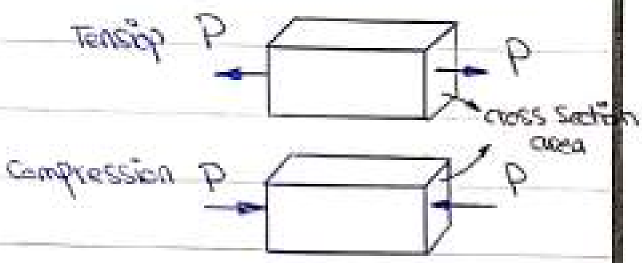
Syllabus :-

1. Simple Stress and Strain.
2. Compound Bars.
3. Shear force and Bending moment diagrams.
4. Bending
5. Shear Stress distribution.
6. Torsion.
7. Thin cylinder.
8. Thick cylinder.
9. Complex stresses.

Refrences :-

1. Mechanics of materials. by :- E.J. Hearn Vol. 1
2. Strength of materials. by :- Singer.
3. Mechanic of materials. by :- F. P. Beer.



Stress $(\sigma) = \frac{P}{A}$

where :-

(σ) :- Normal stress

(i.e) The force is normal to cross-section area

(P) :- Load (N, KN, MN, GN, ...)

(A) :- Cross section area (m^2, mm^2, ...)

Normal strain :- $(\epsilon) = \frac{\delta L}{L}$

where :- δL :- change in length L :- original length

Also :- $E = \frac{\sigma}{\epsilon} \rightarrow \sigma = E.\epsilon$

Where :-

(E) :- Young modulus of elasticity.

$E = \frac{P.L}{A.\delta} \rightarrow \delta = \frac{PL}{AE}$

Poison's Ratio :- (ν)

$\nu = \frac{\delta d/d}{\delta L/L} = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$

For most engineering material

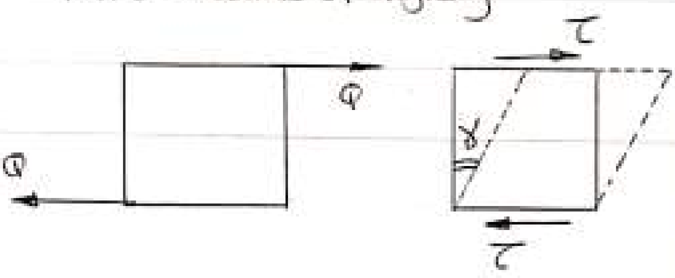
$0.25 < \nu < 0.33$

Shear stress (τ) :- $\tau = \frac{Q}{A}$

where :- (Q): shear force (A):- Area (Parallel with Q)

Also :- $\tau = G\gamma$

where :- (γ):- Shear strain (G):- Modulus of rigidty



Thermal stresses :-

$\delta = \alpha L \Delta t$

where :- (δ) :- change in length due to the effect of temperature.

(Δt) :- Change in temperature.

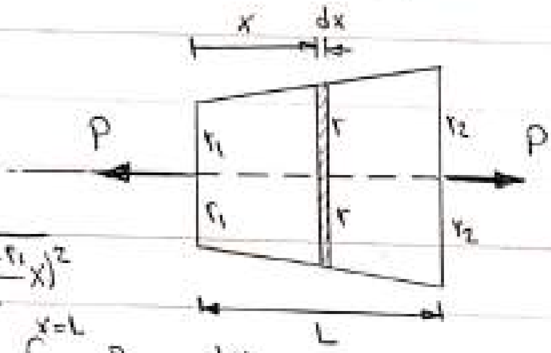
(α) :- Coefficient of linear expansion.

$$\epsilon = \frac{\delta}{L} = \frac{\alpha L \Delta t}{L} = \alpha . \Delta t$$

$$\sigma = E\epsilon = E\alpha . \Delta t$$

Example (1) :- Drive an expression for the total extension of the bar of circular cross-section as shown when its subjected to an axial tensile load (P).

Solution :-



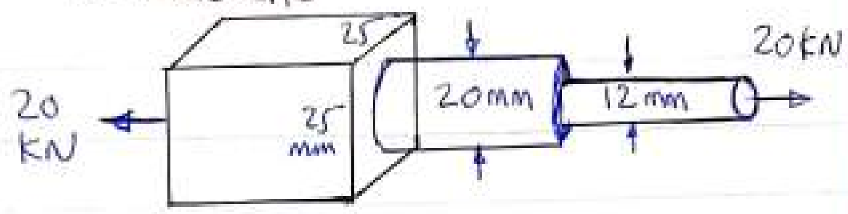
$$r = r_1 + \frac{r_2 - r_1}{L} x$$

$$\sigma = \frac{P}{A} = \frac{P}{\pi r^2} = \frac{P}{\pi (r_1 + \frac{r_2 - r_1}{L} x)^2}$$

For (dx):- $d\delta = \frac{P . dx}{AE}$ $\Rightarrow \delta = \int_{x=0}^{x=L} \frac{P}{\pi E} . \frac{dx}{(r_1 + \frac{r_2 - r_1}{L} x)^2}$

$$\Rightarrow \delta = \frac{P . L}{\pi E r_1 r_2}$$

Example (2):- Find the stress in each section and total extension? $E = 210\ GPa$



Solution:-

① section (AB) :- $\sigma = \frac{P}{A} = \frac{20\times10^3}{25\times25\times10^{-6}} = 32\ \frac{MN}{m^2} = 32\ MPa$

② section (BC) :- $\sigma = \frac{P}{A} = \frac{20\times10^3}{\frac{\pi}{4}(20\times10^{-3})^2} = 63.662\ MPa$

③ section (CD) :- $\sigma = \frac{P}{A} = \frac{20\times10^3}{\frac{\pi}{4}(12\times10^{-3})^2} = 176.838\ MPa.$

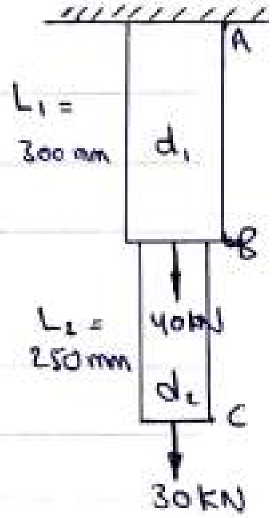
$$\delta_T = \delta_{AB} + \delta_{BC} + \delta_{CD}$$

$$= \frac{P.L_{AB}}{A_{AB}\cdot E} + \frac{P.L_{BC}}{A_{BC}\cdot E} + \frac{P.L_{CD}}{A_{CD}\cdot E} = \frac{P}{E}\left(\frac{L_{AB}}{A_{AB}} + \frac{L_{BC}}{A_{BC}} + \frac{L_{CD}}{A_{CD}}\right)$$

$$= \frac{20\times10^3}{210\times10^9}\left(\frac{0.05}{25\times25\times10^{-6}} + \frac{0.04}{\frac{\pi}{4}(20\times10^{-3})^2} + \frac{0.05}{\frac{\pi}{4}(12\times10^{-3})^2}\right)$$

$$= 0.0618\ mm.$$

Example 8- Two Solid Cylinderical rods AB and BC are welded together and loaded as shown. Knowing that the avarage normal stress not exceed 175 Mpa in rod AB and 150 Mpa in rod BC. Determine the Smallest value of d_1 and d_2



Sol 8- in lecture.

External Load :-

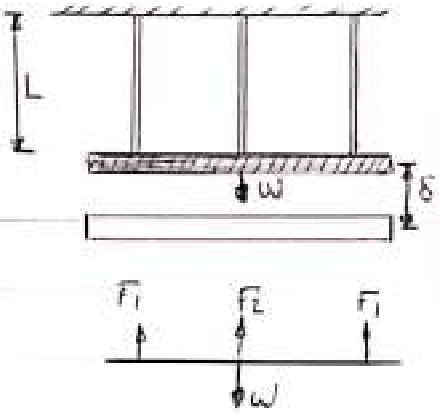
From free body diagram :-

$2F_1 + F_2 = W$

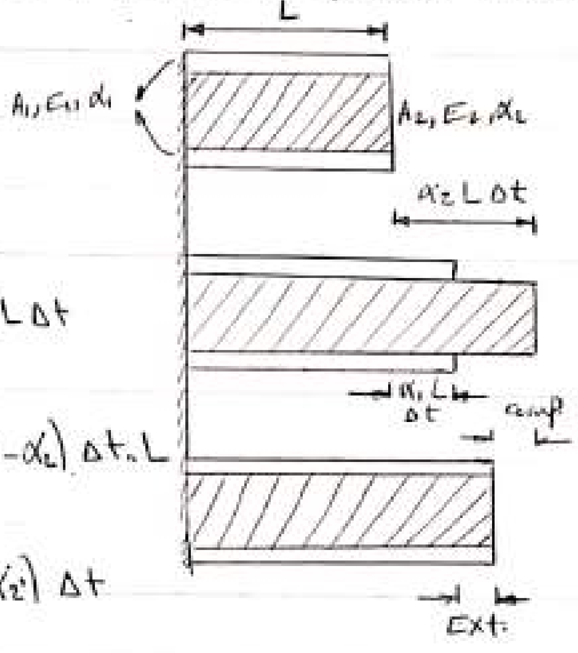
$\delta_1 = \delta_2 \Rightarrow \frac{F_1 \cdot L_1}{A_1 \cdot E_1} = \frac{F_2 \cdot L_2}{A_2 \cdot E_2}$

$L_1 = L_2$

$\Rightarrow \frac{F_1}{A_1 \cdot E_1} = \frac{F_2}{A_2 \, E_2}$



A_1, E_1, A_2 and E_2 are known, thus two equation with two unknown (F_1, F_2)



Temperature change :-

$2F_1 = F_2$

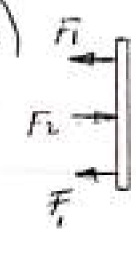
Comp. + Ext. $= \alpha_1 L \Delta t - \alpha_2 L \Delta t$

$\delta = \frac{FL}{AE}$.

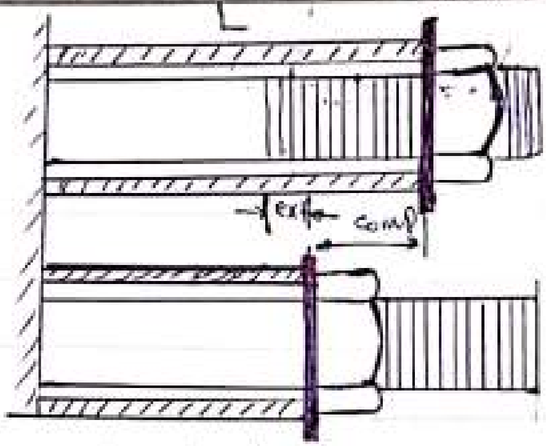
$\Rightarrow \frac{F_1 L}{A_1 E_1} + \frac{F_2 L}{A_2 E_2} = (\alpha_1 - \alpha_2) \, \Delta t \cdot L$

$\Rightarrow \frac{F_1}{A_1 E_1} + \frac{F_2}{A_2 E_2} = (\alpha_1 - \alpha_2) \, \Delta t$

Two equation with two unknowns (F_1 and F_2)



Tube and rod :-



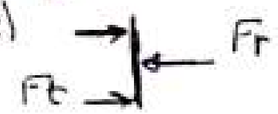
From free body diagram :-

$$F_t = F_r$$

$$\text{Comp.} + \text{Ext.} = d$$

$$\therefore \frac{F_t \cdot L}{A_t E_t} + \frac{F_r \cdot L}{A_r \cdot E_r} = d$$

Two equation with two unknown (F_t and F_R)



Example (1) :- A compound bar consist of four wires of brass, the diameter of each one (2.5 mm) and one wire of steel has a diameter of (1.5 mm) determine :-

1. The stress in each wires if the bar is subjected to load of (500N). All wires have a same length
2. The common deflection if ($L = 0.75$ m). $E_s = 200$ GPa $E_B = 100$ GPa

Solution :- $\sum F_y = 0 \Rightarrow 4F_B + F_s = 500 \rightarrow (1)$

$$\delta_B = \delta_s \Rightarrow \frac{F_B \cdot L}{A_B \cdot E_B} = \frac{F_s \cdot L}{A_s \cdot E_s}$$

$$\therefore \frac{F_B}{\frac{\pi}{4}(2.5\times10^{-3})^2 * 100\times10^9} = \frac{F_s}{\frac{\pi}{4}(1.5\times10^{-3})^2 * 200*10^9}$$

$$\Rightarrow \frac{F_B}{(2.5)^2} = \frac{F_s}{(1.5)^2}$$. Two equation with two unknowns F_b and F_s

$$\Rightarrow F_s = 76 \text{ N} \qquad F_B = 106 \text{ N}$$

$$\sigma_B = \frac{F_B}{A_B} = \frac{106}{\frac{\pi}{4}(2.5)^2 * 10^{-6}} = 21.6 \text{ MPa} \qquad \sigma_s = \frac{F_s}{A_s} = \frac{76}{\frac{\pi}{4}}(1.5\times10^{-3})^2 = 43.2 \text{ MPa}$$

$$\delta_s = \delta_B = \frac{F_s \cdot L}{A_s E_s} = \frac{43.2\times10^6 * 0.75}{200*10^9} = 0.162 \text{ mm}$$

Example (2) :- A compound bar is constructed from three bars 50mm wide by 12mm thick together to form a bar 50mm wide 36mm thick. The middle bar is of aluminum alloy for which E = 70 GPa and the outside of brass with E = 100 MPa. If the bars are initially fastened at 18°C and the temperature of the whole assembly is then raised to 50°C, determine the stresses set up in the brass and the aluminum.

$\alpha_B = 18 \times 10^{-6} \frac{1}{C}$ $\quad \alpha_A = 22 \times 10^{-6} \frac{1}{C}$

Solution :- $2F_B = F_{AL}$

$Ext. + Comp. = (\alpha_{Al} - \alpha_B) L \cdot \Delta t$

$$\frac{F_B \cdot L}{A_B E_B} + \frac{F_{AL} \cdot L}{A_{AL} \cdot E_{AL}} = (\alpha_{AL} - \alpha_B) L \Delta t$$

$$\frac{F_B}{50 \times 12 \times 10^{-6} \times 100 \times 10^9} + \frac{F_{AL}}{50 \times 12 \times 10^{-6} \times 70 \times 10^9} = (22-18) \times 10^{-6} (50-18)$$

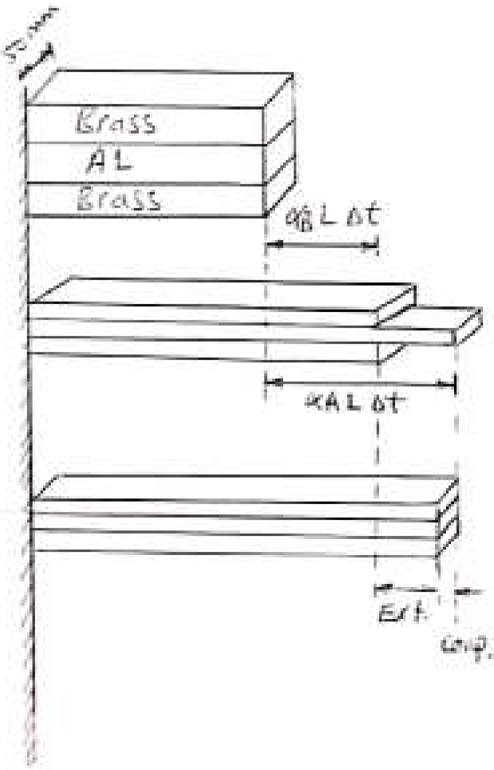
From these equation :-

$F_{AL} = 3984\ N$

$F_B = 1992\ N$

$$\sigma_B = \frac{F_B}{A_B} = \frac{1992}{50 \times 12 \times 10^{-6}} = 3.32\ MPa\ (Ten)$$

$$\sigma_{AL} = \frac{F_{AL}}{A_{AL}} \frac{3984}{50 \times 12 \times 10^{-6}} = 6.64\ MPa\ (comp)$$



* If P = 15 KN (comp) Find σ_B and σ_{AL}

$F_A + 2F_B = 15$

$$\delta_B = \delta_{AL} \Rightarrow \frac{F_{AL} \cdot L}{A_A E_A} = \frac{F_B \cdot L}{A_B \cdot E_B}$$

$\Rightarrow F_B = 5.56\ KN \qquad F_{AL} = 3.89\ KN$

$\sigma_{AL} = \dfrac{F_{AL}}{A_{AL}} = 6.48\ MPa \qquad \sigma_B = 9.26\ MPa$

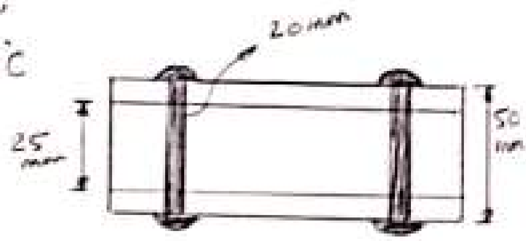
$\Rightarrow \sigma_B\ \text{Total} = -9.26 + 3.32\ MPa \quad (\text{Comp})$

$\sigma_{AL}\ \text{total} = -6.48 - 6.64 = 13\ MPa \quad \text{Comp})$

Example (3) :- A compound bar is constructed from a steel rod of 25 mm diameter surrounded by a copper tube of 50 mm outside diameter and 25 mm inside diameter. The rod and tube are joined by two diameter pins as shown in the figure. Find the stress set up in the pins if, after pinning the temperature is raised by 50 °C.

