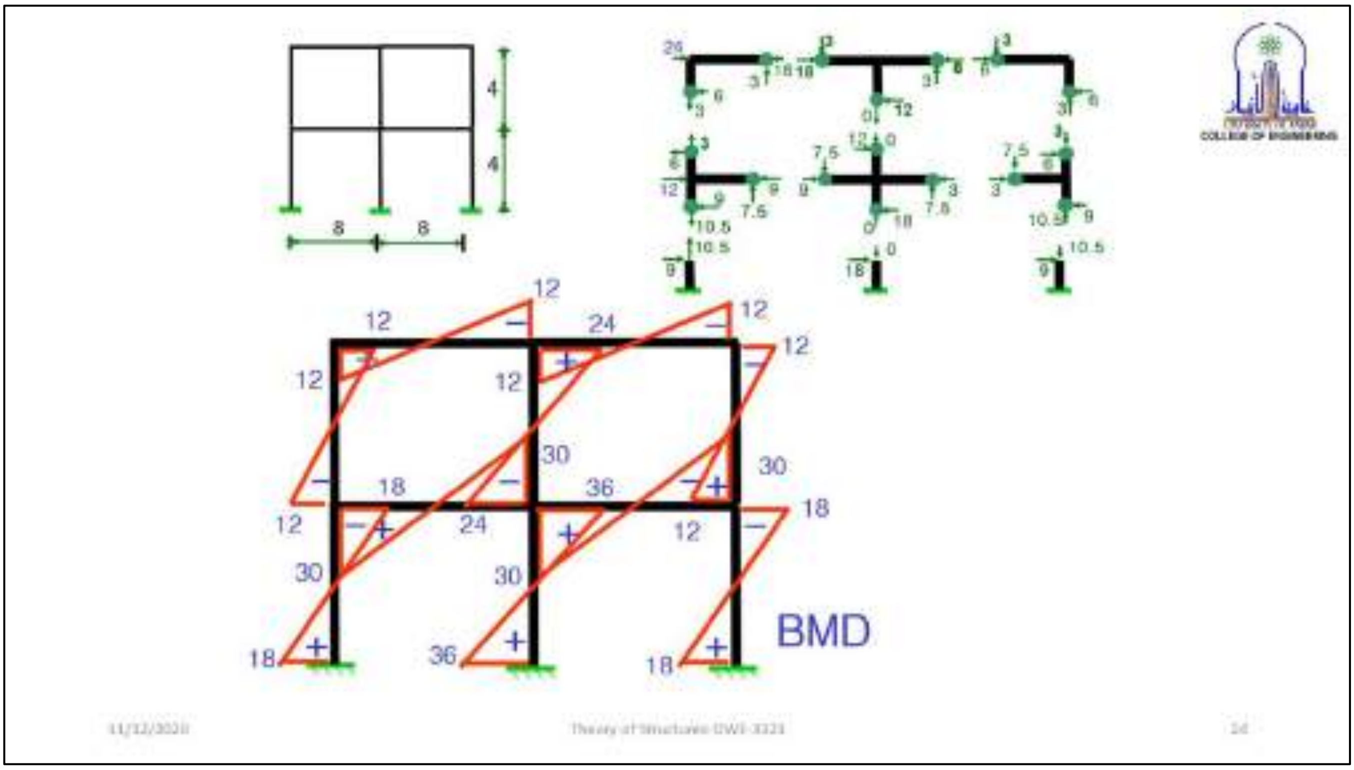

24 t
ΣFx = 0
ΣM@ = 0
ΣFy = 0
a
COLLEGE OF ENGINEERING


24 t
ΣFy = 0
COLLEGE OF ENGINEERING

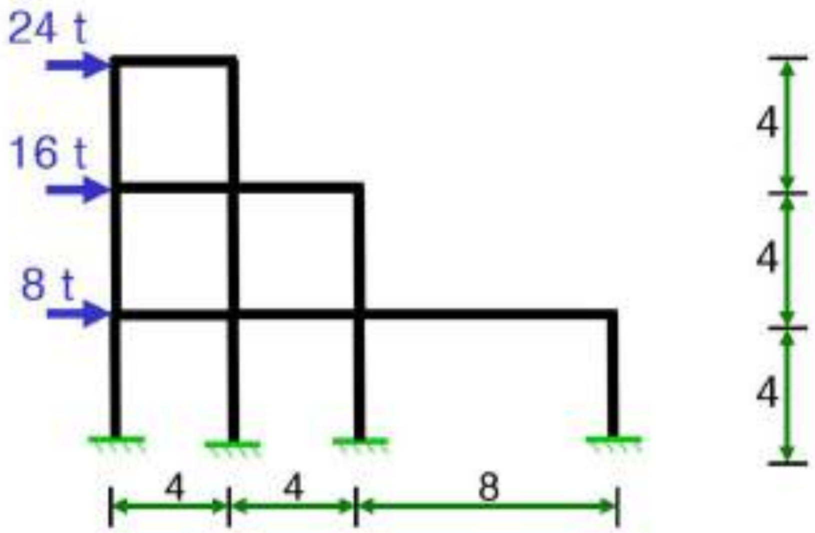

24
18
18
3
6
6
3
3
6
3
12
3
0
6
3
12
9
7.5
9
10.5
12
0
9
7.5
3
7.5
18
0
7.5
6
3
3
9
10.5
9
10.5
18
0
9
10.5


18
6
3
3
10.5
9
3
10.5
NFD


SFD
COLLEGE OF ENGINEERING

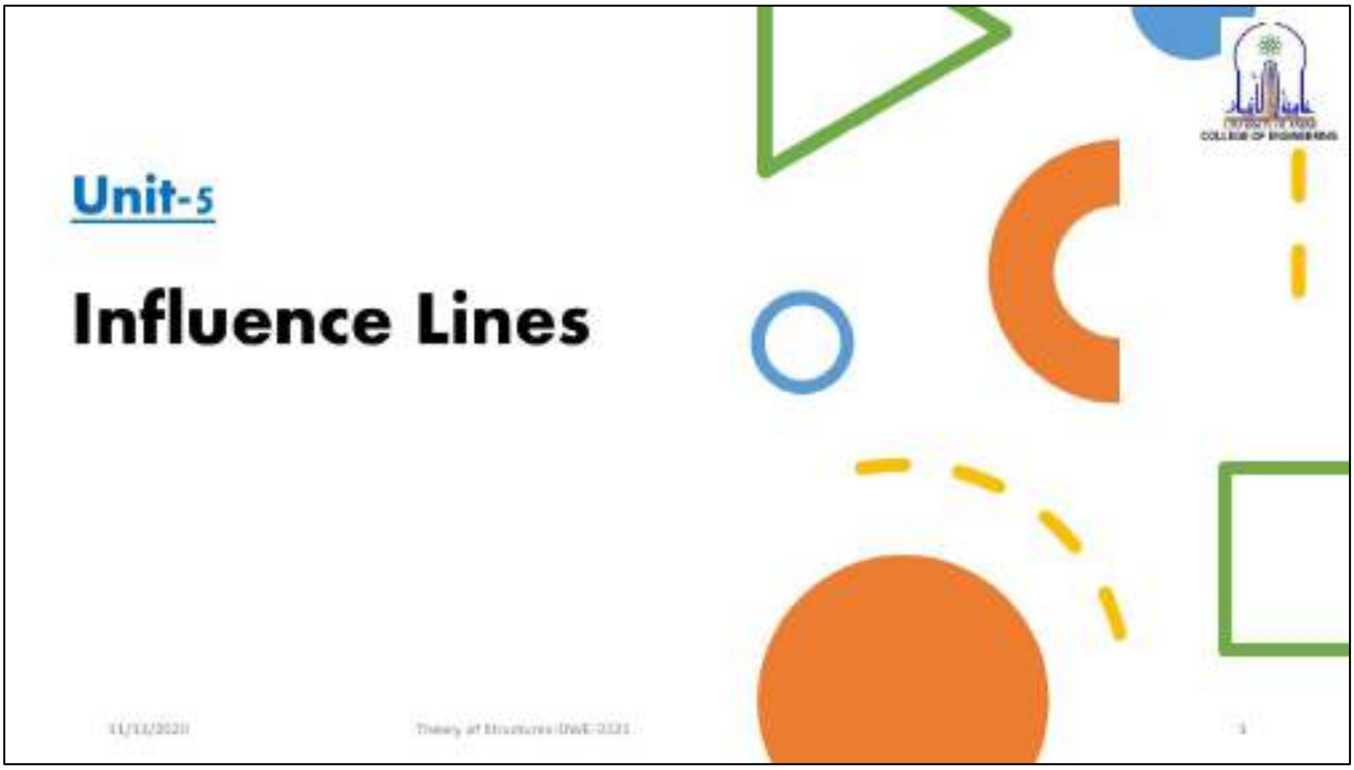

BMD
COLLEGE OF ENGINEERING

Example-2:



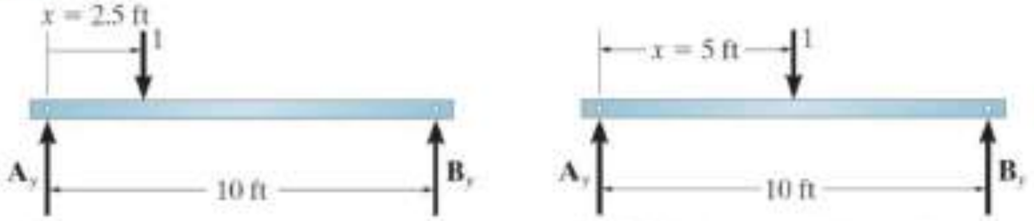
Solve it yourself based on what you have learned in the lecture!

It is the same question for the tutorial session


Unit-5
Influence Lines


Awesome Bridges
Theory of Structures DWE 3321
11/13/2020

Example:

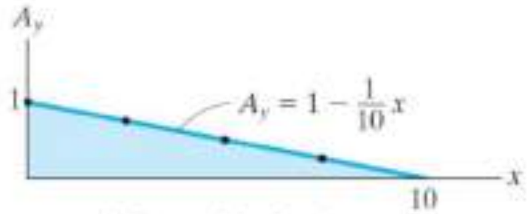


$\circlearrowleft + \Sigma M_B = 0;\ -A_y(10) + 1(7.5) = 0$
$A_y = 0.75$

$\circlearrowleft + \Sigma M_B = 0;\ -A_y(10) + 1(5) = 0$
$A_y = 0.5$

x	A_y
0	1
2.5	0.75
5	0.5
7.5	0.25
10	0

(d)



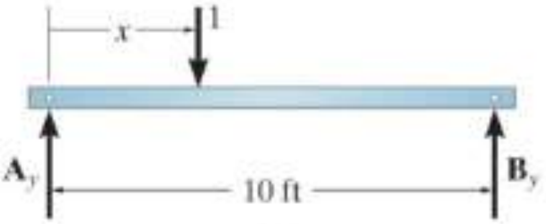
influence line for A_y

(e)

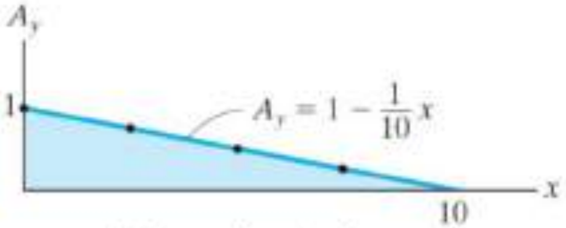
Influence line equation:

$\circlearrowleft + \Sigma M_B = 0;\quad -A_y(10) + (10 - x)(1) = 0$

$A_y = 1 - \frac{1}{10}x$



(f)



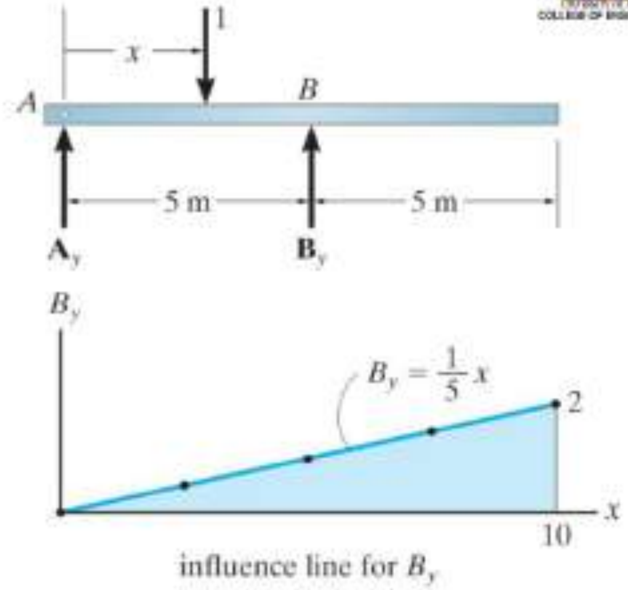
influence line for A_y

(e)

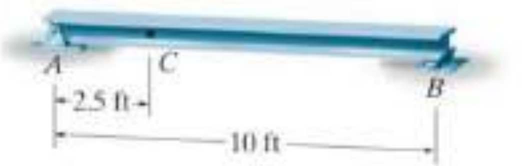
Example: Construct the influence line for the vertical reaction at B of the beam in the figure.

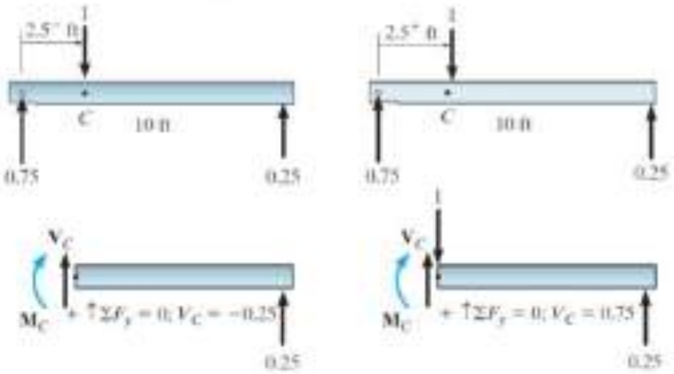
$$\circlearrowleft + \Sigma M_A = 0; \quad B_y(5) - 1(x) = 0$$

$$B_y = \tfrac{1}{5}x$$

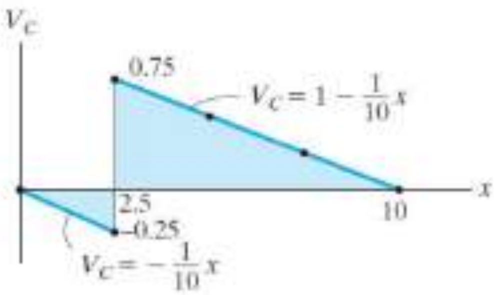


Example: Construct the influence line for the shear at point C of the beam in the figure.

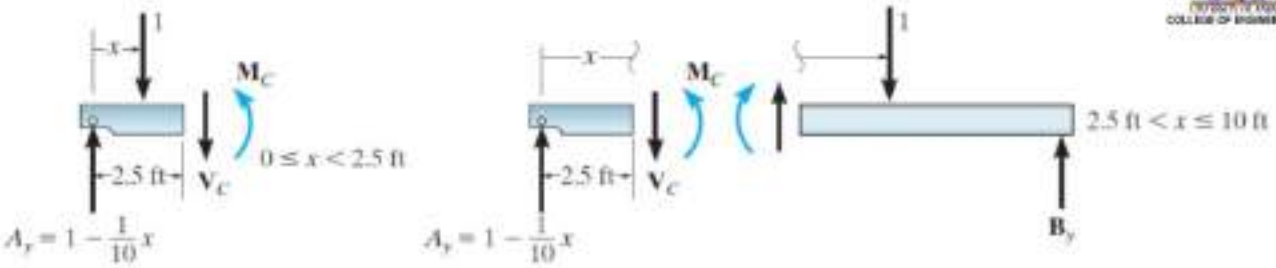


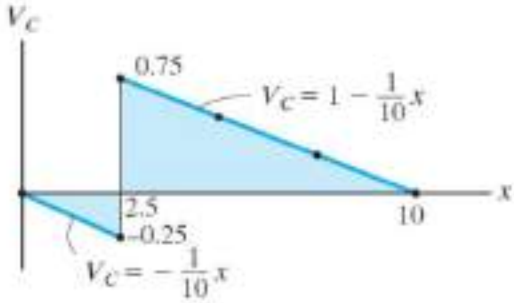


x	V_C
0	0
2.5^-	−0.25
2.5^+	0.75
5	0.5
7.5	0.25
10	0



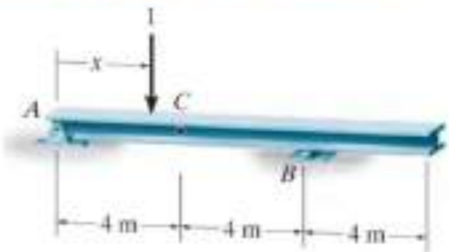
Influence line equation:

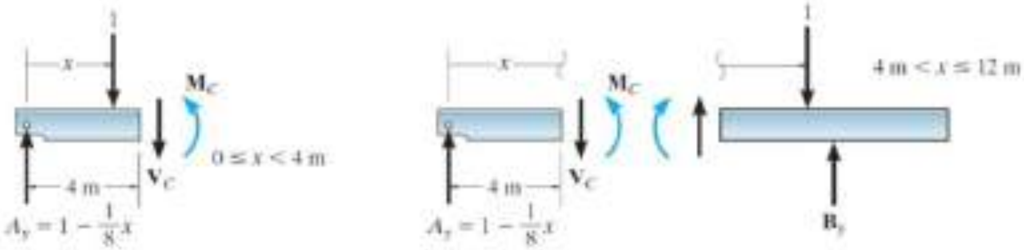




Example: Construct the influence line for the shear at point C of the beam in the figure.

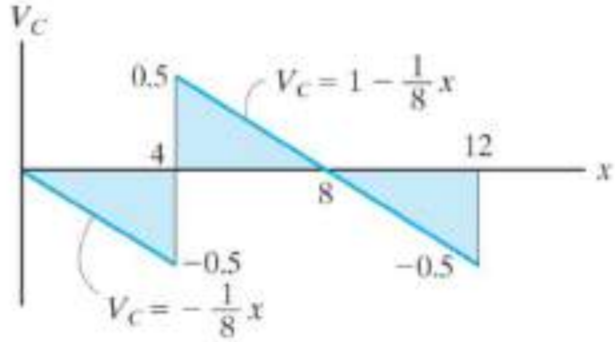
x	V_C
0	0
4^-	−0.5
4^+	0.5
8	0
12	−0.5



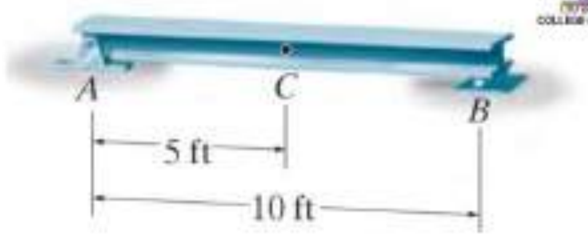


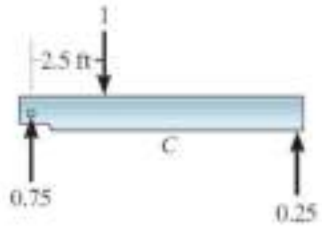
$$V_C = -\tfrac{1}{8}x \qquad 0 \le x < 4\text{ m}$$

$$V_C = 1 - \tfrac{1}{8}x \qquad 4\text{ m} < x \le 12\text{ m}$$

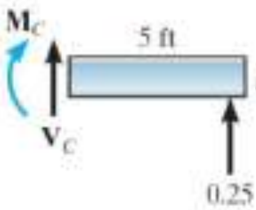


Example: Construct the influence line for the moment at point C of the beam in the figure.



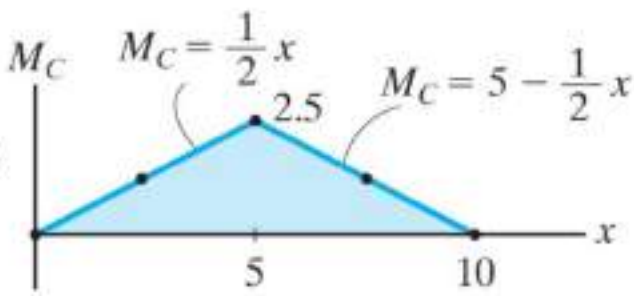


x	M_C
0	0
2.5	1.25
5	2.5
7.5	1.25
10	0



$\zeta + \Sigma M_C = 0;\ -M_C + 0.25\,(5) = 0$

$M_C = 1.25$



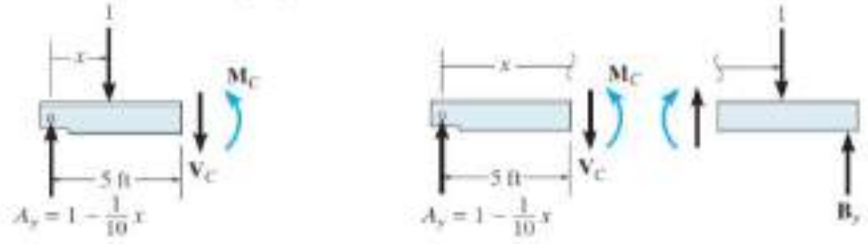
Influence line equation:

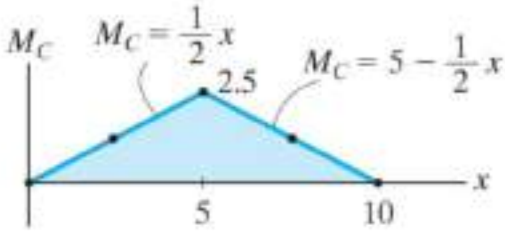
$\circlearrowleft + \Sigma M_C = 0;\quad M_C + 1(5 - x) - \left(1 - \frac{1}{10}x\right)5 = 0$

$M_C = \frac{1}{2}x \qquad 0 \le x < 5\text{ ft}$

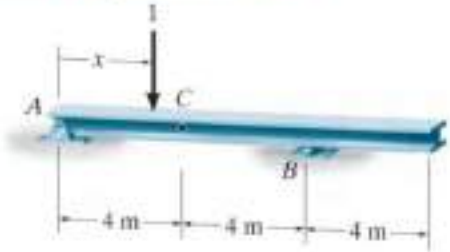
$\circlearrowleft + \Sigma M_C = 0;\quad M_C - \left(1 - \frac{1}{10}x\right)5 = 0$

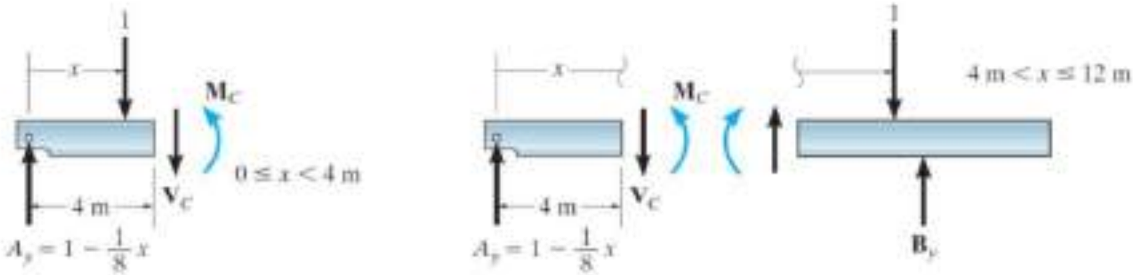
$M_C = 5 - \frac{1}{2}x \qquad 5\text{ ft} < x \le 10\text{ ft}$





Example: Construct the influence line for the moment at point C of the beam in the figure.

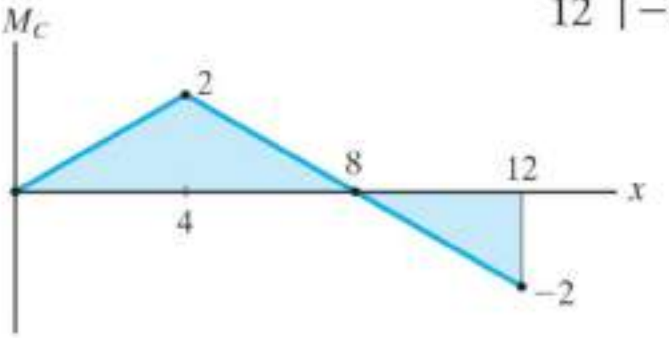




Influence line equation:

$$M_C = \tfrac{1}{2}x \qquad 0 \le x < 4\ \text{m}$$
$$M_C = 4 - \tfrac{1}{2}x \qquad 4\ \text{m} < x \le 12\ \text{m}$$

x	M_C
0	0
4	2
8	0
12	−2



Example: Determine the maximum positive shear that can be developed at point **C** in the beam shown in the figure due to a concentrated moving load of **4000 lb** and a uniform moving load of **2000 lb/ft**.

Concentrated force:

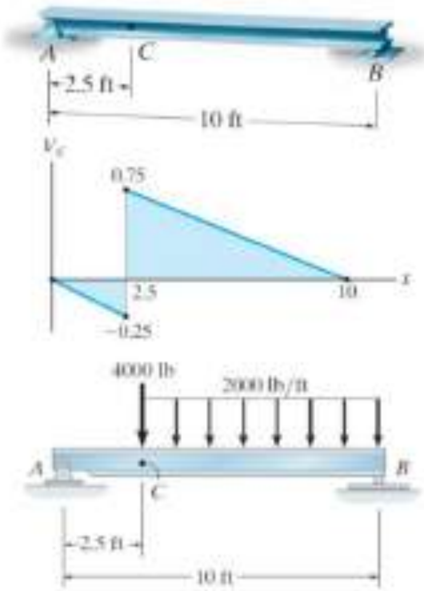
$$V_C = 0.75(4000\ \text{lb}) = 3000\ \text{lb}$$

Uniform load:

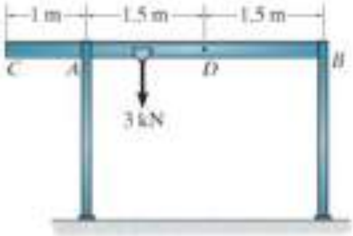
$$V_C = \left[\tfrac{1}{2}(10\ \text{ft} - 2.5\ \text{ft})(0.75)\right]2000\ \text{lb/ft} = 5625\ \text{lb}$$

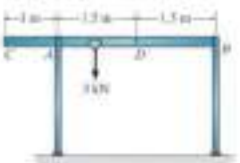
Total maximum load:

$$(V_C)_{\max} = 3000\ \text{lb} + 5625\ \text{lb} = 8625\ \text{lb}$$



Example: The frame structure shown in the figure is used to support a hoist for transferring loads for storage at points underneath it. It is anticipated that the load on the dolly is **3 kN** and the beam CB has a mass of **24 kg/m**. Assume **A** is a pin and **B** is a roller. Determine the maximum vertical support reactions at **A** and **B** and the maximum moment in the beam at **D**.

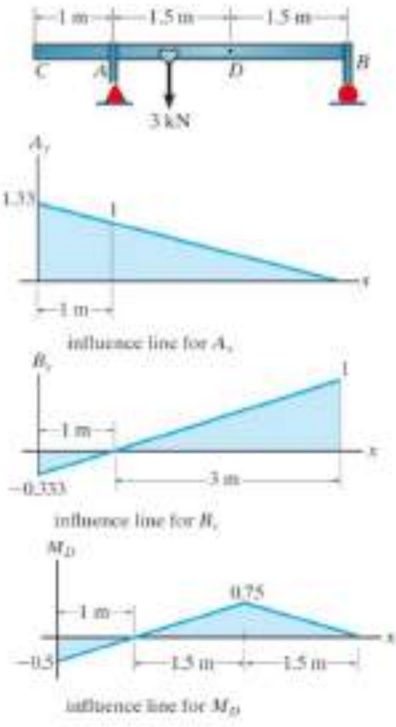




$$(A_y)_{max} = 3000(1.33) + 24(9.81)\left[\tfrac{1}{2}(4)(1.33)\right]$$
$$= 4.63 \text{ kN}$$

$$(B_y)_{max} = 3000(1) + 24(9.81)\left[\tfrac{1}{2}(3)(1)\right] + 24(9.81)\left[\tfrac{1}{2}(1)(-0.333)\right]$$
$$= 3.31 \text{ kN}$$ *Ans.*

$$(M_D)_{max} = 3000(0.75) + 24(9.81)\left[\tfrac{1}{2}(1)(-0.5)\right] + 24(9.81)\left[\tfrac{1}{2}(3)(0.75)\right]$$
$$= 2.46 \text{ kN}\cdot\text{m}$$ *Ans.*



Qualitative Influence Lines

- The Muller-Breslau principle states:

  The ***influence line*** for a function (reaction, shear, moment) is to the same scale as the deflected shape of the beam when the beam is acted on by the function.

  To draw the deflected shape properly, the ability of the beam to resist the applied function must be removed.

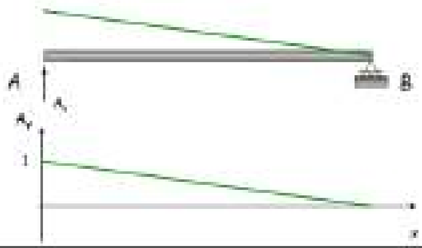
Qualitative Influence Lines

- For example, consider the following simply supported beam.

- Let's try to find the shape of the influence line for the vertical reaction at A.

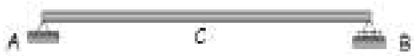
Qualitative Influence Lines

- Remove the ability to resist movement in the vertical direction at A by using a guided roller



Qualitative Influence Lines

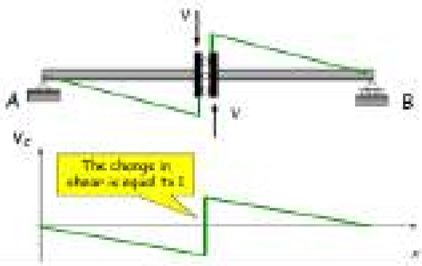
- Consider the following simply supported beam.



- Let's try to find the shape of the influence line for the shear at the mid-point (point C).

Qualitative Influence Lines

- Remove the ability to resist shear at point C



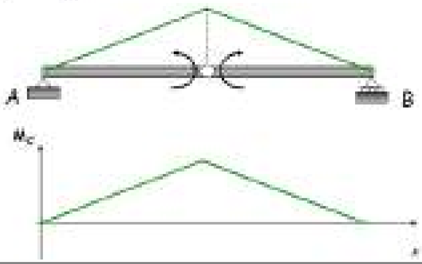
Qualitative Influence Lines

- Consider the following simply supported beam.

- Let's try to find the shape of the influence line for the moment at the mid-point (point C).

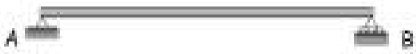
Qualitative Influence Lines

- Remove the ability to resist moment at C by using a hinge



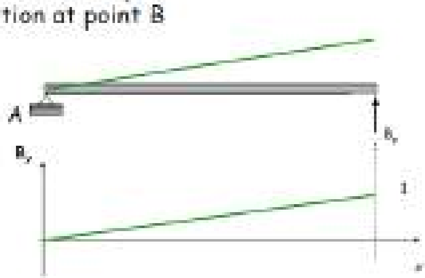
Qualitative Influence Lines

- Sketch the shape of the influence line for the reaction at point B



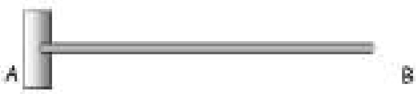
Qualitative Influence Lines

- Sketch the shape of the influence line for the reaction at point B



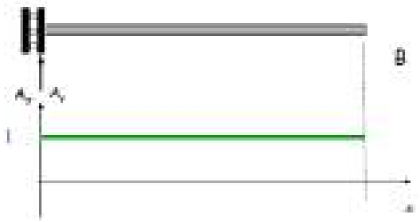
Qualitative Influence Lines

- Sketch the shape of the influence line for the reaction at point A



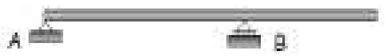
Qualitative Influence Lines

- Sketch the shape of the influence line for the reaction at point A



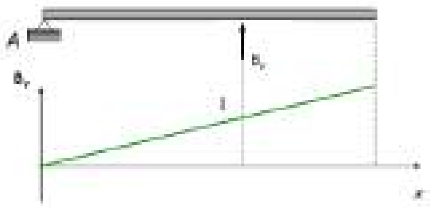
Qualitative Influence Lines

- Sketch the shape of the influence line for the reaction at point B



Qualitative Influence Lines

- Sketch the shape of the influence line for the reaction at point B

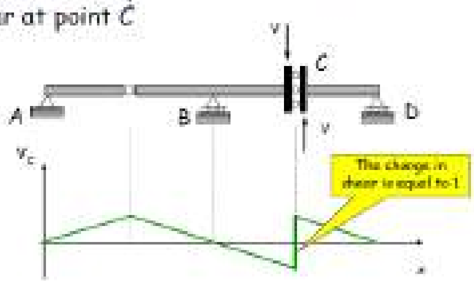


Qualitative Influence Lines

- Sketch the shape of the influence line for the shear at point C

Qualitative Influence Lines

- Sketch the shape of the influence line for the shear at point C

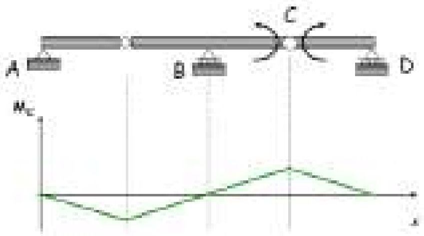


Qualitative Influence Lines

- Sketch the shape of the influence line for the moment at point C

Qualitative Influence Lines

- Sketch the shape of the influence line for the moment at point C

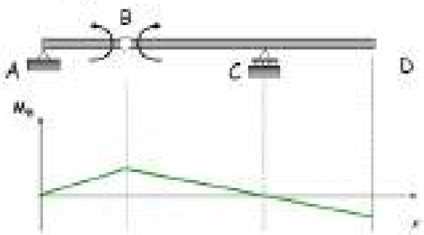


Qualitative Influence Lines

- Sketch the shape of the influence line for the moment at point B

Qualitative Influence Lines

- Sketch the shape of the influence line for the moment at point B

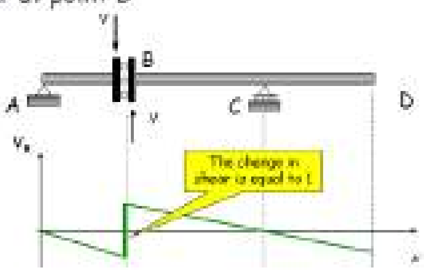


Qualitative Influence Lines

- Sketch the shape of the influence line for the shear at point B

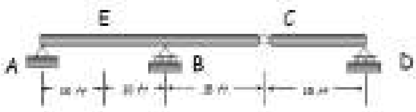
Qualitative Influence Lines

- Sketch the shape of the influence line for the shear at point B



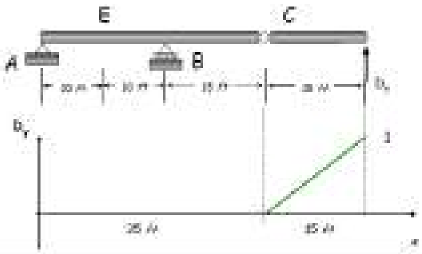
Qualitative Influence Lines

- Draw the influence lines for the vertical reaction at D and the shear at E.



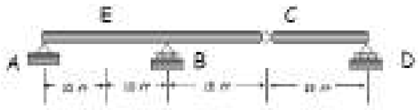
Qualitative Influence Lines

- Draw the influence lines for the vertical reaction at D and the shear at E.



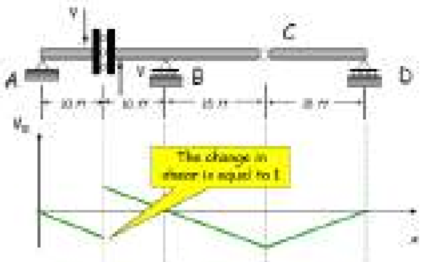
Qualitative Influence Lines

- Draw the influence lines for the vertical reaction at D and the shear at E.



Qualitative Influence Lines

- Draw the influence lines for the vertical reaction at D and the shear at E.