For steel : $E = 210\ GPa \quad \alpha = 11 \times 10^{-6}\ 1/°C$

For copper :- $E = 105\ GPa \quad \alpha = 17 \times 10^{-6}\ 1/°C$



Solutions. $F_T = F_R \rightarrow (1)$

$\delta_T + \delta_R = (\alpha_T - \alpha_R)\, L\, \Delta t$

$\Rightarrow \dfrac{F_T \cdot L}{A_T \cdot E_T} + \dfrac{F_R \cdot L}{A_R \cdot E_R} = (\alpha_T - \alpha_R)\, L \cdot \Delta t$

$\Rightarrow \dfrac{F_T}{\frac{\pi}{4}(50^2 - 25^2) \times 10^{-6} \times 105 \times 10^9} + \dfrac{F_R}{\frac{\pi}{4}(25)^2 \times 10^{-6} \times 210 \times 10^9} = (17 - 11) \times 10^{-6} \times 50$

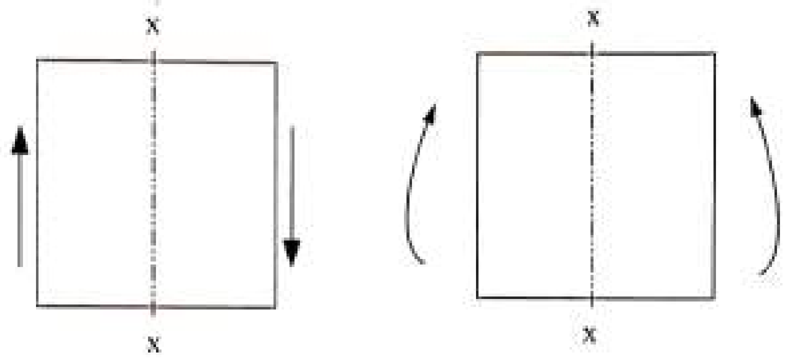
$\Rightarrow 6.46\, F_T \times 10^{-9} + 9.7\, F_R \times 10^{-9} = 300 \times 10^{-6} \rightarrow (2)$

From these equations :-

$F_T = F_R = 18.564\ KN$

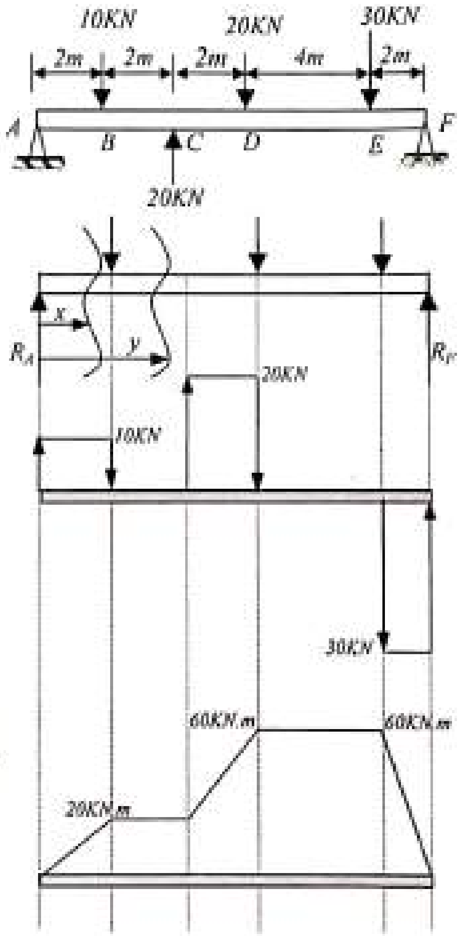
$$\tau = \frac{Q}{2A} = \frac{18.564}{\frac{\pi}{4}(20)^2 \times 10^{-6} \times 2} = 29.55\ MPa.$$

Shear force and bending moment diagram



Positive S.F Positive B.M

Concentrated loads:



$R_A(12) = 10(10) + 20(6) + 30(2) - 20(8) \Rightarrow R_A = 10KN$

$\sum F_Y = 0 \Rightarrow R_A + R_F = 10 + 20 + 30 - 20$

$\Rightarrow R_F = 30KN$

section x;

$M_X = R_A \cdot X = 10x$

(increasing line)

section y;

$M_Y = R_A \cdot Y - 10(Y - 2)$

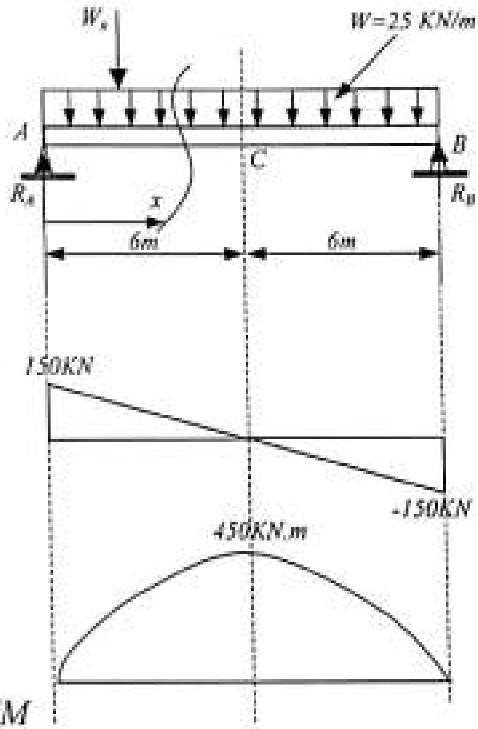
$10 \cdot Y - 10Y + 20 = 20$

(horizontal line)

- *Notes:*

1. *Values of shear force =Slope of bending moment.*
2. *Area of shear force diagram between two points =value difference between bending moment of these points.*

Uniformly distributed loads:



$\sum M_B = 0$

$R_A * 12 = 25 * 12 * 6 \Rightarrow R_A = 150KN$

$\sum FY = 0$

$R_A + R_B = 25 * 12 \Rightarrow R_B = 150KN$

Section x:

S.FX=150-25X(decreasing line)

$B*M_x = RA * x - W * x * \frac{x}{2} = 150X - 125x^2$

Convex parabola

To find $B*M_{max}$

$\frac{dM}{AX} = 0 \Rightarrow 150 - 125X = 0 \Rightarrow X = 6m$

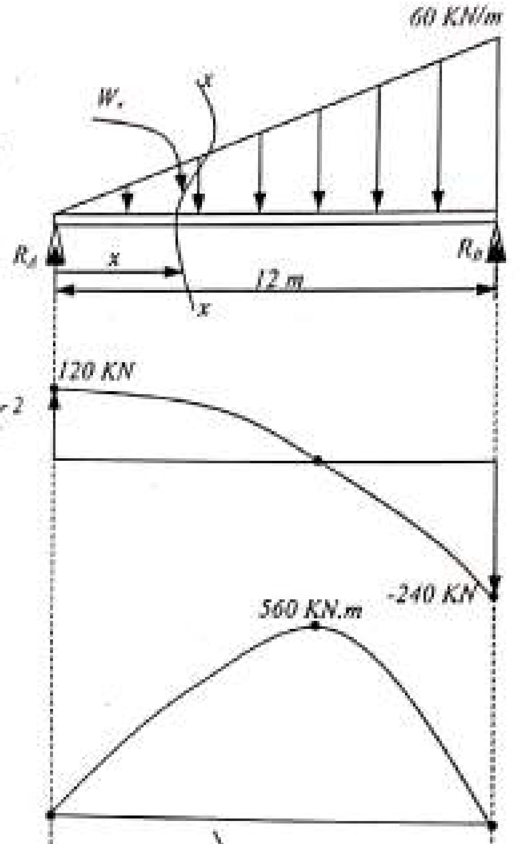
$\Rightarrow B*M_{max} = 450KN.m$

Note: $S*F_x = Q_x = 150 - 125X = \frac{dM}{dx} \Rightarrow Q = \frac{dM}{dx}$

M= Q dx

$IFQ = 0 \Rightarrow M$ is minimum or maximum

∢ Distributed loads of increasing value



$$\sum MB = 0 \Rightarrow RA*12 = \frac{60*12}{2}(300)*4 \therefore RA = 120KN$$

$$\sum FY = 0 \Rightarrow RA + RB = 60*\frac{12}{2} \Rightarrow RB = 240KN$$

SECTION X:

$$Q_X = RA - \frac{1}{2}W_X(X) = 120 - \frac{1}{2}\left(\frac{60}{12}*X\right)X = 120 - 2.5X^2$$

(Convex parabola)

$$Qx = 0 \Rightarrow X = \sqrt{48}$$

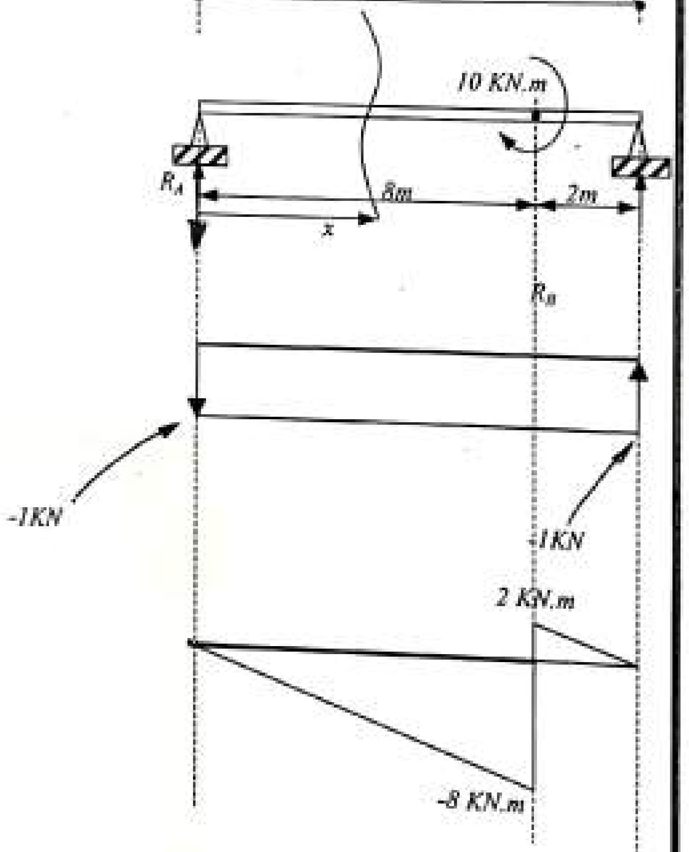
$$Mx = RA*X - \left(\frac{1}{2}(5X)X\right)\frac{X}{3} = 120X - \frac{5}{6}X^3$$

(Convex third degree)

M_{max} at $Qx = 0$ at $x = \sqrt{48}$

$$M_{max} = 120\left(\sqrt{48}\right) - \frac{5}{6}\left(\sqrt{48}\right)^3 = 560KN.m$$

∢ Couples:



$\sum MB = 0 \Rightarrow RA*10 = 10 \Rightarrow RA = 1KN$

$\sum FY = 0 \Rightarrow RA = RB \Rightarrow RB = 1KN$

$Qx = -RA = -1$

$Mx = -RA.X = -X$

$Mc = -8$

Note: point of inflexion occur at (M=0)

Example-1-

$\sum ME = 0 \Rightarrow RA*8+40*2$

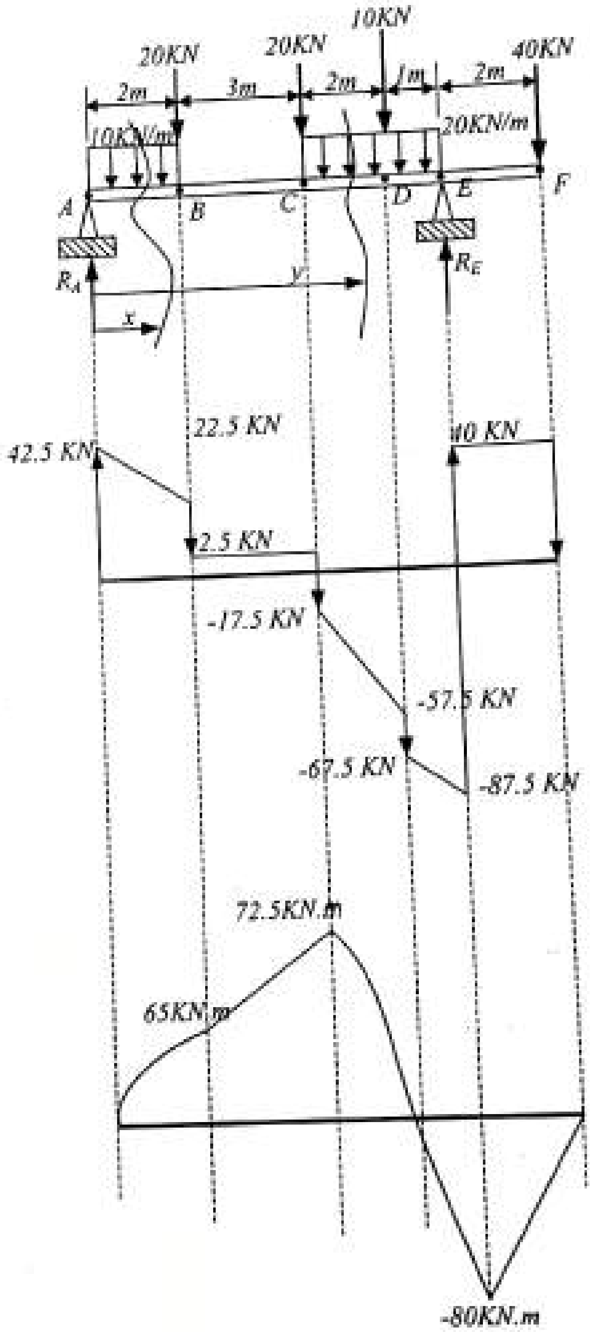
$= 10*2*7+20*6+20*3+10(1)+20(3)(1.5)$

$\Rightarrow RA = 42.5KN$

$\sum FY = 0 \Rightarrow RA = RE = 10(2)+20+20+10+20(3)+40 \Rightarrow RE = 127.5KN$

$QB = 42.5-10(2)-20 = 2.5\,KN$

$QD = 22.5-20-20-20(2)-10 = 67.5KN$



$$MX = 42.5X - 10X\left(\frac{X}{2}\right)$$

$$MB = 42.5(2) - 10\left(\frac{22}{2}\right) = 26\,KNm$$

$$MY = RA(Y) - 10*2(Y-1) - 20(Y-2) -$$

$$20(Y-5) - 20(Y-5)\left(\frac{Y-5}{2}\right) = -90 + 82.5Y - 10Y^2$$

$$MY = 0 \Rightarrow 10Y^2 - 82.5Y + 90 = 0$$

$$\Rightarrow Y = 6.96\ or y = 1.5 (neglect)$$

3

$$MC = -90 + 82.5(5) - 10(5)^2 = 72.5\,KNM$$

$$ME = -90 + 82.5(8) - 10(8)^2 - 10(1) = -80\,KN.M$$

Example 2

C=80KN.M

$\sum MB = 0 \Rightarrow Rc(6) + 10*3*1.5 + 80 = \frac{1}{2}*6*48*2 \Rightarrow Rc = 27\ KN$

$\sum FY = 0 \Rightarrow Rc + RB = 10*3 + \frac{1}{2}*48*6 \Rightarrow RB = 147\ KN$

$QX = -Rc + \frac{1}{2}wx.x = 27.2 + \frac{1}{2}(48)\frac{x}{6}.x = -27.2 + 4x^2$

(Concave parabola)

$Qx = 0 \Rightarrow 4X^2 - 27.2 = 0 \Rightarrow x = 2.61\,m$

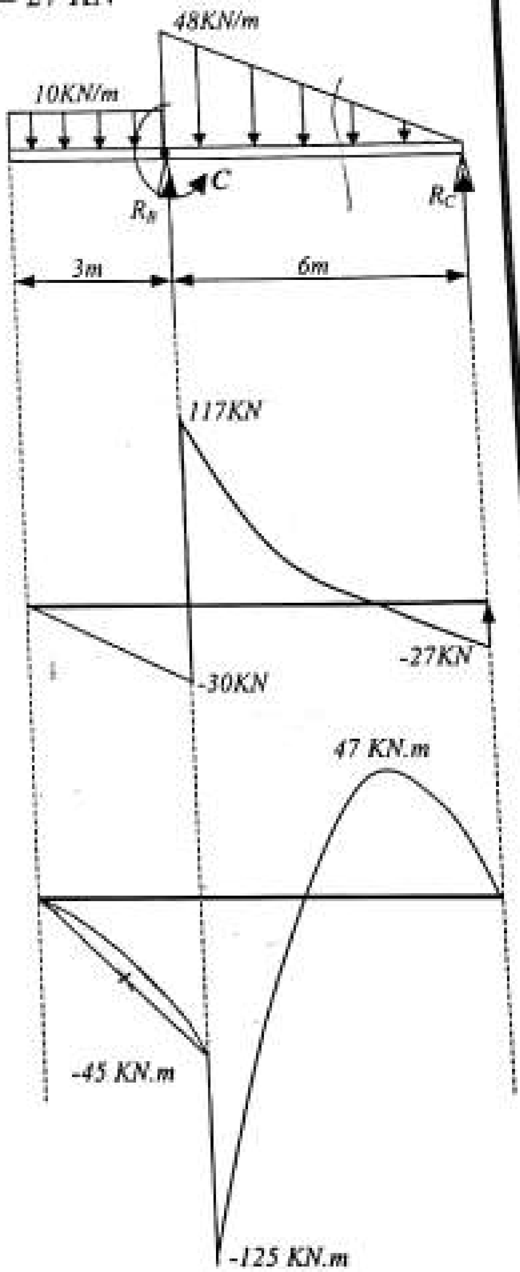
$Mx = 27.2x - \frac{1}{2}w.x.\frac{x}{3} = 27.2x - \frac{1}{2}(8x)*x*\frac{x}{3}27.2x - \frac{3}{4}x^3$