Qualitative Influence Lines

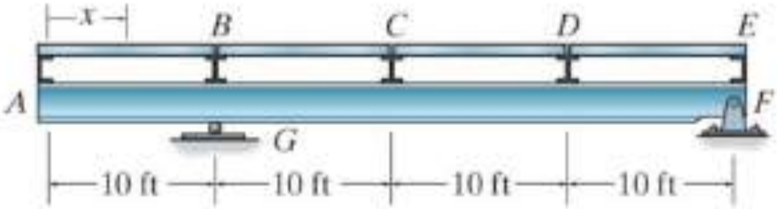
- Draw the influence lines for the vertical reaction at D and the shear at E.
- The change in shear at point E is equal to 1
- The influence lines can be determined by similar triangles the values of

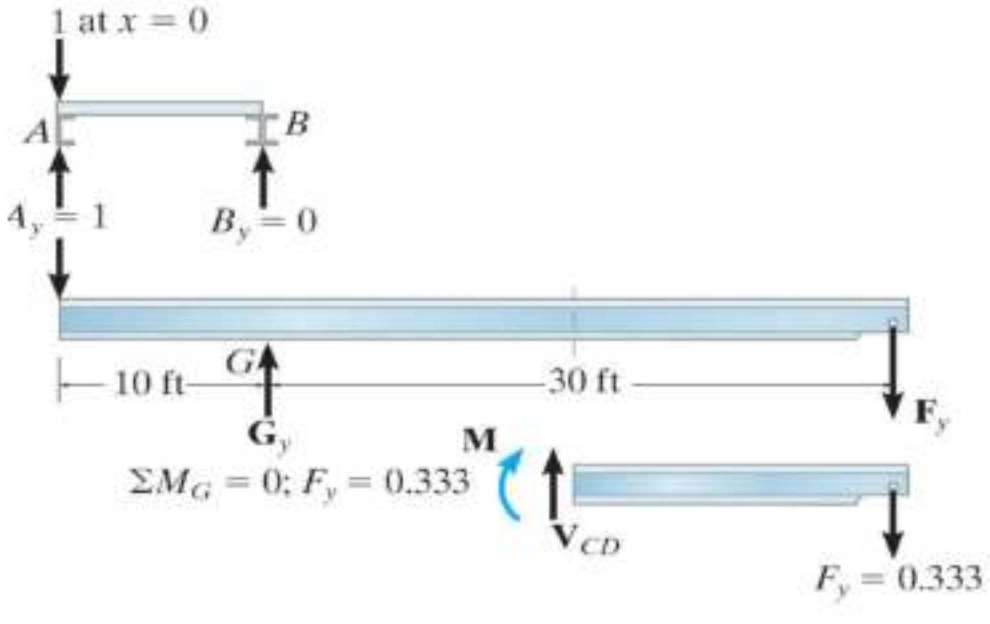
$$\frac{0.5}{10\,ft} = \frac{x}{15\,ft} \Rightarrow x = \frac{0.5(15\,ft)}{10\,ft} = 0.75$$

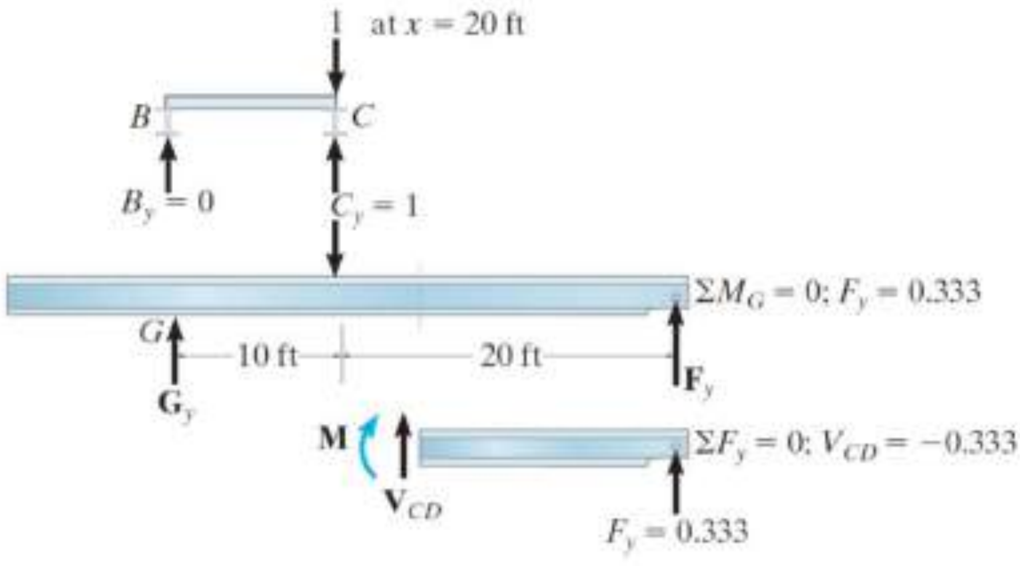
Influence Lines for Floor Girders

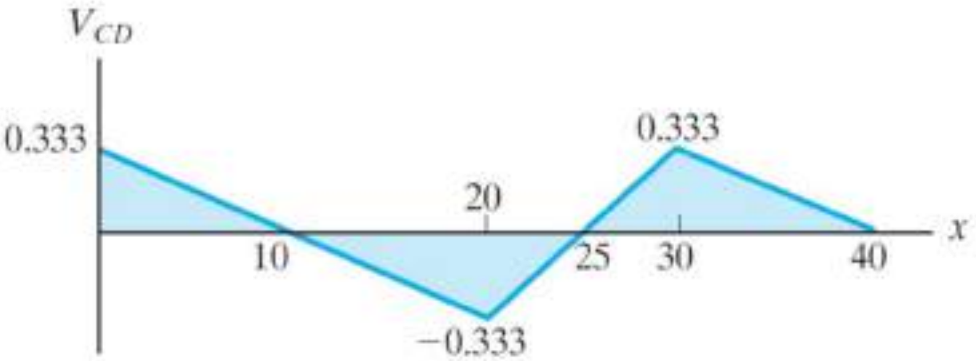
Example: Draw the influence line for the shear in panel ***CD*** of the floor girder in the figure.

x	V_{CD}
0	0.333
10	0
20	−0.333
30	0.333
40	0



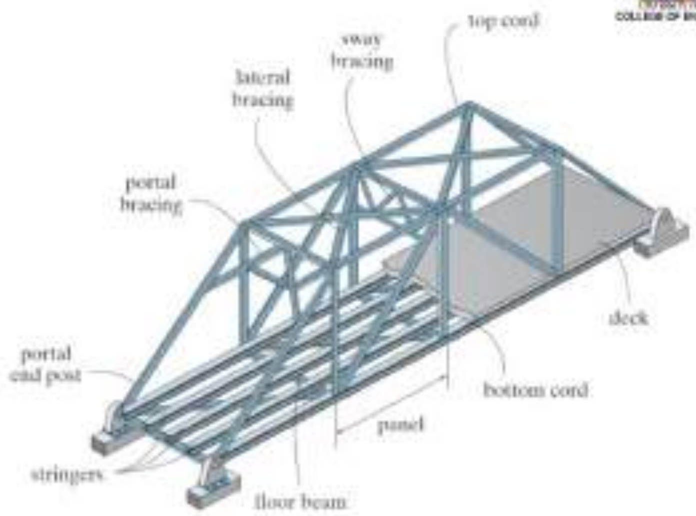

1 at x = 0
A
B
A_y = 1
B_y = 0
G
10 ft
30 ft
G_y
F_y
M
ΣM_G = 0; F_y = 0.333
V_CD
F_y = 0.333


1 at x = 20 ft
B
C
B_y = 0
C_y = 1
ΣM_G = 0; F_y = 0.333
G
10 ft
20 ft
F_y
G_y
M
ΣF_y = 0; V_CD = −0.333
V_CD
F_y = 0.333

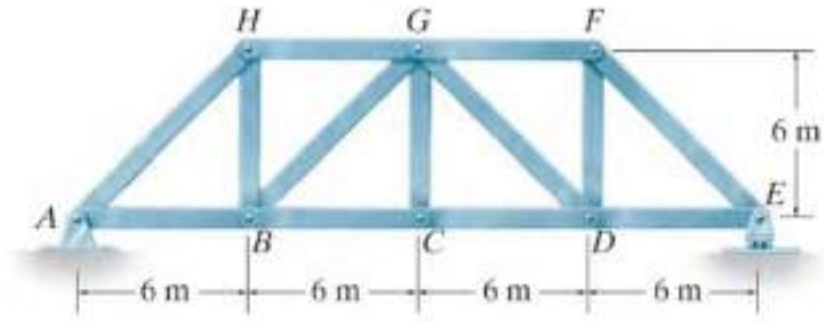


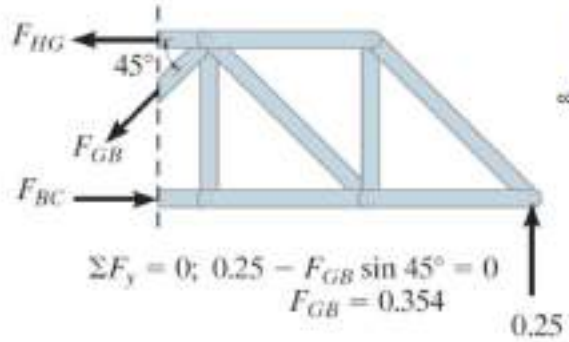
Influence Lines for Trusses

11/13/2020

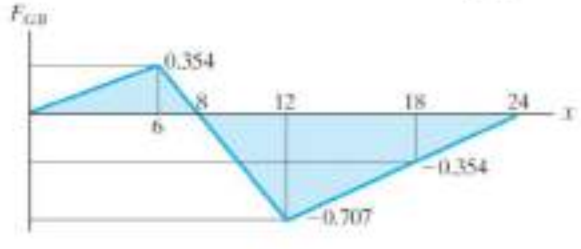


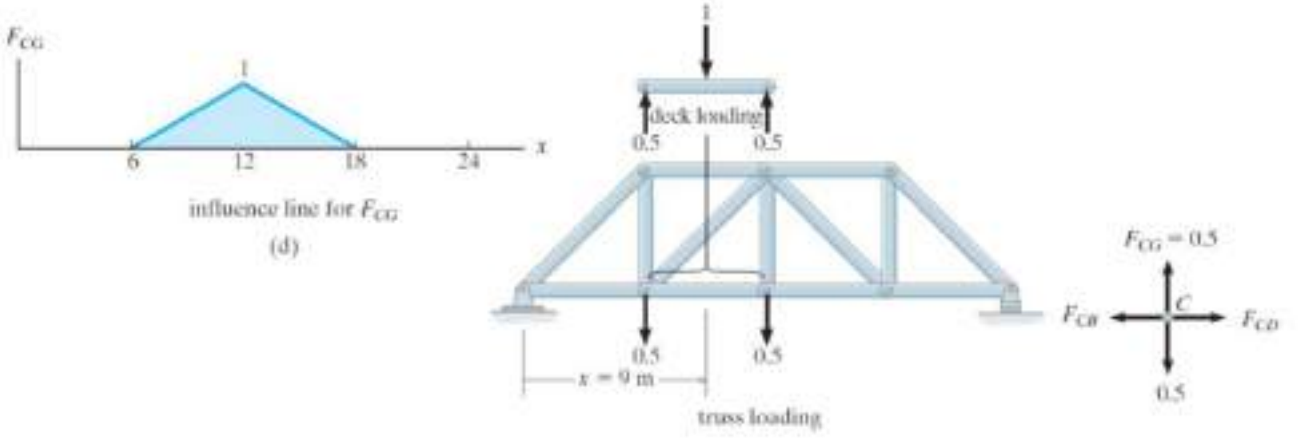
Example: Draw the influence line for the force in members **GB** and **CG** of the bridge truss shown in the figure.



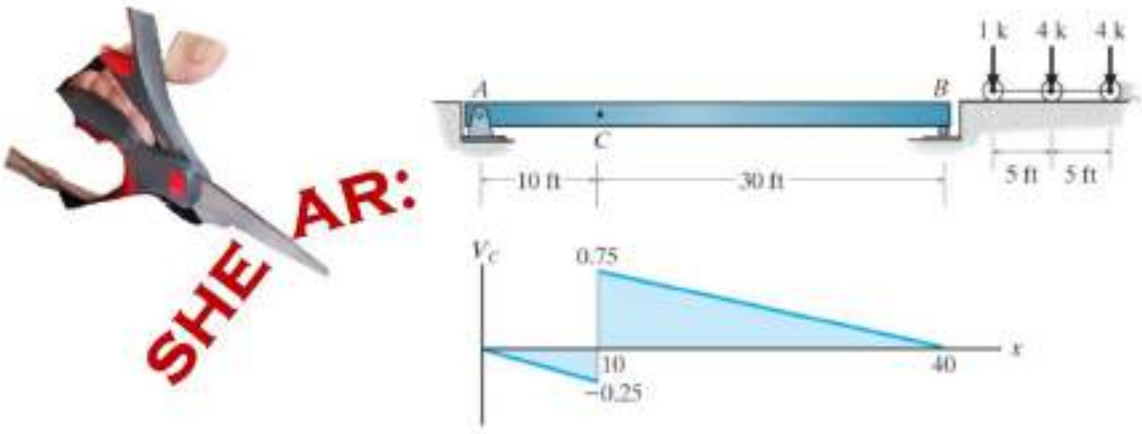


x	F_{GB}
0	0
6	0.354
12	−0.707
18	−0.354
24	0





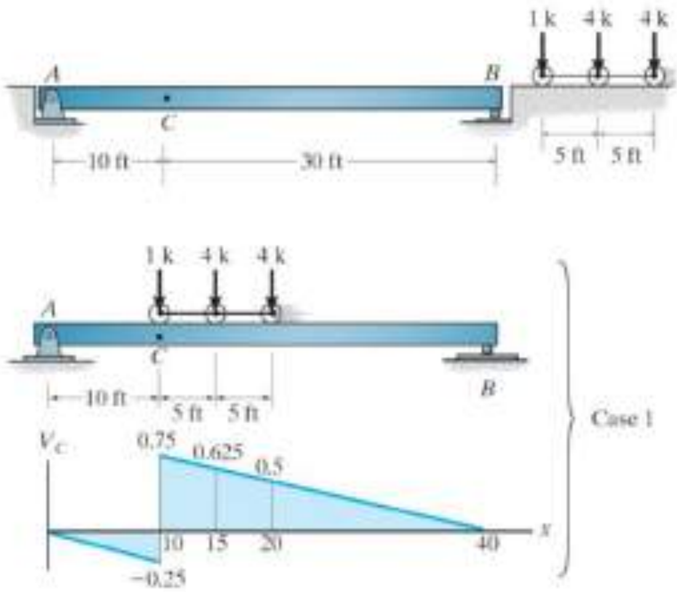
Maximum Influence at Point Due to Series of Concentrated Loads

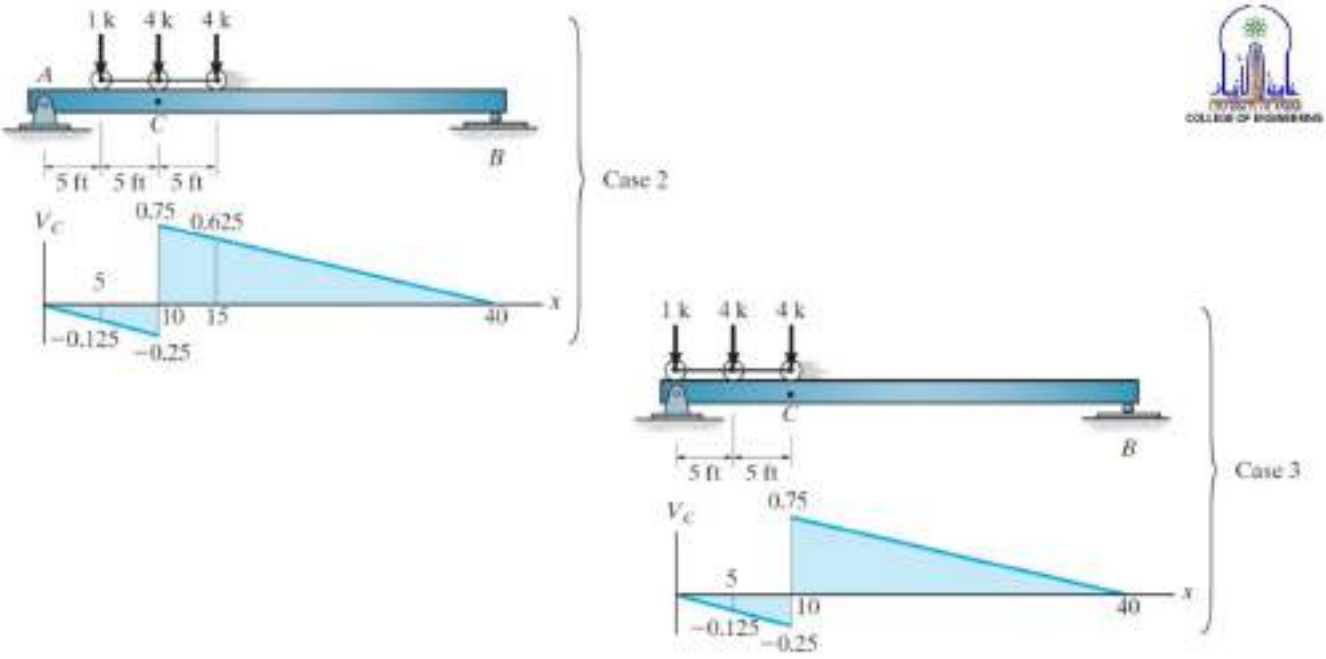


Case 1: $(V_C)_1 = 1(0.75) + 4(0.625) + 4(0.5) = 5.25\text{ k}$

Case 2: $(V_C)_2 = 1(-0.125) + 4(0.75) + 4(0.625) = \boxed{5.375\text{ k}}$

Case 3: $(V_C)_3 = 1(0) + 4(-0.125) + 4(0.75) = 2.5\text{ k}$





When many concentrated loads act on the span, the trial-and-error computations used above can be tedious. Instead, the critical position of the loads can be determined in a more direct manner by finding the change in shear, which occurs when the loads are moved from Case 1 to Case 2, then from Case 2 to Case 3, and so on. As long as each computed is positive, the new position will yield a larger shear in the beam at C than the previous position. Each movement is investigated until a negative change in shear is computed.

$\Delta V = Ps(x_2 - x_1)$ Sloping Line	$\Delta V = P(y_2 - y_1)$ Jump

$$\Delta V_{1-2} = 1(-1) + [1 + 4 + 4](0.025)(5) = +0.125 \text{ k}$$

$$\Delta V_{2-3} = 4(-1) + (1 + 4 + 4)(0.025)(5) = -2.875 \text{ k}$$

Since ΔV2-3 is negative, Case 2 is the position of the critical loading, as determined previously.

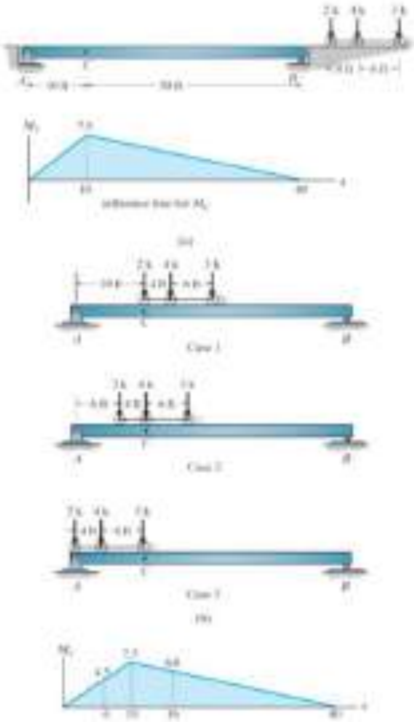
Moment :

$$\Delta M = Ps(x_2 - x_1)$$
Sloping Line

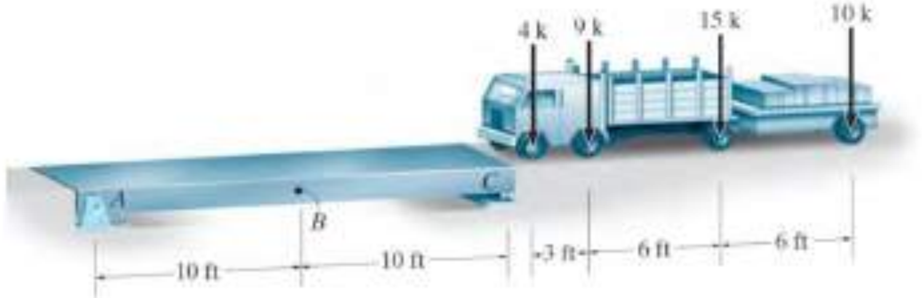
$$\Delta M_{1-2} = -2\left(\frac{7.5}{10}\right)(4) + (4+3)\left(\frac{7.5}{40-10}\right)(4) = 1.0\ \text{k}\cdot\text{ft}$$

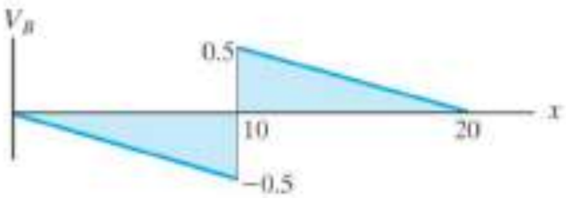
$$\Delta M_{2-3} = -(2+4)\left(\frac{7.5}{10}\right)(6) + 3\left(\frac{7.5}{40-10}\right)(6) = -22.5\ \text{k}\cdot\text{ft}$$

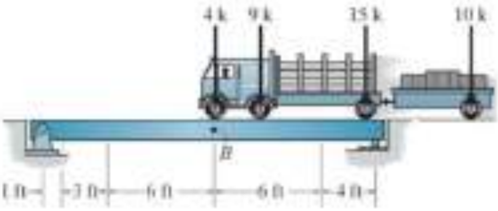
$$(M_C)_{\max} = 2(4.5) + 4(7.5) + 3(6.0) = 57.0\ \text{k}\cdot\text{ft}$$

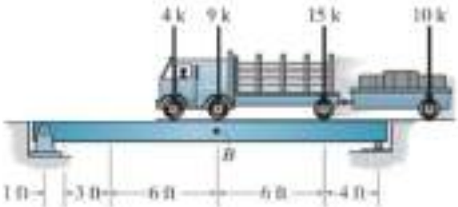


Example: Determine the maximum positive shear created at point **B** in the beam shown in figure due to the wheel loads of the moving truck.

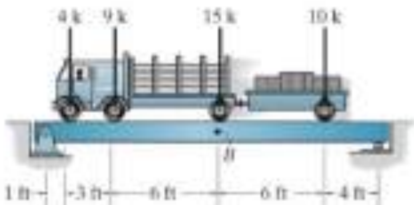






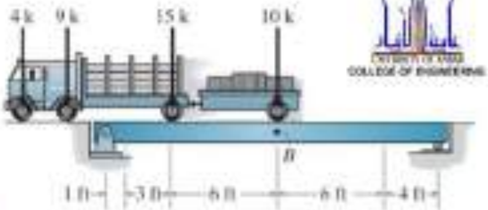


$$\Delta V_B = 4(-1) + (4 + 9 + 15)\left(\frac{0.5}{10}\right)3 = +0.2\text{ k}$$



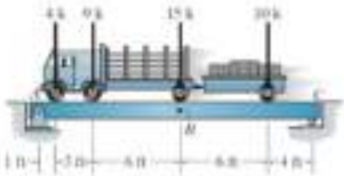
$$\Delta V_B = 9(-1) + (4 + 9 + 15)\left(\frac{0.5}{10}\right)(6) + 10\left(\frac{0.5}{10}\right)(4) = +1.4\text{ k}$$

$$\Delta V_B = 15(-1) + 4\left(\frac{0.5}{10}\right)(1) + 9\left(\frac{0.5}{10}\right)(4) + (15 + 10)\left(\frac{0.5}{10}\right)(6)$$
$$= -5.5 \text{ k}$$

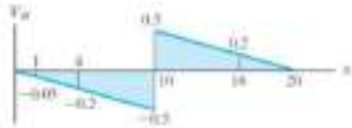


Since ΔVB is negative, Previous case is the position of the critical loading

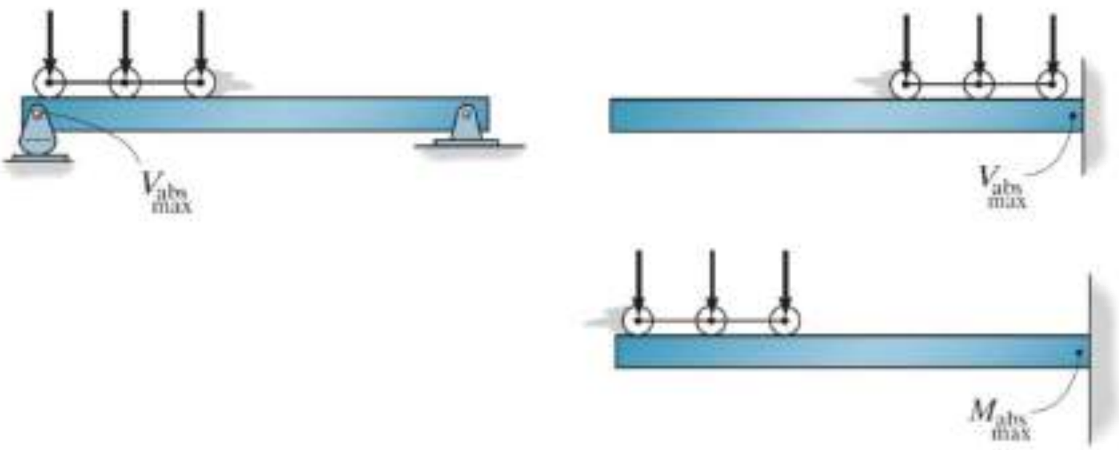
$$(V_B)_{max} = 4(-0.05) + 9(-0.2) + 15(0.5) + 10(0.2)$$
$$= 7.5 \text{ k} \qquad \textit{Ans.}$$

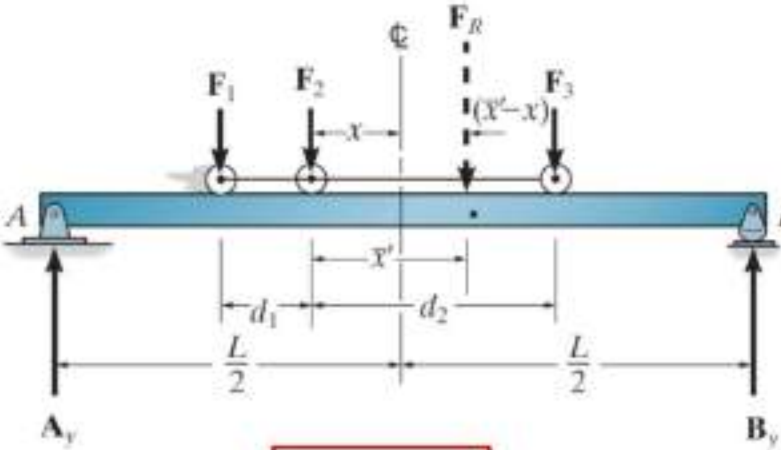


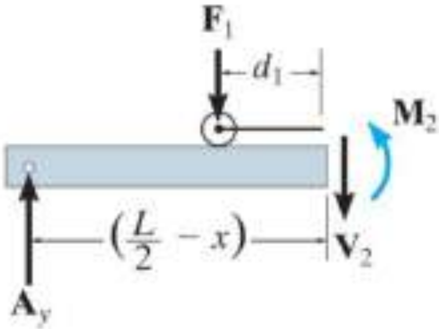
In practice one also has to consider motion of the truck from left to right and then choose the maximum value between these two situations.



Absolute Maximum Shear and Moment







$$x = \frac{\overline{x}'}{2}$$

$$\Sigma M = 0; \quad M_2 = A_y\left(\frac{L}{2} - x\right) - F_1 d_1$$

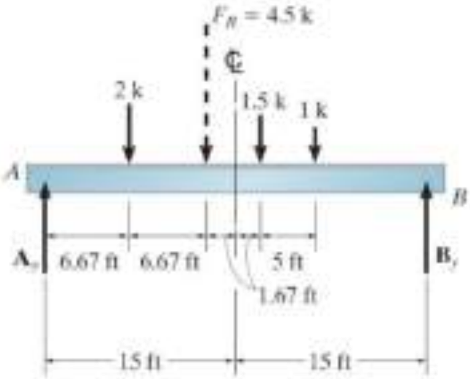
$$= \frac{1}{L}(F_R)\left[\frac{L}{2} - (\overline{x}' - x)\right]\left(\frac{L}{2} - x\right) - F_1 d_1$$

$$= \frac{F_R L}{4} - \frac{F_R \overline{x}'}{2} - \frac{F_R x^2}{L} + \frac{F_R x \overline{x}'}{L} - F_1 d_1$$

For maximum M_2 we require

$$\frac{dM_2}{dx} = \frac{-2F_R x}{L} + \frac{F_R \overline{x}'}{L} = 0$$

Example : Determine the absolute maximum moment in the simply supported bridge deck shown in the figure.

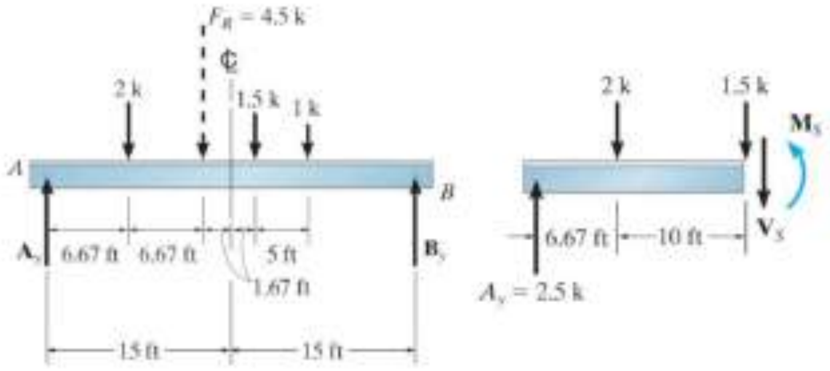


$$+\downarrow F_R = \Sigma F; \qquad F_R = 2 + 1.5 + 1 = 4.5 \text{ k}$$

$$\curvearrowleft + M_{R_C} = \Sigma M_C; \qquad 4.5\overline{x} = 1.5(10) + 1(15)$$

$$\overline{x} = 6.67 \text{ ft}$$

Case-1 :



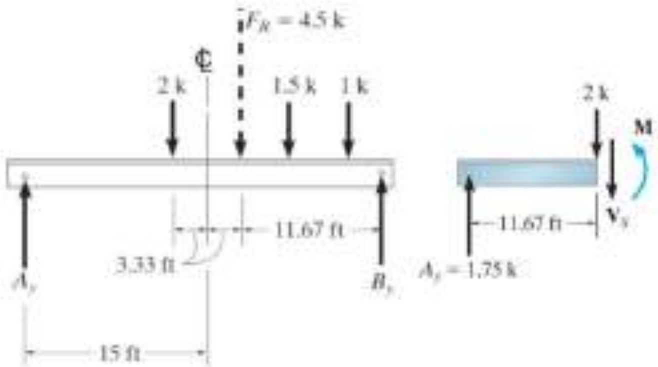
$$\circlearrowleft + \Sigma M_B = 0; \quad -A_y(30) + 4.5(16.67) = 0 \quad A_y = 2.50 \text{ k}$$

$$\circlearrowleft + \Sigma M_S = 0; \quad -2.50(16.67) + 2(10) + M_S = 0$$

$$M_S = 21.7 \text{ k} \cdot \text{ft}$$

Case-2 :

$$M_S = 20.4 \text{ k} \cdot \text{ft}$$

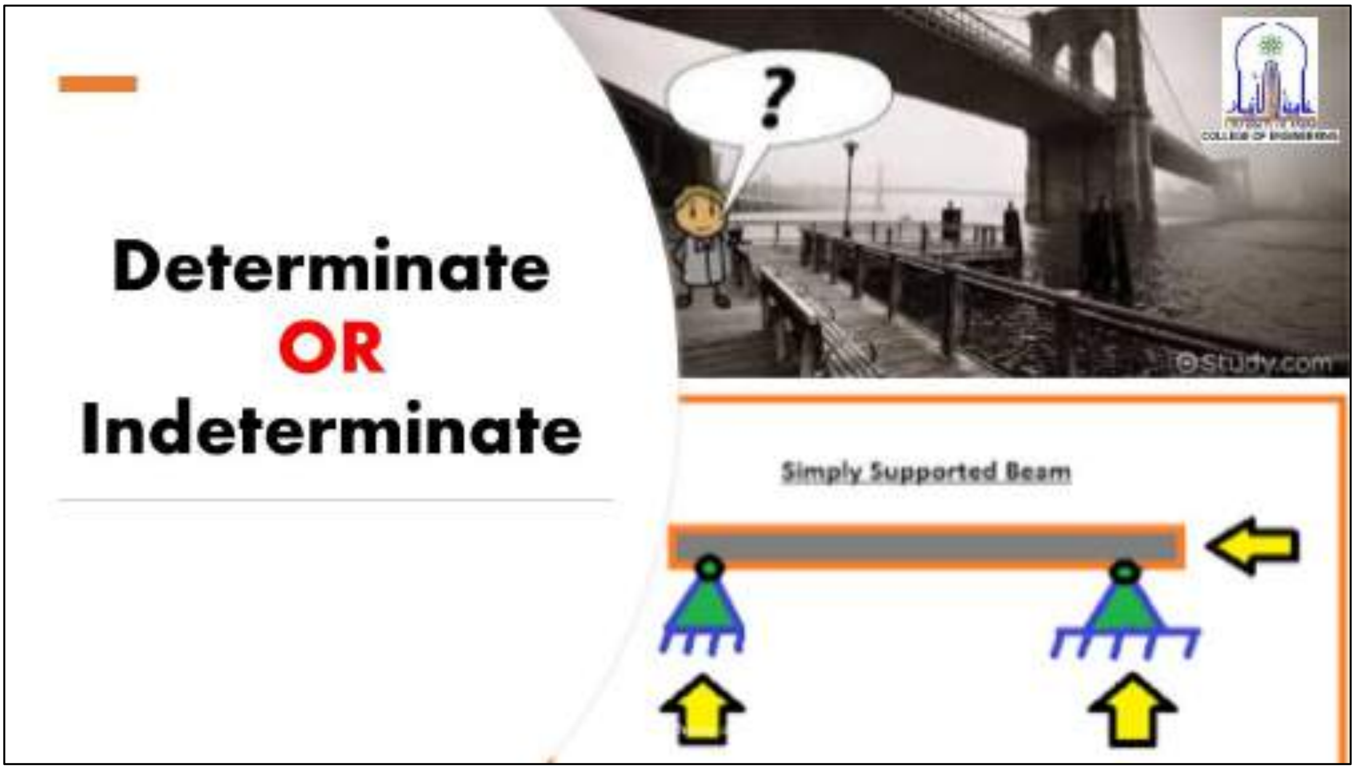


By Comparison the maximum moment is :

$$M_S = 21.7 \text{ k} \cdot \text{ft}$$

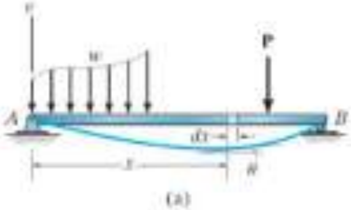
Which occurs under the 1.5 k load when positioned as in the case-1


Unit-6
Deflection
of Statically
Determinate
Structures


Determinate
OR
Indeterminate
?
©Study.com
Simply Supported Beam

Elastic Beam Theory :

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$



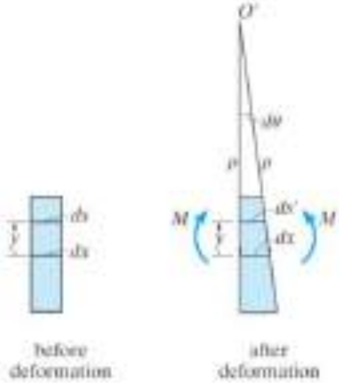
1- Double integration method. X

2- Moment-area theorem. X

3- Conjugate-beam method. X

4- Energy methods:

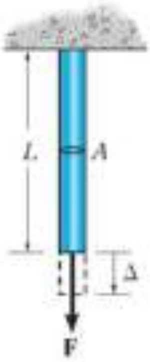
- Method of virtual work. √
- Castigliano theorem. √



External Work and Strain Energy

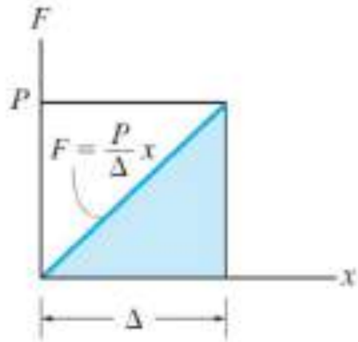
$$U_e = U_i$$

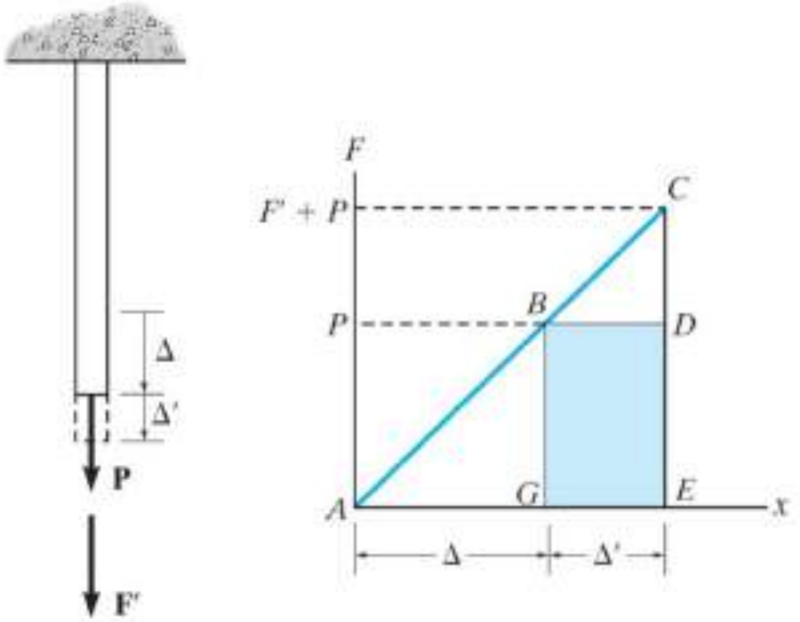
External Work – Force :



$$U_e = \int_0^x F\,dx$$

$$U_e = \frac{1}{2}P\Delta$$

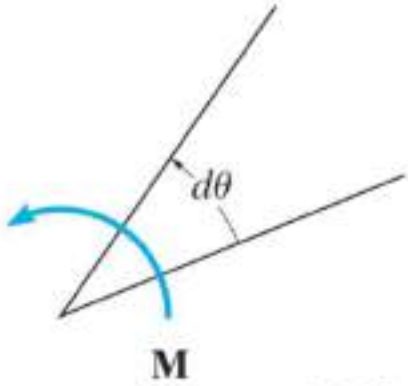




$$U_e' = P\Delta'$$

External Work – Moment :

$$U_e = \int_0^{\theta} M \, d\theta$$



$$U_e = \frac{1}{2} M\theta$$

$$U_e' = M\theta'$$

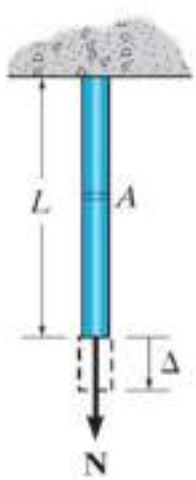
Strain Energy – Axial Force :

$$\Delta = \frac{NL}{AE}$$

$$U_e = U_i$$

$$U_e = \tfrac{1}{2} P\Delta$$

$$U_i = \frac{N^2 L}{2AE}$$

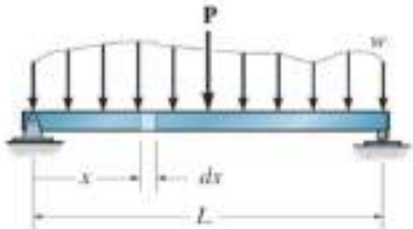


Strain Energy – Bending :

$$d\theta = (M/EI)\,dx$$

$$dU_i = \frac{M^2\,dx}{2EI}$$

$$U_i = \int_0^L \frac{M^2\,dx}{2EI}$$



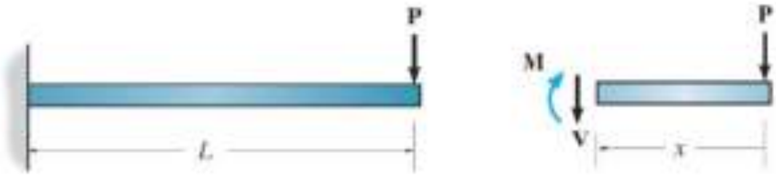
Principle of Work and Energy

$$U_i = \int_0^L \frac{M^2\,dx}{2EI} = \int_0^L \frac{(-Px)^2\,dx}{2EI} = \frac{1}{6}\frac{P^2L^3}{EI}$$

$$U_e = U_i$$

$$\frac{1}{2}P\Delta = \frac{1}{6}\frac{P^2L^3}{EI}$$

$$\Delta = \frac{PL^3}{3EI}$$



Principle of Virtual Work

$\Sigma P\Delta$	=	$\Sigma u\delta$
Work of External Loads		Work of Internal Loads

$1 \cdot \Delta = \Sigma u \cdot dL$ — virtual loadings (1, u); real displacements (Δ, dL)

$1 \cdot \theta = \Sigma u_\theta \cdot dL$ — virtual loadings (1, u_θ); real displacements (θ, dL)

Apply virtual load P' = 1

(a)

Apply real loads P_1, P_2, P_3

(b)

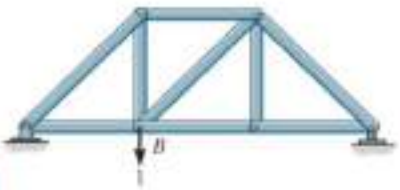
Method of Virtual Work

1- Trusses

A- External Loading: $$1 \cdot \Delta = \sum \frac{nNL}{AE}$$

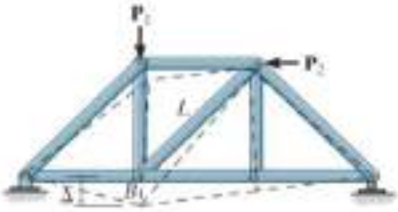
B- Temperature Effect: $$1 \cdot \Delta = \Sigma n \alpha \, \Delta T \, L$$

C- Fabrication Error: $$1 \cdot \Delta = \Sigma n \, \Delta L$$



Apply virtual unit load to *B*

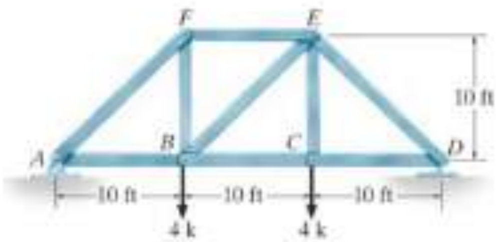
(a)

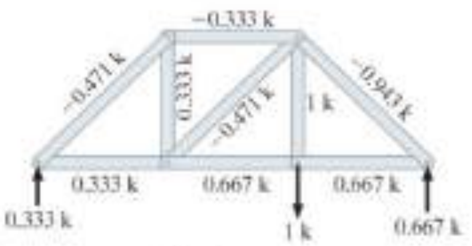


Apply real loads $\mathbf{P}_1$, $\mathbf{P}_2$

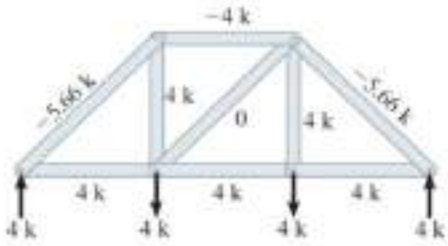
(b)

Example: Determine the vertical displacement of joint **C** of the steel truss shown in the figure. The cross-sectional area of each member is **A** = 0.5 in² and **E** = 29x10³ ksi.





virtual forces **n**



real forces **N**

Member	n (k)	N (k)	L (ft)	nNL ($k^2 \cdot ft$)
AB	0.333	4	10	13.33
BC	0.667	4	10	26.67
CD	0.667	4	10	26.67
DE	−0.943	−5.66	14.14	75.42
FE	−0.333	−4	10	13.33
EB	−0.471	0	14.14	0
BF	0.333	4	10	13.33
AF	−0.471	−5.66	14.14	37.71
CE	1	4	10	40
				Σ246.47

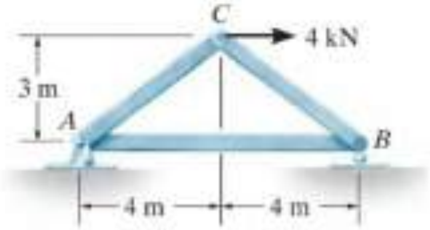
$$1\text{ k} \cdot \Delta_{C_v} = \sum \frac{nNL}{AE} = \frac{246.47\text{ k}^2 \cdot \text{ft}}{AE} \qquad 1\text{ k} \cdot \Delta_{C_v} = \frac{(246.47\text{ k}^2 \cdot \text{ft})(12\text{ in./ft})}{(0.5\text{ in}^2)(29(10^3)\text{ k/in}^2)}$$

$$\Delta_{C_v} = 0.204\text{ in.}$$

Example: The cross-sectional area of each member of the truss shown in the figure is **A** = 400 mm² and **E** = 200 GPa.

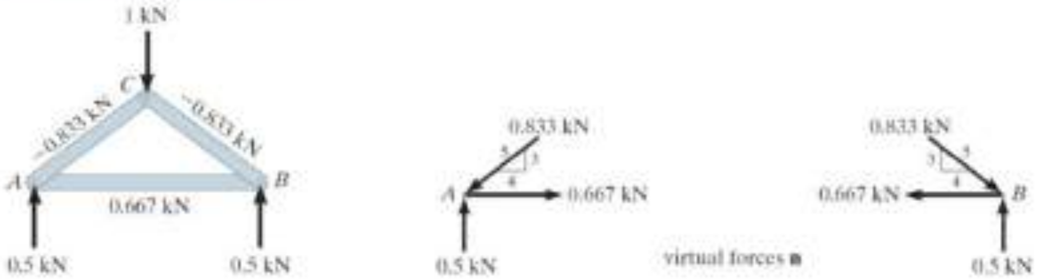
(a) Determine the vertical displacement of joint C if a 4-kN force is applied to the truss at C.

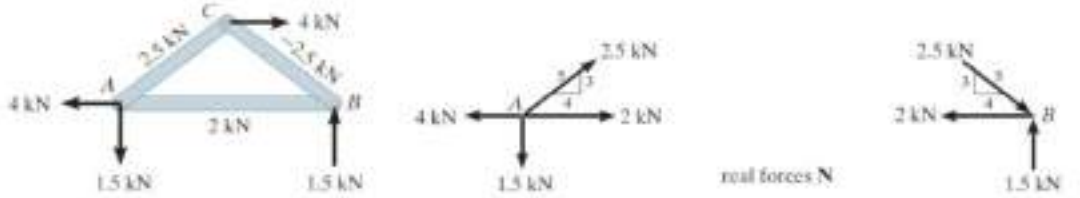
(b) If no loads act on the truss, what would be the vertical displacement of joint ***C*** if member AB were 5 mm too short?



Solution:

Part-A:





Member	n (kN)	N (kN)	L (m)	n NL (kN²·m)
AB	0.667	2	8	10.67
AC	−0.833	2.5	5	−10.41
CB	−0.833	−2.5	5	10.41
				Σ10.67

$$1\text{ kN}\cdot\Delta_{C_v} = \sum\frac{nNL}{AE} = \frac{10.67\text{ kN}^2\cdot\text{m}}{AE}$$

$$1\text{ kN}\cdot\Delta_{C_v} = \frac{10.67\text{ kN}^2\cdot\text{m}}{400(10^{-6})\text{ m}^2(200(10^6)\text{ kN/m}^2)}$$

$$\Delta_{C_v} = 0.000133\text{ m} = 0.133\text{ mm}$$

Part-B:

$$1\cdot\Delta = \Sigma n\,\Delta L$$

$$1\text{ kN}\cdot\Delta_{C_v} = (0.667\text{ kN})(-0.005\text{ m})$$

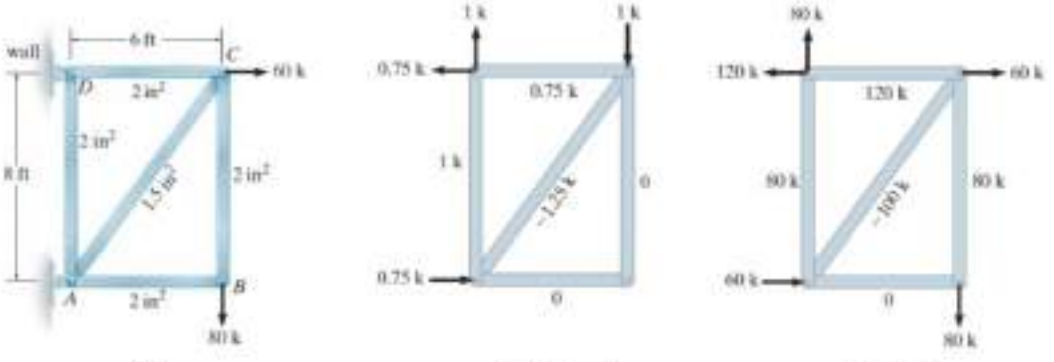
$$\Delta_{C_v} = -0.00333\text{ m} = -3.33\text{ mm}$$

Note:

The negative sign indicates joint ***C*** is displaced *upward,* opposite to the 1-kN vertical load. Note that if the 4-kN load and fabrication error are both accounted for, the resultant displacement is then **Δcv** = 0.133 - 3.33 = -3.20 mm (upward).

Example: Determine the vertical displacement of joint **C** of the steel truss shown in the figure due to radiant heating from the wall, member **AD** is subjected to an increase in temperature of $\Delta T = +120°$ F. Take $\alpha = 0.6\times10^{-5}/°F$ and $E = 29(10^3)$ ksi. The cross-sectional area of each member is indicated in the figure.

Solution:



(a)

virtual forces **n**
(b)

real forces **N**
(c)

Solution:

$$1 \cdot \Delta_{C_v} = \sum \frac{nNL}{AE} + \Sigma n\alpha\,\Delta T\,L$$

$$= \frac{(0.75)(120)(6)(12)}{2[29(10^3)]} + \frac{(1)(80)(8)(12)}{2[29(10^3)]}$$

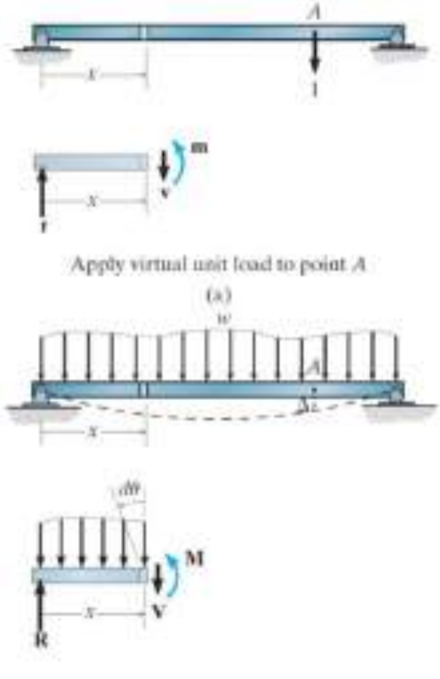
$$+ \frac{(-1.25)(-100)(10)(12)}{1.5[29(10^3)]} + (1)[0.6(10^{-5})](120)(8)(12)$$

$$\Delta_{C_v} = 0.658 \text{ in.} \qquad \textit{Ans.}$$

Method of Virtual Work

2- Beams and Frames:

$$1 \cdot \Delta = \int_0^L \frac{mM}{EI} dx$$



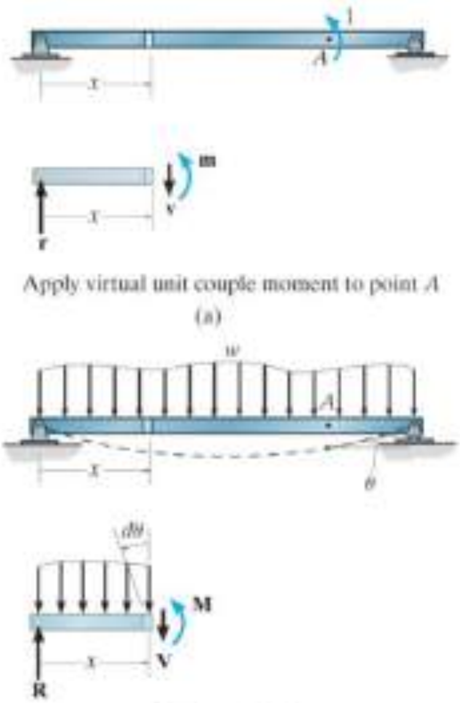
Apply virtual unit load to point A

(a)

Apply real load w

(b)

$$1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI} dx$$

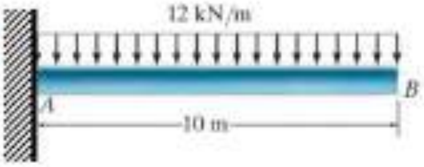


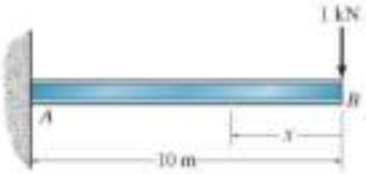
Apply virtual unit couple moment to point A

(a)

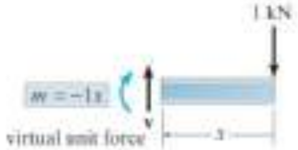
Apply real load w

Example: Determine the displacement of point **B** of the steel beam shown in the figure. Take E = 200 GPa, I = 500x10⁶ mm⁴.



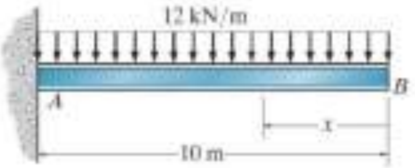


Solution:



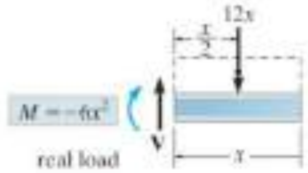
$$1\text{ kN}\cdot\Delta_B = \int_0^L \frac{mM}{EI}dx = \int_0^{10}\frac{(-1x)(-6x^2)\,dx}{EI}$$

$$1\text{ kN}\cdot\Delta_B = \frac{15(10^3)\text{ kN}^2\cdot\text{m}^3}{EI}$$

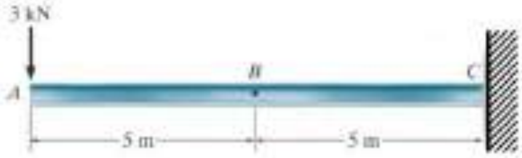


$$\Delta_B = \frac{15(10^3)\text{ kN}\cdot\text{m}^3}{200(10^6)\text{ kN/m}^2(500(10^6)\text{ mm}^4)(10^{-12}\text{ m}^4/\text{mm}^4)}$$

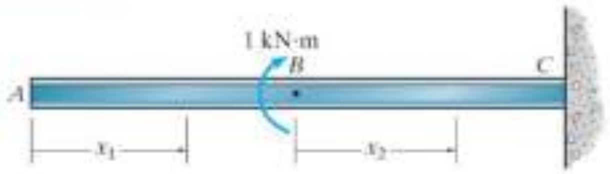
$$= 0.150\text{ m} = 150\text{ mm}$$

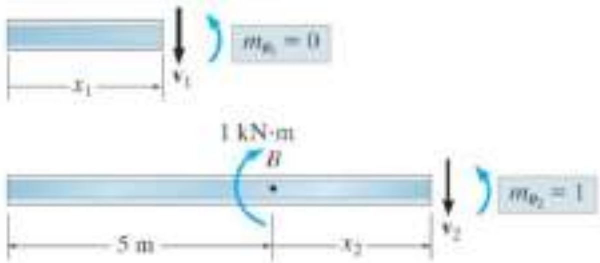


Example: Determine the slop **θ** at point **B** of the steel beam shown in the figure. Take E = 200 GPa, I = 600x10⁶ mm⁴.

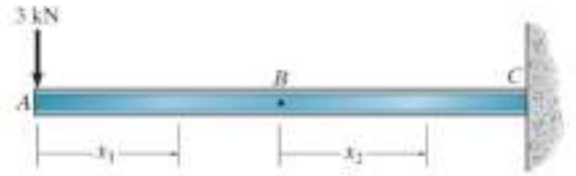


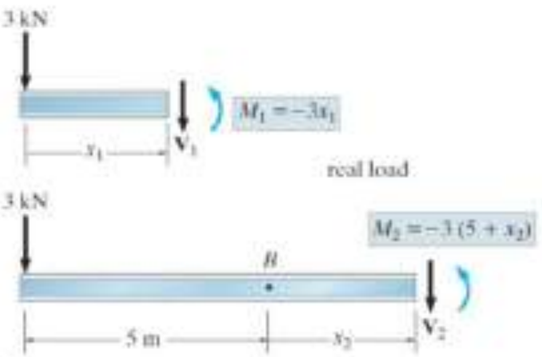
Solution:





virtual unit couple





$$1 \cdot \theta_B = \int_0^L \frac{m_\theta M}{EI} dx$$

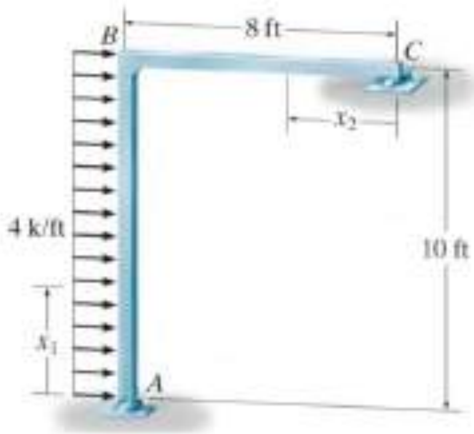
$$= \int_0^5 \frac{(0)(-3x_1)\, dx_1}{EI} + \int_0^5 \frac{(1)[-3(5 + x_2)]\, dx_2}{EI}$$

$$\theta_B = \frac{-112.5 \text{ kN} \cdot \text{m}^2}{EI}$$

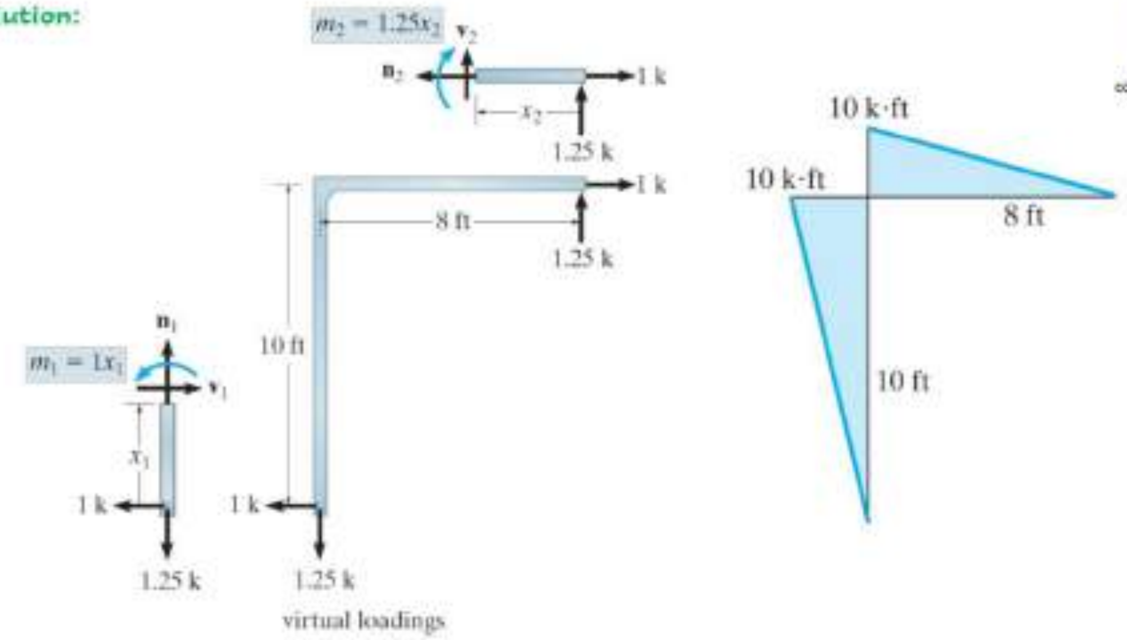
$$(1 \text{ kN} \cdot \text{m}) \cdot \theta_B = \frac{-112.5 \text{ kN}^2 \cdot \text{m}^3}{200(10^6) \text{ kN/m}^2[60(10^6) \text{ mm}^4](10^{-12} \text{ m}^4/\text{mm}^4)}$$

$$\theta_B = -0.00938 \text{ rad} \qquad \textit{Ans.}$$

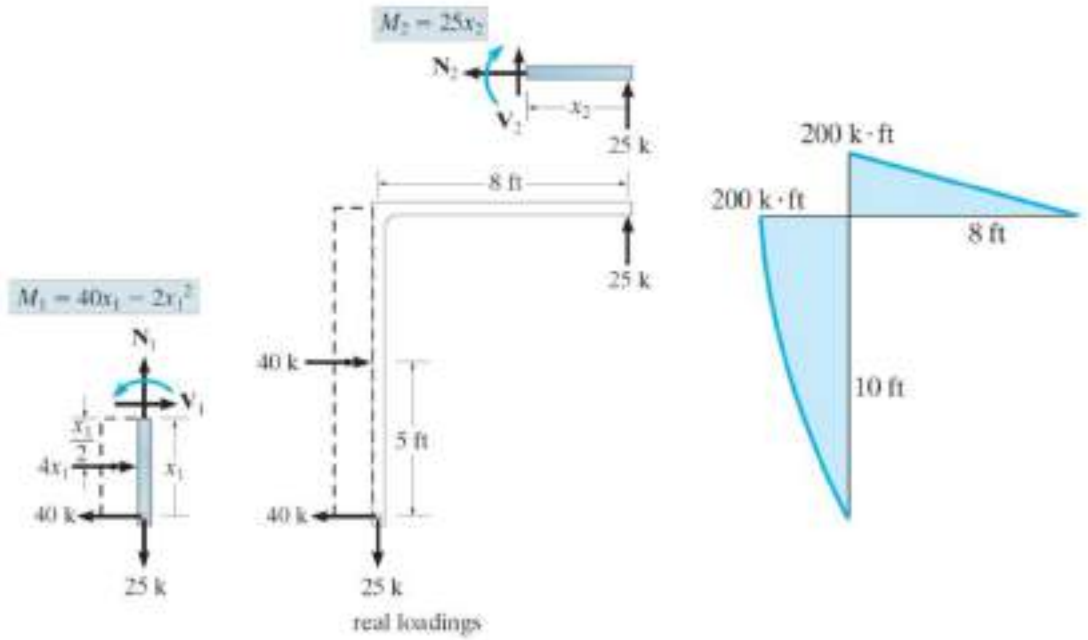
Example: Determine the horizontal displacement at point **C** of the frame shown in the figure. Take E = 29(10^3) ksi, I = 600 in^4 for both members.



Solution:



$m_2 = 1.25x_2$
v_2
n_2
1 k
x_2
1.25 k
10 k·ft
10 k·ft
8 ft
8 ft
1 k
1.25 k
n_1
$m_1 = 1x_1$
v_1
10 ft
x_1
1 k
1 k
1.25 k
1.25 k
virtual loadings
10 ft



$M_2 = 25x_2$
N_2
V_2
x_2
25 k
200 k·ft
200 k·ft
8 ft
8 ft
25 k
$M_1 = 40x_1 - 2x_1^2$
N_1
V_1
40 k
5 ft
10 ft
$\frac{x_1}{2}$
$4x_1$
x_1
40 k
40 k
25 k
25 k
real loadings

$$1 \cdot \Delta_{C_h} = \int_0^L \frac{mM}{EI} dx = \int_0^{10} \frac{(1x_1)(40x_1 - 2x_1^2)\, dx_1}{EI} + \int_0^{8} \frac{(1.25x_2)(25x_2)\, dx_2}{EI}$$

$$\Delta_{C_h} = \frac{8333.3}{EI} + \frac{5333.3}{EI} = \frac{13\,666.7 \text{ k} \cdot \text{ft}^3}{EI} \qquad (1)$$

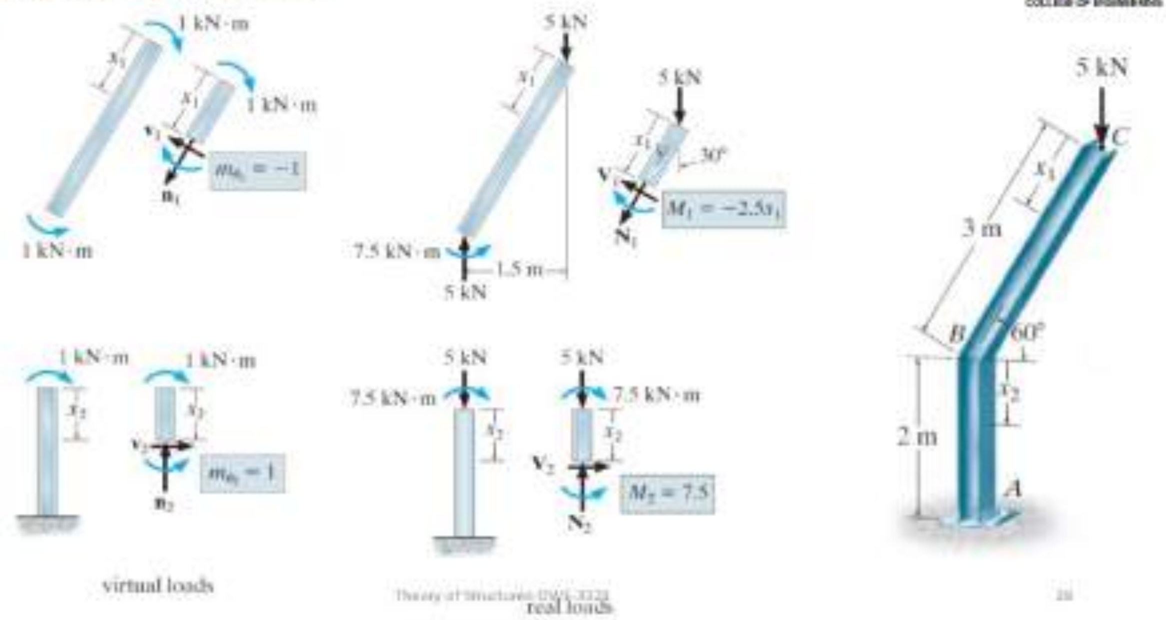
$$\Delta_{C_h} = \frac{13\,666.7 \text{ k} \cdot \text{ft}^3}{[29(10^3) \text{ k/in}^2((12)^2 \text{ in}^2/\text{ft}^2)][600 \text{ in}^4(\text{ft}^4/(12)^4 \text{ in}^4)]}$$

$$= 0.113 \text{ ft} = 1.36 \text{ in.}$$

Ans.

Example: Determine the tangential rotation at point **C** of the frame shown in the figure. Take E = 200 GPa, I = 15x10⁶ mm⁴.

Solution:



$$1 \cdot \theta_C = \int_0^L \frac{m_\theta M}{EI} dx = \int_0^3 \frac{(-1)(-2.5x_1)\, dx_1}{EI} + \int_0^2 \frac{(1)(7.5)\, dx_2}{EI}$$

$$\theta_C = \frac{11.25}{EI} + \frac{15}{EI} = \frac{26.25 \text{ kN} \cdot \text{m}^2}{EI}$$

or

$$\theta_C = \frac{26.25 \text{ kN} \cdot \text{m}^2}{200(10^6) \text{ kN/m}^2[15(10^6) \text{ mm}^4](10^{-12} \text{ m}^4/\text{mm}^4)}$$

$$= 0.00875 \text{ rad}$$

Ans.

Theory of Structures CIVE-3323 29

Castigliano's Theorem

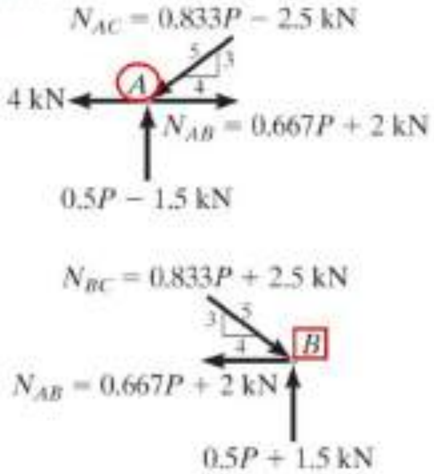
1- Trusses :

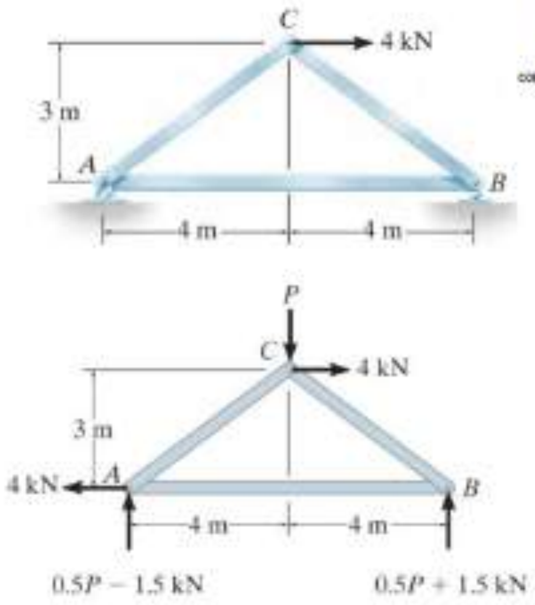
$$\Delta = \sum N \left(\frac{\partial N}{\partial P} \right) \frac{L}{AE}$$

Theory of Structures CIVE-3323 30

Example: Determine the vertical displacement at Joint **C** of the truss shown in the figure. The cross-sectional area of each member is A = 400 mm² and E = 200 GPa.

Solution:





Member	N	$\frac{\partial N}{\partial P}$	$N\,(P = 0)$	L	$N\left(\frac{\partial N}{\partial P}\right)L$
AB	$0.667P + 2$	0.667	2	8	10.67
AC	$-(0.833P - 2.5)$	−0.833	2.5	5	−10.42
BC	$-(0.833P + 2.5)$	−0.833	−2.5	5	10.42

$\Sigma = 10.67$ kN·m

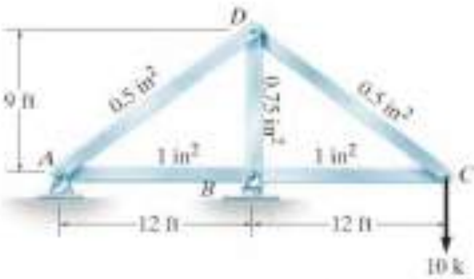
$$\Delta_{C_v} = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE} = \frac{10.67 \text{ kN}\cdot\text{m}}{AE}$$

Substituting $A = 400 \text{ mm}^2 = 400(10^{-6}) \text{ m}^2$, $E = 200 \text{ GPa} = 200(10^9)$ Pa, and converting the units of N from kN to N, we have

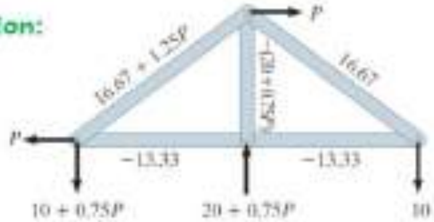
$$\Delta_{C_v} = \frac{10.67(10^3) \text{ N}\cdot\text{m}}{400(10^{-6}) \text{ m}^2(200(10^9) \text{ N/m}^2)} = 0.000133 \text{ m} = 0.133 \text{ mm}$$

Ans.

Example: Determine the horizontal displacement at Joint **D** of the truss shown in the figure. The cross-sectional area of each member is indicated in the figure and E = 29(10³) ksi.