(Convex third degree)

$M_{max} = 27.2x(2.61) - \frac{3}{4}(2.61)^3 = 47.3\,KN.m$

$M = 0 \Rightarrow \frac{3}{4}x^3 - 27.2x = 0 \Rightarrow x = 4.5\,m$

$Mc = 27.2(6) - \frac{3}{4}(6)^3 = -125\,KN.m$

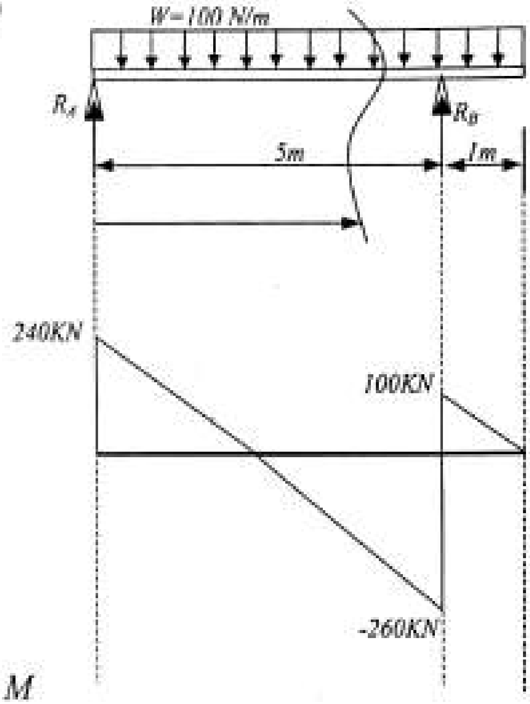


Example: A simply supported beam (ABC) has length of (6m) and loaded as shown below. The weight of the beam 100N/m. find:

(1) The reactions at A and B

(2)Draw shear force diagrams and from it find the value and position of maximum bending moment

Solution:



$$\sum MB = 0 \Rightarrow RA*5 - 100*5*2.5 + 100*1*0.5 = 0$$

$$\Rightarrow RA = 240\,N$$

$$\sum FY = 0 \Rightarrow RA + RB = 100*6 \Rightarrow RB = 360N$$

$$Mx = RA*x - 100x*\frac{x}{2} = 240x - 50x^2$$

$$Rx = RA - 100x \Rightarrow Rx = 240 - 100x \Rightarrow$$

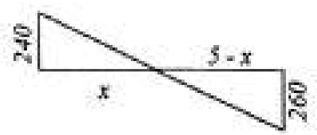
$$M_{max} atR = 0 \Rightarrow 240 - 100x = 0 \Rightarrow x = 2.4\,m$$

$$M_{max} = 240(2.4)*50(2.4)^2 = 288N.M$$

From (S.F.D)

$$\frac{240}{x} = \frac{260}{5-x} \Rightarrow x = 2.4\,m$$

$$B.M_{max} = areaof(S.F.D) = \frac{1}{2}*240*2.4 = 288\,N.M$$

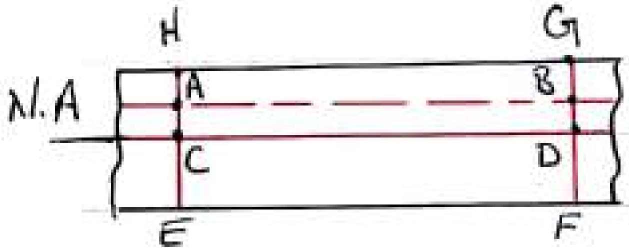


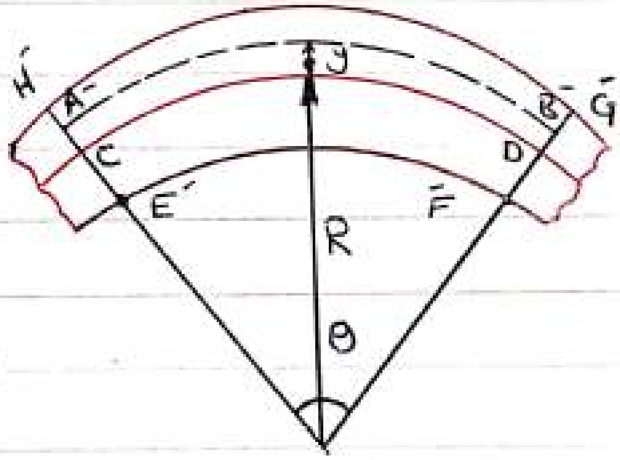
Chapter Four :- Bending.

* Simple bending theory :-

* Assumptions :-

1. The beam is initially straight and unstressed.
2. The material is homogenous and isotropic.
3. The Elastic limit is nowhere exceeded.
4. Young's modulus is the same in tension and compression.
5. Plane cross-section remains plane after bending.
6. Every cross-section is symmetrical about the plane of bending.
7. the loading is pure bending.





The strain of the line AB

$$\epsilon = \frac{A'B' - AB}{AB}$$

before bending AB = CD

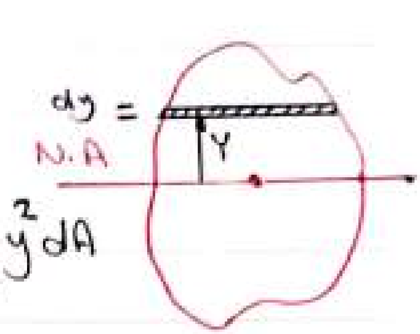
and CD on the nutral axis (No strain)

$$\therefore e = \frac{A'B' - CD}{CD} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

$$\text{But } \epsilon = \boxed{\frac{\sigma}{E} = \frac{y}{R}} \rightarrow (1)$$

Also :- $F = \sigma \cdot A$

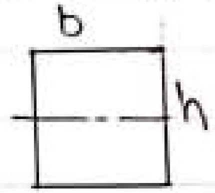
$$\Rightarrow dF = \sigma \cdot dA \Rightarrow dM = Y \cdot dF = \sigma Y \cdot dA$$



$$\therefore M = \int_A \sigma Y dA = \int_A \frac{E}{R} Y^2 dA = \frac{E}{R}\int_A y^2 dA$$

$$\therefore M = \frac{E}{R} \cdot I \rightarrow (2)$$ I = Second moment of inertia

$$I \text{ (rectangular)} = \frac{bh^3}{12}$$



$$I \text{ (circle)} = \frac{\pi}{4} R^4$$ where (R) = radius of circle

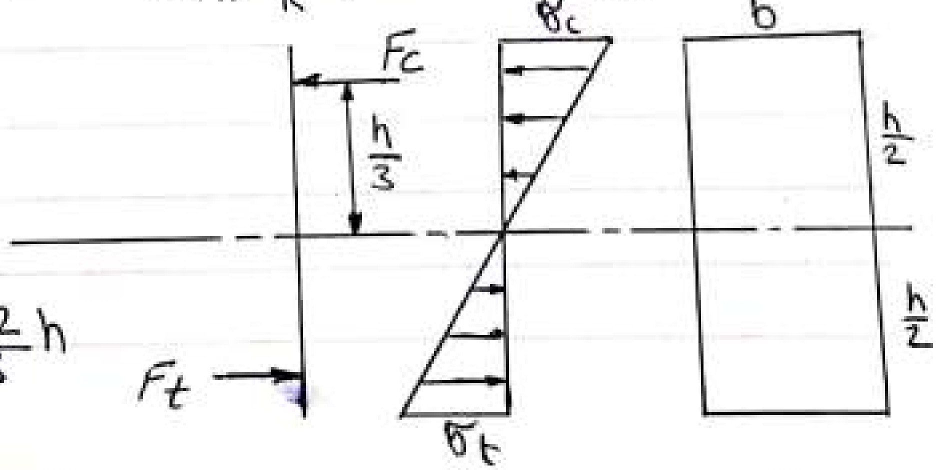
* Always the nutral axis (N.A) passes through the Centroid because assumption (7) means that there is no resultant force across the Section. i.e

$$F = \int_A \sigma dA = 0 \Rightarrow \int_A \frac{E}{R} Y dA = 0 \Rightarrow \int_A dA = 0$$

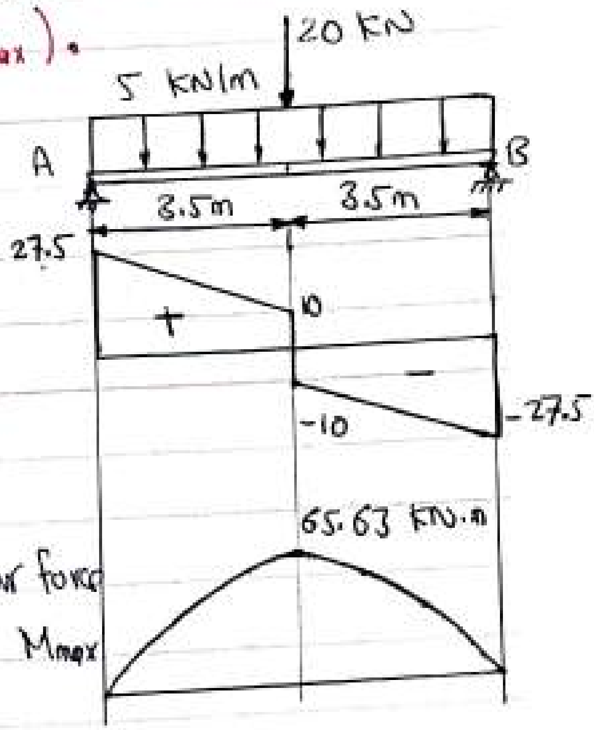
$$F_c = F_t$$

$$M = F_t \cdot \frac{2}{3} h$$

$$= \left(\frac{1}{2}\sigma_t \cdot \frac{h}{2} b\right)\frac{2}{3} h$$



Example (1) :- the beam loaded as shown, Find the maximum bending stress (σ_{max}).



Solution :-

$\sum M_B = 0$

$R_A * 7 = 5 * 7 * 3.5 + 20 * 3.5$

$\Rightarrow R_A = 27.5 \ KN$

$\sum F_y = 0 \Rightarrow R_A + R_B = 5 * 7 + 20$

$\Rightarrow R_B = 27.5 \ KN$

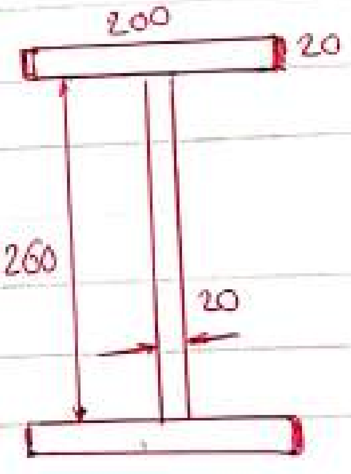
Mmax occures at Zero shear force

There are two ways to find Mmax

① $M_{max} = 27.5 * 3.5 - 5 * 3.5 * \frac{3.5}{2} = 65.62 \ KN.m$

② M_{max} = are of shear force between A and the Centre

$= \frac{1}{2} * 17.5 * 3.5 + 10 * 3.5 = 65.625 \ KN.m$



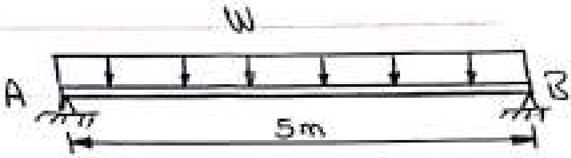
$Y_{max} = 150 \ mm$

$$I_{N.A} = \frac{200 (300)^3}{12} - 2 * \frac{90 * 260^3}{12}$$

$$= 1.86 * 10^8 \ mm^4$$

$$\sigma = \frac{65.625 * 10^3 * 150 * 10^{-3}}{1.86 * 10^{-4}} = 51.8 \ Mpa.$$

Example ② :- Find (w) if maximum stress in tension is 160 MPa and in compression is 80 MPa.



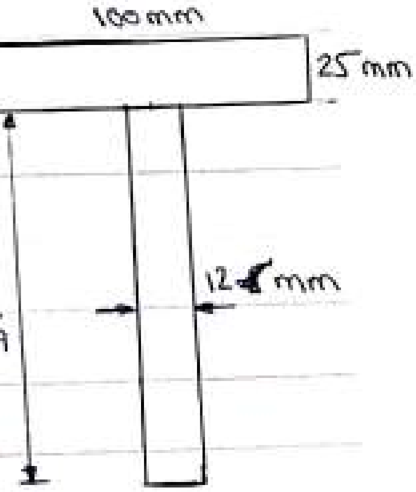
Solution :-

Because of the Symmetry

$R_A = R_B = \frac{1}{2} wL$

Maximum bending moment occurs at the Centre $= 2.5w * 2.5 - 2.5w \cdot \frac{2.5}{2}$

$= 3.125 w$



$$\bar{y} = \frac{\Sigma AY}{\Sigma A} = \frac{(100*25*137.5 + 125*12*62.5)*10^{-9}}{(100*25 + 125*12)*10^{-6}}$$

$= 109.4$ mm (From the base)

$$I_{N.A} = \left[\frac{100*40.6^3}{3} - \frac{88*15.6^3}{3} + \frac{12*109.4^3}{3}\right]*10^{-12} = 7.36*10^{-6}\ m^4$$

$$M = \frac{\sigma . I}{Y} = \frac{80*10^6 * 7.36*10^{-6}}{40.6*10^{-3}} = 14.5\ KN.m$$

$$M = \frac{160*10^6 * 7.36*10^{-6}}{109.4*10^{-3}} = 10.76\ KN.m$$

$\therefore M_{max} = 10.76$ KN.m $= 3.125 w$

$\Rightarrow w = 3.44$ KN/m

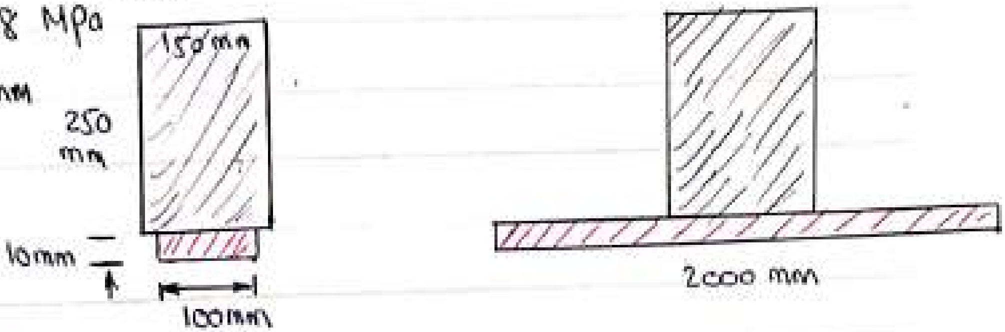
Example ② :- Determine the concentrated load that can be applied at the centre of a simply supported span 6 m long if $\frac{E_s}{E_w} = 20$. $(\sigma_{max})_s = 120$ Mpa

$(\sigma_{max})_w = 8$ Mpa

$\bar{Y} = 170.2$ mm

From top

$I = 416 \times 10^{-6}$ m^4



Solution :-

$$E_s t_s = E_w t_w \Rightarrow \frac{E_s}{E_w} t_s = 20 \times 100 = 2000 \text{ mm}$$

$$\sigma_w = \frac{M.Y}{I} \Rightarrow M = \frac{\sigma.I}{Y} \Rightarrow M = \frac{8 \times 10^6 \times 416 \times 10^{-6}}{170.2 \times 10^{-3}} = 19.55 \text{ kN.m}$$

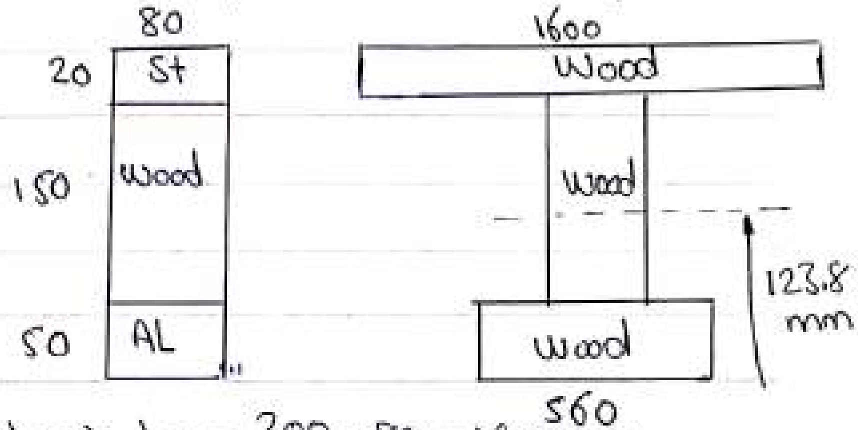
$$\sigma_w = \sigma_s . \frac{E_w}{E_s} = 120 \times \frac{1}{20} = 6 \text{ Mpa.}$$

$$M = \frac{6 \times 10^6 \times 416 \times 10^{-6}}{89.78 \times 10^{-3}} = 27.8 \text{ kN.m}$$

$$M_{max} = 19.55 = \frac{W}{2} \times 3 \Rightarrow W = 13.03 \text{ kN}$$

Example (3): Find the maximum bending moment for a composite beam that has the following properties.
$(\sigma_{max})_s = 120\,Mpa$, $(\sigma_{max})_{Al} = 80\,Mpa$, $(\sigma_{max})_w = 10\,Mpa$
$E_s = 200\,Gpa$, $E_{AL} = 70\,Gpa$ and $E_w = 10\,Gpa$.

Solution:



$$E_s t_s = E_w t_w \Rightarrow t_w = \frac{200}{10} * 80 = 1600\,mm$$

$$E_w t_w = E_{AL} \cdot t_{AL} \Rightarrow t_w = \frac{70}{10} * 80 = 560\,mm$$

$$\bar{y} = \frac{\Sigma AY}{\Sigma A} = \frac{20*1600*210 + 150*80*125 + 560*50*25}{20*1600 + 150*80 + 560*50} = 123.8\,mm$$

$$I = \frac{560*123.8^3}{3} - \frac{480*73.8^3}{3} + \frac{1600*96.2^3}{3} - \frac{1520*76.2^3}{3} = 540.5*10^{-6}\,m^4$$

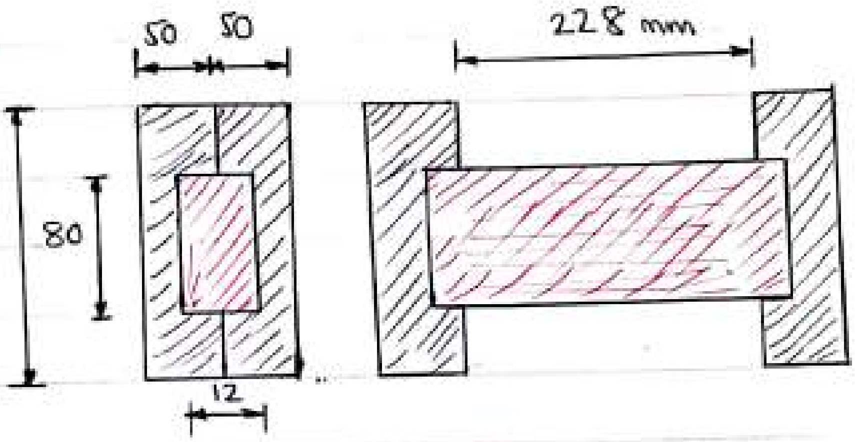
$$M = \frac{10*10^6 * 540.5*10^{-6}}{76.2*10^{-3}} = 70.8\,KN.m$$

$$\sigma_w = \frac{E_w}{E_s} \cdot \sigma_s = \frac{10}{200} * 120 = 6\,MPa \Rightarrow M = \frac{6*10^6 * 540*10^{-6}}{96.2*10^{-3}} = 33.7\,KN.m$$

$$\sigma_w = \frac{E_w}{E_{al}} \cdot \sigma_{al} = \frac{10}{70} * 8 = 11.4\,MPa \Rightarrow M = \frac{11.4*10^6 * 540*10^{-6}}{123.8*10^{-3}} = 49.5\,KN.m$$

$$\therefore M_{max} = 33.7\,KN.m$$

Example ① :-



$M_{max} = ?$

For wood :-

$\sigma_{max} = 12\ MPa$

$E = 10\ GPa$

For steel :-

$\sigma_{max} = 150\ MPa \quad E = 200\ GPa.$

Solution :-

$$E_s t_s = E_w t_w \Rightarrow t_w = \frac{E_s}{E_w} \cdot t_s = \frac{200}{10} * 12 = 240\ mm$$

$$I_{N.A} = 2 * \frac{50 * 200^3}{12} + \frac{228 * 80^3}{12} = 76.36 * 10^6\ mm^4 = 76.36 * 10^{-6}\ m^4$$

$$M_{max} = \frac{\sigma_{max} \cdot I}{Y_{max}} = \frac{12 * 10^6 * 76.36 * 10^{-6}}{100 * 10^{-3}} = 9200\ N.m$$

$$\frac{\sigma_s}{\sigma_w} = \frac{E_s}{E_w} \Rightarrow \sigma_w = \frac{E_w}{E_s} \cdot \sigma_s = \frac{10 * 10^9}{200 * 10^9} * 150 * 10^6$$

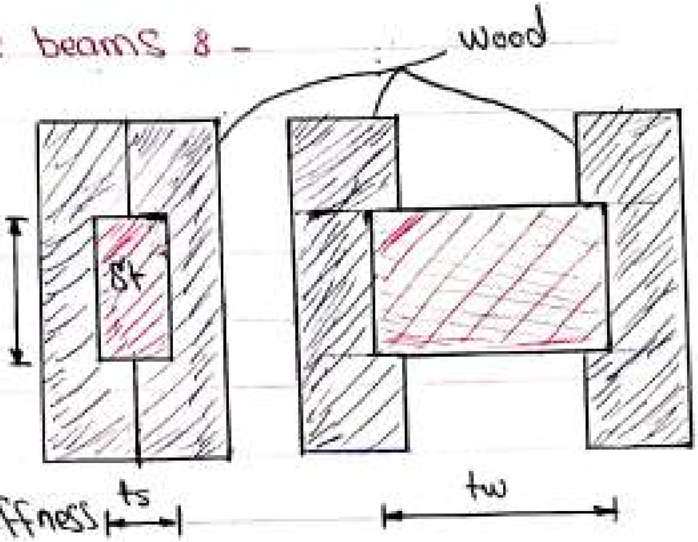
$$= 7.5\ MPa.$$

$$M_{max} = \frac{7.5 * 10^6 * 76.36 * 10^{-6}}{40 * 10^{-3}} = 14.3175\ KN.m$$

$\therefore M_{max} = 9200\ N.m$

* Bending of Composite beams :-

(1) Flitched beams :-



$M = EI \cdot \frac{1}{R}$

$\therefore M \propto \frac{1}{R}$

where EI = constand

and called Flexural stiffness

Equivalent section (EI) constant

i.e $E_s I_s = E_w I_w \Rightarrow E_s \cdot \frac{t_s \cdot h^3}{12} = E_w \cdot \frac{t_w h^3}{12}$

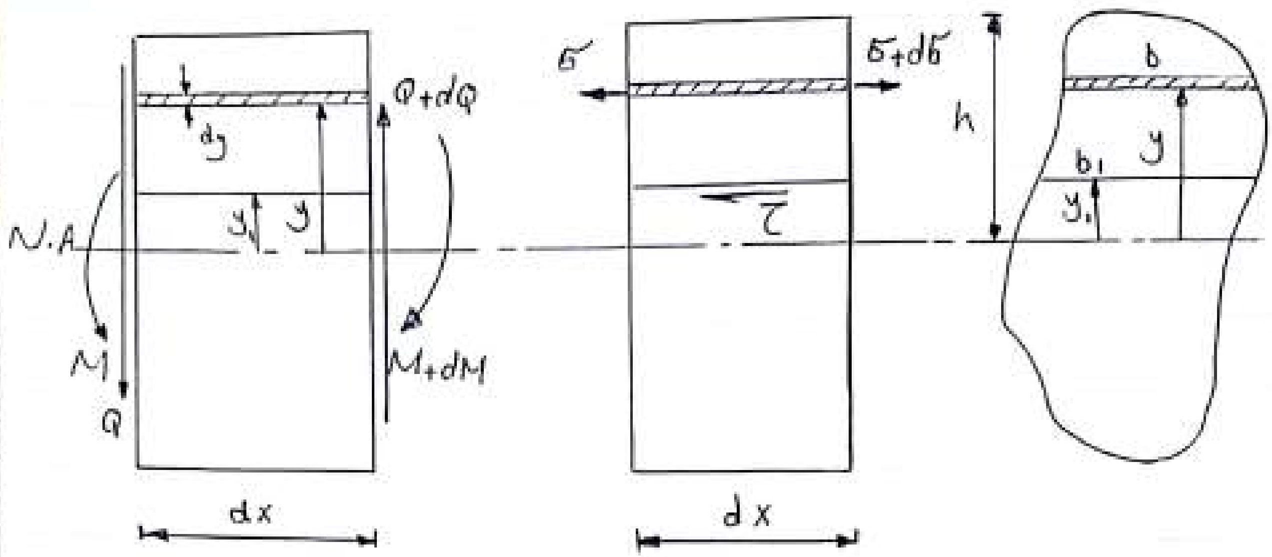
h is constant thus

$$E_s \cdot t_s = E_w \cdot t_w \rightarrow (1)$$

$$\text{and } R_w = R_s \Rightarrow \frac{E_s \cdot Y_s}{\sigma_s} = \frac{E_w Y_w}{\sigma_w}$$

$$y_s = y_w \Rightarrow \frac{E_s}{\sigma_s} = \frac{E_w}{\sigma_w} \rightarrow (2)$$

Equations (1) and (2) are very important to solve any proplem for this type of beams.



$$\sigma = \frac{M \cdot y}{I}, \qquad \sigma + d\sigma = \frac{(M+dM)\,y}{I} \Rightarrow d\sigma = \frac{dM \cdot y}{I}$$

$$\tau \cdot b_1 \cdot dx = \int_{y_1}^{h} d\sigma \cdot b \cdot dy = \int_{y_1}^{h} \frac{dM}{I}\, b \cdot y \cdot dy$$

But $dM = Q\,dx$ $\Rightarrow \tau \cdot b_1 \cdot dx = \int_{y_1}^{h} \frac{Q \cdot dx}{I}\, b y \cdot dy$

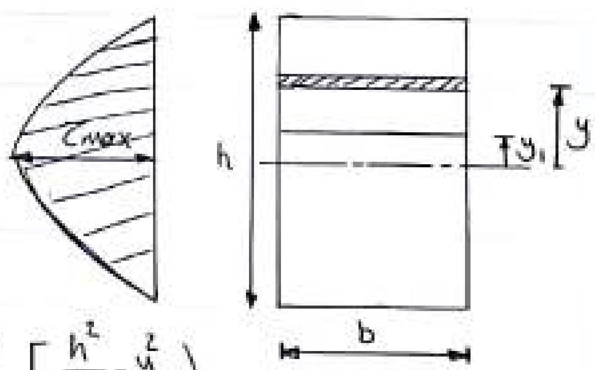
$$\Rightarrow \tau \cdot b_1 = \frac{Q}{I}\int_{y_1}^{h} b \cdot y \cdot dy = \frac{Q}{I}\int_{y_1}^{h} y \cdot dA$$

$$\Rightarrow \tau = \frac{Q A \bar{y}}{I b}$$

* Rectangular Sections.

$b = b_1$

$$\tau \cdot b = \frac{Q}{I}\int_{y_1}^{h/2} b \cdot y \cdot dy$$

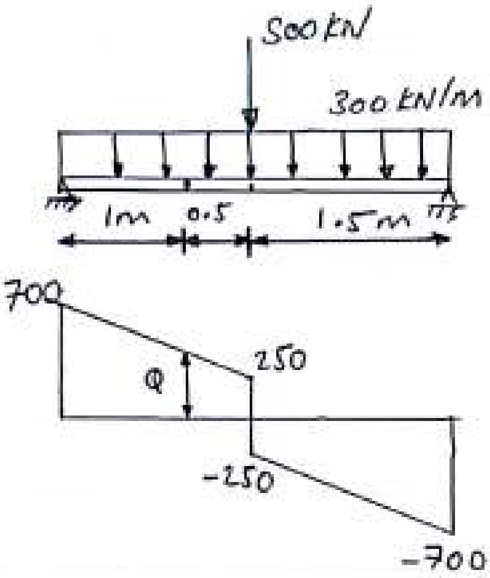


$$I = \frac{bh^3}{12}$$

$$\tau \cdot b = \frac{12\,Q}{b h^3}\left[\frac{y^2}{2}\right]_{y_1}^{h/2} = \frac{6Q}{bh^3}\left[\frac{h^2}{4} - y_1^2\right)$$

at $y_1 = 0 \Rightarrow \tau = \tau_{max} = \frac{3}{2}\,\frac{Q}{b \cdot h} = \frac{3}{2}\,\tau_{mean}$.

Example ① :- find shear stress at E and D for the cross-section as shown.



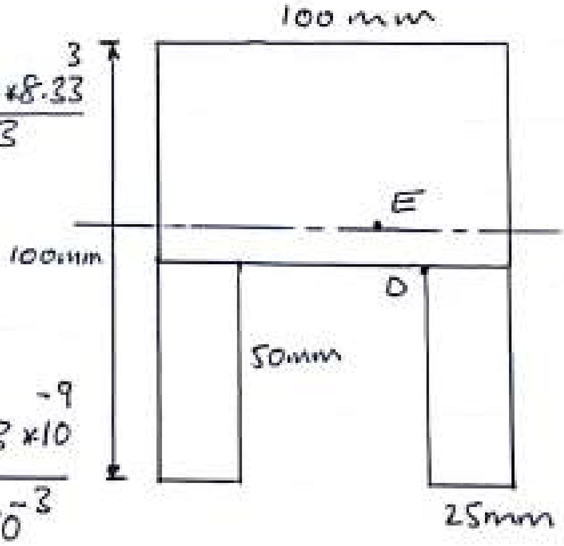
Sol :- $R_A = R_B = 700\,kN$

From S.F.D $\Rightarrow Q = 700 - 300*1$

$= 400\,kN$

$$\bar{y} = \frac{\Sigma A y}{\Sigma A} = \frac{100*100*50 - 50*50*25}{100*100 - 50*50}$$

$\Rightarrow \bar{y} = 58.33\ mm$



$$I = \frac{100*41.67^3}{3} + 2*\frac{25*58.33^3}{3} + \frac{50*8.33^3}{3}$$

$= 5.72*10^6\ mm^4$

$$\tau_E = \frac{Q A \bar{y}}{I \cdot b}$$

$$= \frac{400*10^3*(41.6*100)*20.83*10^{-9}}{5.72*10^{-6}*100*10^{-3}}$$

$= 60.6\ MPa.$

$$\tau_D = \frac{400*10^3*(50*100)*16.66*10^{-9}}{5.72*10^{-6}*100*10^{-3}} = 58.4\ Mpa.$$

$$\tau_D = \frac{400*10^3*50*100*16.66*10^{-9}}{5.72*10^{-6}*50*10^{-3}} = 116.8\ MPa.$$

* Circular Section :-

$$\tau . b_1 = \frac{Q}{I}\int_{y_1}^{h} b . y . dy$$

$$\Rightarrow \tau . 2x_1 = \frac{Q}{I}\int_{y_1}^{R} 2x . y\, dy$$

$$\Rightarrow \tau . 2\sqrt{R^2-y_1^2} = \frac{4Q}{\pi R^4}\int_{y_1}^{R} 2\sqrt{R^2-y^2}\; y\, dy$$

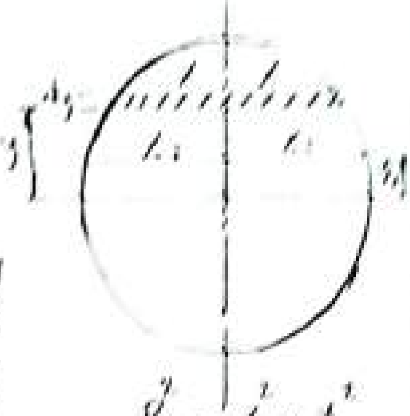
$$\Rightarrow \tau . 2\left(R^2-y_1^2\right)^{1/2} = -\frac{4Q}{\pi R^4}\left[(R^2-y^2)^{3/2} . \frac{2}{3}\right]_{y_1}^{R}$$

$$\Rightarrow \tau . 2(R^2-y_1^2)^{1/2} = \frac{8Q}{3\pi R^4}(R^2-y_1^2)^{3/2}$$

$$\Rightarrow \tau = \frac{4Q}{3\pi R^4}(R^2-y_1^2)$$

at $y_1 = 0 \Rightarrow \tau_{max} = \frac{4}{3}\frac{Q}{\pi R^2}$

$$= \frac{4}{3}\tau_{mean}.$$



$R^2 = x_1^2 + y_1^2$

$R^2 = x^2 + y^2$

* I - Section :- (web)