Solution:



Member	N	$\frac{\partial N}{\partial P}$	$N\,(P=0)$	L	$N\left(\frac{\partial N}{\partial P}\right)L$
AB	-13.33	0	-13.33	12	0
BC	-13.33	0	-13.33	12	0
CD	16.67	0	16.67	15	0
DA	$16.67 + 1.25P$	1.25	16.67	15	312.50
BD	$-(20 + 0.75P)$	-0.75	-20	9	135.00

$$\Delta_{D_h} = \sum N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE} = 0 + 0 + 0$$

$$+ \frac{312.50\ \text{k}\cdot\text{ft}(12\ \text{in./ft})}{(0.5\ \text{in}^2)[29(10^3)\ \text{k/in}^2]}$$

$$+ \frac{135.00\ \text{k}\cdot\text{ft}(12\ \text{in./ft})}{(0.75\ \text{in}^2)[29(10^3)\ \text{k/in}^2]}$$

$$= 0.333\ \text{in.}$$

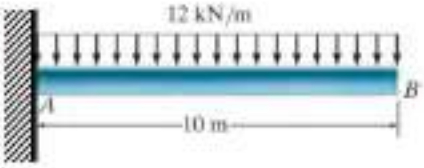
Castigliano's Theorem

2- Beams and Frames:

$$\Delta = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI}$$

$$\theta = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI}$$

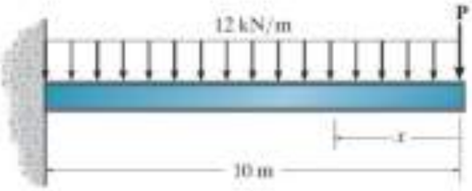
Example: Determine the displacement of point B of the steel beam shown in the figure. Take E = 200 GPa, I = 500x10⁶ mm⁴.



Solution:

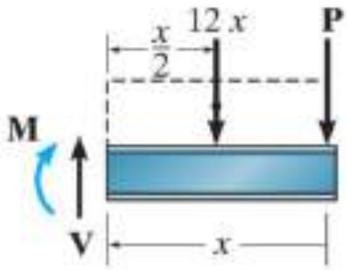
$\circlearrowleft + \Sigma M = 0; \qquad -M - (12x)\left(\frac{x}{2}\right) - Px = 0$

$$M = -6x^2 - Px \qquad \frac{\partial M}{\partial P} = -x$$



Setting $P = 0$, its actual value, yields

$$M = -6x^2 \qquad \frac{\partial M}{\partial P} = -x$$



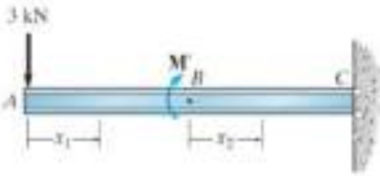
$$\Delta_B = \int_0^L M\left(\frac{\partial M}{\partial P}\right)\frac{dx}{EI} = \int_0^{10}\frac{(-6x^2)(-x)\,dx}{EI} = \frac{15(10^3)\text{ kN}\cdot\text{m}^3}{EI}$$

or

$$\Delta_B = \frac{15(10^3)\text{ kN}\cdot\text{m}^3}{200(10^6)\text{ kN/m}^2[500(10^6)\text{ mm}^4](10^{-12}\text{ m}^4/\text{mm}^4)}$$
$$= 0.150\text{ m} = 150\text{ mm} \qquad \textit{Ans.}$$

Example: Determine the slop θ at point B of the steel beam shown in the figure. Take E = 200 GPa, I = 600x10⁶ mm⁴.

Solution:



(b)

For x_1:

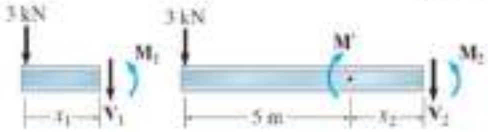
$\circlearrowleft + \Sigma M = 0; \qquad M_1 + 3x_1 = 0$

$$M_1 = -3x_1$$
$$\frac{\partial M_1}{\partial M'} = 0$$

For x_2:

$\circlearrowleft + \Sigma M = 0; \qquad M_2 - M' + 3(5 + x_2) = 0$

$$M_2 = M' - 3(5 + x_2)$$
$$\frac{\partial M_2}{\partial M'} = 1$$



$$\theta_B = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI}$$

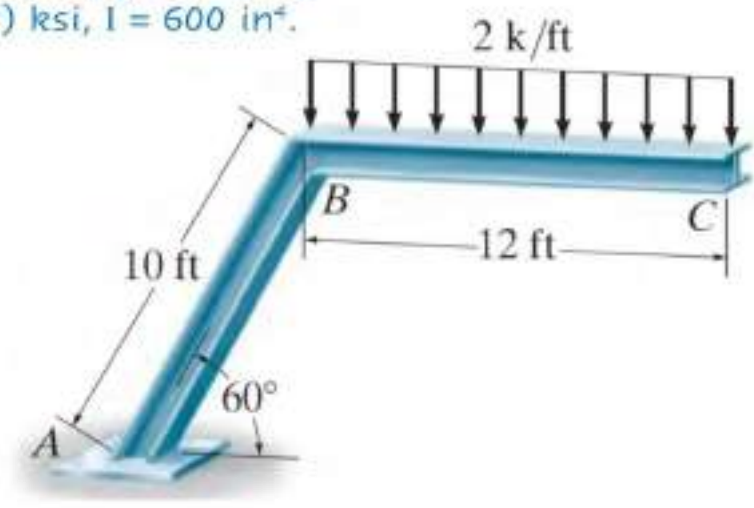
$$= \int_0^5 \frac{(-3x_1)(0)\,dx_1}{EI} + \int_0^5 \frac{-3(5 + x_2)(1)\,dx_2}{EI} = -\frac{112.5\text{ kN}\cdot\text{m}^2}{EI}$$

or

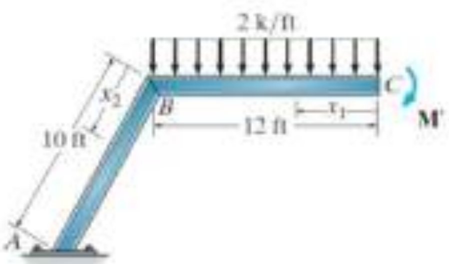
$$\theta_B = \frac{-112.5\text{ kN}\cdot\text{m}^2}{200(10^6)\text{ kN/m}^2[60(10^6)\text{ mm}^4](10^{-12}\text{ m}^4/\text{mm}^4)}$$

$$= -0.00938\text{ rad}$$ *Ans.*

Example: Determine the slop at point **C** of the steel frame shown in the figure. Take $E = 29(10^3)$ ksi, $I = 600$ in^4.



Solution:



For x_1:

$\curvearrowleft + \Sigma M = 0;$ $$-M_1 - 2x_1\left(\frac{x_1}{2}\right) - M' = 0$$

$$M_1 = -\left(x_1^2 + M'\right)$$

$$\frac{\partial M_1}{\partial M'} = -1$$

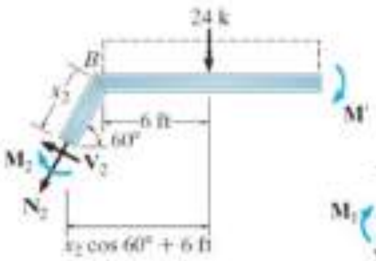
For x_2:

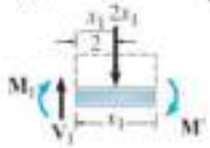
$\curvearrowleft + \Sigma M = 0;$ $$-M_2 - 24(x_2 \cos 60^\circ + 6) - M' = 0$$

$$M_2 = -24(x_2 \cos 60^\circ + 6) - M'$$

$$\frac{\partial M_2}{\partial M'} = -1$$

Theory of Structures CWE-3323





$$\theta_C = \int_0^L M\left(\frac{\partial M}{\partial M'}\right)\frac{dx}{EI}$$

$$= \int_0^{12} \frac{(-x_1^2)(-1)\,dx_1}{EI} + \int_0^{10} \frac{-24(x_2 \cos 60^\circ + 6)(-1)\,dx_2}{EI}$$

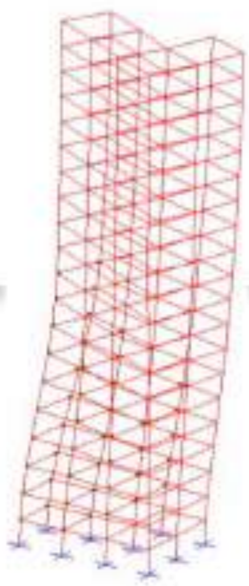
$$= \frac{576 \text{ k}\cdot\text{ft}^2}{EI} + \frac{2040 \text{ k}\cdot\text{ft}^2}{EI} = \frac{2616 \text{ k}\cdot\text{ft}^2}{EI}$$

$$\theta_C = \frac{2616 \text{ k}\cdot\text{ft}^2(144 \text{ in}^2/\text{ft}^2)}{29(10^3)\text{ k/in}^2(600 \text{ in}^4)} = 0.0216 \text{ rad}$$ *Ans.*

Theory of Structures CWE-3323

Unit-7

Analysis of Indeterminate Structures Using Force Methods



Indeterminate Structures

Analysis of Indeterminate Structures

1- Force (Flexibility) Methods: Classical Methods

- Consistent Deformation Method. √
- Castigliano's Second theorem. X

2- Displacement (Stiffness) methods:

- Slope Deflection Method. √
- Moment Distribution Method. √
- Direct Stiffness Method. (maybe)

Force Method VS. Displacement Methods

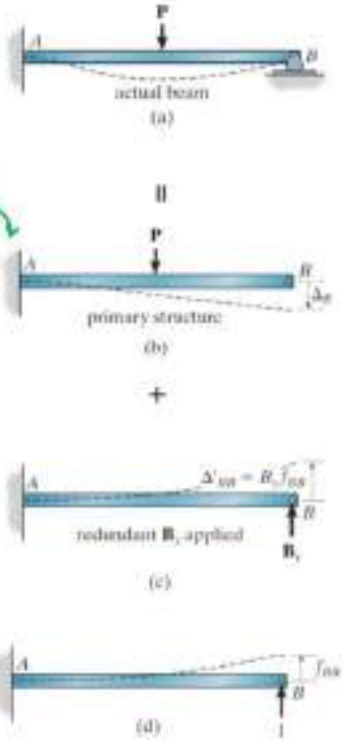
	Unknowns	Equations Used for Solution	Coefficients of the Unknowns
Force Method	Forces	Compatibility and Force Displacements	Flexibility Coefficients
Displacement Method	Displacements	Equilibrium and Force Displacement	Stiffness Coefficients

Consistent Deformation Method:

BEAMS

General Analysis Procedure:

Primary Structures

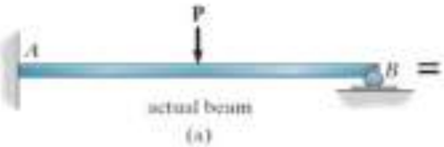


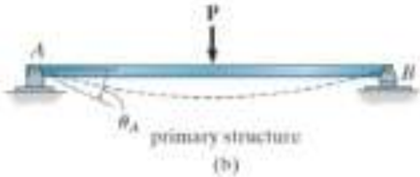
$$0 = -\Delta_B + \Delta'_{BB}$$

$$\Delta'_{BB} = B_y f_{BB}$$

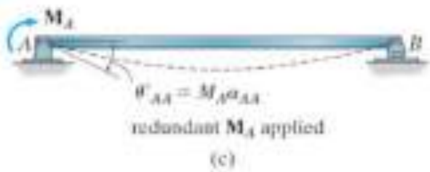
$$0 = -\Delta_B + B_y f_{BB}$$

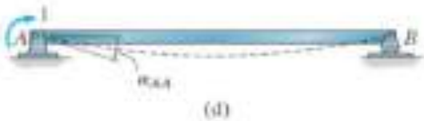
$$\theta'_{AA} = M_A \alpha_{AA}$$





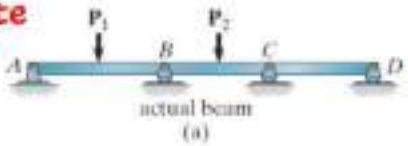
+

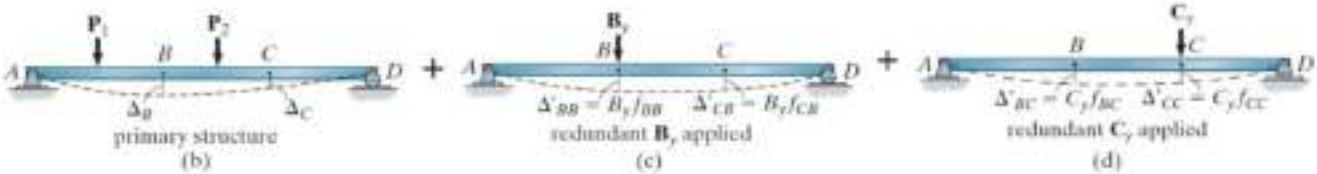




$$0 = \theta_A + M_A \alpha_{AA}$$

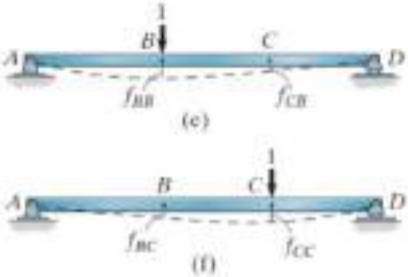
2nd Degree Indeterminate Structure:





$$0 = \Delta_B + B_y f_{BB} + C_y f_{BC}$$

$$0 = \Delta_C + B_y f_{CB} + C_y f_{CC}$$



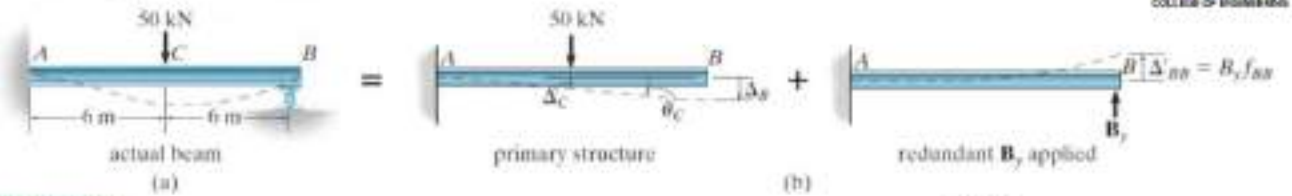
Procedure for Analysis:

Principle of Superposition: Determine the number of degrees *n* to which the structure is indeterminate. Then specify the *n* unknown redundant forces or moments that must be removed from the structure in order to make it statically determinate and stable. Using the principle of superposition, draw the statically indeterminate structure and show it to be equal to a series of corresponding statically *determinate* structures.

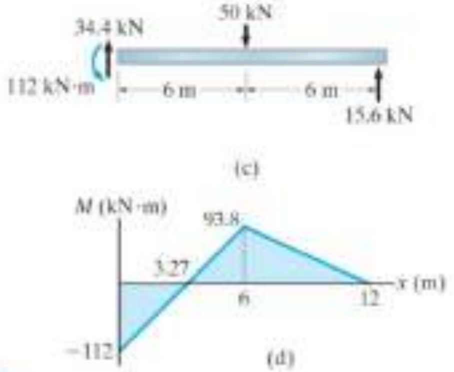
Compatibility Equations: Write a compatibility equation for the displacement or rotation at each point where there is a redundant force or moment. These equations should be expressed in terms of the unknown redundants and their corresponding flexibility coefficients obtained from unit loads or unit couple moments that are collinear with the redundant forces or moments.

Equilibrium Equations: Draw a free-body diagram of the structure. Since the redundant forces and/or moments have been calculated, the remaining unknown reactions can be determined from the equations of equilibrium.

Example : Determine the reaction at the roller support *B* of the beam shown in the figure, *EI* is constant.



Solution :



$$0 = -\Delta_B + B_y f_{BB}$$

$$\Delta_B = \frac{P(L/2)^3}{3EI} + \frac{P(L/2)^2}{2EI}\left(\frac{L}{2}\right)$$

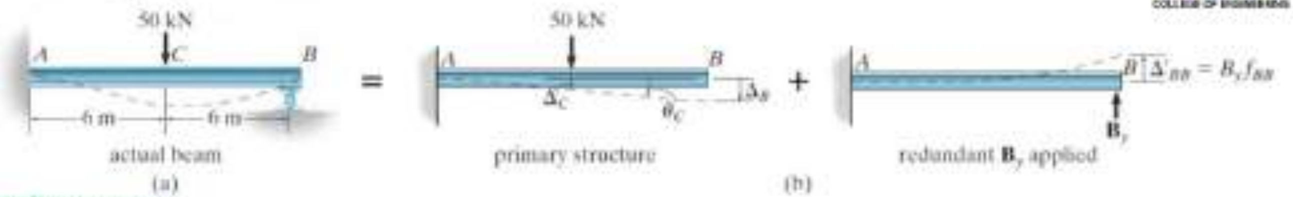
$$= \frac{(50\text{ kN})(6\text{ m})^3}{3EI} + \frac{(50\text{ kN})(6\text{ m})^2}{2EI}(6\text{ m}) = \frac{9000\text{ kN}\cdot\text{m}^3}{EI}\downarrow$$

$$f_{BB} = \frac{PL^3}{3EI} = \frac{1(12\text{ m})^3}{3EI} = \frac{576\text{ m}^3}{EI}\uparrow$$

Substituting these results into Eq. (1) yields

$$(+\uparrow) \qquad 0 = -\frac{9000}{EI} + B_y\left(\frac{576}{EI}\right) \qquad B_y = 15.6\text{ kN} \qquad \textit{Ans.}$$

Example : Determine the reaction at the roller support *B* of the beam shown in the figure, *EI* is constant.



2nd Solution :

$$0 = -\Delta_B + B_y f_{BB}$$

$$\Delta_B = \int_0^L \frac{Mm}{EI}dx = \int_0^6 \frac{Mm}{EI}dx + \int_6^{12} \frac{Mm}{EI}dx = \int_0^6 \frac{0.0\times(-x)}{EI}dx + \int_6^{12} \frac{(-50(x-6)\times(-x)}{EI}dx$$

$$\Delta_B = 0.0 + \frac{1}{EI}\int_6^{12}(50x^2 - 300x)dx$$

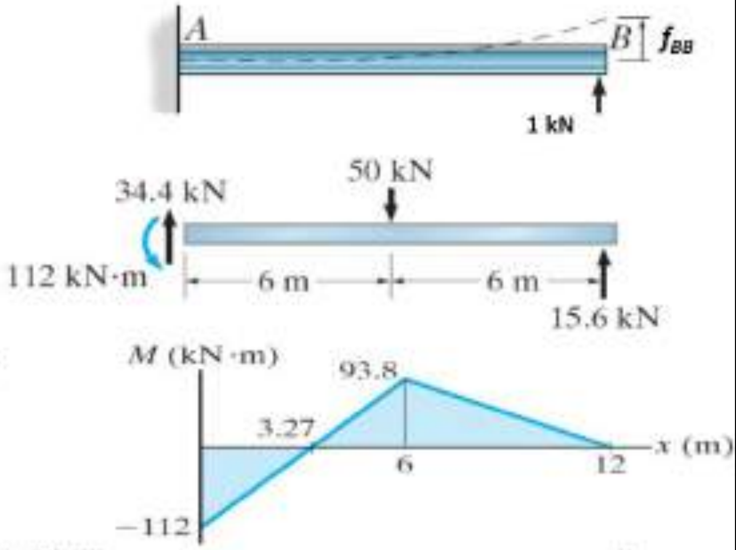
$$\Delta_B = \frac{1}{EI}\left[\frac{50x^3}{3} - \frac{300x^2}{2}\right]_6^{12} = \frac{9000\ kN.m^3}{EI}\downarrow$$

$$f_{BB} = \int_0^L \frac{m.m}{EI} dx$$

$$f_{BB} = \int_0^{12} \frac{(x)\times(x)}{EI} dx = \int_0^{12} \frac{x^2}{EI} dx$$

$$f_{BB} = \frac{1}{EI} \int_0^{12} \frac{x^3}{3} dx = \frac{1}{EI} \left[\frac{x^3}{3}\right]_0^{12}$$

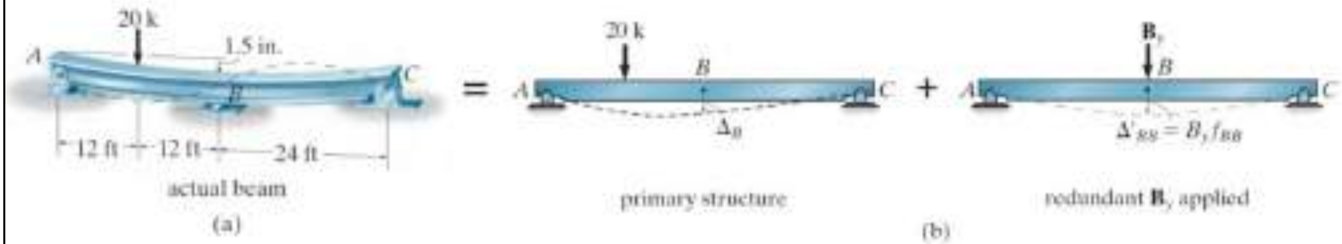
$$f_{BB} = \frac{1}{EI} \frac{1726}{3} = \frac{576 kN.m^3}{EI} \uparrow$$



$$0 = -\frac{9000}{EI} + B_y\left(\frac{576}{EI}\right) \qquad B_y = 15.6 \text{ kN}$$

Example : Draw the shear and moment diagrams for the beam shown in the figure. The support at **B** settles 1.5 in. Take E = 29(10^3) ksi and I = 750 in^4.

Solution :



$$(+\downarrow) \qquad 1.5 \text{ in.} = \Delta_B + B_y f_{BB}$$

$$\Delta_B = \frac{Pbx}{6LEI}(L^2 - b^2 - x^2) = \frac{20(12)(24)}{6(48)EI}[(48)^2 - (12)^2 - (24)^2]$$

$$= \frac{31{,}680 \text{ k} \cdot \text{ft}^3}{EI}$$

$$f_{BB} = \frac{PL^3}{48EI} = \frac{1(48)^3}{48EI} = \frac{2304 \text{ k} \cdot \text{ft}^3}{EI}$$

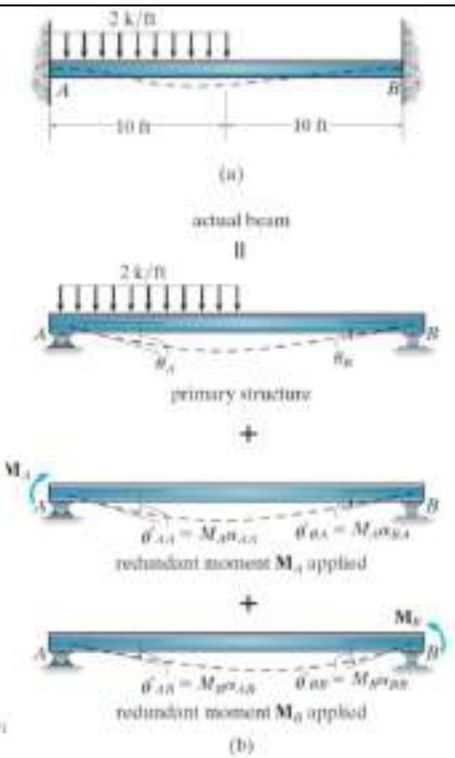
$$1.5 \text{ in.} (29(10^3) \text{ k/in}^2)(750 \text{ in}^4)$$

$$= 31{,}680 \text{ k} \cdot \text{ft}^3 (12 \text{ in./ft})^3 + B_y(2304 \text{ k} \cdot \text{ft}^3)(12 \text{ in./ft})^3$$

$$B_y = -5.56 \text{ k}$$

Note : The negative sign indicates that By acts upward on the beam.

Example : Draw the shear and moment diagrams for the beam shown in the figure. *EI* is constant. Neglect the effects of axial load.



Solution :

$$0 = \theta_A + M_A\alpha_{AA} + M_B\alpha_{AB}$$

$$0 = \theta_B + M_A\alpha_{BA} + M_B\alpha_{BB}$$

Using direct equations:
Check Hibbeler

$$\theta_A = \frac{3wL^3}{128EI} = \frac{3(2)(20)^3}{128EI} = \frac{375}{EI}$$

$$\theta_B = \frac{7wL^3}{384EI} = \frac{7(2)(20)^3}{384EI} = \frac{291.7}{EI}$$

$$\alpha_{AA} = \frac{ML}{3EI} = \frac{1(20)}{3EI} = \frac{6.67}{EI}$$

$$\alpha_{BB} = \frac{ML}{3EI} = \frac{1(20)}{3EI} = \frac{6.67}{EI}$$

$$\alpha_{AB} = \frac{ML}{6EI} = \frac{1(20)}{6EI} = \frac{3.33}{EI}$$

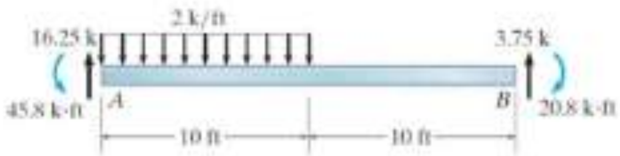
Note that $\alpha_{BA} = \alpha_{AB}$ a consequence of Maxwell's theorem of reciprocal displacements.

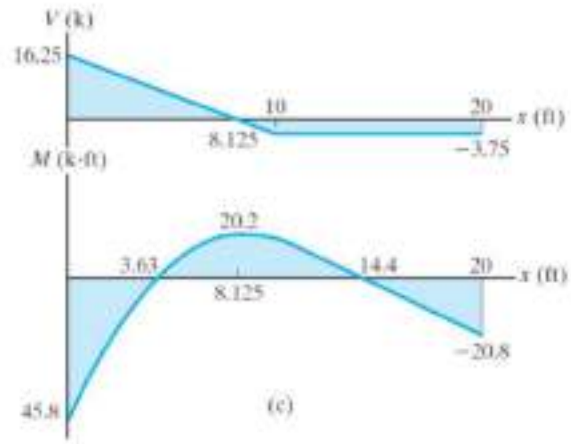
Substitute data in Eqs. (1) and (2):

$$0 = \frac{375}{EI} + M_A\left(\frac{6.67}{EI}\right) + M_B\left(\frac{3.33}{EI}\right)$$

$$0 = \frac{291.7}{EI} + M_A\left(\frac{3.33}{EI}\right) + M_B\left(\frac{6.67}{EI}\right)$$

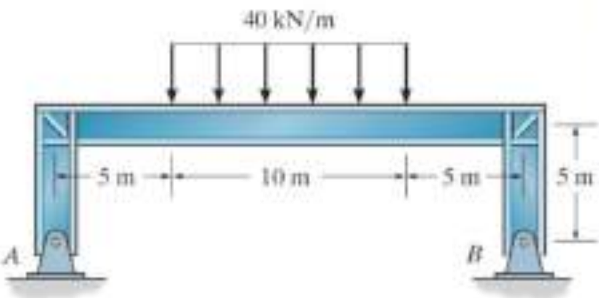
$$M_A = -45.8 \text{ k}\cdot\text{ft} \qquad M_B = -20.8 \text{ k}\cdot\text{ft}$$



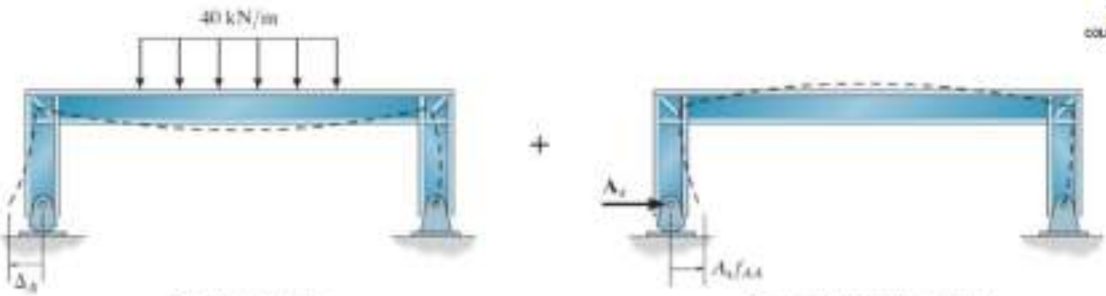


Consistent Deformation Method : FRAMES

Example : The frame, or bent, shown in the photo is used to support the bridge deck. Assuming EI is constant, Determine the support reactions.



Solution :



Primary structure   Redundant force A_x applied

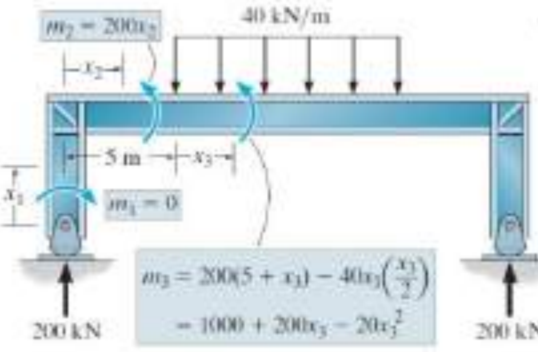
Compatibility Equation :

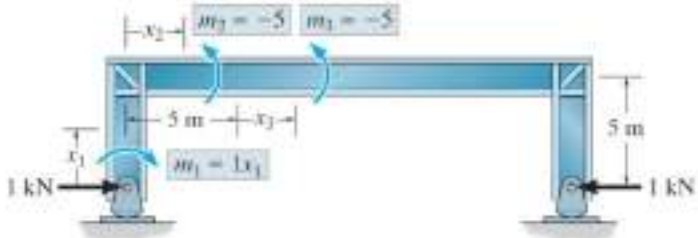
$$(\overset{+}{\rightarrow}) \qquad 0 = \Delta_A + A_x f_{AA}$$

$$\Delta_A = \int_0^L \frac{Mm}{EI}dx = 2\int_0^5 \frac{(0)(1x_1)dx_1}{EI} + 2\int_0^5 \frac{(200x_2)(-5)dx_2}{EI}$$

$$+ 2\int_0^5 \frac{(1000 + 200x_3 - 20x_3^2)(-5)dx_3}{EI}$$

$$= 0 - \frac{25000}{EI} - \frac{66\,666.7}{EI} = -\frac{91\,666.7}{EI}$$





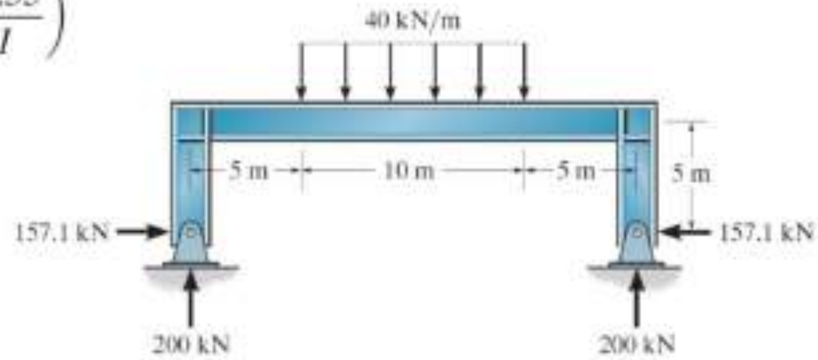
$$f_{AA} = \int_0^L \frac{mm}{EI}dx = 2\int_0^5 \frac{(1x_1)^2 dx_1}{EI} + 2\int_0^5 (5)^2 dx_2 + 2\int_0^5 (5)^2 dx_3$$

$$= \frac{583.33}{EI}$$

$$0 = \frac{-91\,666.7}{EI} + A_x\left(\frac{583.33}{EI}\right)$$

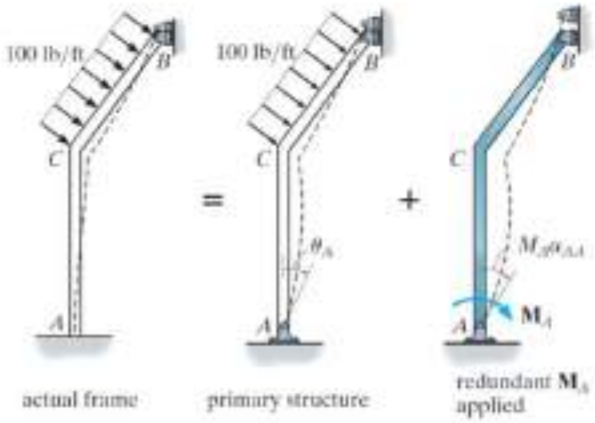
$$A_x = 157 \text{ kN}$$

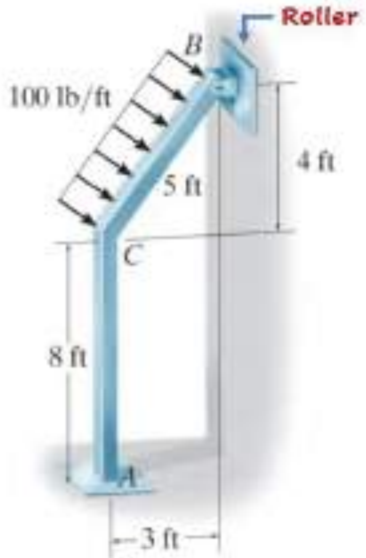
Using Equilibrium Equation ⟹



Example : The Determine the moment at the fixed support **A** for the frame shown in the figure. EI is constant.

Solution





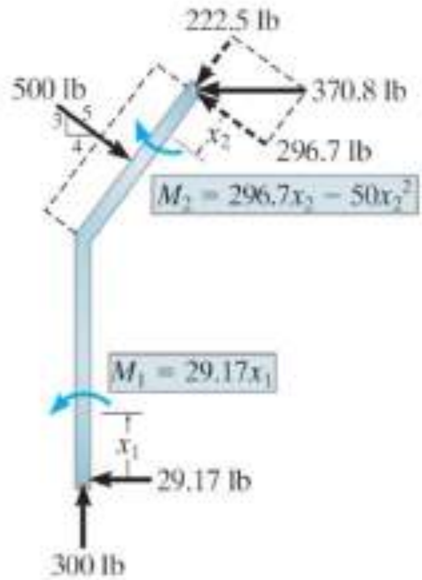
Compatibility Equation : $(\curvearrowleft +) \quad 0 = \theta_A + M_A\alpha_{AA}$

$$\theta_A = \sum \int_0^L \frac{M m_\theta \, dx}{EI}$$

$$= \int_0^8 \frac{(29.17x_1)(1 - 0.0833x_1)\, dx_1}{EI}$$

$$+ \int_0^5 \frac{\left(296.7x_2 - 50x_2^2\right)(0.0667x_2)\, dx_2}{EI}$$

$$= \frac{518.5}{EI} + \frac{303.2}{EI} = \frac{821.8}{EI}$$

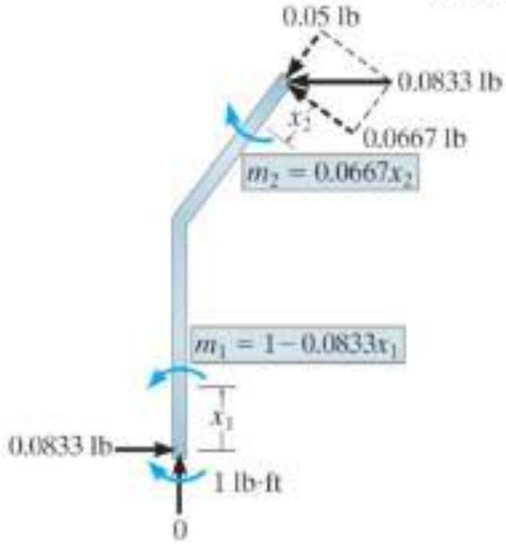


$$\alpha_{AA} = \sum \int_0^L \frac{m_\theta m_\theta}{EI} dx$$

$$= \int_0^8 \frac{(1 - 0.0833x_1)^2\, dx_1}{EI} + \int_0^5 \frac{(0.0667x_2)^2\, dx_2}{EI}$$

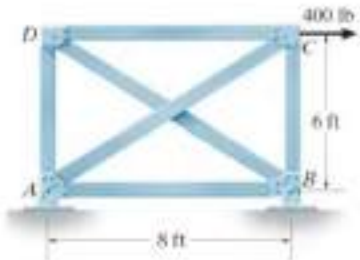
$$= \frac{3.85}{EI} + \frac{0.185}{EI} = \frac{4.04}{EI}$$

$$0 = \frac{821.8}{EI} + M_A\left(\frac{4.04}{EI}\right) \qquad M_A = -204 \text{ lb}\cdot\text{ft}$$

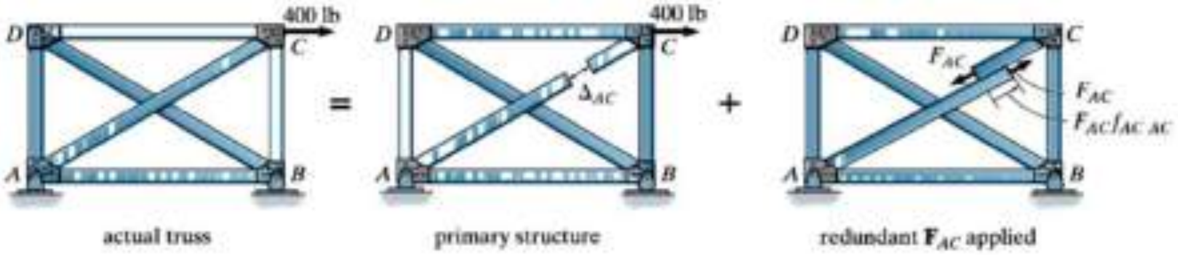


Consistent Deformation Method : TRUSSES

Example : The Determine the force in member AC of the truss shown in the figure. ***AE*** is the same for all the members.