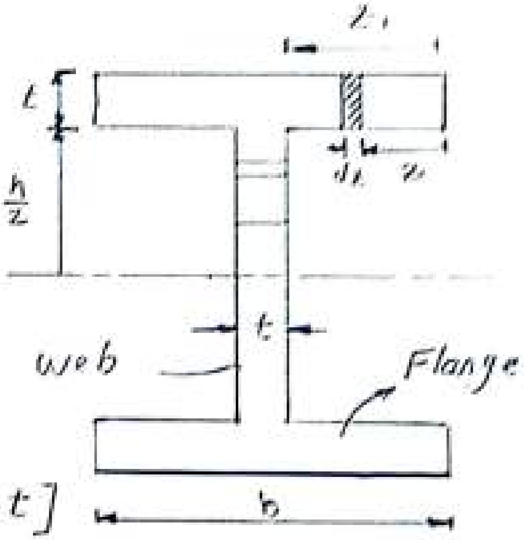
$$\tau . b_1 = \frac{Q}{I}\int_{y_1}^{h} y\, dA$$

$$\Rightarrow \tau . t = \frac{Q}{I}\left[\int_{y_1}^{h/2} y\, t . dy + \int_{h/2}^{h/2+t} y\, b dy\right]$$

$$\Rightarrow \tau = \frac{Q}{2I}\left(\frac{h^2}{4} + bh + bt - y_1^2\right]$$

at $y_1 = 0$

$$\Rightarrow \tau_{max} = \frac{Q}{2I}\left(\frac{h^2}{4} + bh + bt\right]$$



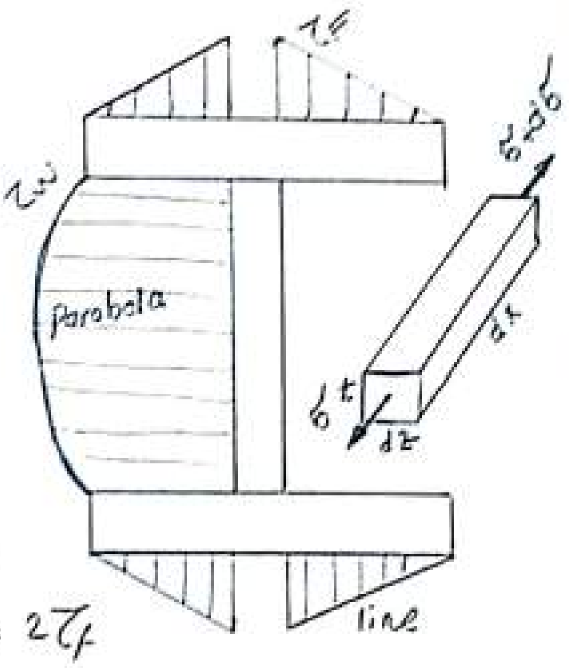
Flange :- $\tau . t . dx = \int_0^{z_1} d\sigma\, t . dz$

But $d\sigma = \frac{dM}{I}\left(\frac{h}{2}+\frac{t}{2}\right)$

$\therefore \tau = \frac{Q}{2I}(h+t)\, z_1$

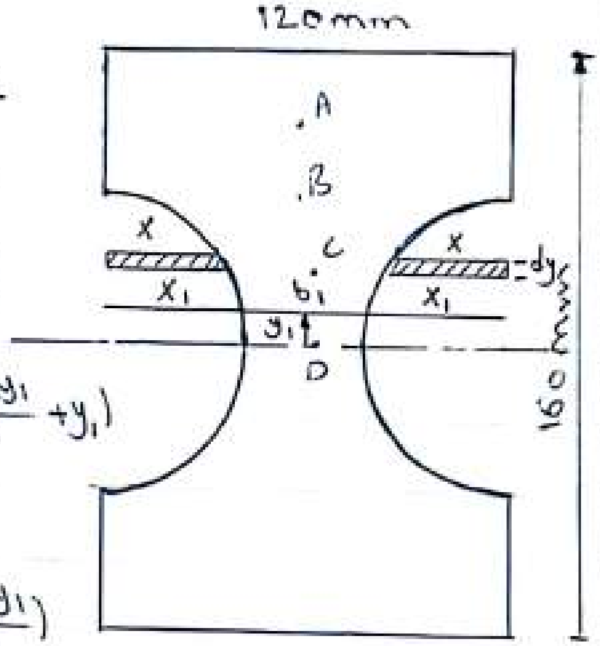
$$\tau_f = \frac{Q}{2I}(h+t) . \frac{b}{2}$$

$$\tau_w = \frac{Q}{2I}(h+t)\, b$$



Equilibrium of the junction common to the flange and the webs: $\tau_w = 2\tau_f$

Example ② If $Q = 200\,kN$ find the shear stress at points A, B, C and D and also find the ratio between the maximum shear stress and mean shear stress.



Sol :- $I = \left[\frac{120 * 160^3}{12} - \frac{\pi}{64}(80)^4\right] * 10^{-12}$

$= 38.95 * 10^{-6}\ m^4$

$\tau (120 - 2x_1) = \frac{Q}{I}(A\bar{y}_{rec.} - A\bar{y}_{cir.})$

$\Rightarrow \tau (120 - 2x_1) = \frac{Q}{I}\left[120(80 - y_1) * \left(\frac{80 - y_1}{2} + y_1\right) - \int_{y_1}^{R} 2xy\,dy\right.$

$\Rightarrow \tau (120 - 2x_1) = \frac{Q}{I}\left[120(80 - y_1)\left(\frac{80 + y_1}{2}\right) - \int_{y_1}^{R} \sqrt{R^2 - y^2}\, 2y\,dy\right.$

$\Rightarrow 2\tau (60 - \sqrt{R^2 - y_1^2}) = \frac{Q}{I}\left[60(80^2 - y_1^2) - \left[-\frac{2}{3}(R^2 - y^2)^{3/2}\right]_{y_1}^{R}\right.$

$= \frac{Q}{I}\left[60(80^2 - y_1^2) - \left(-\frac{2}{3}\left[(R^2 - R^2) - (R^2 - y_1^2)\right]\right)\right]$

$= \frac{Q}{I}\left(60(80^2 - y_1^2) - \left[\frac{2}{3}(R^2 - y_1^2)^{3/2}\right]\right.$

$\Rightarrow \tau (60 - \sqrt{R^2 - y_1^2}) = \frac{Q}{I}\left[\left(30(80^2 - y_1^2) - \frac{1}{3}(R^2 - y_1^2)^{3/2}\right]\right.$

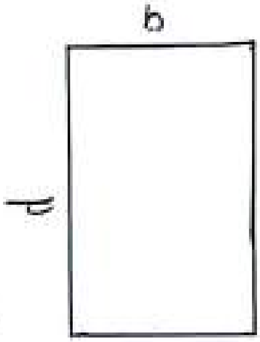
$\Rightarrow \tau (60 - \sqrt{40^2 - y_1^2}) = \frac{200}{38.95 * 10^6}\left[30(80^2 - y_1^2) - \frac{1}{3}(40^2 - y_1^2)^{3/2}\right]$

at $y_1 = 0 \Rightarrow \tau_D = 43.816$ MPa , at $y_1 = 20$ mm $\Rightarrow \tau_C = 33.6$

at $y_1 = 40 \Rightarrow \tau_B = 12.3$ at $y_1 = 60 \Rightarrow \tau_A = 7.2$ MPa

$\tau_m = -$

Example (3) :- If a simply supported beam carry a concentrated load (W) at the center and the beam has a rectangular cross section with depth (d). At what distance be the shear stress equal to the mean shear stress?



Sol :- $\tau_m = \frac{Q}{A} = \frac{W}{2bd}$

$\tau = \frac{Q \cdot A \cdot \bar{y}}{I \cdot b}$, $A = b\left(\frac{d}{2} - y_1\right)$

$\bar{y} = \frac{d/2 - y_1}{2} + y_1 = \frac{d/2 + y_1}{2}$

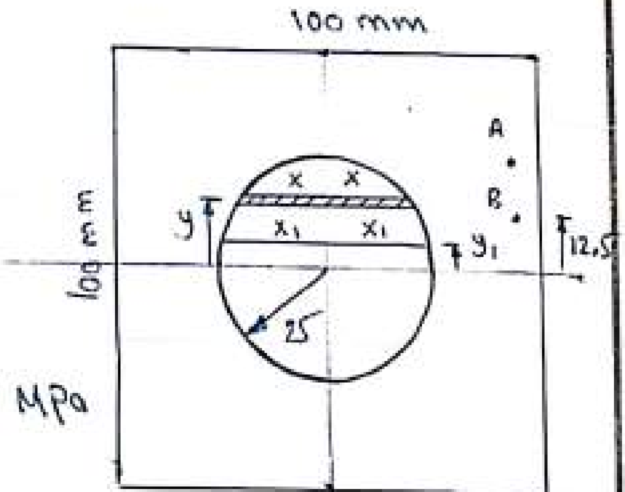
$I = \frac{bd^3}{12}$

$\Rightarrow \tau = \frac{W \cdot b\left(\frac{d}{2} - y_1\right)\left(\frac{d}{2} + y_1\right) * 12}{2 * 2\ bd^3 \cdot b} = \frac{3W\left(\frac{d^2}{4} - y_1^2\right)}{d^3 \cdot b}$

$\tau = \tau_{mean} \Rightarrow \frac{3W\left(\frac{d^2}{4} - y_1^2\right)}{d^3 \cdot b} = \frac{W}{2bd}$

$\Rightarrow 6\left(\frac{d^2}{4} - y_1^2\right) = d^2 \Rightarrow 1.5d^2 - 6y_1^2 = d^2 \Rightarrow 6y_1^2 = \frac{d^2}{2} \Rightarrow y_1 = \frac{d}{\sqrt{12}}$

Example (4) :- $Q = 140$ kN.



Sol :-

$I = \left[\frac{100 * 100^3}{12} - \frac{\pi}{64}(50)^4\right]$

$= 8.02 \times 10^6\ mm^4$

$\tau_A = \frac{140 * (100 * 25) * 37.5}{8.02 \times 10^6 \times 100} = 16.4$ MPa

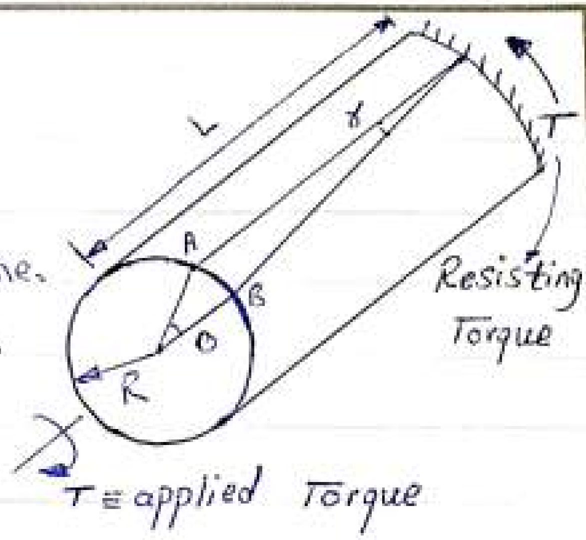
$\tau \cdot b_1 = \frac{Q}{I}\int_{y_1}^{h} y\,dA \Rightarrow \tau(100 - 2X_1) = \frac{Q}{I}\left[100(50 - y_1)\left(50 - \frac{50 - y_1}{2}\right) - \int_{y_1}^{25} y(2x\,dy)\right]$

But :- $25^2 = X^2 + y^2 = x_1^2 + y_1^2$

* Simple torsion theory :-

* Assumptions :-

1. The material is homogeneous.
2. The load is within the elastic zone.
3. Circular sections remain circular.
4. Cross-section remain plane.
5. Cross-section rotate as if rigid.



* Derivation :- $\overset{\frown}{AB} = R\theta = \gamma L$ where :-

$R \equiv$ Radius of circular shaft.

$\theta \equiv$ Angle of twist (unit radian).

$\gamma \equiv$ Angle of distortion (shear strain).

$L \equiv$ Length of the twisted shaft.

But :- $\gamma = \dfrac{\tau}{G} \Rightarrow \dfrac{\tau}{G} \cdot L = R.\theta \Rightarrow \dfrac{\tau}{R} = \dfrac{G.\theta}{L} = \dfrac{\bar{\tau}}{r}$

Where :-

$\tau \equiv$ shear stress at radius (R).

$G \equiv$ Modulus of rigidity.

$\bar{\tau} \equiv$ shear stress at radius (r).

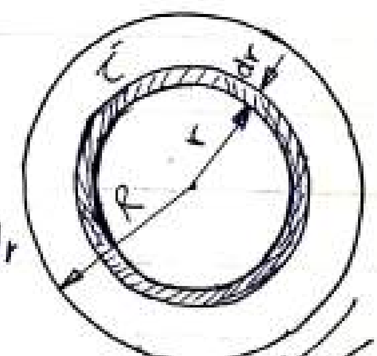
$dF = \bar{\tau} \times 2\pi r\, dr$

$dT = r.dF = 2\pi r^2 \bar{\tau}\, dr$

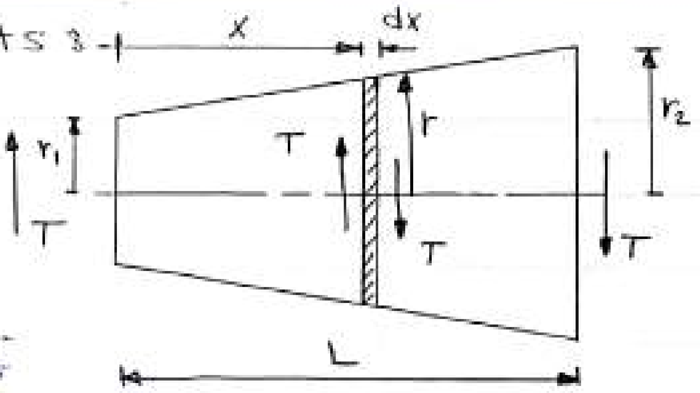
$\therefore \int dT = \int_0^R 2\pi r^2 \bar{\tau}\, dr = \int_0^R 2\pi \left(\dfrac{G.\theta}{L} r\right) r^2\, dr$

$\Rightarrow T = \dfrac{G.\theta}{L} \left(\dfrac{\pi}{2} R^4\right)$

$\therefore T = \dfrac{G.\theta}{L} J \Rightarrow \dfrac{T}{J} = \dfrac{G.\theta}{L}$ Where



* Torsion of tapered shafts :-



$$r = r_1 + \frac{r_2 - r_1}{L} x$$

$$\frac{G.\theta}{L} = \frac{T}{J} \Rightarrow \frac{G.d\theta}{dx} = \frac{T}{\frac{\pi}{2} r^4}$$

$$\therefore \theta = \int d\theta = \int_0^L \frac{T}{G} \cdot \frac{dx}{\frac{\pi}{2}\left(r_1 + \frac{r_2 - r_1}{L} x\right)^4}$$

$$\Rightarrow \theta = \frac{T.L}{G} \cdot \frac{2}{3\pi} \frac{r_1^2 + r_1 r_2 + r_2^2}{r_1^3 . r_2^3}$$

if $r_1 = r_2 \Rightarrow \theta = \frac{T.L}{G} \cdot \frac{1}{\frac{\pi}{2} R^4} = \frac{T.L}{G.J}$

* Power transmitted by shafts :-

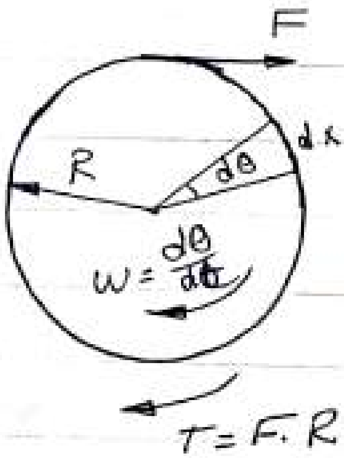
$$\text{Power} = \frac{F.dx}{dt} = \frac{T}{R} \cdot \frac{R d\theta}{dt}$$

$\therefore$ Power $= T.\omega$ where :-

P ≡ Transmitted power (watt)

T ≡ Transmitted Torque (N.m)

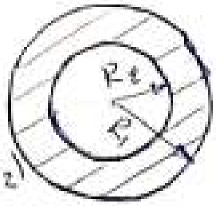
ω ≡ Angular speed (rad/sec)



J = Polar Second moment of area.

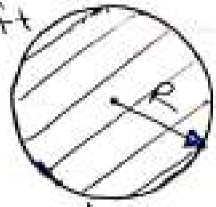
hollow shaft

$J = \frac{\pi}{2}(R_1^4 - R_2^4)$



Solid shaft

$J = \frac{\pi}{2} R^4$

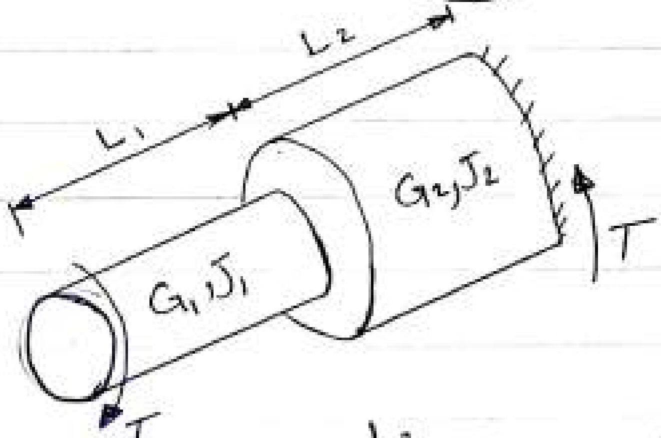


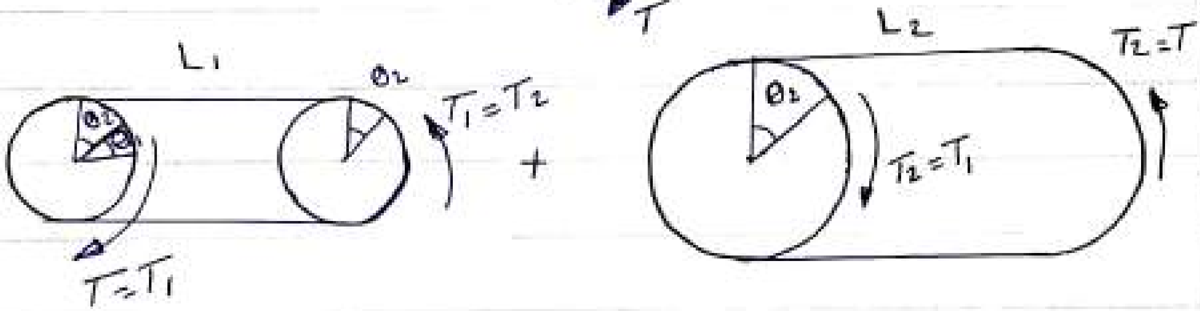
* Compound shafts :-

① Series Connection :-

Total $\theta = \theta_1 + \theta_2$

$T = T_1 = T_2$

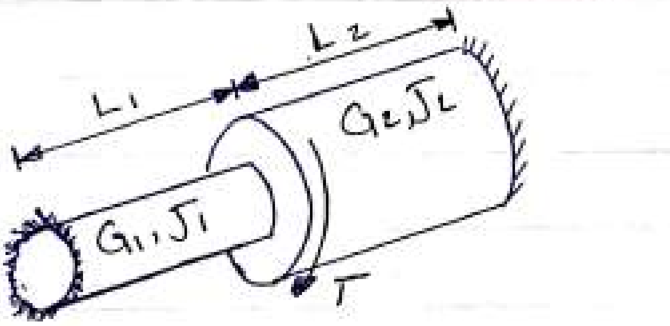


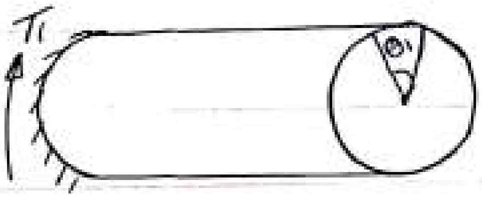


② Parallel Conection :-

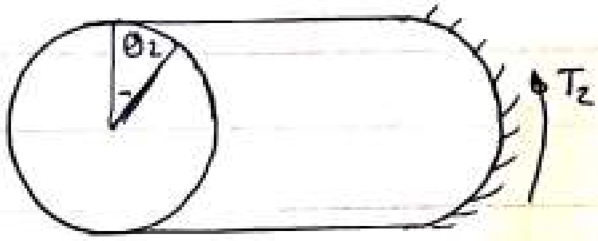
Total $T = T_1 + T_2$

$\theta = \theta_1 = \theta_2$





$\theta_1 = \theta_2$



Solution :-

$$\theta_1 = \theta_2 \Rightarrow \frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2} \Rightarrow \frac{1.8\, T_1}{\frac{\pi}{2} * 25^4} = \frac{1.2\, T_2}{\frac{\pi}{2} * 12.5^4}$$

$$\left.\begin{array}{l} \Rightarrow T_1 = 10.67\, T_2 \rightarrow (1) \\ T_1 + T_2 = 680 \rightarrow (2) \end{array}\right] \Rightarrow T_1 = 622 \;,\; T_2 = 58\ N.m$$

$$\tau_1 = \frac{T_1 R_1}{J_1} = \frac{622 * 25 * 10^{-3}}{\frac{\pi}{2} * 25^4 * 10^{-12}} = 25.33\ MPa$$

$$\tau_2 = \frac{T_2 R_2}{J_2} = \frac{58 * 12.5 * 10^{-3}}{\frac{\pi}{2} * 12.5^4 * 10^{-12}} = 19\ MPa.$$

$$\theta = \theta_1 = \theta_2 = \frac{T_1 L_1}{J_1 G_1} = \frac{622 * 1.8}{\frac{\pi}{2} * 25^4 * 10^{-12} * 80 * 10^9} = 0.0228\ rad = 1.3^\circ$$

H.W :- A flanged coupling having six bolts placed at a pitch circle diameter of 180mm connects two lengths of solid steel shafting of the same diameter. The shaft is required to transmit 80 kW at 240 rev/min. Assuming the allowable intensities of shearing stress in the shaft and bolts are 75 MPa and 55 MPa respectively and the maximum torque is 1.4 times the mean torque, find :-

(a) the diameter of the shaft.

(b) the diameter of the bolts.

Example ① :- Determine the dimensions of a hollow shaft with a diameter ratio 3:4 which is to transmit 60 KW at 200 rev/min. The maximum shear stress in the shaft is limited to 70 MPa and the angle of twist to 3.8 in a length of 4m. For the shaft material G = 80 GPa.

Solution :- $P = T \cdot \omega \Rightarrow T = P/\omega$ $\qquad \omega = \frac{2\pi}{60} * 200$

$$\Rightarrow T = \frac{60 * 10^3}{\frac{2\pi}{60} * 200} = 2860 \text{ N.m}$$

$$J = \frac{T \cdot R}{\tau} \Rightarrow \frac{\pi}{2}\left[R^4 - (0.75R)^4\right] = \frac{2860 \cdot R}{70 * 10^6} \Rightarrow R = 33.65 \text{ mm}$$

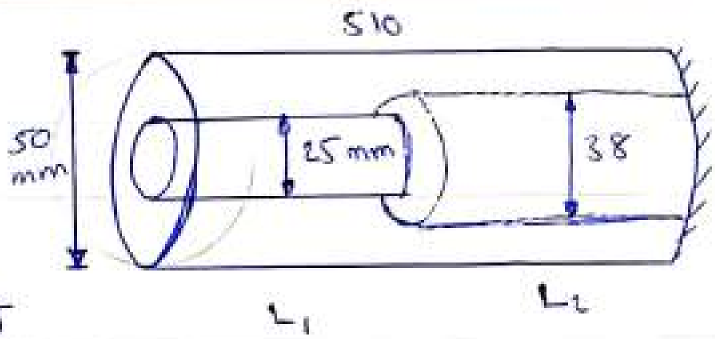
$$\theta = 3.8 * \frac{\pi}{180} \Rightarrow J = \frac{T \cdot L}{G \cdot \theta} \Rightarrow \frac{\pi}{2}\left[R^4 - (0.75R)^4\right] = \frac{2860 * 4}{80 * 10^9 * 3.8 * \frac{\pi}{180}}$$

$$\Rightarrow R = 37.65 \text{ mm} \qquad \therefore R = 37.65 \text{ mm. (max)}$$

Example ② :- A shaft is (510mm) long and (50 mm) external diameter. For part of its length it is bored to a diameter of 25mm and for the rest to 38mm diameter such that the angle of twist in both parts is the same. Find :-

① The maximum power transmitted at a speed of 210 rev/min if the shear stess is not exceed (700 MPa).

② The total angle of twist (G = 80 GPa).



Solution 8- ①

$J = \frac{\pi}{2}(25^4 - 19^4)$
$= 0.41 \times 10^6 \ mm^4$

$\frac{T.R}{J} = \tau \Rightarrow T = \frac{\tau.J}{R}$
$= \frac{70 \times 10^6 \times 0.41}{25 \times 10^{-3}} = 1150 \ N.m$

$Power = T.W = 1150 \times \frac{2\pi \times 210}{60} = 25200$ watt

② $\theta_1 = \theta_2 \Rightarrow \frac{T_1 L_1}{J_1 G_1} = \frac{T_2 L_2}{G_2 J_2} \Rightarrow \frac{L_1}{J_1} = \frac{L_2}{J_2}$

$\Rightarrow \frac{L_1}{L_2} = \frac{J_1}{J_2} = \frac{\frac{\pi}{32}(50^4 - 25^4)}{\frac{\pi}{32}(50^4 - 38^4)} \Rightarrow L_1 = 1.43 L_2 \rightarrow (2)$

$L_1 + L_2 = 510 \rightarrow (2) \Rightarrow L_2 = 210 \quad L_1 = 300$ mm

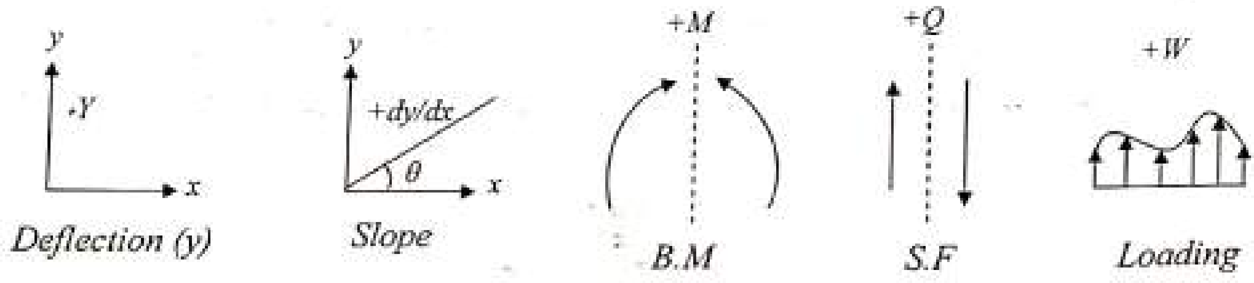
$\theta = \theta_1 + \theta_2 = 2\theta_1$
$= \frac{2 T_1 L_1}{G_1 J_1} = \frac{2 \times 1150 \times 300 \times 10^{-3}}{80 \times 10^9 \times \frac{\pi}{32}(50^4 - 25^4)} = 0.93 \ rad \times 10^{-3}$

=

Example ③ :- A circular bar (ABC), 3 m long is rigidly fixed at its ends (A) and (C). The portion (AB) is 1.8 m long and of 50 mm diameter, and (BC) is 1.2 m long and of 25 mm diameter. If a twisting moment of 680 N.m is applied at B, find the value of resisting moment at (A) and (C) and the maximum stress in each section of the shaft. what will be the angle of twist of each portion? $G = 80 \ GN/m^2$.

Slope and deflection of beams

Convention

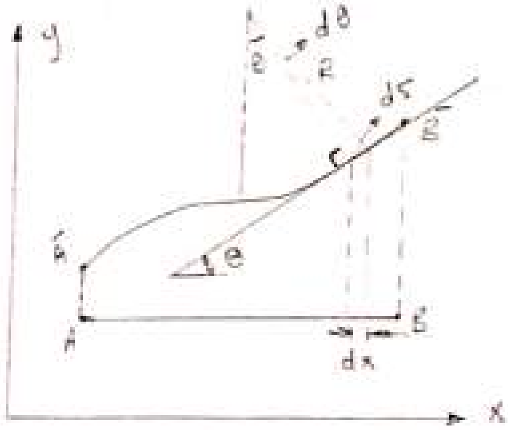


Relations between (loading ,shear force ,bending moment ,slope and deflection)

$\theta = \frac{d_y}{d_x} \equiv$ slope Since: θ is small thus $d_s = d_x$ but $d_s = Rd\theta \Rightarrow Rd\theta = dx \Rightarrow \frac{d\theta}{dx} = \frac{1}{R}$

$\therefore \frac{d^2y}{dx^2} = \frac{1}{R} = \frac{M}{EI} \Rightarrow M = EI\frac{d^2y}{dx^2}$ and $Q = \frac{dM}{dx} \Rightarrow Q = EI\frac{d^3y}{dx^3}$

Finally: $Q = \int w dx \Rightarrow w = \frac{dQ}{dx} \Rightarrow W = EI\left(\frac{d^4y}{dx^4}\right)$



Macaulays method (direct integration)

$M = EI\frac{d^2y}{dx^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{EI}\cdot M$ Integrating

$\frac{dy}{dx} = \frac{1}{EI}\int M_{dx} + A$

$Y = \frac{1}{EI}\int M_{dx} + Ax + B$ Where A and B constants to be found from boundary conditions

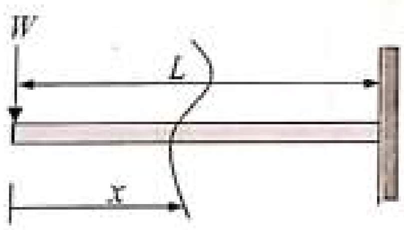
Type of loading:

(1) Cantilever with concentrated load:

$EIy'' = M = -wx$

$EIy' = -\frac{1}{2}wx^2 + A \quad \rightarrow (1)$

$EIy = -\frac{1}{6}wx^3 + Ax + B \quad \rightarrow (2)$



Boundary conditions

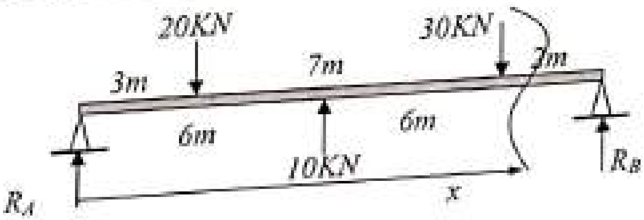
2. $At\ x=0 \Rightarrow M=0 \Rightarrow B=0$

3. $At\ x=L \Rightarrow y'=0 \Rightarrow C=\dfrac{wL^3}{4}$

4. $At\ x=L \Rightarrow y=0 \Rightarrow D=\dfrac{23}{120}wL^4$

$$y=\frac{1}{EI}\left(-\frac{wx^4}{24}-\frac{wx^5}{60L}+\frac{wL^3x}{4}-\frac{23}{120}wL^4\right)$$

- Simply supported beam with concentrated loads:



$$\sum M_B=0 \Rightarrow R_A=15KN$$
$$\sum Fy=0 \Rightarrow R_B=25KN$$

$$EIy''=M=15x-20(x-3)+10(x-6)-30(x-10)$$
$$EIy'=\frac{15}{2}x^2-10(x-3)^2+5(x-6)^2-15(x-10)^2+A \quad \rightarrow(1)$$
$$EIy=\frac{5}{2}x^3-\frac{10}{3}(x-3)^3+\frac{5}{3}(x-6)^3-5(x-10)^3+Ax+B \quad \rightarrow(2)$$

Boundary conditions

1. $At x=0 \Rightarrow y=0 \Rightarrow B=0$

Note: (x-3) = 0 for $x \le 3$
(x-6) = 0 for $x \le 6$
(x-10) =0 for $x \le 10$

2. $At x=12 \Rightarrow y=0 \Rightarrow 0=\frac{5}{2}(12)^3-\frac{10}{3}(9)^3\frac{5}{3}(6)^3-5(2)^3+12A \Rightarrow A=-184.2$

$$EIy=\frac{5}{2}x^3-\frac{10}{3}(x-3)^3+\frac{5}{3}(x-6)^3-5(x-10)^3-184.2x$$

Note : Unit used are (KN,m)
The deflection at the center x=6m $\Rightarrow (x-6)=0$
$(x-10)=0$

$$EIy=\frac{5}{2}(6)^3-\frac{10}{3}(3)^3-184.2(6)=-655.2KN.m^3$$

For E= 208 GPa and $I=82*10^{-6}m^4$
$Y=38.4*10^{-3}m=38.4\ mm$

1. $At x = L \Rightarrow y' = 0 \Rightarrow 0 = \frac{-1}{2} wL^2 + A \Rightarrow A = \frac{1}{2} wL^2$
2. $At x = L \Rightarrow y = 0 \Rightarrow 0 = \frac{-1}{6} wL^3 + \frac{1}{2} wL^2 \cdot L + B \Rightarrow B = \frac{-1}{3} wL^3$

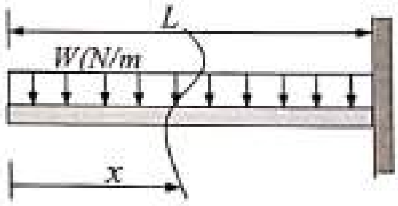
$\therefore EIy = -\frac{1}{6} wx^3 + \frac{1}{2} wL^2 x - \frac{1}{3} wL^3$

$At x = 0 \Rightarrow y = y_{max} \Rightarrow EIy_{max} = -\frac{1}{3} wL^3 \Rightarrow y_{max} = -\frac{wL^3}{3EI}$

- *cantilever beam with uniformly distributed load (U.D.L):*

$EIy'' = M = -wx.\frac{x}{2}$

$EIy' = -\frac{1}{6} wx^3 + A$



$EIy = -\frac{1}{24} wx^4 + Ax + B$

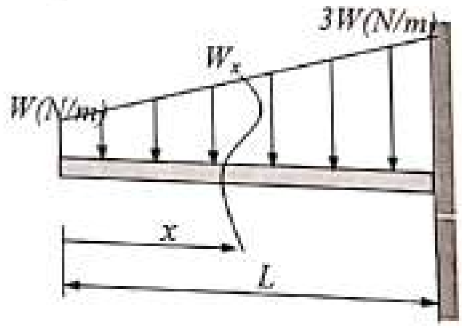
Boundary conditions

1. $At x = L \Rightarrow y' = 0 \Rightarrow -\frac{1}{6} wL^3 + A \Rightarrow A = \frac{1}{6} wL^3$
2. $At x = L \Rightarrow y = 0 \Rightarrow -\frac{1}{24} wL^4 + \frac{1}{6} wL^3 .L + B \Rightarrow B = -\frac{1}{8} wL^4$