Solution : $0 = \Delta_{AC} + F_{AC} f_{AC\,AC}$



$$\Delta_{AC} = \sum \frac{nNL}{AE}$$

$$= 2\left[\frac{(-0.8)(400)(8)}{AE}\right] + \frac{(-0.6)(0)(6)}{AE} + \frac{(-0.6)(300)(6)}{AE}$$

$$+ \frac{(1)(-500)(10)}{AE} + \frac{(1)(0)(10)}{AE}$$

$$= -\frac{11\,200}{AE}$$

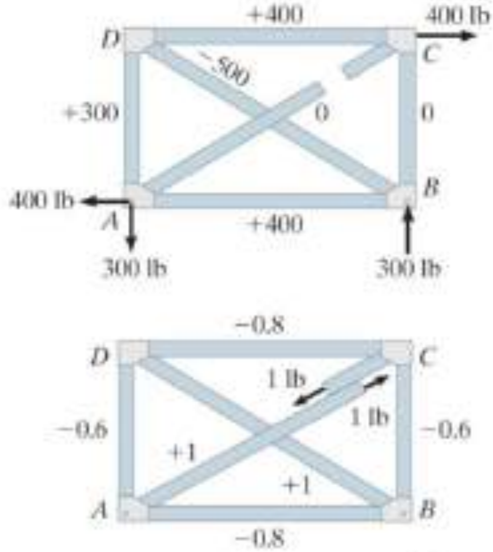
$$f_{AC\,AC} = \sum \frac{n^2L}{AE}$$

$$= 2\left[\frac{(-0.8)^2(8)}{AE}\right] + 2\left[\frac{(-0.6)^2(6)}{AE}\right] + 2\left[\frac{(1)^2 10}{AE}\right]$$

$$= \frac{34.56}{AE}$$

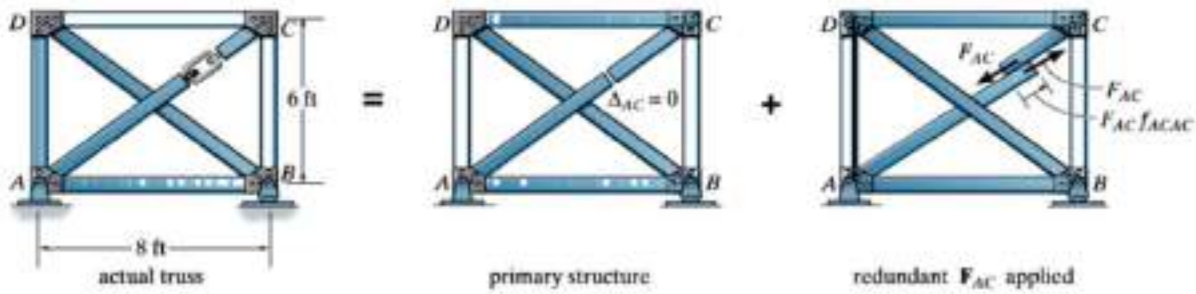
$$0 = -\frac{11\,200}{AE} + \frac{34.56}{AE} F_{AC}$$

$$F_{AC} = 324 \text{ lb (T)}$$



Example : Determine the force in each member of the truss shown in the figure if the turnbuckle on member ***AC*** is used to shorten the member by 0.5 in. Each bar has a cross-sectional area of 0.2 in², and ***E*** = 2911062 psi.

Solution :

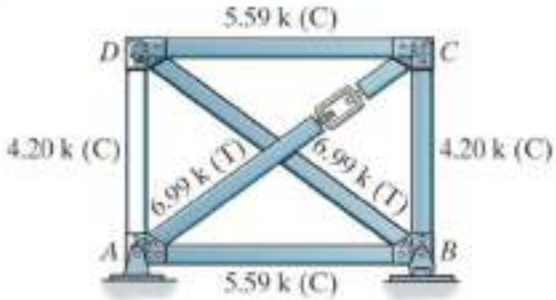


$$f_{AC\,AC} = \frac{34.56}{AE}$$ **From previous example**

$$0.5 \text{ in.} = 0 + \frac{34.56}{AE} F_{AC}$$

$$0.5 \text{ in.} = 0 + \frac{34.56 \text{ ft}(12 \text{ in./ft})}{(0.2 \text{ in}^2)[29(10^6) \text{ lb/in}^2]} F_{AC}$$

$$F_{AC} = 6993 \text{ lb} = 6.99 \text{ k (T)}$$

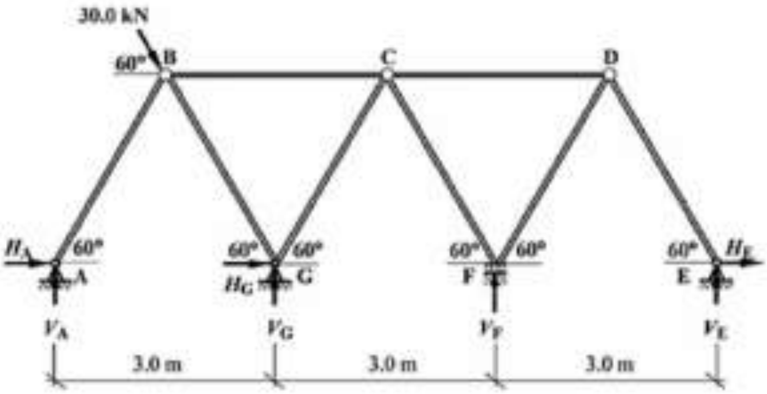


Example : Using the data given, determine the member forces and support reactions for the pinjointed frame shown in the figure. The cross-sectional area of all members is equal to 140 mm². Assume E = 205 kN/mm².

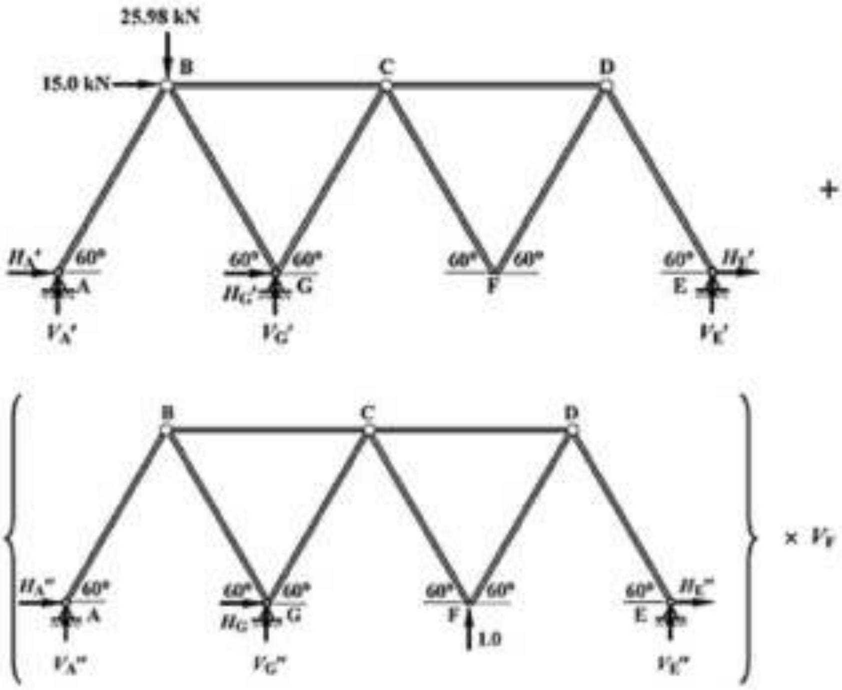
Solution :

All member lengths L=3.0 m
AE = (140×205)= 28.7×10³ kN
Sin60°=0.866, Cos60°=0.5

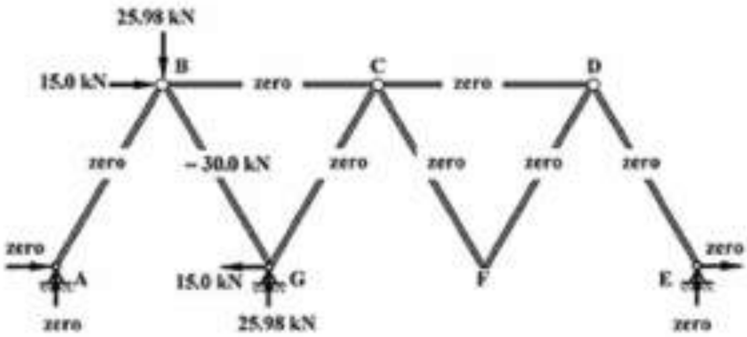
30 Sin60° = 25.98 kN ↓
30 Cos60° = 15.00 kN →



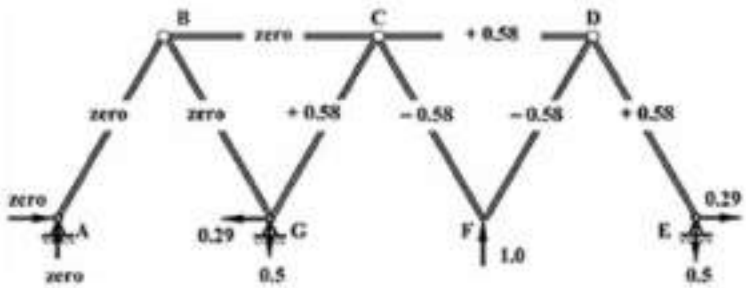
Compatibility :



P - forces :



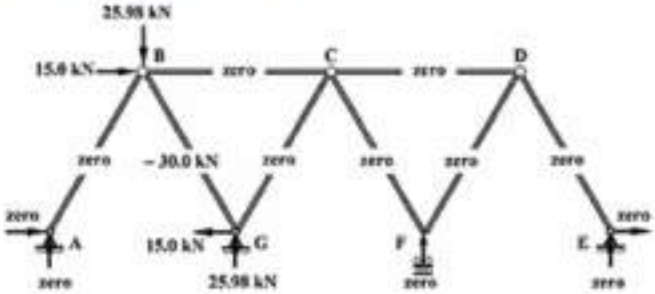
U - forces :



$$\text{i.e. } \sum \frac{PL}{AE}u + \left(\sum \frac{uL}{AE}u\right) \times V_F = 0$$

$$V_F = -\sum \frac{PL}{AE}u \Big/ \sum \frac{uL}{AE}u = 0/0.18 = \text{zero}$$

Final member forces :



Member	Length (mm)	AE (kN)	P-force (kN)	PL/AE (mm)	u	$(PL/AE) \times u$ (mm)	$(uL/AE) \times u$ (mm)
AB	3000	28.7×10^3	0	0	0	0	0
BC	3000	28.7×10^3	0	0	0	0	0
CD	3000	28.7×10^3	0	0	+0.58	0	0.035
DE	3000	28.7×10^3	0	0	+0.58	0	0.035
DF	3000	28.7×10^3	0	0	−0.58	0	0.035
CF	3000	28.7×10^3	0	0	−0.58	0	0.035
CG	3000	28.7×10^3	0	0	+0.58	0	0.035
BG	3000	28.7×10^3	−30.00	−3.14	0	0	0
						Σ=zero	Σ=+0.18

Theory of Structures

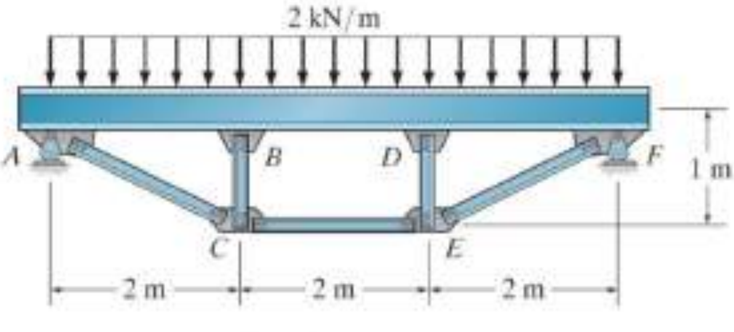
Consistent Deformation Method :

COMPOSITE STRUCTURES

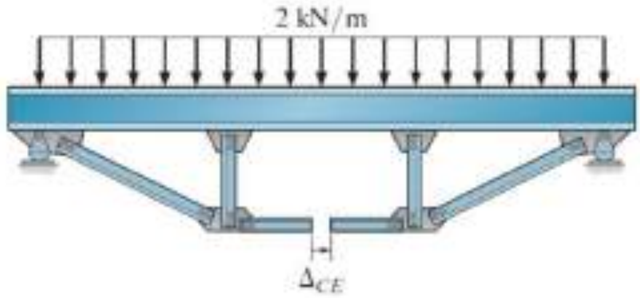
Theory of Structures CWE-3323

Example : The simply supported queen-post trussed beam shown in the photo is to be designed to support a uniform load of 2 kN/m. The dimensions of the structure are shown in the figure. Determine the force developed in member *CE*. Neglect the thickness of the beam and assume the truss members are pin connected to the beam. Also, neglect the effect of axial compression and shear in the beam. The cross-sectional area of each strut is 400 mm², and for the beam $I = 20(10^6)$ mm⁴. Take $E = 200$ GPa.

Solution :



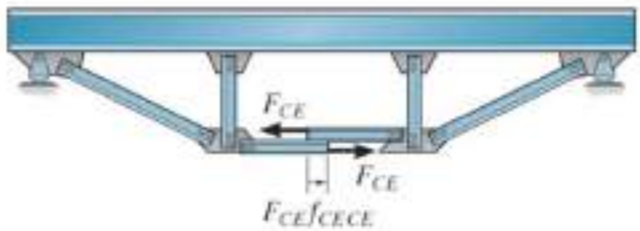
Actual structure



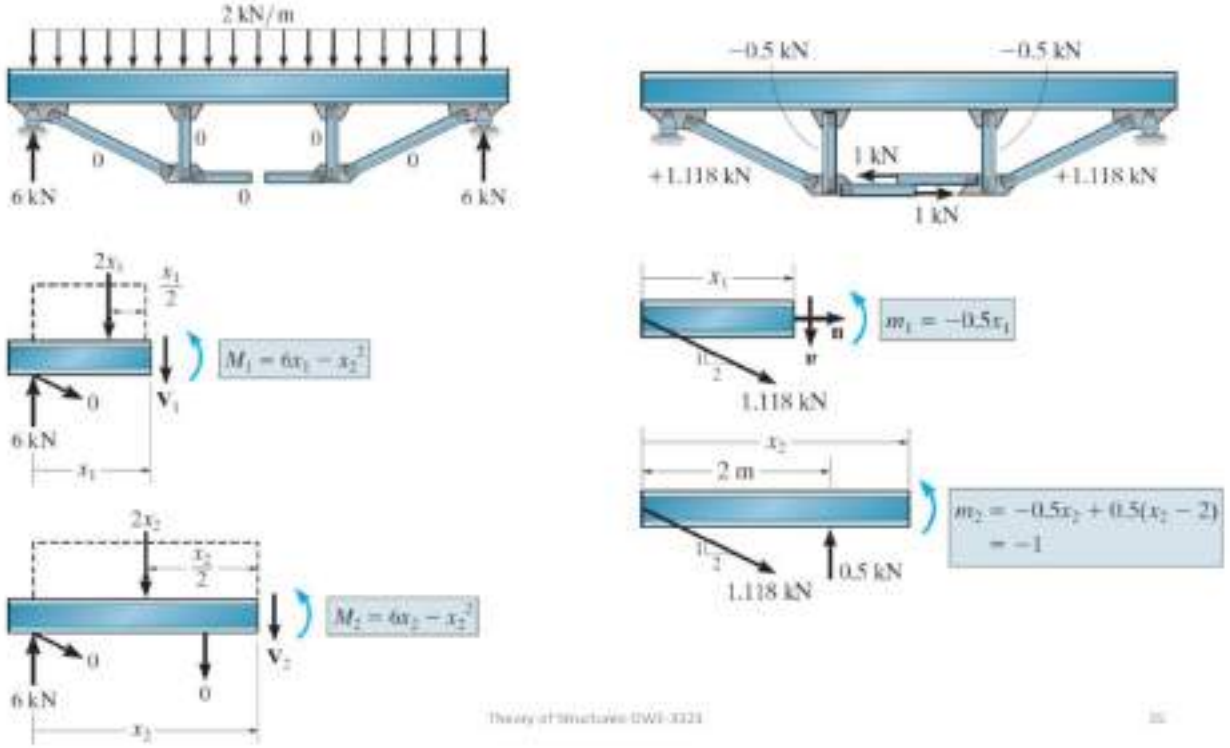
Primary structure

$$0 = \Delta_{CE} + F_{CE} f_{CE\,CE}$$

+



Redundant $\mathbf{F}_{CE}$ applied



$$\Delta_{CE} = \int_0^L \frac{Mm}{EI}dx + \sum \frac{nNL}{AE} = 2\int_0^2 \frac{(6x_1 - x_1^2)(-0.5x_1)dx_1}{EI}$$

$$+ 2\int_2^3 \frac{(6x_2 - x_2^2)(-1)dx_2}{EI} + 2\left(\frac{(1.118)(0)(\sqrt{5})}{AE}\right)$$

$$+ 2\left(\frac{(-0.5)(0)(1)}{AE}\right) + \left(\frac{1(0)2}{AE}\right)$$

$$= -\frac{12}{EI} - \frac{17.33}{EI} + 0 + 0 + 0$$

$$= \frac{-29.33(10^3)}{200(10^9)(20)(10^{-6})} = -7.333(10^{-3})\text{ m}$$

$$f_{CECE} = \int_0^L \frac{m^2 dx}{EI} + \sum \frac{n^2 L}{AE} = 2\int_0^2 \frac{(-0.5x_1)^2 dx_1}{EI} + 2\int_2^3 \frac{(-1)^2 dx_2}{EI}$$

$$+ 2\left(\frac{(1.118)^2(\sqrt{5})}{AE}\right) + 2\left(\frac{(-0.5)^2(1)}{AE}\right) + \left(\frac{(1)^2(2)}{AE}\right)$$

$$= \frac{1.3333}{EI} + \frac{2}{EI} + \frac{5.590}{AE} + \frac{0.5}{AE} + \frac{2}{AE}$$

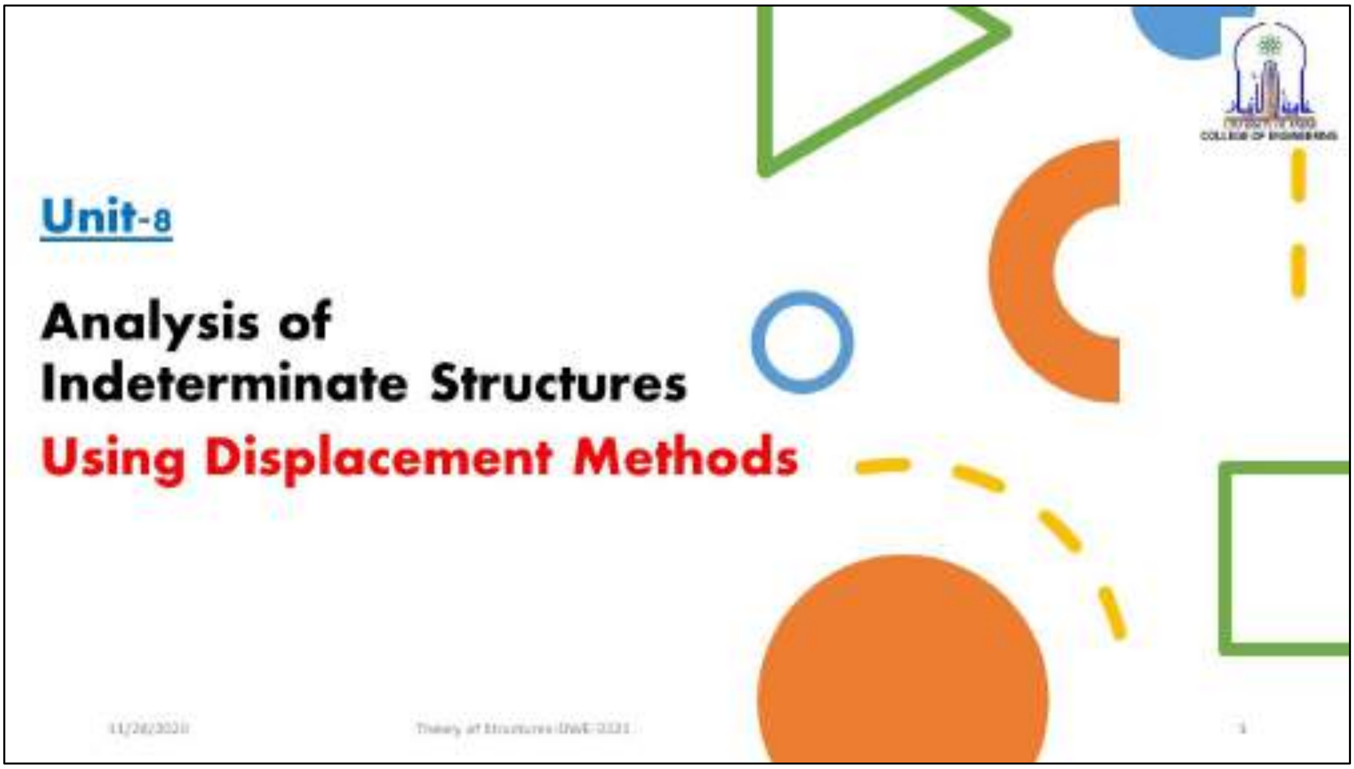
$$= \frac{3.333(10^3)}{200(10^9)(20)(10^{-6})} + \frac{8.090(10^3)}{400(10^{-6})(200(10^9))}$$

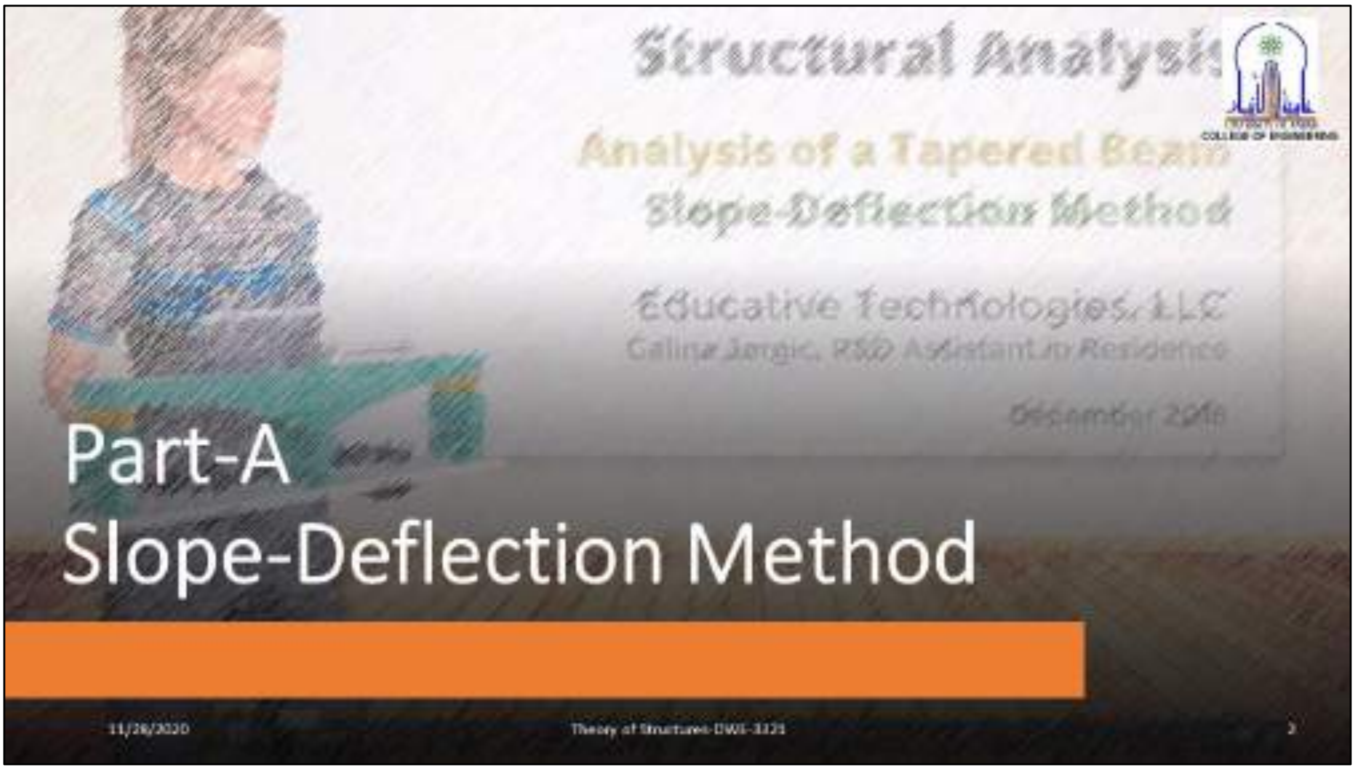
$$= 0.9345(10^{-3})\ \text{m/kN}$$

$$0 = -7.333(10^{-3})\ \text{m} + F_{CE}(0.9345(10^{-3})\ \text{m/kN})$$

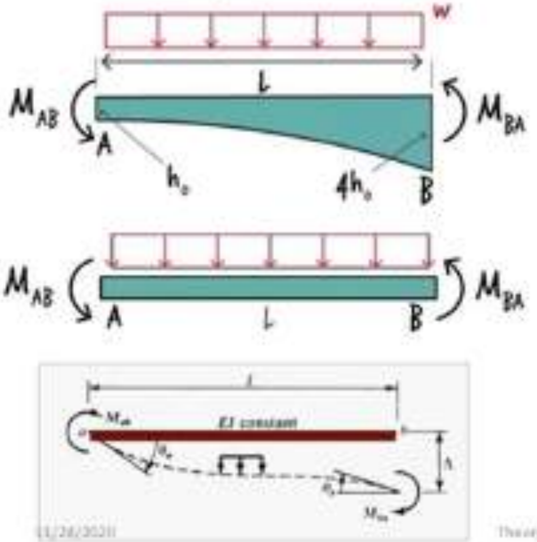
$$F_{CE} = 7.85\ \text{kN}$$

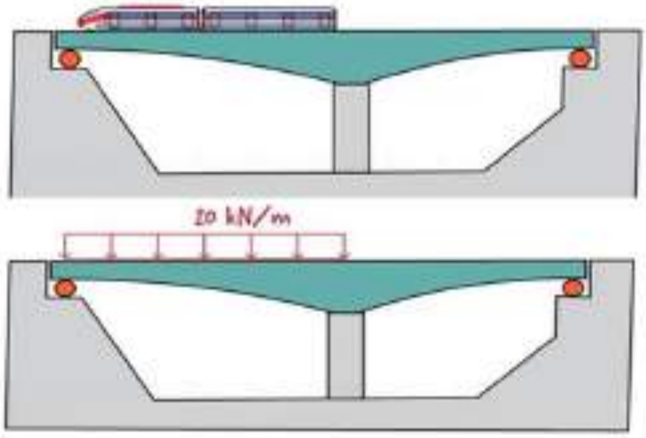
Theory of Structures CWE-3321 37


Unit-8
Analysis of
Indeterminate Structures
Using Displacement Methods

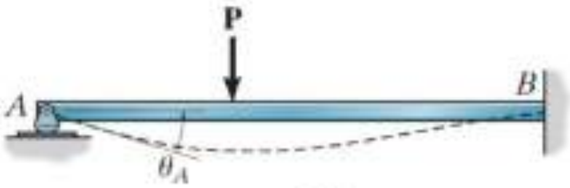

Structural Analysis
Slope-Deflection Method
Part-A
Slope-Deflection Method

Degrees of Freedom : When a structure is loaded, specified points on it, called *nodes*, will undergo unknown displacements. These displacements are referred to as the *degrees of freedom* for the structure, and in the displacement method of analysis it is important to specify these degrees of freedom since they become the unknowns when the method is applied.

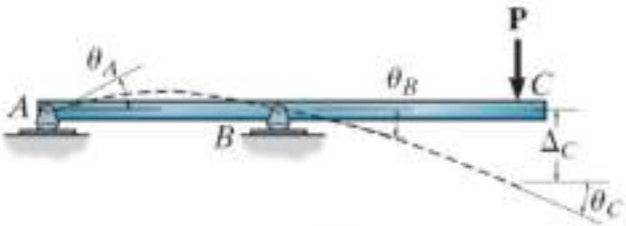




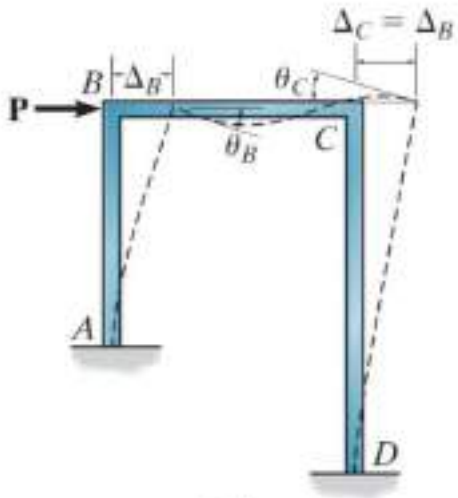
BEAMs & FRAMEs :



(a)

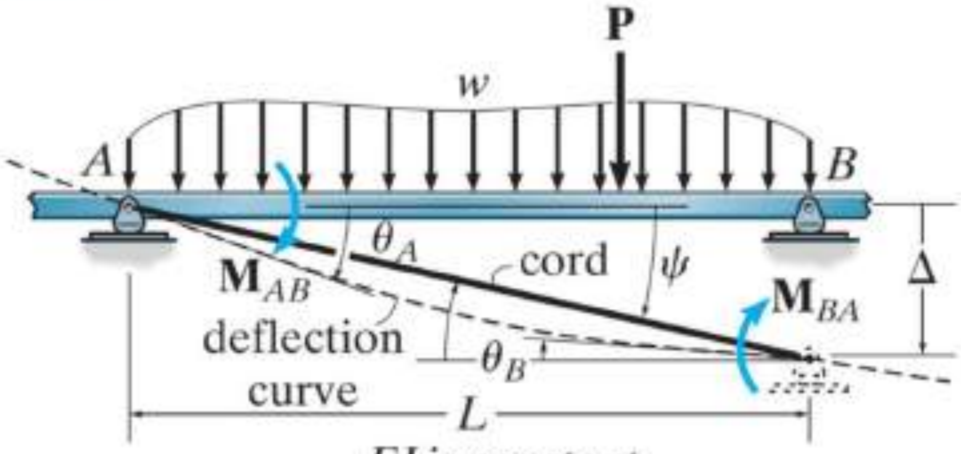


(b)



(c)

General Case :



Angular Displacement at A, θ_A :

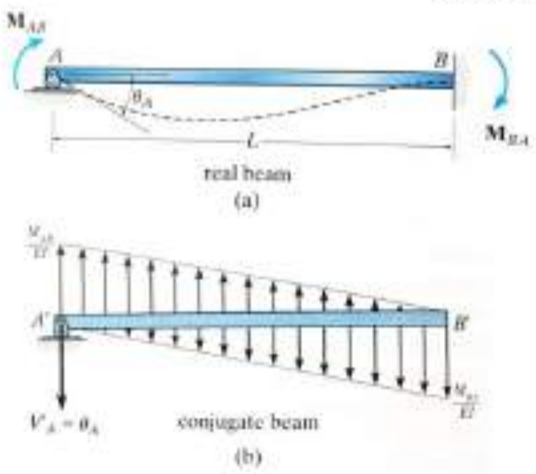
Using Conjugate Beam Method

$\circlearrowleft + \Sigma M_{A'} = 0;\quad \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{2L}{3} = 0$

$\circlearrowleft + \Sigma M_{B'} = 0;\quad \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{2L}{3} + \theta_A L = 0$

$$M_{AB} = \frac{4EI}{L}\theta_A$$

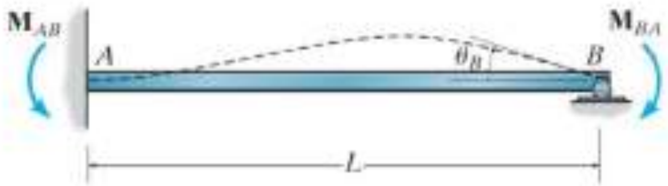
$$M_{BA} = \frac{2EI}{L}\theta_A$$



Angular Displacement at B, θ_B :

$$M_{BA} = \frac{4EI}{L}\theta_B$$

$$M_{AB} = \frac{2EI}{L}\theta_B$$

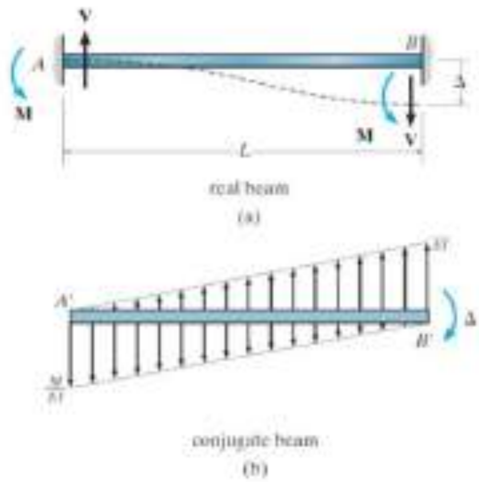


Relative Linear Displacement, Δ:

Using Conjugate Beam Method

$$\circlearrowleft + \Sigma M_{B'} = 0; \quad \left[\frac{1}{2}\frac{M}{EI}(L)\left(\frac{2}{3}L\right)\right] - \left[\frac{1}{2}\frac{M}{EI}(L)\left(\frac{1}{3}L\right)\right] - \Delta = 0$$

$$M_{AB} = M_{BA} = M = \frac{-6EI}{L^2}\Delta$$

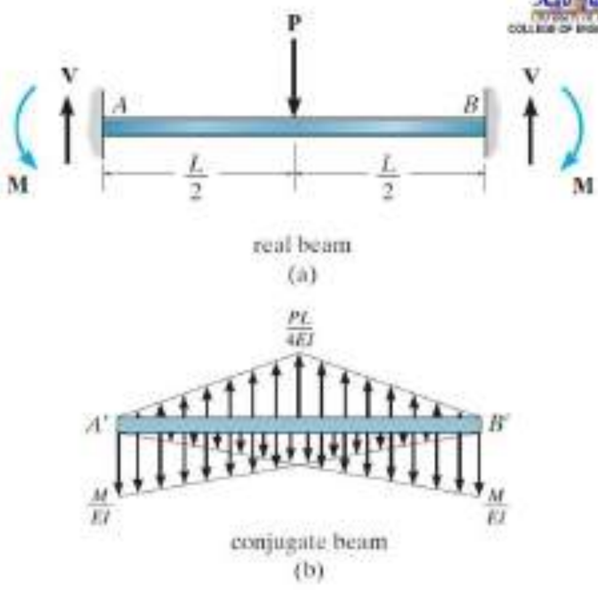


Fixed-End Moments:

Using Conjugate Beam Method

$$+\uparrow \Sigma F_y = 0; \qquad \left[\frac{1}{2}\left(\frac{PL}{4EI}\right)L\right] - 2\left[\frac{1}{2}\left(\frac{M}{EI}\right)L\right] = 0$$

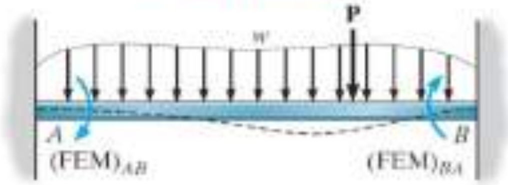
$$M = \frac{PL}{8}$$



real beam
(a)

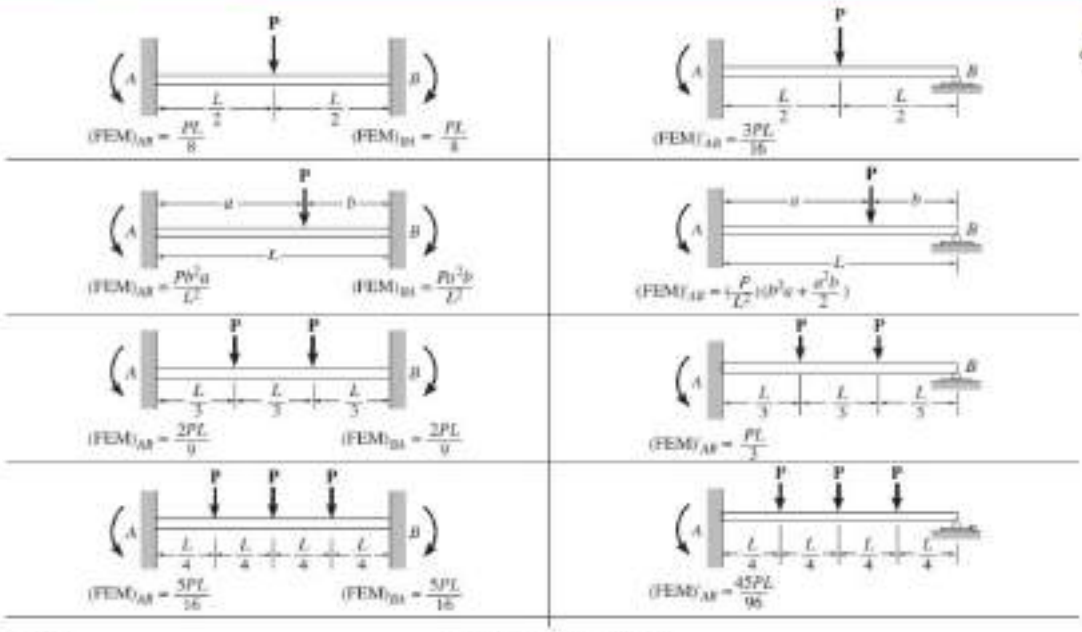
conjugate beam
(b)

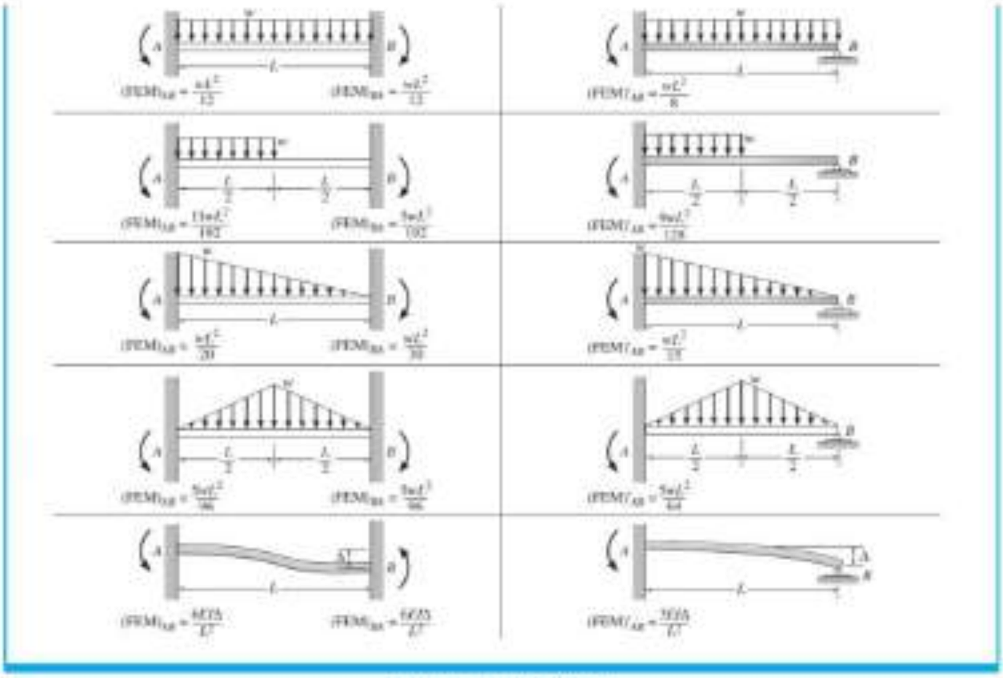
General Case



$$M_{AB} = (\text{FEM})_{AB} \qquad M_{BA} = (\text{FEM})_{BA}$$

Fixed End Moments





Slope-Deflection Equations :

If we add all the effects of θ_A, θ_B and Δ we get the following slope-deflection equations :

$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta}{L}\right)\right] + (\text{FEM})_{AB}$$

$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta}{L}\right)\right] + (\text{FEM})_{BA}$$

Since these two equations are similar, the result can be expressed as a single equation. Referring to one end of the span as the near end (**N**) and the other end as the far end (**F**), and letting the member stiffness be represented as $k = I/L$ and the span's cord rotation as ψ (psi) $= \Delta/L$ we can write

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

For Internal Span or End Span with Far End Fixed

where

M_N = internal moment in the near end of the span; this moment is *positive clockwise* when acting on the span.