$\therefore y = \frac{1}{EI}\left(-\frac{1}{24} wx^4 + \frac{1}{6} wL^3 x - \frac{1}{8} wL^4\right)$

$At x = 0 \Rightarrow y = y_{max} = -\frac{wL^4}{8EI} \qquad y'_{max} = \frac{wL^3}{6EI}$

- *cantilever beam with increasing (U.D.L):*



$wx = EIy'''' = -\left(w + \frac{2w}{L} x\right)$

$EIy''' = -wx - \frac{w}{L} x^2 + A = Q \qquad \rightarrow (1)$

$EIy'' = -\frac{1}{2} wx^2 - \frac{w}{3L} x^3 + Ax + B = M \qquad \rightarrow (2)$

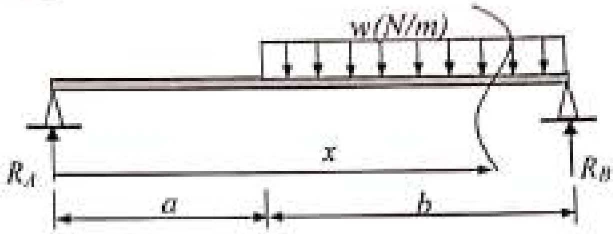
$EIy' = -\frac{1}{6} wx^3 - \frac{w}{12L} x^4 + \frac{1}{2} Ax^2 + Bx + C = \theta \quad \rightarrow (3)$

$EIy = -\frac{1}{24} wx^4 - \frac{w}{60L} x^5 + \frac{1}{6} Ax^3 + \frac{1}{2} Bx^2 + Cx + D \quad \rightarrow (4)$

Boundary conditions

1. $At\ x=0 \Rightarrow Q = 0 \Rightarrow A = 0$

Simply supported beam with (U.D.L):



$\sum M_B = 0 \quad \sum Fy = 0$

R_A and R_B are found

$$\therefore EIy'' = M = R_A.x - w(x-a).\frac{(x-a)}{2}$$

$$= R_A.x - \frac{1}{2}w(x-a)^2$$

$$EIy' = \frac{1}{2}R_A x^2 - \frac{1}{6}w(x-a)^3 + A$$

$$EIy = \frac{1}{6}R_A x^3 - \frac{1}{24}w(x-a)^4 + Ax + B$$

Boundary conditions:

1. At $x = 0 \Rightarrow y = 0 \Rightarrow B = 0$
2. At $x = a + b = L \Rightarrow y = 0 \Rightarrow B = 0 = \frac{1}{6}RaL^3 - \frac{1}{24}w(x-a)^4 + AL \Rightarrow A =$

Simply supported with (U.D.L) over a part of the beam:

$$EIy'' = M = RA.x - w(x-a).\frac{(x-a)}{2} + w(x-b).\frac{(x-b)}{2}$$

$$= R_A.x - \frac{1}{2}w(x-a)^2 + \frac{1}{2}w(x-b)^2$$

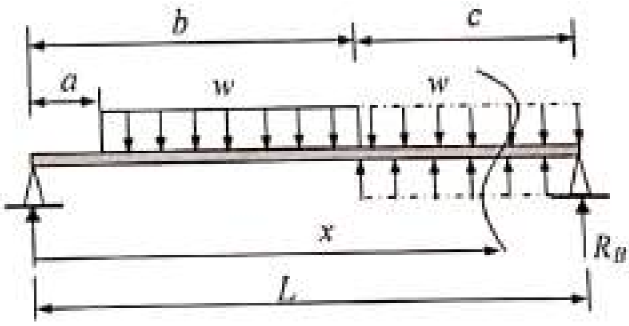
$$EIy' = \frac{1}{2}R_A x^2 - \frac{1}{6}w(x-a)^3 + \frac{1}{6}w(x-b)^3 + A$$

$$EIy = \frac{1}{6}R_A x^3 - \frac{1}{24}w(x-a)^4 + \frac{1}{24}w(x-b)^4 + Ax + B$$

Boundary conditions:

1. At $x = o \Rightarrow y = 0$
2. At $x = L \Rightarrow y = 0$

From these equation A and B are found

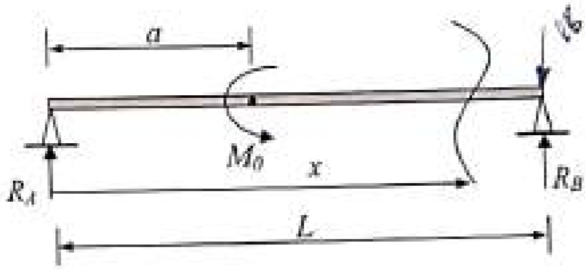


4. Simply supported beam with a couple :

$\sum M_B = 0$

$\sum Fy = 0$

R_A and R_B are found



$$EIy'' = M = R_A.x - M_0(x-a)^0$$

$$EIy' = \frac{1}{2}R_A x^2 - \frac{1}{2}M_0(x-a) + A$$

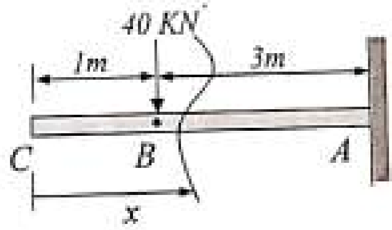
$$EIy = \frac{1}{6}R_A x^3 - \frac{1}{2}M_0(x-a)^2 + Ax + B$$

Boundary conditions :

1. $At\ x = 0 \Rightarrow y = 0 \Rightarrow B = 0$

2. $At\ x = L \Rightarrow y = 0 \Rightarrow 0 = \frac{1}{6}R_A L^3 - \frac{1}{2}M_0(L-a) + AL \Rightarrow A =$

Example (1): Find y_c for the beam shown in the figure?



EI= 65 MN/m^2

Solution:

$$EIy'' = M = -40(x-1)$$

$$EIy' = -20(x-1)^2 + A \rightarrow (1)$$

$$EIy = -\frac{20}{3}(x-1)^3 + Ax + B \rightarrow (2)$$

Boundary conditions:

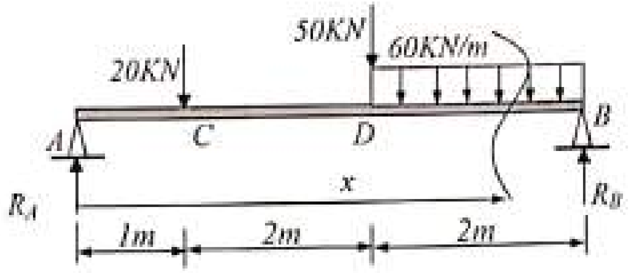
1. $At\ x = 4 \Rightarrow y' = 0 = -20(3)^2 + A$

$\Rightarrow A = 180$

2. $At\ x = 4 \Rightarrow y = 0 \Rightarrow 0 = -\frac{20}{3}(3)^3 + 180(4) + B \Rightarrow B = -540$

$$At\ x = 0 \Rightarrow y_C = \frac{1}{EI}\left(-0 + 0 - 540 = \frac{-540}{65*10^3}\right) = -8.31*10^{-3}\,m$$

Example: (2) find y_d and y_{max} for the beam shown in the figure if E= 200GN/m² and I=83*10⁻⁶ m⁴ ?



Solution:

$$\sum M = o \Rightarrow RA = 60KN$$

$$\sum Fy = 0 \Rightarrow RB = 130KN$$

$$EIy'' = M = 60x - 20(x-1) - 50(x-3) - \frac{1}{2}.60(x-3)^2$$

$$EIy' = M = 30x^2 - 10(x-1)^2 - 25(x-3)^2 - 10(x-3)^3 + A$$

$$EIy = M = 10x^3 - \frac{10}{3}(x-1)^3 - \frac{25}{3}(x-3)^3 - \frac{10}{4}(x-3)^4 + Ax + B$$

Boundary conditions:

1. $At\ x = 5 \Rightarrow y = 0 \Rightarrow 0 = 10(50)^3 - \frac{10}{3}(4)^3 - \frac{25}{3}(2)^4 + 5A \Rightarrow A = -186$

2. $At\ x = 3 \Rightarrow EIy_d = 10(3)^3 - \frac{10}{3}(2)^3 - 186*3 \Rightarrow y_d = -19mm$

② at x = 0 y = 0 ⇒ B = 0

$$EIy'_D = 30(3)^2 - 10(2)^2 - 186 = 44$$

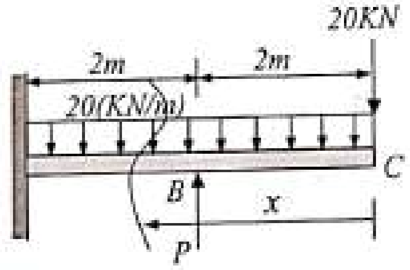
$$EIy'_C = 30(1)^2 - 186 = -156$$

y max occurs between C and D $(1 \overset{<}{\succ} x \overset{<}{\succ} 3)$

$$EIy' = 30x^2 - 10(x-1)^2 - 186 = 0 \Rightarrow x = 2.67m$$

$$\therefore EIy_{max} = 321.8KN.m^3 \Rightarrow y_{max} = \frac{321.8}{200*10^6*83*10^{-6}} = -19.4mm$$

Example (3): Find y_C and y'_C if p=0 and EI= 20 MN/m²



Solution:

$$EIy'' = M = -20x - 20x.\frac{x}{2} = -20x - 10x^2$$

$$EIy' = -10x^2 - \frac{10}{3}x^3 + A$$

$$EIy = -\frac{10}{3}x^3 - \frac{10}{12}x^4 + Ax + B$$

Boundary conditions : (1) $At\ x = 4m \Rightarrow y' = 0 = -10(4)^2 - \frac{10}{3}(4)^3 + A \Rightarrow A = 373.3$

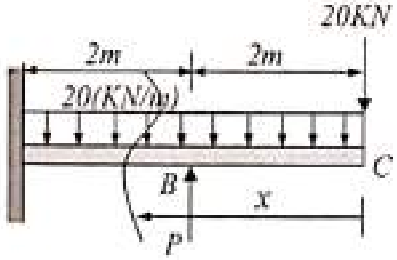
(2) At x =4m

$$\Rightarrow y = 0 = -\frac{10}{3}(4)^3 - \frac{5}{6}(4)^4 + 4*373.3 \Rightarrow B = -1066.6$$

$$y'_c \, at x = 0 = \frac{1}{20*10^3}(373.3) = 18.667*10^{-3}\, rad$$

$$y_c \, at x = 0 = \frac{-1}{20*10^3}(1066.6) = -53*10^{-3}\, m$$

Example: (3B): Find the value of the force P to reduce the deflection of point c to the half?



Solution :

$$EIy'' = M = -20x - 20x.\frac{x}{2} = -20x - 10x^2 + P(x-2)$$

$$EIy' = -10x^2 - \frac{10}{3}x^3 + \frac{1}{2}P(x-2)^2 + A$$

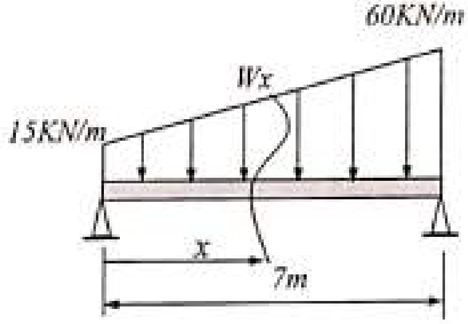
$$EIy = -\frac{10}{3}x^3 - \frac{10}{12}x^4 + \frac{1}{6}P(x-2)^3 + Ax + B$$

Boundary conditions:

1. $At\, x = 0 \Rightarrow y = 26.5*10^{-3}\, m \Rightarrow B = -20*10^3*26.5*10^{-3} = -530$
2. $At\, x = 4\, m \Rightarrow y = 0 = -\frac{10}{3}(4)^3 - \frac{5}{6}(4)^4 + \frac{1}{6}P(2)^3 + 4A - 530 \rightarrow (1)$
3. $At\, x = 4\, m \Rightarrow y' = 0 = -10(4)^2 - \frac{10}{3}(4)^3 + \frac{1}{2}P(2)^2 + A \rightarrow (2)$

From (1) and (2) P= 80 KN

Example (4): Find y at the mid-point if $d = 450\ mm$ $\sigma_{max} = 100\ MN/m^2$ and $E = 210\ GN/m^2$



Solution : $EIy'''' = Wx = -15 - \frac{45x}{L}$

$$EIy''' = -15x - \frac{45x^2}{2L} + A = Qx$$

$$EIy'' = -\frac{15}{2}x^2 - \frac{45}{6L}x^3 + Ax + B = Mx$$

$$EIy' = -\frac{15}{6}x^3 - \frac{45}{24L}x^4 + \frac{1}{2}Ax^2 + Bx + C$$

$$EIy = -\frac{15}{24}x^4 - \frac{45}{120L}x^5 + \frac{1}{6}Ax^3 + \frac{1}{2}Bx^2 + Cx + D$$

Boundary conditions:

1. $At\, x = 0 \Rightarrow y = 0 \Rightarrow D = 0$

2. $At\ x = 0 \Rightarrow M = 0 \Rightarrow B = 0$

3. $At\ x = 7m \Rightarrow M = 0 = -7.5(7)^2 - \frac{45}{7*6}(7)^3 + 7A \Rightarrow A = 105$

4. $At\ x = 7m \Rightarrow y = 0 = -\frac{15}{24}(7)^4 - \frac{45}{120*7}(7)^5 + \frac{1}{6}(105)(7)^3 + 7C \Rightarrow C = -514.5$

$$\therefore EIy = -\frac{15}{24}x^4 - \frac{45}{120L}x^5 + \frac{105}{6}x^3 - 514.5x$$

$$EIy'' = -15x - \frac{45}{2L}x^2 + 105$$

$At\ x = 3.5\ m \quad EI\ y = -1172.35$

$$\sigma_{max} = \frac{M_{max} \cdot y_{max}}{I}$$

$M_{max}\ Occur\ at\ Q_x = 0 \Rightarrow -15x - \frac{45}{2L}x^2 + 105 = 0$

$\Rightarrow x^2 + 4.667x - 32.667 = 0 \Rightarrow x_1 = -8.5m(Neglect)$

$x_2 = 3.8m$

at x = 3.5 m
mid point

Substituting this value in equation of M_x

$$M_{max} = -\frac{15}{2}(3.84)^2 - \frac{45}{6*7}(3.84)^3 + 105(3.84) = 232KN.m$$

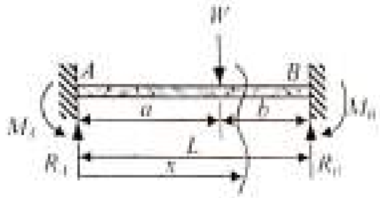
$$100*10^6 = \frac{232*10^3 * \frac{450}{2} * 10^{-3}}{I} \Rightarrow I = 552*10^{-6}\ m^4$$

$$552*10^{-6} * 210*10^6\ y = -1172.36 \Rightarrow y = -10.67mm$$

INDETERMINATE BEAMS

DEFINITIONS: *They are beams with extra supports. The reactions at these supports can not be determined using the equations of equilibrium only, rather the deflection and slope of beams must be concerned.*

1. *Built – in beams with concentrated loads :*



$$\sum Fy=0 \Rightarrow R_A+R_B=W \rightarrow (1)$$

$$\sum M_B=0 \Rightarrow -M_A+R_A.L-W.b+M_B=0 \rightarrow (2)$$

$$EIy''=M_x=-M_A x^0+R_A x-W(x-a)$$

$$EIy'=-M_A x+\frac{1}{2}R_A x^2-\frac{1}{2}W(x-a)^2+A$$

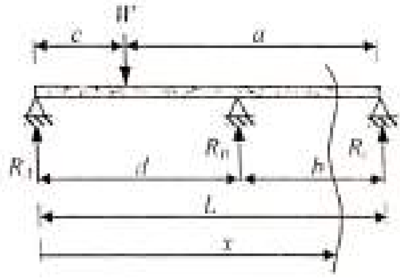
$$EIy=-\frac{1}{2}M_A x^2+\frac{1}{6}R_A x^3-\frac{1}{6}W(x-a)^3+Ax+B$$

Boundary conditions:

1. At $x=0 \Rightarrow y=0 \Rightarrow B=0$
2. At $x=0 \Rightarrow y'=0 \Rightarrow A=0$
3. At $x=L \Rightarrow y'=0=-M_A L+\frac{1}{2}R_A L^2-\frac{1}{2}W(L-a)^2 \rightarrow (3)$
4. At $x=L \Rightarrow y=0=-\frac{1}{2}M_A L^2+\frac{1}{6}R_A L^3-\frac{1}{6}W(L-a)^3 \rightarrow (4)$

Four equations with Four unknown (R_A, R_B, M_A, M_B)

2. *Simply supported beam with concentrated loads:*



$$\sum Fy=0 \Rightarrow R_A+R_B+R_C=W \rightarrow (1)$$

$$\sum M_C=0 \Rightarrow R_A.L-W.a+R_B.b=0 \rightarrow (2)$$

$$EIy''=M=R_A x-W(x-c)+R_B(x-d)$$

$$EIy'=\frac{1}{2}R_A x^2-\frac{1}{2}W(x-c)^2+\frac{1}{2}R_B(x-d)^2+A$$

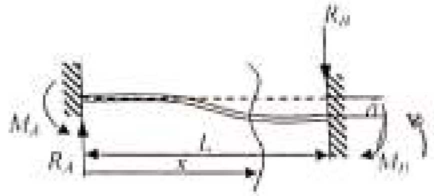
$$EIy=\frac{1}{6}R_A x^3-\frac{1}{6}W(x-c)^3+\frac{1}{6}R_B(x-d)^3+Ax+B$$

Boundary conditions:

1. At $x=0 \Rightarrow y=0 \Rightarrow B=0$
2. At $x=d \Rightarrow y=0=\frac{1}{6}R_A d^3-\frac{1}{6}W(d-c)^3+Ad \rightarrow (3)$
3. At $x=L \Rightarrow y=0=\frac{1}{6}R_A L^3-\frac{1}{6}W(L-c)^3+\frac{1}{6}R_B(L-d)^3+AL \rightarrow (4)$

Four equations with Four unknown (R_A, R_B, R_C, A)

3. *Built – in beams with movement of support:*

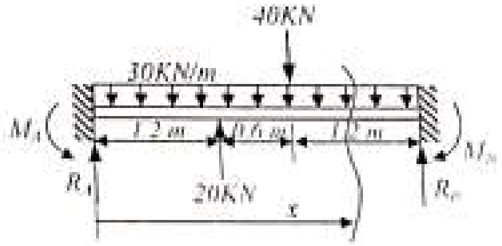


$$\sum Fy=0 \quad \Rightarrow R_A=R_B \quad \to (1)$$
$$\sum MB=0 \quad \Rightarrow M_A+M_B=R_A.L \quad \to (2)$$
$$EIy''=M=-M_Ax^0+R_Ax$$
$$EIy'=-M_Ax+\frac{1}{2}R_Ax^2+A$$
$$EIy=-\frac{1}{2}M_Ax^2+\frac{1}{6}R_Ax^3+Ax+B$$

Boundary conditions:

1. At $x=0 \quad \Rightarrow y=0 \quad \Rightarrow B=0$
2. At $x=0 \quad \Rightarrow y'=0 \quad \Rightarrow A=0$
3. At $x=L \quad \Rightarrow y=\delta \Rightarrow EI\delta=-\frac{1}{2}M_AL^2+\frac{1}{6}R_AL^3 \quad \to (3)$
4. At $x=L \quad \Rightarrow y'=0 \Rightarrow 0=-M_AL+\frac{1}{2}R_AL^2 \quad \to (4)$

Four equations with Four unknown (R_A, R_B, M_A, M_B)



Example (1): $I=42*10^{-6}\ m^4 \qquad y_{max}=100\,mm$

Find σ_{max}.