E, k = modulus of elasticity of material and span stiffness $k = I/L$.

θ_N, θ_F = near- and far-end slopes or angular displacements of the span at the supports; the angles are measured in *radians* and are *positive clockwise*.

ψ = span rotation of its cord due to a linear displacement, that is, $\psi = \Delta/L$; this angle is measured in *radians* and is *positive clockwise*.

$(\text{FEM})_N$ = fixed-end moment at the near-end support; the moment is *positive clockwise* when acting on the span; refer to the table on the inside back cover for various loading conditions.

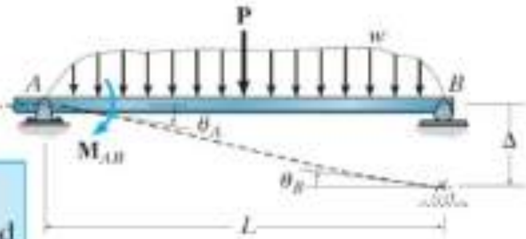
Pin-Supported End Span :

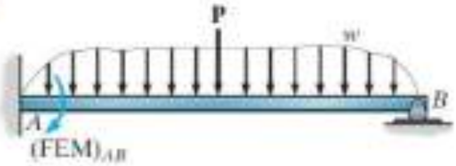
$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

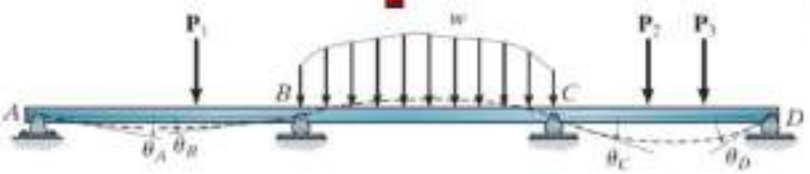
$$0 = 2Ek(2\theta_F + \theta_N - 3\psi) + 0$$

$$M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N$$

Only for End Span with Far End Pinned or Roller Supported







Procedure for Analysis :

Degrees of Freedom

Slope-Deflection Equations

Equilibrium Equations

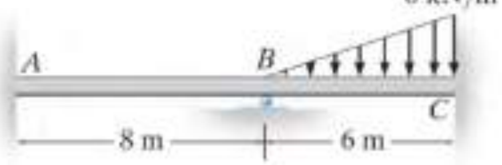
Example : Draw the shear and moment diagrams for the beam shown in the figure. EI is constant.

Solution :

Degrees of Freedom = 1

$$(\text{FEM})_{BC} = -\frac{wL^2}{30} = -\frac{6(6)^2}{30} = -7.2 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{20} = \frac{6(6)^2}{20} = 10.8 \text{ kN}\cdot\text{m}$$



Slope-Deflection Equations

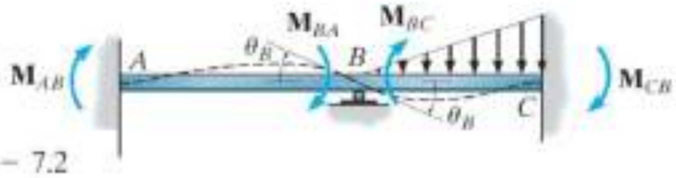
$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{8}\right)[2(0) + \theta_B - 3(0)] + 0 = \frac{EI}{4}\theta_B$$

$$M_{BA} = 2E\left(\frac{I}{8}\right)[2\theta_B + 0 - 3(0)] + 0 = \frac{EI}{2}\theta_B$$

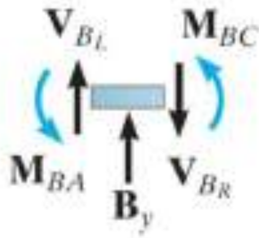
$$M_{BC} = 2E\left(\frac{I}{6}\right)[2\theta_B + 0 - 3(0)] - 7.2 = \frac{2EI}{3}\theta_B - 7.2$$

$$M_{CB} = 2E\left(\frac{I}{6}\right)[2(0) + \theta_B - 3(0)] + 10.8 = \frac{EI}{3}\theta_B + 10.8$$



Equilibrium Equations

$\circlearrowleft + \Sigma M_B = 0; \qquad M_{BA} + M_{BC} = 0$



$$\frac{EI}{2}\theta_B + \left(\frac{2EI}{3}\theta_B - 7.2\right) = 0 \Rightarrow \theta_B = \frac{6.17}{EI} \Rightarrow$$

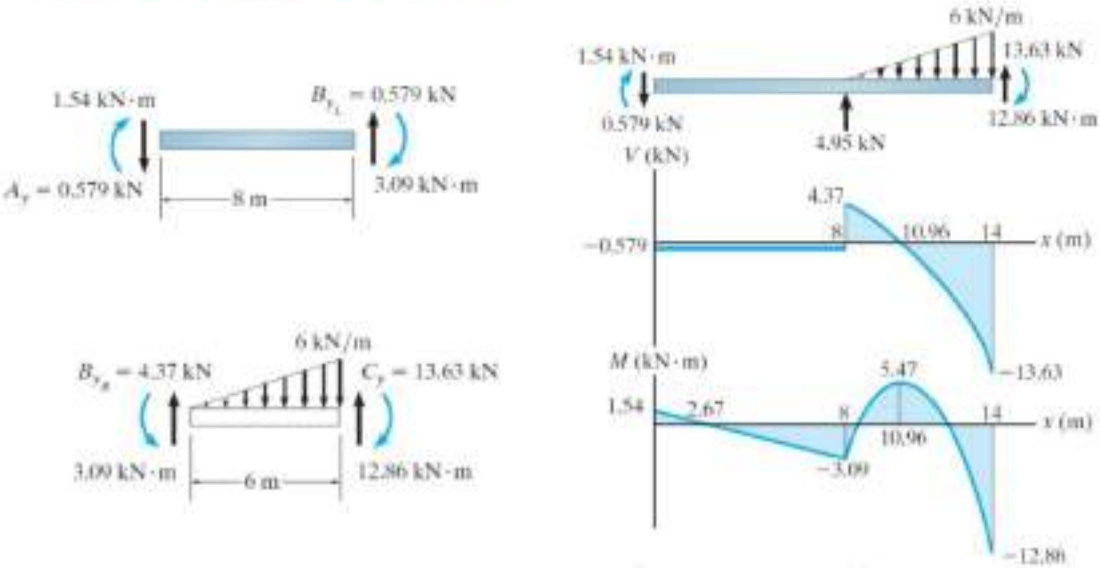
$M_{AB} = 1.54\ \text{kN} \cdot \text{m}$

$M_{BA} = 3.09\ \text{kN} \cdot \text{m}$

$M_{BC} = -3.09\ \text{kN} \cdot \text{m}$

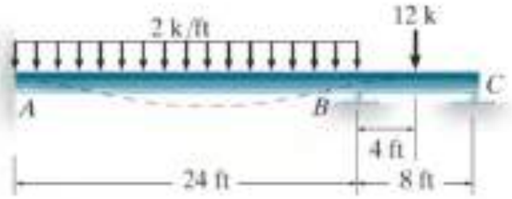
$M_{CB} = 12.86\ \text{kN} \cdot \text{m}$

Shear and Bending Moment Diagrams :



Example : Draw the shear and moment diagrams for the beam shown in the figure. *EI* is constant.

Solution :



$$(FEM)_{AB} = -\frac{wL^2}{12} = -\frac{1}{12}(2)(24)^2 = -96\ \text{k}\cdot\text{ft}$$

$$(FEM)_{BA} = \frac{wL^2}{12} = \frac{1}{12}(2)(24)^2 = 96\ \text{k}\cdot\text{ft}$$

$$(FEM)_{BC} = -\frac{3PL}{16} = -\frac{3(12)(8)}{16} = -18\ \text{k}\cdot\text{ft}$$

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (FEM)_N$$

$$M_{AB} = 2E\left(\frac{I}{24}\right)[2(0) + \theta_B - 3(0)] - 96$$

$$M_{AB} = 0.08333EI\theta_B - 96$$

$$M_{BA} = 2E\left(\frac{I}{24}\right)[2\theta_B + 0 - 3(0)] + 96$$

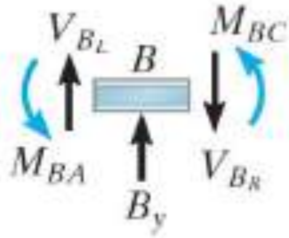
$$M_{BA} = 0.1667EI\theta_B + 96$$

$$M_N = 3E\left(\frac{I}{L}\right)(\theta_N - \psi) + (FEM)_N$$

$$M_{BC} = 3E\left(\frac{I}{8}\right)(\theta_B - 0) - 18$$

$$M_{BC} = 0.375EI\theta_B - 18$$

$\circlearrowleft + \Sigma M_B = 0;$ $\quad M_{BA} + M_{BC} = 0$

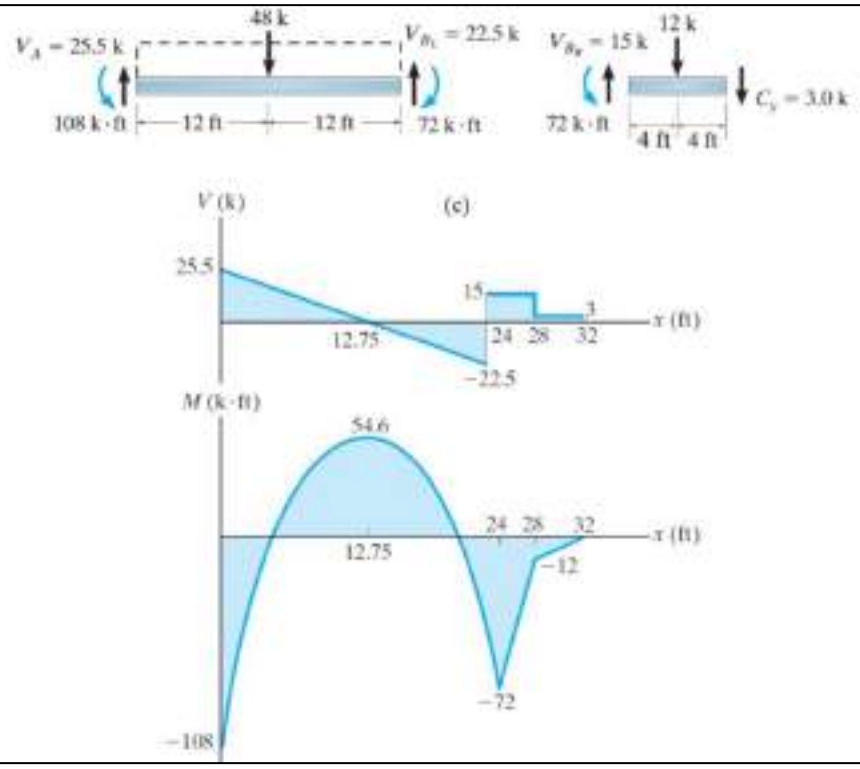


$$\theta_B = -\frac{144.0}{EI}$$

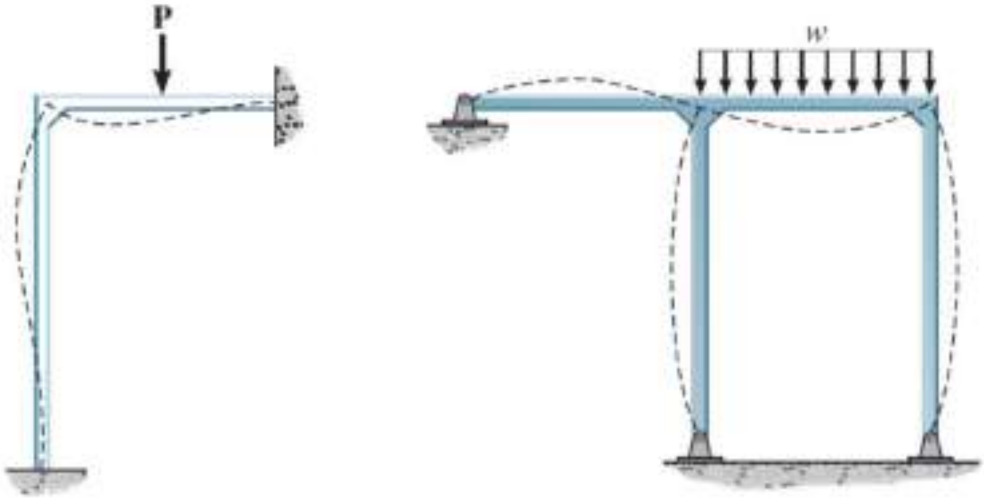
$$M_{AB} = -108.0\ \text{k}\cdot\text{ft}$$

$$M_{BA} = 72.0\ \text{k}\cdot\text{ft}$$

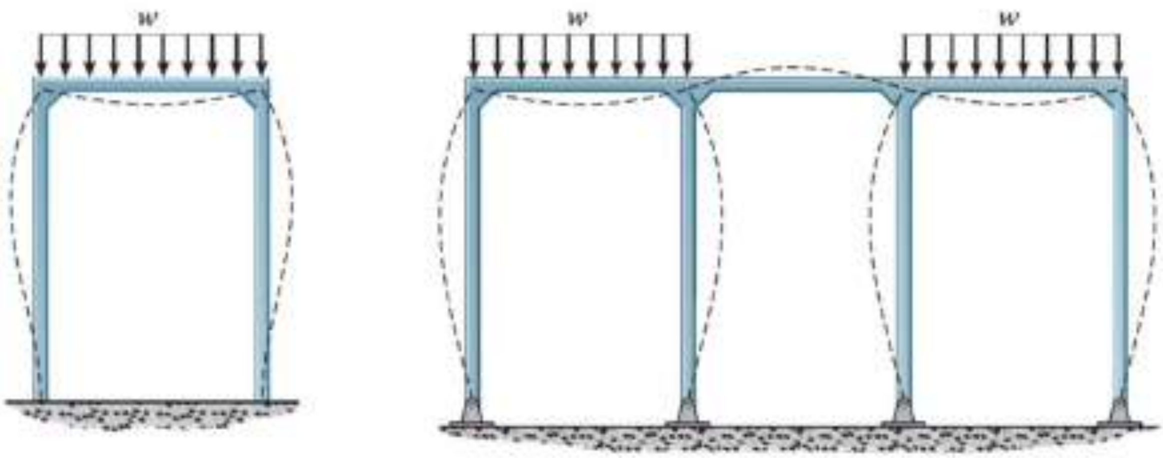
$$M_{BC} = -72.0\ \text{k}\cdot\text{ft}$$



Analysis of FRAMEs – No Sidesway

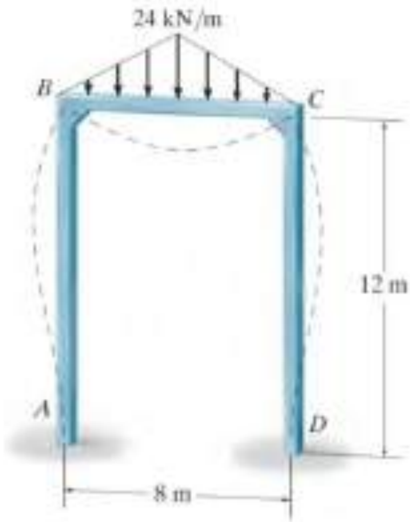


Symmetric Frames :



Example : Draw the shear and moment diagrams for the frame shown in the figure. *EI* is constant.

Solution :



$$(\text{FEM})_{BC} = -\frac{5wL^2}{96} = -\frac{5(24)(8)^2}{96} = -80 \text{ kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{5wL^2}{96} = \frac{5(24)(8)^2}{96} = 80 \text{ kN}\cdot\text{m}$$

Note that $\theta_A = \theta_D = 0$ and $\psi_{AB} = \psi_{BC} = \psi_{CD} = 0$, since no sidesway will occur.

$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2E\left(\frac{I}{12}\right)[2(0) + \theta_B - 3(0)] + 0$$

$$M_{AB} = 0.1667EI\theta_B$$

$$M_{BA} = 2E\left(\frac{I}{12}\right)[2\theta_B + 0 - 3(0)] + 0$$

$$M_{BA} = 0.333EI\theta_B$$

$$M_{BC} = 2E\left(\frac{I}{8}\right)[2\theta_B + \theta_C - 3(0)] - 80$$

$$M_{BC} = 0.5EI\theta_B + 0.25EI\theta_C - 80$$

$$M_{CB} = 2E\left(\frac{I}{8}\right)[2\theta_C + \theta_B - 3(0)] + 80$$

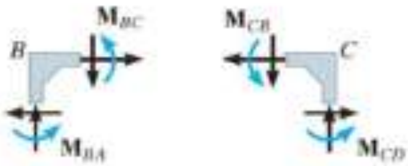
$$M_{CB} = 0.5EI\theta_C + 0.25EI\theta_B + 80$$

$$M_{CD} = 2E\left(\frac{I}{12}\right)[2\theta_C + 0 - 3(0)] + 0$$

$$M_{CD} = 0.333EI\theta_C$$

$$M_{DC} = 2E\left(\frac{I}{12}\right)[2(0) + \theta_C - 3(0)] + 0$$

$$M_{DC} = 0.1667EI\theta_C$$

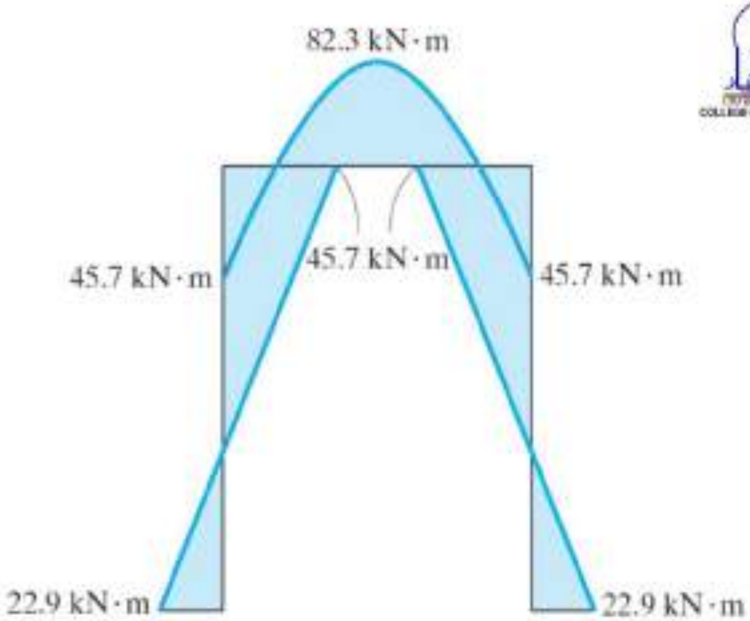


$$M_{BA} + M_{BC} = 0$$
$$M_{CB} + M_{CD} = 0$$

$$0.833EI\theta_B + 0.25EI\theta_C = 80$$
$$0.833EI\theta_C + 0.25EI\theta_B = -80$$

$$\theta_B = -\theta_C = \frac{137.1}{EI}$$

$$M_{BA} = 45.7 \text{ kN} \cdot \text{m}$$
$$M_{BC} = -45.7 \text{ kN} \cdot \text{m}$$
$$M_{CB} = 45.7 \text{ kN} \cdot \text{m}$$
$$M_{CD} = -45.7 \text{ kN} \cdot \text{m}$$
$$M_{DC} = -22.9 \text{ kN} \cdot \text{m}$$

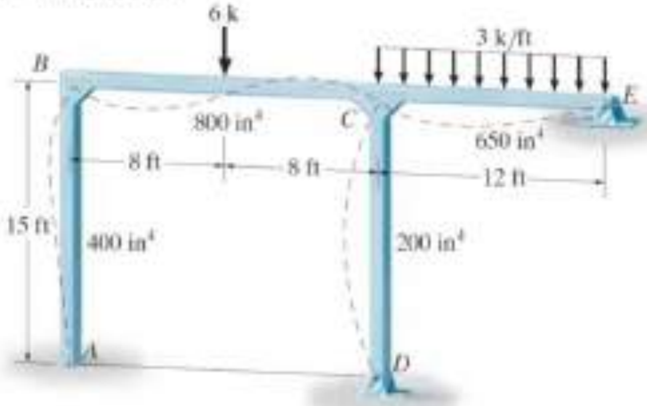


Example : Determine the internal moments at each joint of the frame shown in the figure. The moment of inertia for each member is given in the figure. Take E = 29(10³) ksi.

Solution :

$$k_{AB} = \frac{400}{15(12)^4} = 0.001286\ \text{ft}^3 \qquad k_{CD} = \frac{200}{15(12)^4} = 0.000643\ \text{ft}^3$$

$$k_{BC} = \frac{800}{16(12)^4} = 0.002411\ \text{ft}^3 \qquad k_{CE} = \frac{650}{12(12)^4} = 0.002612\ \text{ft}^3$$



$$(\text{FEM})_{BC} = -\frac{PL}{8} = -\frac{6(16)}{8} = -12\ \text{k}\cdot\text{ft}$$

$$(\text{FEM})_{CB} = \frac{PL}{8} = \frac{6(16)}{8} = 12\ \text{k}\cdot\text{ft}$$

$$(\text{FEM})_{CE} = -\frac{wL^2}{8} = -\frac{3(12)^2}{8} = -54\ \text{k}\cdot\text{ft}$$

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$$M_N = 2Ek(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

$$M_{AB} = 2[29(10^3)(12)^2](0.001286)[2(0) + \theta_B - 3(0)] + 0$$

$$M_{AB} = 10740.7\theta_B$$

$$M_{BA} = 2[29(10^3)(12)^2](0.001286)[2\theta_B + 0 - 3(0)] + 0$$

$$M_{BA} = 21\,481.5\theta_B$$

$$M_{BC} = 2[29(10^3)(12)^2](0.002411)[2\theta_B + \theta_C - 3(0)] - 12$$

$$M_{BC} = 40\,277.8\theta_B + 20\,138.9\theta_C - 12$$

$$M_{CB} = 2[29(10^3)(12)^2](0.002411)[2\theta_C + \theta_B - 3(0)] + 12$$

$$M_{CB} = 20\,138.9\theta_B + 40\,277.8\theta_C + 12$$

$$M_N = 3Ek(\theta_N - \psi) + (\text{FEM})_N$$

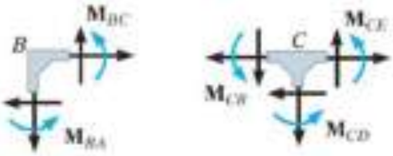
$$M_{CD} = 3[29(10^3)(12)^2](0.000643)[\theta_C - 0] + 0$$

$$M_{CD} = 8055.6\theta_C$$

$$M_{CE} = 3[29(10^3)(12)^2](0.002612)[\theta_C - 0] - 54$$

$$M_{CE} = 32\,725.7\theta_C - 54$$

28



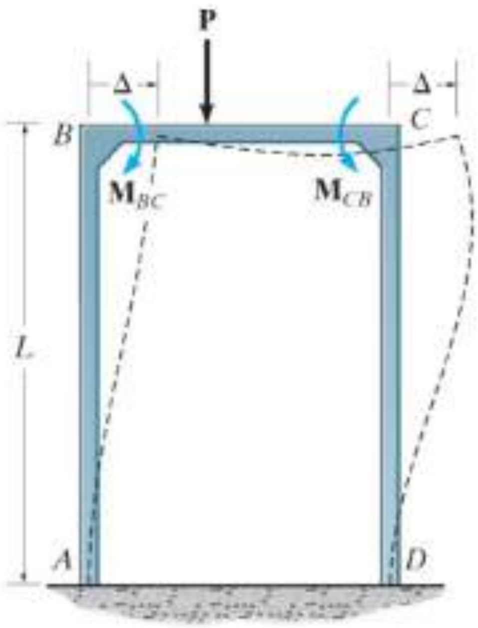
$$M_{BA} + M_{BC} = 0$$
$$M_{CB} + M_{CD} + M_{CE} = 0$$

$$61\,759.3\theta_B + 20\,138.9\theta_C = 12$$
$$20\,138.9\theta_B + 81\,059.0\theta_C = 42$$

$$\theta_B = 2.758(10^{-5})\text{ rad}$$
$$\theta_C = 5.113(10^{-4})\text{ rad}$$

$$M_{AB} = 0.296\text{ k}\cdot\text{ft}$$
$$M_{BA} = 0.592\text{ k}\cdot\text{ft}$$
$$M_{BC} = -0.592\text{ k}\cdot\text{ft}$$
$$M_{CB} = 33.1\text{ k}\cdot\text{ft}$$
$$M_{CD} = 4.12\text{ k}\cdot\text{ft}$$
$$M_{CE} = -37.3\text{ k}\cdot\text{ft}$$

Analysis of FRAMEs – Sidesway



Example : Determine the moments at each joint of the frame shown in the figure. EI is constant.

Solution :

$$M_{AB} = 2E\left(\frac{I}{12}\right)\left[2(0) + \theta_B - 3\left(\frac{18}{12}\psi_{DC}\right)\right] + 0 = EI(0.1667\theta_B - 0.75\psi_{DC})$$

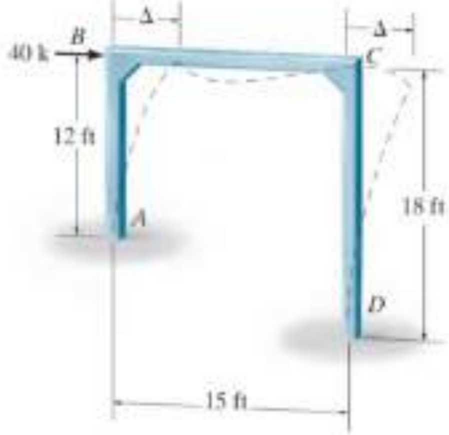
$$M_{BA} = 2E\left(\frac{I}{12}\right)\left[2\theta_B + 0 - 3\left(\frac{18}{12}\psi_{DC}\right)\right] + 0 = EI(0.333\theta_B - 0.75\psi_{DC})$$

$$M_{BC} = 2E\left(\frac{I}{15}\right)[2\theta_B + \theta_C - 3(0)] + 0 = EI(0.267\theta_B + 0.133\theta_C)$$

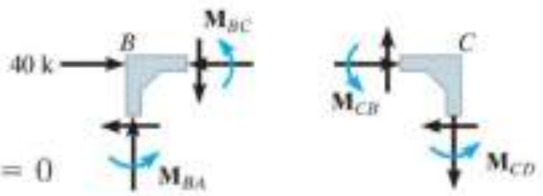
$$M_{CB} = 2E\left(\frac{I}{15}\right)[2\theta_C + \theta_B - 3(0)] + 0 = EI(0.267\theta_C + 0.133\theta_B)$$

$$M_{CD} = 2E\left(\frac{I}{18}\right)[2\theta_C + 0 - 3\psi_{DC}] + 0 = EI(0.222\theta_C - 0.333\psi_{DC})$$

$$M_{DC} = 2E\left(\frac{I}{18}\right)[2(0) + \theta_C - 3\psi_{DC}] + 0 = EI(0.111\theta_C - 0.333\psi_{DC})$$



$$M_{BA} + M_{BC} = 0$$
$$M_{CB} + M_{CD} = 0$$



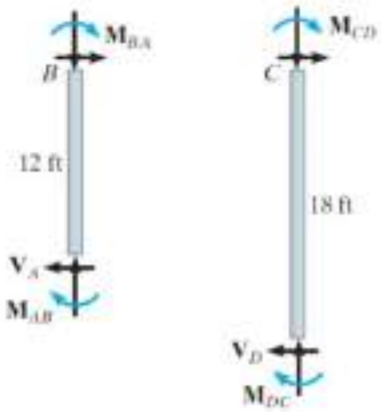
$\overset{+}{\rightarrow} \Sigma F_x = 0;$ $\quad 40 - V_A - V_D = 0$

$\Sigma M_B = 0;$ $\quad V_A = -\dfrac{M_{AB} + M_{BA}}{12}$

$\Sigma M_C = 0;$ $\quad V_D = -\dfrac{M_{DC} + M_{CD}}{18}$

Thus,

$$40 + \frac{M_{AB} + M_{BA}}{12} + \frac{M_{DC} + M_{CD}}{18} = 0$$



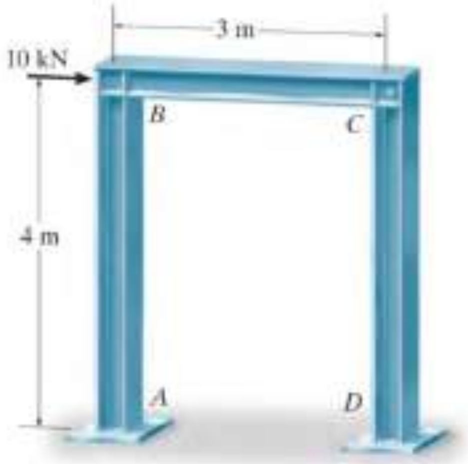
$$0.6\theta_B + 0.133\theta_C - 0.75\psi_{DC} = 0$$
$$0.133\theta_B + 0.489\theta_C - 0.333\psi_{DC} = 0$$
$$0.5\theta_B + 0.222\theta_C - 1.944\psi_{DC} = -\frac{480}{EI}$$

$EI\theta_B = 438.81 \qquad EI\theta_C = 136.18 \qquad EI\psi_{DC} = 375.26$

$$M_{AB} = -208 \text{ k}\cdot\text{ft}$$
$$M_{BA} = -135 \text{ k}\cdot\text{ft}$$
$$M_{BC} = 135 \text{ k}\cdot\text{ft}$$
$$M_{CB} = 94.8 \text{ k}\cdot\text{ft}$$
$$M_{CD} = -94.8 \text{ k}\cdot\text{ft}$$
$$M_{DC} = -110 \text{ k}\cdot\text{ft}$$

Example : Determine the moments at each joint of the frame shown in the figure. The supports at **A** and **D** are fixed and joint **C** is assumed pin connected. EI is constant for each member.

Solution :



Slope-Deflection Equations :

$$M_N = 2E\left(\frac{I}{L}\right)(2\theta_N + \theta_F - 3\psi) + (\text{FEM})_N$$

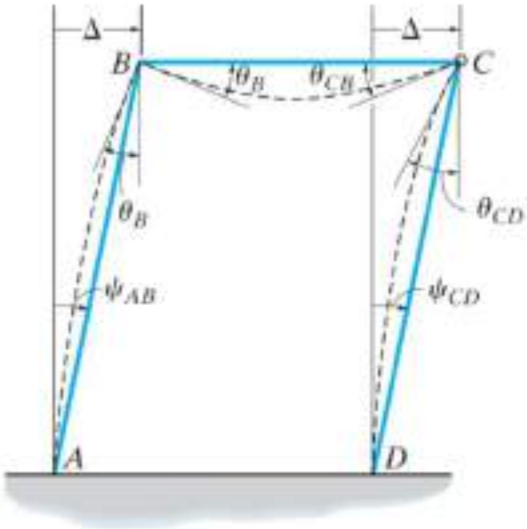
$$M_{AB} = 2E\left(\frac{I}{4}\right)[2(0) + \theta_B - 3\psi] + 0$$

$$M_{BA} = 2E\left(\frac{I}{4}\right)(2\theta_B + 0 - 3\psi) + 0$$

$$M_N = 3E\left(\frac{I}{L}\right)(\theta_N - \psi) + (\text{FEM})_N$$

$$M_{BC} = 3E\left(\frac{I}{3}\right)(\theta_B - 0) + 0$$

$$M_{DC} = 3E\left(\frac{I}{4}\right)(0 - \psi) + 0$$



Equilibrium Equations :

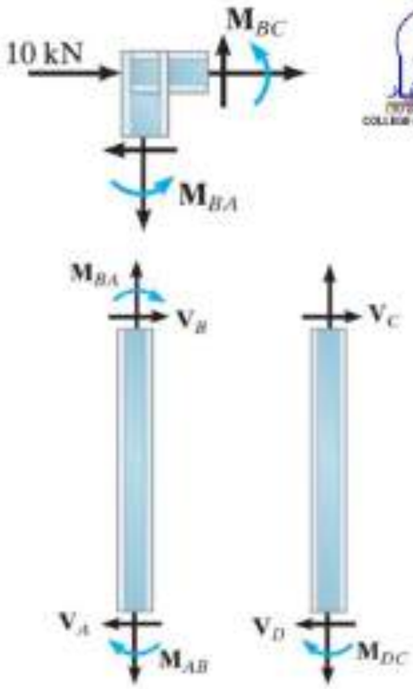
$$M_{BA} + M_{BC} = 0$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 10 - V_A - V_D = 0$$

$$\Sigma M_B = 0; \qquad V_A = -\frac{M_{AB} + M_{BA}}{4}$$

$$\Sigma M_C = 0; \qquad V_D = -\frac{M_{DC}}{4}$$

$$10 + \frac{M_{AB} + M_{BA}}{4} + \frac{M_{DC}}{4} = 0$$



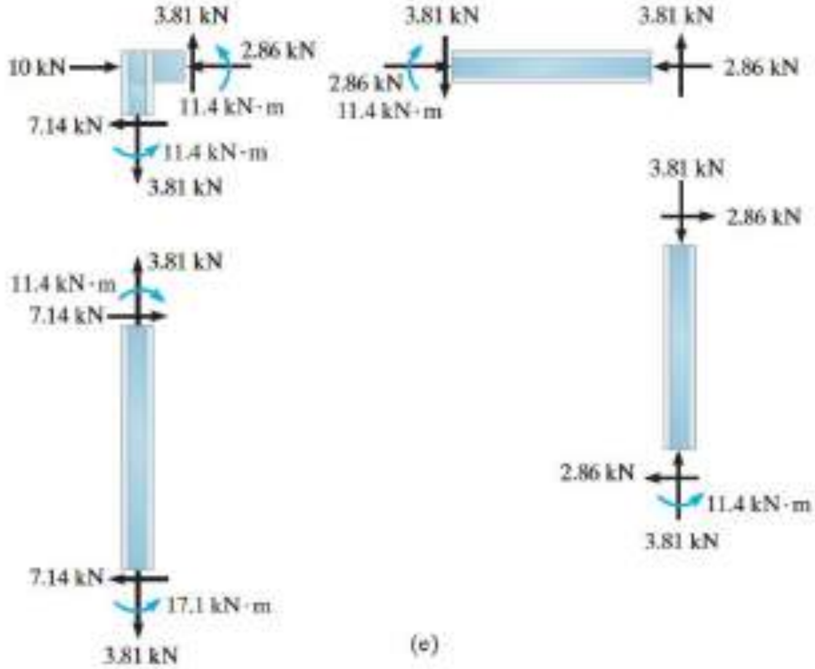
Substituting :

$$\theta_B = \frac{3}{4}\psi$$

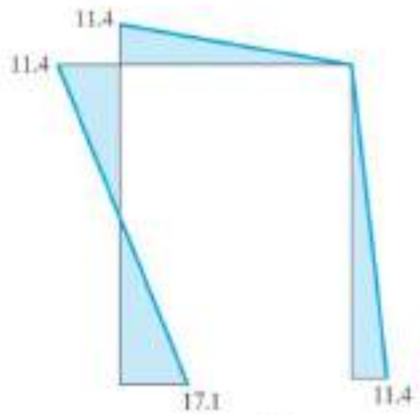
$$10 + \frac{EI}{4}\left(\frac{3}{2}\theta_B - \frac{15}{4}\psi\right) = 0 \quad \Rightarrow \quad \theta_B = \frac{240}{21EI} \qquad \psi = \frac{320}{21EI}$$

$$M_{AB} = -17.1 \text{ kN}\cdot\text{m}, \quad M_{BA} = -11.4 \text{ kN}\cdot\text{m}$$

$$M_{BC} = 11.4 \text{ kN}\cdot\text{m}, \quad M_{DC} = -11.4 \text{ kN}\cdot\text{m}$$



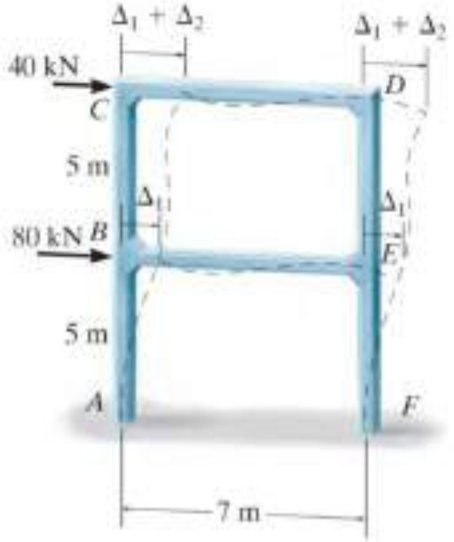
(e)



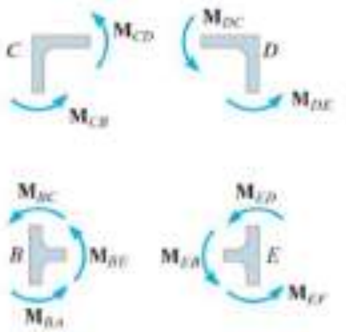
(f)

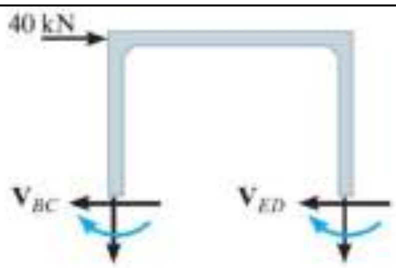
Example : Explain how the moments in each joint of the two-story frame shown in the figure are determined. EI is constant.

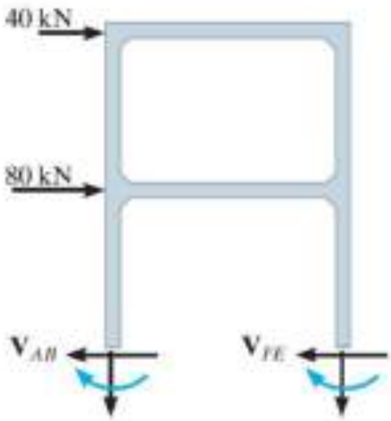
Solution :



$$M_{AB} = 2E\left(\frac{I}{5}\right)[2(0) + \theta_B - 3\psi_1] + 0$$
$$M_{BA} = 2E\left(\frac{I}{5}\right)[2\theta_B + 0 - 3\psi_1] + 0$$
$$M_{BC} = 2E\left(\frac{I}{5}\right)[2\theta_B + \theta_C - 3\psi_2] + 0$$
$$M_{CB} = 2E\left(\frac{I}{5}\right)[2\theta_C + \theta_B - 3\psi_2] + 0$$
$$M_{CD} = 2E\left(\frac{I}{7}\right)[2\theta_C + \theta_D - 3(0)] + 0$$
$$M_{DC} = 2E\left(\frac{I}{7}\right)[2\theta_D + \theta_C - 3(0)] + 0$$
$$M_{BE} = 2E\left(\frac{I}{7}\right)[2\theta_B + \theta_E - 3(0)] + 0$$
$$M_{EB} = 2E\left(\frac{I}{7}\right)[2\theta_E + \theta_B - 3(0)] + 0$$
$$M_{ED} = 2E\left(\frac{I}{5}\right)[2\theta_E + \theta_D - 3\psi_2] + 0$$
$$M_{DE} = 2E\left(\frac{I}{5}\right)[2\theta_D + \theta_E - 3\psi_2] + 0$$
$$M_{FE} = 2E\left(\frac{I}{5}\right)[2(0) + \theta_E - 3\psi_1] + 0$$
$$M_{EF} = 2E\left(\frac{I}{5}\right)[2\theta_E + 0 - 3\psi_1] + 0$$







$$M_{BA} + M_{BE} + M_{BC} = 0$$

$$M_{CB} + M_{CD} = 0$$

$$M_{DC} + M_{DE} = 0$$

$$M_{EF} + M_{EB} + M_{ED} = 0$$

$\xrightarrow{+} \Sigma F_x = 0;\qquad 40 - V_{BC} - V_{ED} = 0$

$$40 + \frac{M_{BC} + M_{CB}}{5} + \frac{M_{ED} + M_{DE}}{5} = 0$$

$\xrightarrow{+} \Sigma F_x = 0;\qquad 40 + 80 - V_{AB} - V_{FE} = 0$

$$120 + \frac{M_{AB} + M_{BA}}{5} + \frac{M_{EF} + M_{FE}}{5} = 0$$

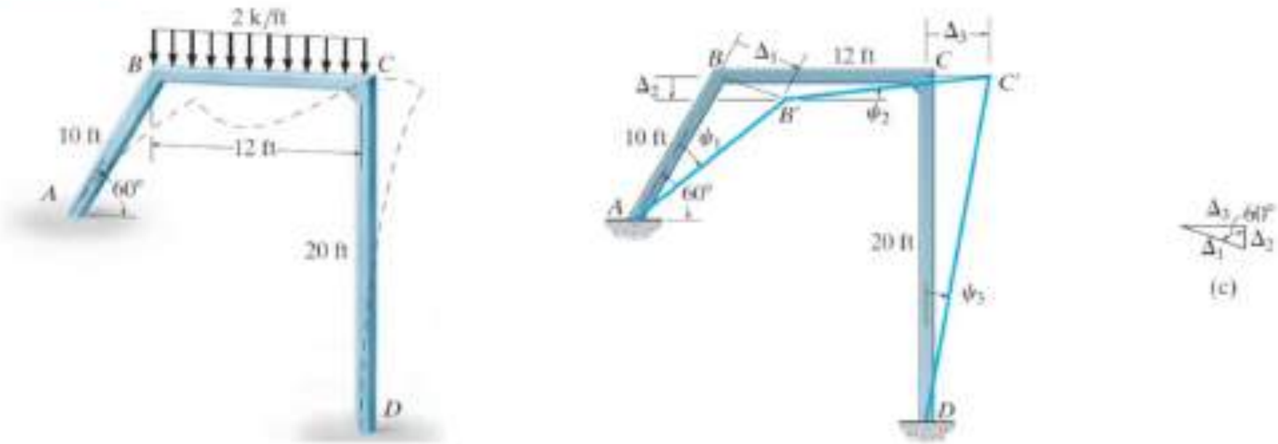
Substituting the 12 slope-deflection equations in these 6 equilibrium equations will lead to a system of 6-equatins 6-unknowns which can be solve algebraically to find :

$$\psi_1, \psi_2, \theta_B, \theta_C, \theta_D, \text{ and } \theta_E$$

Then Moments can be found and drawn

Example : Determine the moments at each joint of the frame shown in the figure. EI is constant for each member.

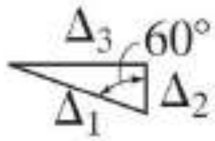
Solution :



$$(FEM)_{BC} = -\frac{wL^2}{12} = -\frac{2(12)^2}{12} = -24\ \text{k}\cdot\text{ft}$$

$$(FEM)_{CB} = \frac{wL^2}{12} = \frac{2(12)^2}{12} = 24\ \text{k}\cdot\text{ft}$$

$$\psi_1 = \frac{\Delta_1}{10} \qquad \psi_2 = -\frac{\Delta_2}{12} \qquad \psi_3 = \frac{\Delta_3}{20}$$



But: $\Delta_2 = 0.5\Delta_1$ and $\Delta_3 = 0.866\Delta_1$

$$\psi_2 = -0.417\psi_1 \qquad \psi_3 = 0.433\psi_1$$

$$M_{AB} = 2E\left(\frac{I}{10}\right)[2(0) + \theta_B - 3\psi_1] + 0 \quad (1)$$

$$M_{BA} = 2E\left(\frac{I}{10}\right)[2\theta_B + 0 - 3\psi_1] + 0 \quad (2)$$

$$M_{BC} = 2E\left(\frac{I}{12}\right)[2\theta_B + \theta_C - 3(-0.417\psi_1)] - 24 \quad (3)$$

$$M_{CB} = 2E\left(\frac{I}{12}\right)[2\theta_C + \theta_B - 3(-0.417\psi_1)] + 24 \quad (4)$$

$$M_{CD} = 2E\left(\frac{I}{20}\right)[2\theta_C + 0 - 3(0.433\psi_1)] + 0 \quad (5)$$

$$M_{DC} = 2E\left(\frac{I}{20}\right)[2(0) + \theta_C - 3(0.433\psi_1)] + 0 \quad (6)$$

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$$M_{BA} + M_{BC} = 0 \quad (7)$$

$$M_{CD} + M_{CB} = 0 \quad (8)$$

$\circlearrowleft + \Sigma M_O = 0;$

$$M_{AB} + M_{DC} - \left(\frac{M_{AB} + M_{BA}}{10}\right)(34) - \left(\frac{M_{DC} + M_{CD}}{20}\right)(40.78) - 24(6) = 0$$

$$-2.4M_{AB} - 3.4M_{BA} - 2.04M_{CD} - 1.04M_{DC} - 144 = 0 \quad (9)$$

$$0.733\theta_B + 0.167\theta_C - 0.392\psi_1 = \frac{24}{EI}$$

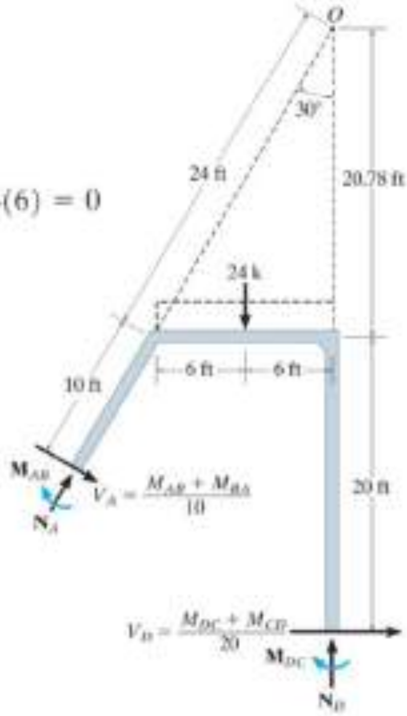
$$0.167\theta_B + 0.533\theta_C + 0.0784\psi_1 = -\frac{24}{EI}$$

$$-1.8400\theta_B - 0.512\theta_C + 3.880\psi_1 = \frac{144}{EI}$$

$$EI\theta_B = 87.67 \qquad EI\theta_C = -82.3 \qquad EI\psi_1 = 67.83$$

$M_{AB} = -23.2\ \text{k}\cdot\text{ft}$ $\quad M_{BC} = 5.63\ \text{k}\cdot\text{ft}$ $\quad M_{CD} = -25.3\ \text{k}\cdot\text{ft}$

$M_{BA} = -5.63\ \text{k}\cdot\text{ft}$ $\quad M_{CB} = 25.3\ \text{k}\cdot\text{ft}$ $\quad M_{DC} = -17.0\ \text{k}\cdot\text{ft}$



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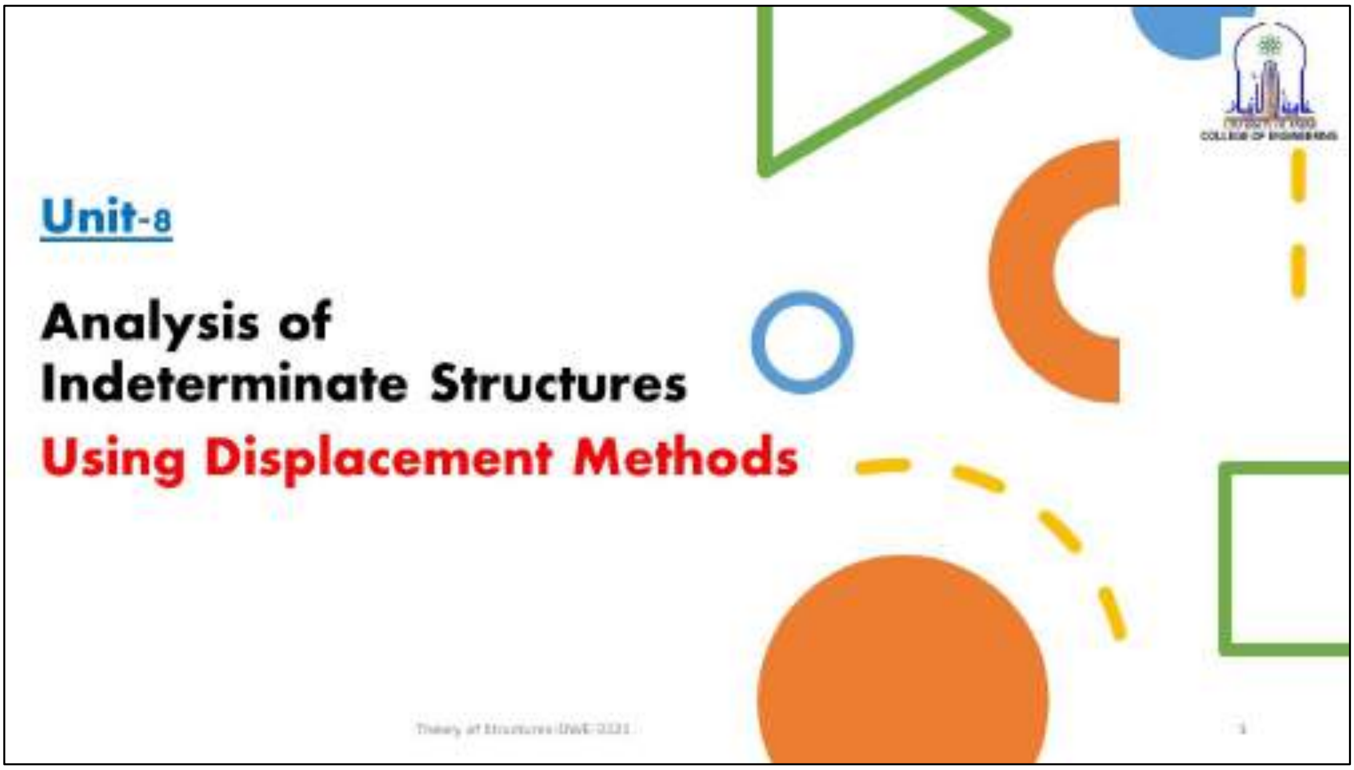
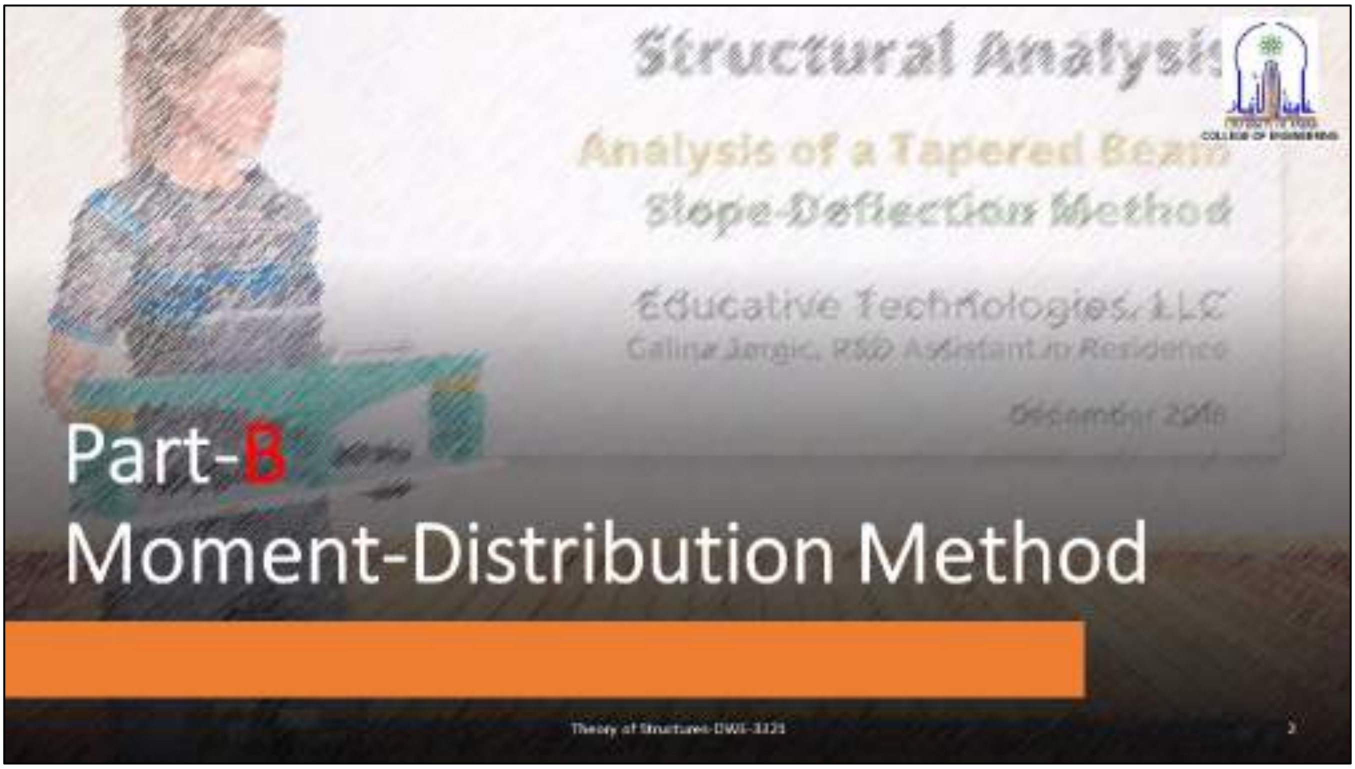
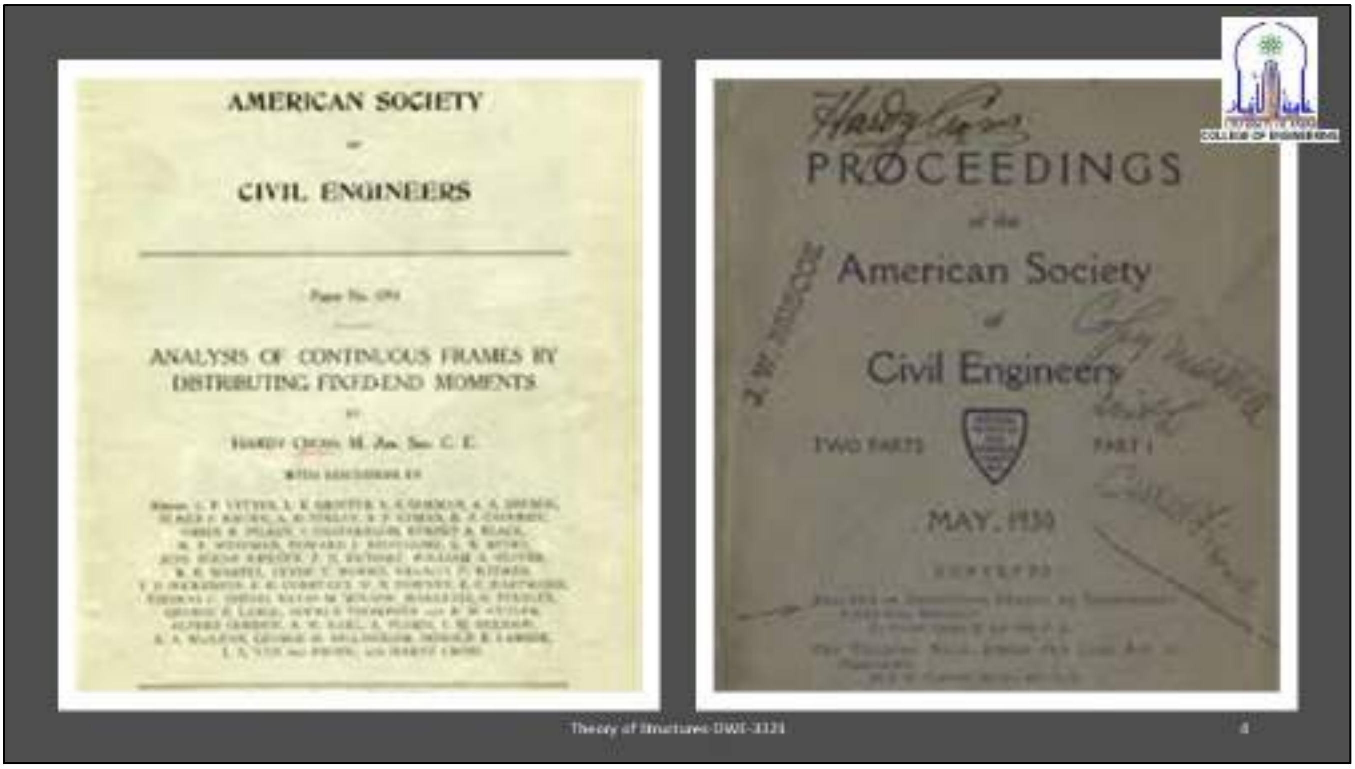

Unit-8
Analysis of
Indeterminate Structures
Using Displacement Methods

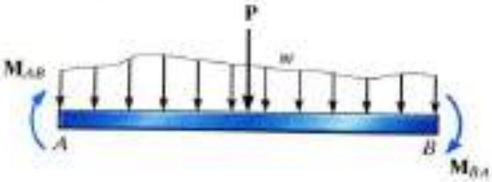

Structural Analysis
Analysis of a Tapered Beam
Slope-Deflection Method
Educative Technologies, LLC
Part-B
Moment-Distribution Method

Moment-Distribution Method

The method of analysing beams and frames using moment distribution was developed by Hardy Cross, in 1930. At the time this method was first published it attracted immediate attention, and it has been recognized as one of the most notable advances in structural analysis during the twentieth century.

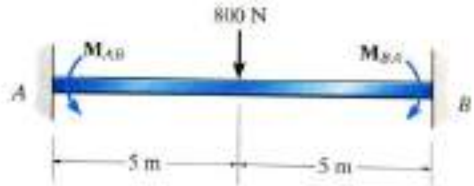


Sign Convention :



Clockwise moments that act on the member are considered ***positive***, whereas ***counterclockwise*** moments are ***negative***

Fixed-End Moments (FEMs) :

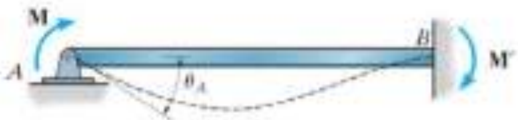


FEM = PL/8 = 800(10)/8 = 1000 N.m.

Noting the action of these moments on the beam and applying our sign convention, it is seen that

$M_{AB} = -1000$ N.m $\quad M_{BA} = +1000$ N.m

Member Stiffness Factor :



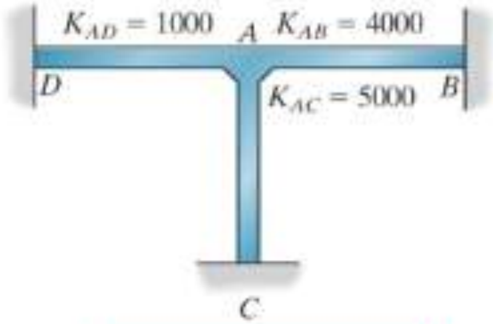
$$M = (4EI/L)\,\theta_A$$

$$K = \frac{4EI}{L}$$

Far End Fixed

K is referred to as the stiffness factor at ***A*** and can be defined as the amount of moment ***M*** required to rotate the end ***A*** of the beam $\theta_A = 1$ rad.

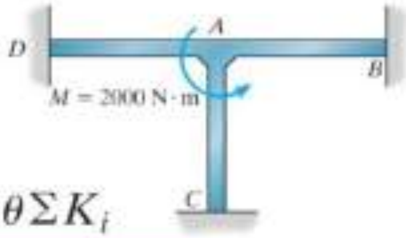
Joint Stiffness Factor :



$$K_T = \Sigma K$$

$$K_T = \Sigma K = 4000 + 5000 + 1000 = 10\,000.$$

Distribution Factor (DF) :

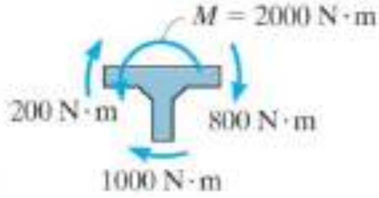


$$M_i = K_i\theta$$

$$M = M_1 + M_n = K_1\theta + K_n\theta = \theta\Sigma K_i$$

$$\mathrm{DF}_i = \frac{M_i}{M} = \frac{K_i\theta}{\theta\Sigma K_i}$$

$$\mathrm{DF} = \frac{K}{\Sigma K}$$



$\mathrm{DF}_{AB} = 4000/10\,000 = 0.4 \quad M_{AB} = 0.4(2000) = 800\ \mathrm{N\cdot m}$

$\mathrm{DF}_{AC} = 5000/10\,000 = 0.5 \quad M_{AC} = 0.5(2000) = 1000\ \mathrm{N\cdot m}$

$\mathrm{DF}_{AD} = 1000/10\,000 = 0.1 \quad M_{AD} = 0.1(2000) = 200\ \mathrm{N\cdot m}$

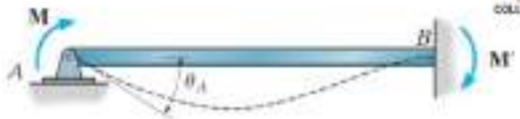
Member Relative Stiffness Factor :

Most of the time E is identical for all members, so it can be omitted from the equation :

$$K_R = \frac{I}{L}$$

Far End Fixed

Carry-Over Factor :



$$M_{AB} = (4EI/L)\,\theta_A$$

$$M_{BA} = (2EI/L)\,\theta_A$$

Solving for θ_A and equating the equations leads to the fact that :

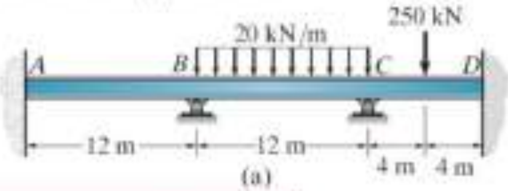
$$M_{BA} = M_{AB}/2$$

$$\mathbf{M'} = \boxed{\tfrac{1}{2}}\mathbf{M}$$

COF

Example : Determine the internal moments at each support of the beam shown in the figure. EI is constant.

Solution :



(a)

$$K_{AB} = \frac{4EI}{12} \qquad K_{BC} = \frac{4EI}{12} \qquad K_{CD} = \frac{4EI}{8}$$

Therefore,

$$DF_{AB} = DF_{DC} = 0 \qquad DF_{BA} = DF_{BC} = \frac{4EI/12}{4EI/12 + 4EI/12} = 0.5$$

$$DF_{CB} = \frac{4EI/12}{4EI/12 + 4EI/8} = 0.4 \quad DF_{CD} = \frac{4EI/8}{4EI/12 + 4EI/8} = 0.6$$

The fixed-end moments are

$$(FEM)_{BC} = -\frac{wL^2}{12} = \frac{-20(12)^2}{12} = -240 \text{ kN}\cdot\text{m} \qquad (FEM)_{CB} = \frac{wL^2}{12} = \frac{20(12)^2}{12} = 240 \text{ kN}\cdot\text{m}$$

$$(FEM)_{CD} = -\frac{PL}{8} = \frac{-250(8)}{8} = -250 \text{ kN}\cdot\text{m} \qquad (FEM)_{DC} = \frac{PL}{8} = \frac{250(8)}{8} = 250 \text{ kN}\cdot\text{m}$$

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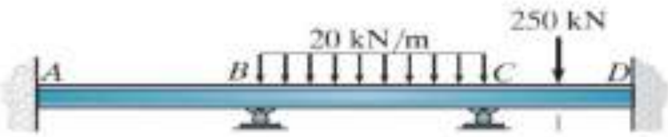

Joint	A	B		C		D	1
Member	AB	BA	BC	CB	CD	DC	2
DF	0	0.5	0.5	0.4	0.6	0	3
FEM			−240	240	−250	250	4
Dist.		120	120	4	6		5
CO	60		2	60		3	6
Dist.		−1	−1	−24	−36		7
CO	−0.5		−12	−0.5		−18	8
Dist.		6	6	0.2	0.3		9
CO	3		0.1	3		0.2	10
Dist.		−0.05	−0.05	−1.2	−1.8		11
CO	−0.02		−0.6	−0.02		−0.9	12
Dist.		0.3	0.3	0.01	0.01		13
ΣM	62.5	125.2	−125.2	281.5	−281.5	234.3	14

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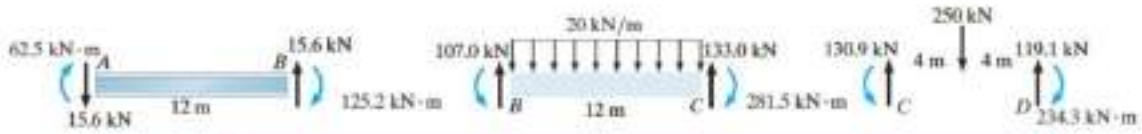
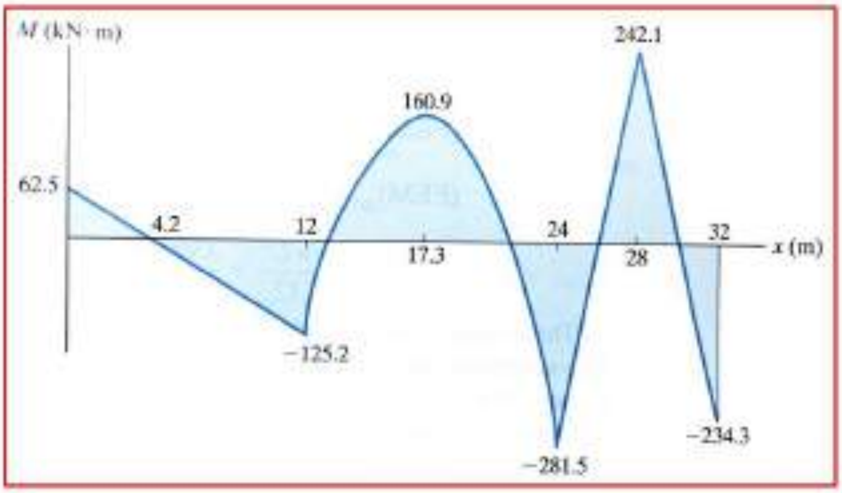




Example : Determine the internal moments at each support of the beam shown in the figure. *EI* is constant and The moment of inertia of each span is indicated

Solution :

$$K_{BC} = \frac{4E(750)}{20} = 150E \qquad K_{CD} = \frac{4E(600)}{15} = 160E$$

$$\text{DF}_{BC} = 1 - (\text{DF})_{BA} = 1 - 0 = 1$$

$$\text{DF}_{CB} = \frac{150E}{150E + 160E} = 0.484$$

$$\text{DF}_{CD} = \frac{160E}{150E + 160E} = 0.516$$

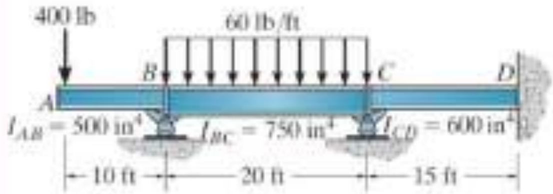
$$\text{DF}_{DC} = \frac{160E}{\infty + 160E} = 0$$

Due to the overhang,

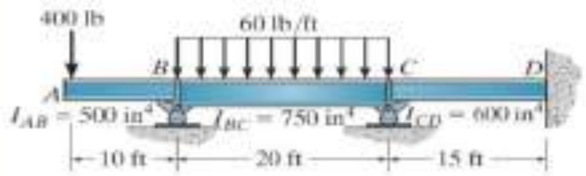
$$(\text{FEM})_{BA} = 400\text{ lb}(10\text{ ft}) = 4000\text{ lb}\cdot\text{ft}$$

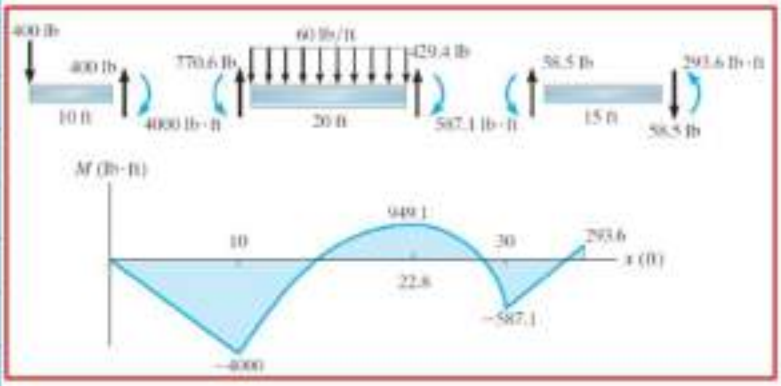
$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{60(20)^2}{12} = -2000\text{ lb}\cdot\text{ft}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{60(20)^2}{12} = 2000\text{ lb}\cdot\text{ft}$$



Joint	B		C		D
Member		BC	CB	CD	DC
DF	0	1	0.484	0.516	0
FEM	4000	−2000	2000		
Dist.		−2000	−968	−1032	
CO		−484	−1000		−516
Dist.		484	484	516	
CO		242	242		258
Dist.		−242	−117.1	−124.9	
CO		−58.6	−121		−62.4
Dist.		58.6	58.6	62.4	
CO		29.3	29.3		31.2
Dist.		−29.3	−14.2	−15.1	
CO		−7.1	−14.6		−7.6
Dist.		7.1	7.1	7.6	
CO		3.5	3.5		3.8
Dist.		−3.5	−1.7	−1.8	
CO		−0.8	−1.8		−0.9
Dist.		0.8	0.9	0.9	
CO		0.4	0.4		0.4
Dist.		−0.4	−0.2	−0.2	
CO		−0.1	−0.2		−0.1
Dist.		0.1	0.1	0.1	
ΣM	4000	−4000	587.1	−587.1	−293.6

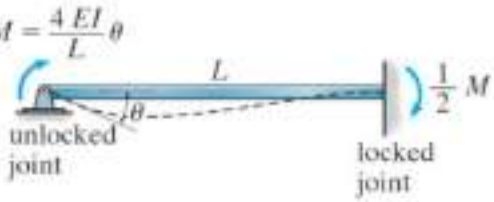




Stiffness Factor Modifications :

$M = \frac{4EI}{L}\theta$

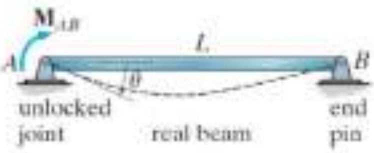
Typical Scenario ! ➡



Member Pin Supported at Far End :

$$\circlearrowleft + \Sigma M_{B'} = 0; \quad V'_A(L) - \frac{1}{2}\left(\frac{M}{EI}\right)L\left(\frac{2}{3}L\right) = 0$$

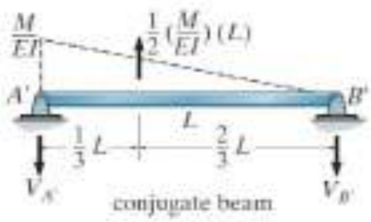
$$V'_A = \theta = \frac{ML}{3EI} \Rightarrow \boxed{M = \frac{3EI}{L}\theta}$$



$$K = \frac{3EI}{L}$$

Far End Pinned
or Roller Supported

∴ **k** would have to be modified by ¾ to model the case of having the far end pin connected.



Symmetric Beam and Loading :

$$\circlearrowleft + \Sigma M_{C'} = 0; \quad -V_{B'}(L) + \frac{M}{EI}(L)\left(\frac{L}{2}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{2EI}$$

or

$$M = \frac{2EI}{L}\theta$$

$$K = \frac{2EI}{L}$$

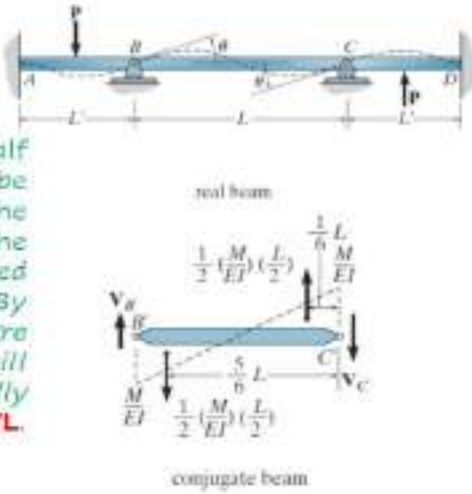
Symmetric Beam and Loading

∴ Thus, moments for only half the beam can be distributed provided the stiffness factor for the centre span is computed using **K = 2EI/L** . By comparison, the centre span's stiffness factor will be one **1/2** that usually determined using **K = 4EI/L** .

Symmetric Beam with Antisymmetric Loading :

$$\circlearrowleft + \Sigma M_{C'} = 0; \; -V_{B'}(L) + \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{5L}{6}\right) - \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{6}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{6EI} \quad \Rightarrow \quad M = \frac{6EI}{L}\theta$$

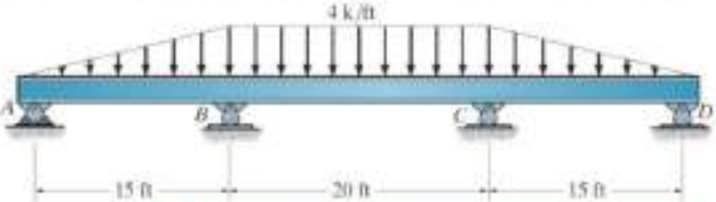


$$K = \frac{6EI}{L}$$

Symmetric Beam with Antisymmetric Loading

∴ Thus, moments for only half the beam can be distributed provided the stiffness factor for the centre span is computed using **K = 6EI/L** . By comparison, the centre span's stiffness factor will be one **1.5** that usually determined using **K = 4EI/L**.

Example : Determine the internal moments at each support of the beam shown in the figure. *EI* is constant.



Solution :

$$K_{AB} = \frac{3EI}{15}$$

$$K_{BC} = \frac{2EI}{20}$$

$$\mathrm{DF}_{AB} = \frac{3EI/15}{3EI/15} = 1$$

$$\mathrm{DF}_{BA} = \frac{3EI/15}{3EI/15 + 2EI/20} = 0.667$$

$$\mathrm{DF}_{BC} = \frac{2EI/20}{3EI/15 + 2EI/20} = 0.333$$

$$(\mathrm{FEM})_{BA} = \frac{wL^2}{15} = \frac{4(15)^2}{15} = 60\ \mathrm{k \cdot ft}$$

$$(\mathrm{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{4(20)^2}{12} = -133.3\ \mathrm{k \cdot ft}$$

Joint	*A*	*B*	
Member	*AB*	*BA*	*BC*
DF	1	0.667	0.333
FEM Dist.		60 48.9	−133.3 24.4
ΣM	0	108.9	−108.9

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Example : Determine the internal moments at each support of the beam shown in the figure. *The moments of inertia for the two spans are indicated.*

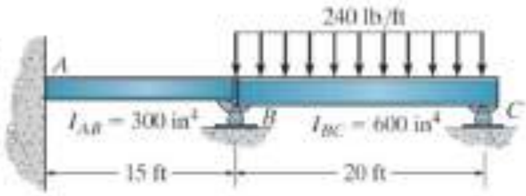
Solution :

$$K_{AB} = \frac{4EI}{L} = \frac{4E(300)}{15} = 80E \qquad \mathrm{DF}_{AB} = \frac{80E}{\infty + 80E} = 0$$

$$K_{BC} = \frac{3EI}{L} = \frac{3E(600)}{20} = 90E \qquad \mathrm{DF}_{BA} = \frac{80E}{80E + 90E} = 0.4706$$

$$\mathrm{DF}_{BC} = \frac{90E}{80E + 90E} = 0.5294$$

$$\mathrm{DF}_{CB} = \frac{90E}{90E} = 1$$



$$(\mathrm{FEM})_{BC} = -\frac{wL^2}{8} = \frac{-240(20)^2}{8} = -12\,000\ \mathrm{lb \cdot ft}$$

Joint	*A*	*B*		*C*
Member	*AB*	*BA*	*BC*	*CB*
DF	0	0.4706	0.5294	1
FEM Dist.		 5647.2	−12 000 6352.8	
CO	2823.6			
ΣM	2823.6	5647.2	−5647.2	0

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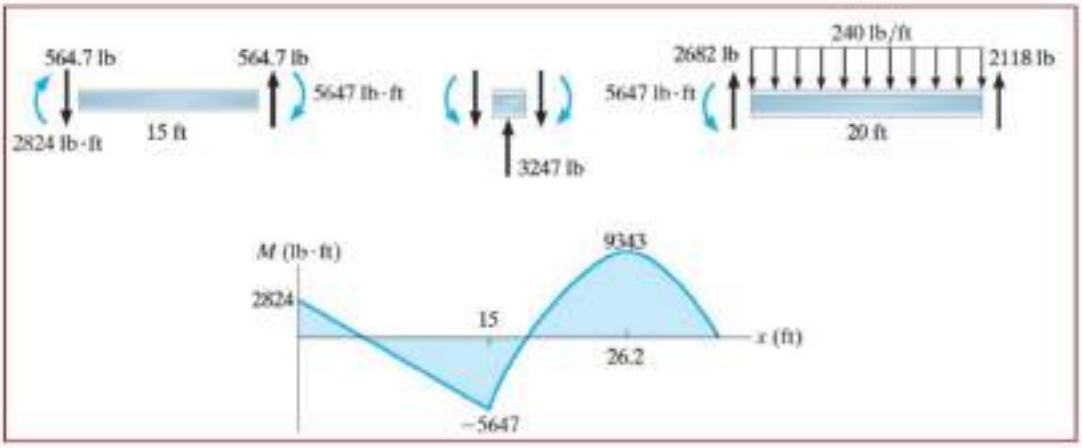


Moment Distribution for Frames: NO SIDESWAY

Example : Determine the internal moments at the joints of the frame shown in the figure. *There is a pin at **E** and **D** and a fixed support at **A**. EI is constant.*

Solution :

$K_{AB} = \frac{4EI}{15} \quad K_{BC} = \frac{4EI}{18} \quad K_{CD} = \frac{3EI}{15} \quad K_{CE} = \frac{3EI}{12}$

$DF_{AB} = 0$

$DF_{BA} = \frac{4EI/15}{4EI/15 + 4EI/18} = 0.545$

$DF_{BC} = 1 - 0.545 = 0.455$

$DF_{CB} = \frac{4EI/18}{4EI/18 + 3EI/15 + 3EI/12} = 0.330$

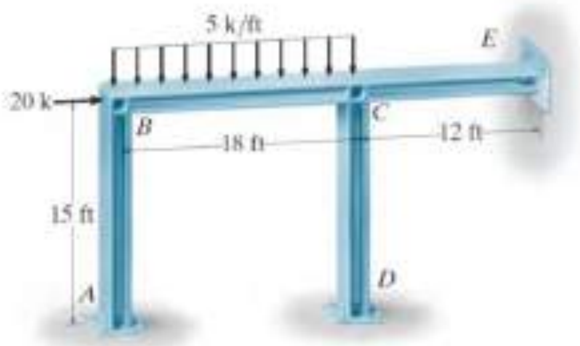
$DF_{CD} = \frac{3EI/15}{4EI/18 + 3EI/15 + 3EI/12} = 0.298$

$DF_{CE} = 1 - 0.330 - 0.298 = 0.372$

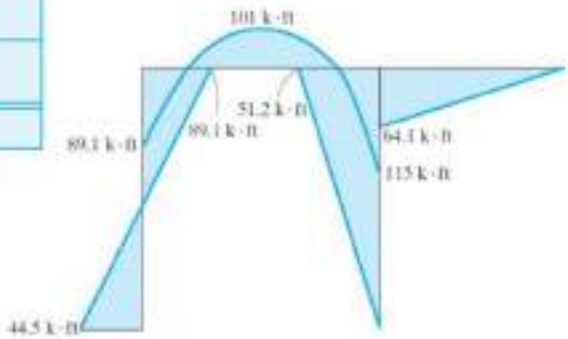
$DF_{DC} = 1 \qquad DF_{EC} = 1$

$(FEM)_{BC} = \frac{-wL^2}{12} = \frac{-5(18)^2}{12} = -135\ \text{k}\cdot\text{ft}$

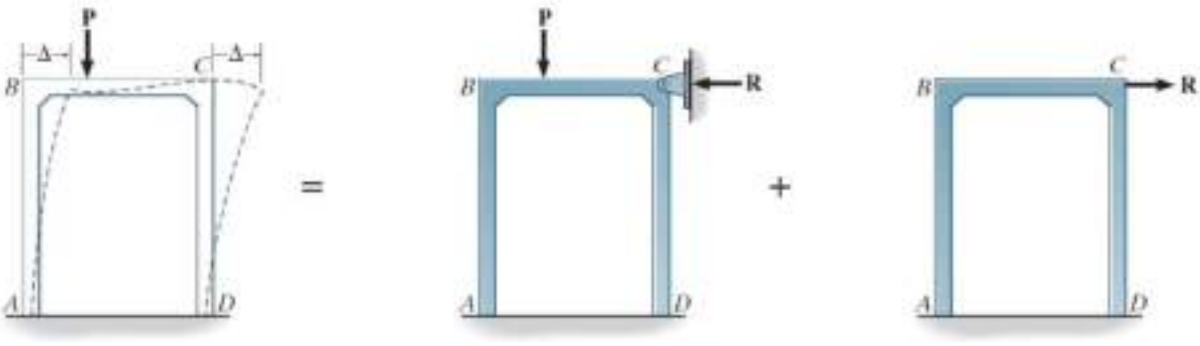
$(FEM)_{CB} = \frac{wL^2}{12} = \frac{5(18)^2}{12} = 135\ \text{k}\cdot\text{ft}$



Joint	A	B		C			D	E
Member	AB	BA	BC	CB	CD	CE	DC	EC
DF	0	0.545	0.455	0.330	0.298	0.372	1	1
FEM			−135	135				
Dist.		73.6	61.4	−44.6	−40.2	−50.2		
CO	36.8		−22.3	30.7				
Dist.		12.2	10.1	−10.1	−9.1	−11.5		
CO	6.1		−5.1	5.1				
Dist.		2.8	2.3	−1.7	−1.5	−1.9		
CO	1.4		−0.8	1.2				
Dist.		0.4	0.4	−0.4	−0.4	−0.4		
CO	0.2		−0.2	0.2				
Dist.		0.1	0.1	−0.1	0.0	−0.1		
ΣM	44.5	89.1	−89.1	115	−51.2	−64.1		

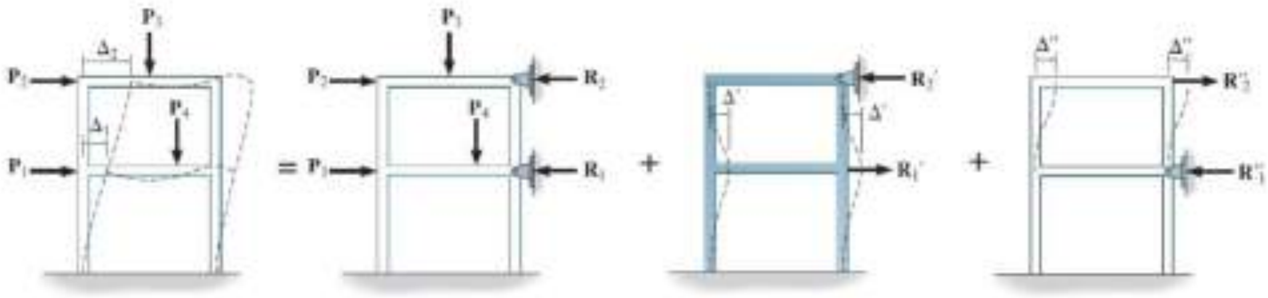


Moment Distribution for Frames: SIDESWAY



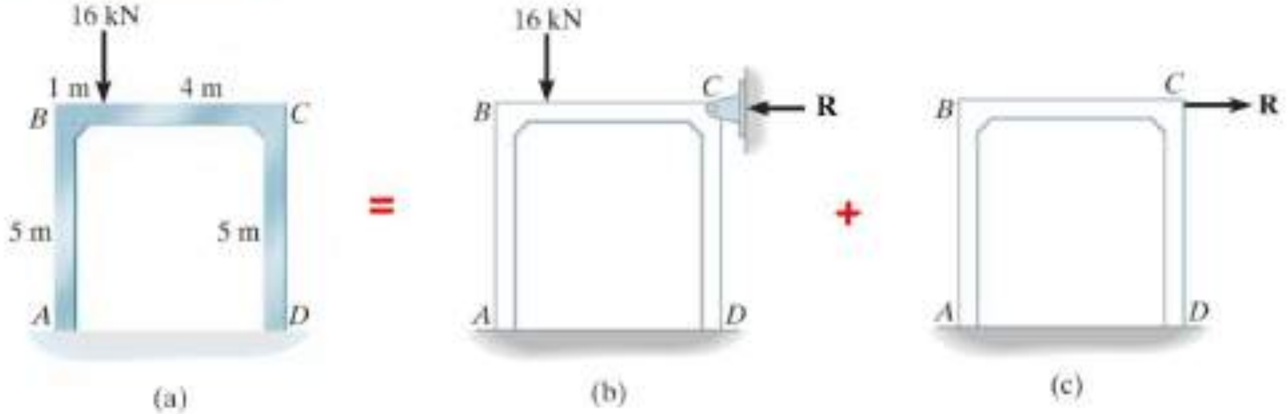
artificial joint applied
(no sidesway)

artificial joint removed
(sidesway)



Example : Determine the moment at the joints of the frame shown in the figure. *EI is constant.*

No-Sway Solution :

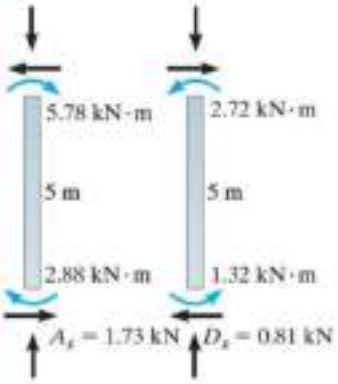


$$(\text{FEM})_{BC} = -\frac{16(4)^2(1)}{(5)^2} = -10.24 \text{ kN}\cdot\text{m}$$

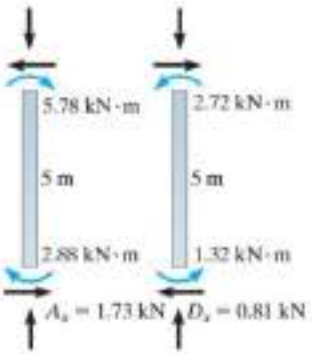
$$(\text{FEM})_{CB} = \frac{16(1)^2(4)}{(5)^2} = 2.56 \text{ kN}\cdot\text{m}$$

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.5	0.5	0.5	0.5	0
FEM			−10.24	2.56		
Dist.		5.12	5.12	−1.28	−1.28	
CO	2.56		−0.64	2.56		−0.64
Dist.		0.32	0.32	−1.28	−1.28	
CO	0.16		−0.64	0.16		−0.64
Dist.		0.32	0.32	−0.08	−0.08	
CO	0.16		−0.04	0.16		−0.04
Dist.		0.02	0.02	−0.08	−0.08	
ΣM	2.88	5.78	−5.78	2.72	−2.72	−1.32

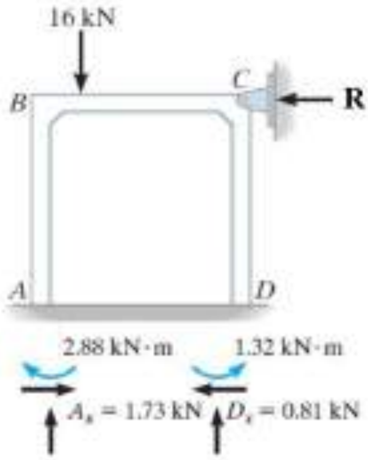
(d)



(e)



⇒



$$A_x = \frac{5.78 + 2.88}{5} = 1.73\ kN$$

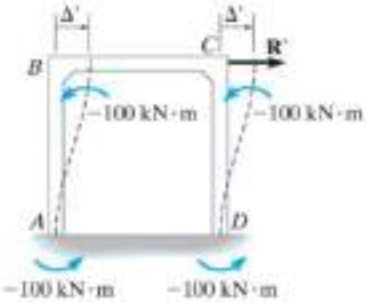
$$D_x = \frac{2.72 + 1.32}{5} = 0.81\ kN$$

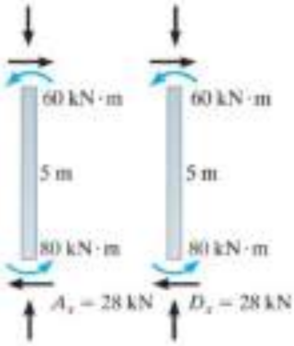
$\Sigma F_x = 0; \quad R = 1.73 \text{ kN} - 0.81 \text{ kN} = 0.92 \text{ kN}$

Sway Solution : We will arbitrarily assume the FEM to be 100 kN.m

$$M = \frac{6EI\Delta}{L^2} \Rightarrow (\text{FEM})_{AB} = (\text{FEM})_{BA} = (\text{FEM})_{CD} = (\text{FEM})_{DC} = -100\ \text{kN}\cdot\text{m}$$

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.5	0.5	0.5	0.5	0
FEM	−100	−100			−100	−100
Dist.		50	50	50	50	
CO	25		25	25		25
Dist.		−12.5	−12.5	−12.5	−12.5	
CO	−6.25		−6.25	−6.25		−6.25
Dist.		3.125	3.125	3.125	3.125	
CO	1.56		1.56	1.56		1.56
Dist.		−0.78	−0.78	−0.78	−0.78	
CO	−0.39		−0.39	−0.39		−0.39
Dist.		0.195	0.195	0.195	0.195	
ΣM	−80.00	−60.00	60.00	60.00	−60.00	−80.00





$$\Sigma F_x = 0; \qquad R' = 28 + 28 = 56.0\ \text{kN}$$

Total / Final Solution = NoSway + Modified Sway Solutions

$$M_{AB} = 2.88 + \tfrac{0.92}{56.0}(-80) = 1.57\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{BA} = 5.78 + \tfrac{0.92}{56.0}(-60) = 4.79\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{BC} = -5.78 + \tfrac{0.92}{56.0}(60) = -4.79\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

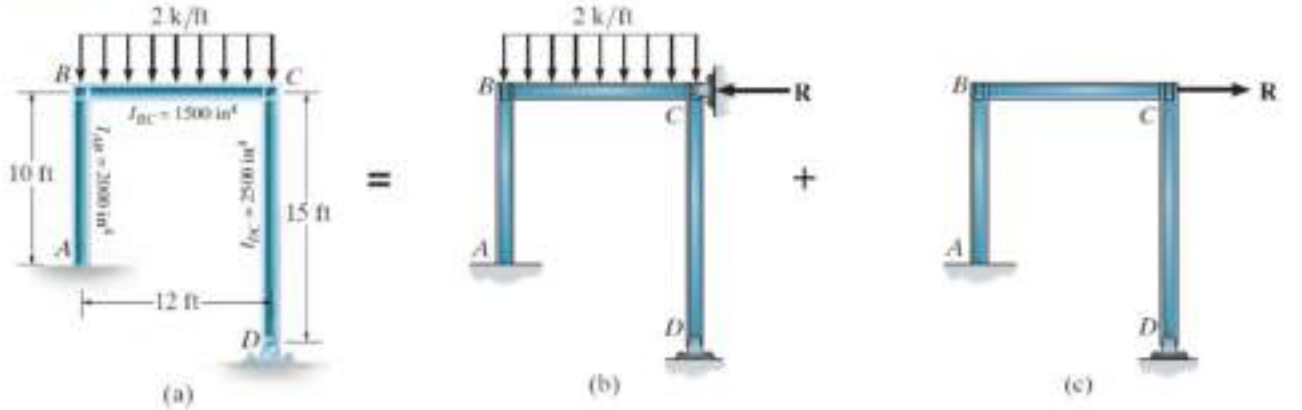
$$M_{CB} = 2.72 + \tfrac{0.92}{56.0}(60) = 3.71\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{CD} = -2.72 + \tfrac{0.92}{56.0}(-60) = -3.71\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{DC} = -1.32 + \tfrac{0.92}{56.0}(-80) = -2.63\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

Example : Determine the moment at the joints of the frame shown in the figure. *The moment of inertia is indicated.*

Solution :



No-Sway Solution :

$$FEM_{BC} = -\frac{w\,l^2}{12} = -\frac{2 \times 12^2}{12} = -24\ k.ft \qquad FEM_{CB} = \frac{w\,l^2}{12} = \frac{2 \times 12^2}{12} = 24\ k.ft$$

$$K_{AB} = \frac{4E(2000)}{10} = 800\ E \qquad K_{BC} = \frac{4E(1500)}{12} = 500\ E \qquad K_{CD} = \frac{3E(2500)}{15} = 500\ E$$

$DF_{AB} = 0$

$$DF_{BA} = \frac{800E}{800E + 500E} = 0.615$$

$$DF_{BC} = \frac{500E}{800E + 500E} = 0.385$$

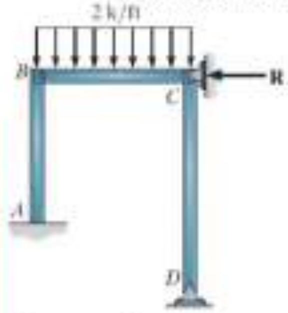
$$DF_{CB} = \frac{500E}{500E + 500E} = 0.5$$

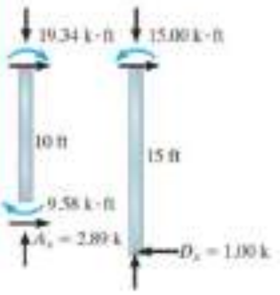
$$DF_{CB} = \frac{500E}{500E + 500E} = 0.5$$

$DF_{DC} = 1$

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.615	0.385	0.5	0.5	1
FEM			−24	24		
Dist.		14.76	9.24	−12	−12	
CO	7.38		−6	4.62		
Dist.		3.69	2.31	−2.31	−2.31	
CO	1.84		−1.16	1.16		
Dist.		0.713	0.447	−0.58	−0.58	
CO	0.357		−0.29	0.224		
Dist.		0.18	0.11	−0.11	−0.11	
ΣM	9.58	19.34	−19.34	15.00	−15.00	0

$\Sigma F_x = 0; \qquad R = 2.89 - 1.00 = 1.89\ k$



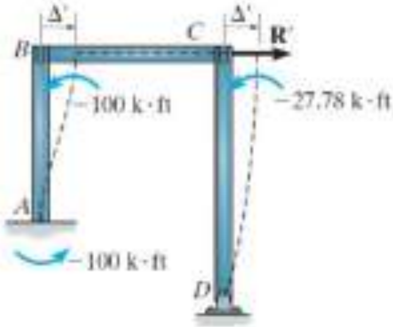


Sway Solution :

$$(FEM)_{AB} = (FEM)_{BA} = -\frac{6EI\Delta}{L^2} = -\frac{6E(2000)\Delta'}{(10)^2}$$

$$(FEM)_{CD} = -\frac{3EI\Delta}{L^2} = -\frac{3E(2500)\Delta'}{(15)^2}$$

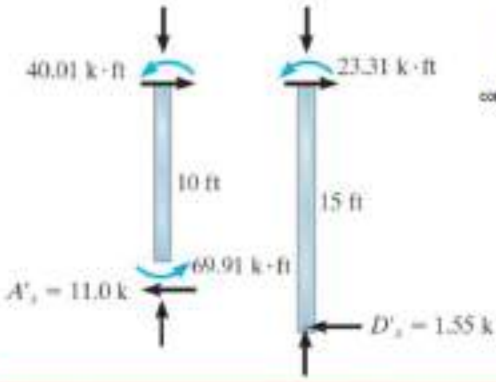
Assuming the FEM for AB is -100 k.ft, the corresponding FEM at C, causing the same $\acute{\Delta}$ is found by comparison, i.e.,



$$\Delta' = -\frac{(-100)(10)^2}{6E(2000)} = -\frac{(FEM)_{CD}(15)^2}{3E(2500)}$$

$$(FEM)_{CD} = -27.78 \text{ k}\cdot\text{ft}$$

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.615	0.385	0.5	0.5	1
FEM	−100	−100			−27.78	
Dist.		61.5	38.5	13.89	13.89	
CO	30.75		6.94	19.25		
Dist.		−4.27	−2.67	−9.625	−9.625	
CO	−2.14		−4.81	−1.34		
Dist.		2.96	1.85	0.67	0.67	
CO	1.48		0.33	0.92		
Dist.		−0.20	−0.13	−0.46	−0.46	
ΣM	−69.91	−40.01	40.01	23.31	−23.31	0



$$\Sigma F_x = 0; \qquad R' = 11.0 + 1.55 = 12.55 \text{ k}$$

$$M_{AB} = 9.58 + \left(\tfrac{1.89}{12.55}\right)(-69.91) = -0.948 \text{ k}\cdot\text{ft}$$

$$M_{BA} = 19.34 + \left(\tfrac{1.89}{12.55}\right)(-40.01) = 13.3 \text{ k}\cdot\text{ft}$$

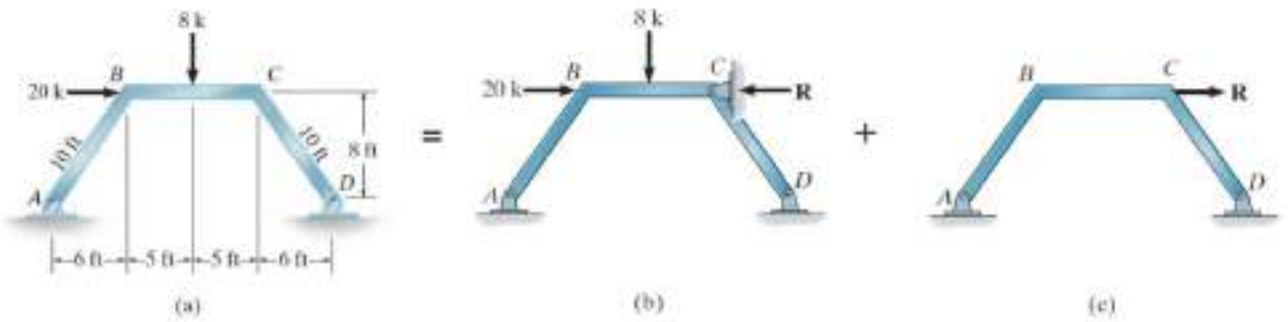
$$M_{BC} = -19.34 + \left(\tfrac{1.89}{12.55}\right)(40.01) = -13.3 \text{ k}\cdot\text{ft}$$

$$M_{CB} = 15.00 + \left(\tfrac{1.89}{12.55}\right)(23.31) = 18.5 \text{ k}\cdot\text{ft}$$

$$M_{CD} = -15.00 + \left(\tfrac{1.89}{12.55}\right)(-23.31) = -18.5 \text{ k}\cdot\text{ft}$$

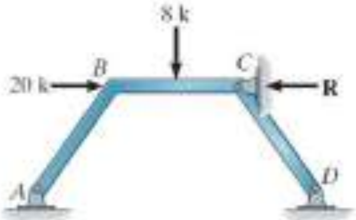
Example : Determine the moment at the joints of the frame shown in the figure. *EI is constant.*

Solution :

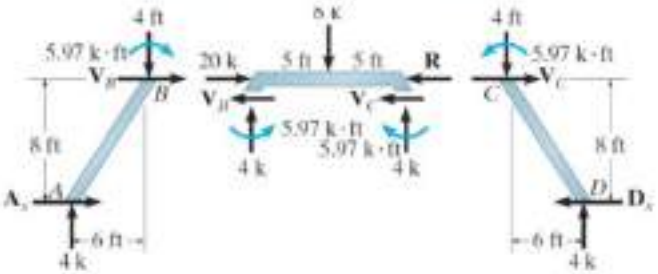


No-Sway Solution :

$$(\text{FEM})_{BC} = -\frac{8(10)}{8} = -10\ \text{k}\cdot\text{ft} \qquad (\text{FEM})_{CB} = \frac{8(10)}{8} = 10\ \text{k}\cdot\text{ft}$$



Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	1	0.429	0.571	0.571	0.429	1
FEM			−10	10		
Dist.		4.29	5.71	−5.71	−4.29	
CO			−2.86	2.86		
Dist.		1.23	1.63	−1.63	−1.23	
CO			−0.82	0.82		
Dist.		0.35	0.47	−0.47	−0.35	
CO			−0.24	0.24		
Dist.		0.10	0.13	−0.13	−0.10	
ΣM	0	5.97	−5.97	5.97	−5.97	0



$\circlearrowleft + \Sigma M_B = 0;\qquad -5.97 + A_x(8) - 4(6) = 0 \qquad A_x = 3.75\ \text{k}$

$\circlearrowleft + \Sigma M_C = 0;\qquad 5.97 - D_x(8) + 4(6) = 0 \qquad D_x = 3.75\ \text{k}$

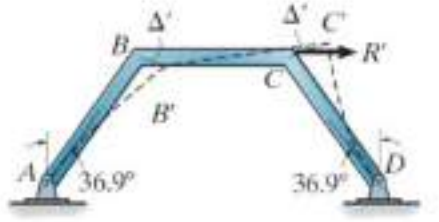
Thus, for the entire frame,

$\Sigma F_x = 0;\qquad R = 3.75 - 3.75 + 20 = 20\ \text{k}$

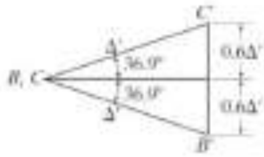
Sway Solution :

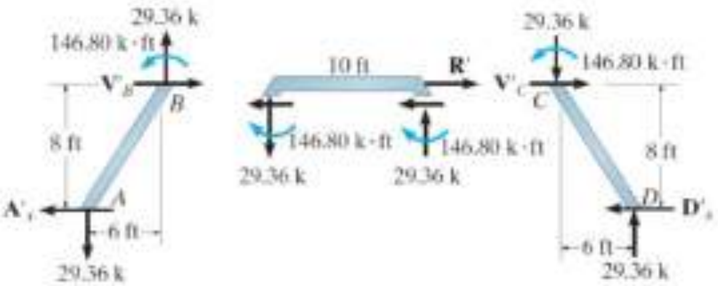
$(FEM)_{BA} = (FEM)_{CD} = -3EI\Delta'/(10)^2$, $(FEM)_{BC} = (FEM)_{CB} = 6EI(1.2\Delta')/(10)^2$.

If we arbitrarily assign a value of $(FEM)_{BA} = (FEM)_{CD} = -100$ k·ft, then equating Δ' in the above formulas yields $(FEM)_{BC} = (FEM)_{CB} = 240$ k·ft.



Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	1	0.429	0.571	0.571	0.429	1
FEM		−100	240	240	−100	
Dist.		−60.06	−79.94	−79.94	−60.06	
CO			−39.97	−39.97		
Dist.		17.15	22.82	22.82	17.15	
CO			11.41	11.41		
Dist.		−4.89	−6.52	−6.52	−4.89	
CO			−3.26	−3.26		
Dist.		1.40	1.86	1.86	1.40	
CO			0.93	0.93		
Dist.		−0.40	−0.53	−0.53	−0.40	
ΣM	0	−146.80	146.80	146.80	−146.80	0





$\circlearrowleft + \Sigma M_B = 0; \quad -A'_x(8) + 29.36(6) + 146.80 = 0 \qquad A'_x = 40.37$ k

$\circlearrowleft + \Sigma M_C = 0; \quad -D'_x(8) + 29.36(6) + 146.80 = 0 \qquad D'_x = 40.37$ k

Thus, for the entire frame,

$\Sigma F_x = 0; \qquad R' = 40.37 + 40.37 = 80.74$ k

$$M_{BA} = 5.97 + \left(\tfrac{20}{80.74}\right)(-146.80) = -30.4 \text{ k}\cdot\text{ft}$$

$$M_{BC} = -5.97 + \left(\tfrac{20}{80.74}\right)(146.80) = 30.4 \text{ k}\cdot\text{ft}$$

$$M_{CB} = 5.97 + \left(\tfrac{20}{80.74}\right)(146.80) = 42.3 \text{ k}\cdot\text{ft}$$

$$M_{CD} = -5.97 + \left(\tfrac{20}{80.74}\right)(-146.80) = -42.3 \text{ k}\cdot\text{ft}$$