Solution:

$$\sum Fy=0 \ \Rightarrow R_A+R_B+20=40+30*3 \to (1)$$
$$\sum MB=0 \ \Rightarrow RA*3-MA+20*1.8-40*1.2-30*3*1.5+MB=0 \ \to (2)$$
$$EIy''=M=-M_Ax^0+R_Ax+20(x-1.2)-40(x-1.8)-30x*\frac{x}{2}$$
$$EIy'=-M_Ax+\frac{1}{2}R_Ax^2+10(x-1.2)^2-20(x-1.8)^2-5x^3+A$$
$$EIy=-\frac{1}{2}M_Ax^2+\frac{1}{6}R_Ax^3+\frac{10}{3}(x-1.2)^3-\frac{20}{3}(x-1.8)^3-\frac{5}{4}x^4+Ax+B$$

Boundary conditions:

1. At $x=0 \quad \Rightarrow y=0 \quad \Rightarrow B=0$
2. At $x=0 \quad \Rightarrow y'=0 \quad \Rightarrow A=0$
3. At $x=3m \quad \Rightarrow y'=0=-M_A(3)+\frac{1}{2}R_A(3)^2+10(1.8)^2-20(1.2)^2-5(3)^3 \ \to (3)$

4. $At\ x=3m \quad \Rightarrow y=0=-\frac{1}{2}M_A(3)^2+\frac{1}{6}R_A(3)^3+\frac{10}{3}(1.8)^3-\frac{20}{3}(1.28)^3-\frac{5}{4}(3)^4 \rightarrow (4)$

From equations (3) and (4) $\Rightarrow M_A = 25.4\ KN.m \qquad R_A = 46.1\ KN$

Substituting $R_A = 46.1\ KN$ in eq.(1) $R_B = 63.9\ KN$

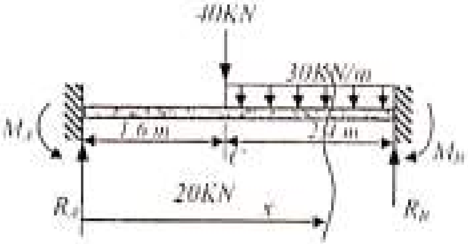
Substituting R_A and M_A in eq.(2) $M_B = 34.9\ KN.m$

Depending on shear force diagram the maximum bending moment occur at one of three points (A,B,at x=1.8m)

$At\ x = 1.8\ m \Rightarrow M = -24.4 + 46.1(1.8) + 20(0.6) - 15(1.8)^2 \ = 21.04\ KN.m$

$\therefore M_{max} = 34\ KN.m$ at B

$$\sigma_{max} = \frac{M_{max} Y_{max}}{I} = \frac{34*10^3*100*10^{-3}}{42*10^{-6}} \quad = 81*10^6 \frac{N}{m^2}$$



Example(2): $EI=14MN/m^2$

Find RA, RB, MA, MB and yc

Sol:

$\sum Fy=0 \quad \Rightarrow R_A + R_B = 40 + 2.4(30) \quad \rightarrow (1)$

$\sum MB=0$

$$\Rightarrow -M_A + R_A(4) - 40(2.4) - 30(2.4)\frac{2.4}{2} + M_B = 0 \qquad \rightarrow (2)$$

$$EI\ y'' = M = -M_A x^0 + R_A x - -40(x-1.6) - \frac{1}{2}(30)(x-1.6)^2$$

$$EI\ y' = -M_A x + \frac{1}{2}R_A x^2 - -20(x-1.6)^2 - 5(x-1.6)^3 + A$$

$$EI\ y = -\frac{1}{2}M_A x^2 + \frac{1}{6}R_A x^3 - -\frac{20}{3}(x-1.6)^3 - \frac{5}{4}(x-1.6)^4 + Ax + B$$

1. $At\ x=0 \quad \Rightarrow y=0 \qquad \Rightarrow B=0$
2. $At\ x=0 \quad \Rightarrow y'=0 \qquad \Rightarrow A=0$
3. $At\ x=4m \quad \Rightarrow y'=0=-M_A(4)+\frac{1}{2}R_A(4)^2+20(2.4)^2-5(2.4)^3 \rightarrow (3)$
4. $At\ x=3m \quad \Rightarrow y=0=-\frac{1}{2}M_A(4)^2+\frac{1}{6}R_A(4)^3+\frac{20}{3}(2.4)^3-\frac{5}{4}(2.4)^4 \rightarrow (4)$

From equations (3) and (4) $\Rightarrow RA=44.1KN, MA=42.12\ KN.m$

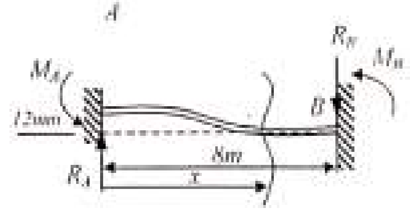
Substituting these values in equation (1) $\Rightarrow RB=67.9\ KN$

Substituting these values in equation (2) $\Rightarrow MB=48.12\ KN.m$

$$\therefore EI\,Yc = -\frac{1}{2}(42.12)(1.6)^2 + \frac{1}{6}(44.1)(1.6)^3$$

$$\therefore Yc = \frac{-23.75}{14*10^3} = -1.7*10^{-3}\,m = -1.7\,mm$$

Example : 3: $E = 210\,GN/m^2$ $I = 90*10^{-6}\,m^4$ *find* R_A, R_B, M_A *and* M_B



$$\sum Fy = 0 \quad \Rightarrow R_A = R_B \quad \rightarrow (1)$$
$$\sum MB = 0 \quad \Rightarrow M_A + M_B = R_A.8 \quad \rightarrow (2)$$
$$EIy'' = M_x = -M_A x^0 + R_A x$$
$$EIy' = -M_A x + \frac{1}{2}R_A x^2 + A$$
$$EIy = -\frac{1}{2}M_A x^2 + \frac{1}{6}R_A x^3 + Ax + B$$

Boundary conditions:

1. $At\ x = 0 \quad \Rightarrow y' = 0 \quad \Rightarrow A = 0$
2. $At\ x = 0 \quad \Rightarrow y = 12\,mm \quad \Rightarrow 90*10^{-6}*210*10^6*0.012 = B = 226.8$
3. $At\ x = 8m \quad \Rightarrow y' = 0 \ = -\frac{1}{2}R_A(8)^2 - M_A(8) \ \Rightarrow R_A = \frac{M_A}{4} \rightarrow (3)$
4. $At\ x = 8m \quad \Rightarrow y = 0 \Rightarrow 0 = \frac{1}{6}R_A(8)^3 - \frac{1}{2}M_A(8)^2 + 226.8$

$$\Rightarrow 85.33\,R_A - 32\,M_A + 226.8 = 0 \quad \rightarrow (4)$$

From equations (3) and (4) $\Rightarrow R_A = 5.3156\,KN \qquad M_A = 21.26\,KN.m$

Substituting these values in (1) and (2) to get :

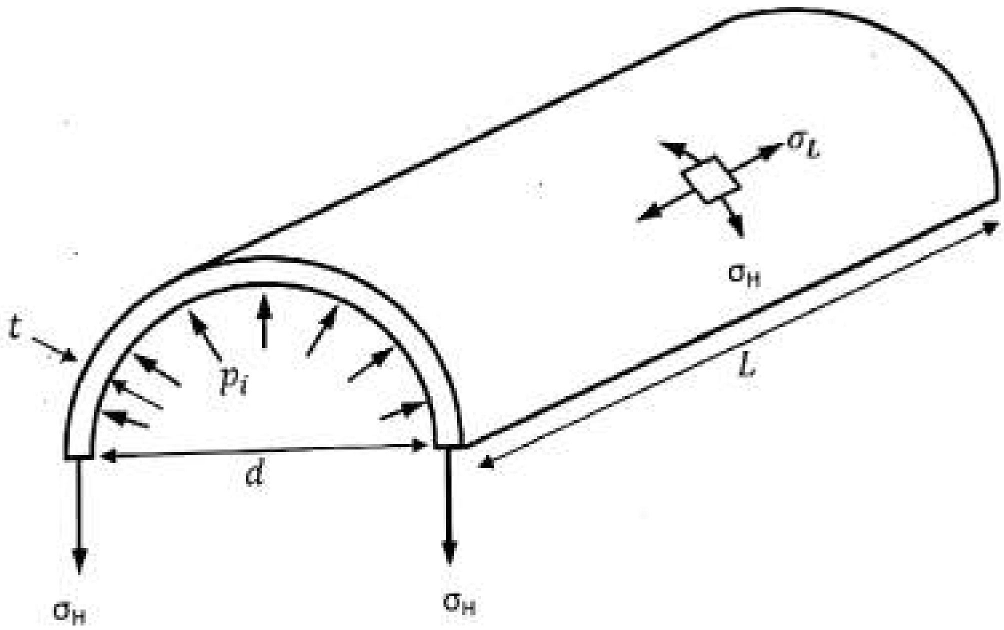
$R_B = 5.3156\,KN \qquad M_B = 21.26\,KN.m$

Thin Cylinders and spheres

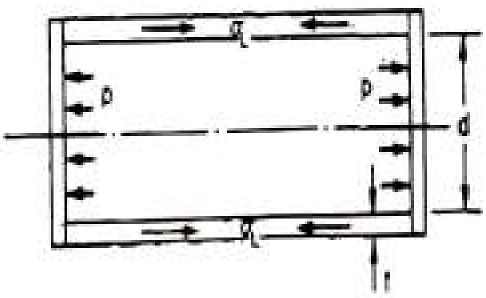
When the thickness of the wall of the cylinderis less than(1/20)of the diameter of cylinder then the cylinder is considered as thin cylinder.Otherwise it is termed as thick cylinder.

Equilibrium of half of the cylinder: $P.d.L = 2\,\sigma_H.\,T.L$

L=Length of the cylinder d= Diameter of cylinder t = thickness of cylinder P= Internal Pressure due to fluid. Circumferential Stress or Hoop Stress (σ_H). Longitudinal Stress (σ_L)



$\sigma_H = \frac{P.d}{2.t}$



Longitudinal stress: Consider now the cylinder as shown. Total force on the end of the cylinder owing to internal pressure: pressure x area = p x $\pi d^2/4$ = $\sigma_L.\pi.D.t$

$$\sigma_L = \frac{P.d}{4.t}$$

Change in Length:

The change in length of the cylinder may be determined from the longitudinal strain, i.e. neglecting the radial stress.

$$\text{Longitudinal strain} = \frac{1}{E}[\sigma_L - \nu\sigma_H]$$

and

$$\text{change in length} = \text{longitudinal strain} \times \text{original length}$$

$$= \frac{1}{E}[\sigma_L - \nu\sigma_H]L$$

$$= \frac{pd}{4tE}[1 - 2\nu]L$$

Change in Diameter:

$$\epsilon_H = \frac{1}{E}(\sigma_H - \nu\sigma_L)$$

$$\frac{\pi(D + \Delta D)}{\pi D} = \frac{1}{E}\left(\frac{PD}{2t} - \nu\frac{PD}{4t}\right)$$

$$\Delta D = \frac{PD^2}{4tE}(2 - \nu)$$

Change in Internal volume:

$$V = \frac{\pi}{4}D^2L \quad \rightarrow \Delta V = \frac{\pi}{4}(D^2\Delta L + L.D.\Delta D)$$

$$\Delta V = \frac{\pi}{4}D^2L\left(\frac{\Delta L}{L} + 2\frac{\Delta D}{D}\right)$$

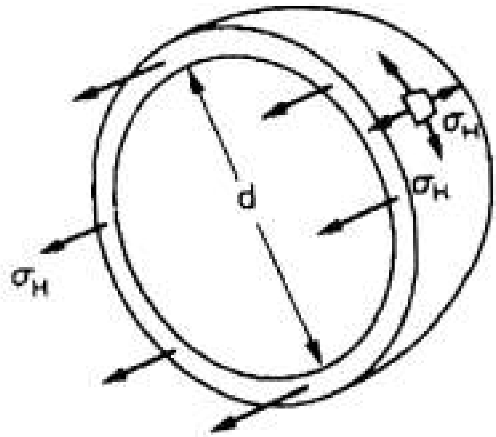
$$\Delta V = V\left[\frac{PD}{4tE}(1-2\nu) + \frac{PD}{4tE}(2-\nu)\right]$$

$$\Delta V = V\left[\frac{PD}{4tE}(5-4\nu)\right]$$

Thin shperes under internal pressure:

Equilibrium of half of the sphere:

Total force on the end of the cylinder owing to internal pressure: pressure x area = p x $\pi d^2/4 = \sigma_H.\pi.D.t$



$$\sigma_H = \frac{P.d}{4.t}$$

$$\epsilon_H = \frac{1}{E}(\sigma_H - \nu\sigma_L)$$

$$\frac{\pi(D+\Delta D)}{\pi D} = \frac{1}{E}\left(\frac{PD}{4t} - \nu\frac{PD}{4t}\right)$$

$$\Delta D = \frac{PD^2}{4tE}(1-\nu)$$

$$V = \frac{\pi}{6}D^3 \quad \rightarrow \Delta V = \frac{\pi}{6}(3D^2\Delta D)$$

$$\Delta V = V.\left(3\frac{\Delta D}{D}\right) = \Delta D = \frac{3PD}{4tE}(1-\nu)$$

Vessels subjected to fluid pressure :

If a fluid is used as the pressurisation medium the fluid itself will change in volume as pressure is increased and this must be taken into account when calculating the amount of fluid which must be pumped into the cylinder in order to raise the pressure by a specified amount, the cylinder being initially full of fluid at atmospheric pressure. Now the bulk modulus of a fluid is defined as follows:

$$\text{bulk modulus } K = \frac{\text{volumetric stress}}{\text{volumetric strain}}$$

where, in this case, volumetric stress = pressure p

and $$\text{volumetric strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{\delta V}{V}$$

$\therefore$ $$K = \frac{p}{\delta V/V} = \frac{pV}{\delta V}$$

i.e. $$\textbf{change in volume of fluid under pressure} = \frac{pV}{K}$$

The extra fluid required to raise the pressure must, therefore, take up this volume together with the increase in internal volume of the cylinder itself.

$\therefore$ extra fluid required to raise *cylinder* pressure by p

$$= \frac{pd}{4tE}[5-4\nu]V + \frac{pV}{K}$$

Similarly, for *spheres*, the extra fluid required is

$$= \frac{3pd}{4tE}[1-\nu]V + \frac{pV}{K}$$

Example: (a) A sphere, 1m internal diameter and 6mm wall thickness, is to be pressure-tested for safety purposes with water as the pressure medium. Assuming that the sphere is initially filled with water at atmospheric pressure, what extra volume of water is required to be pumped in to produce a pressure of 3 MN/m^2 gauge? For water, $K = 2.1\ GN/m^2$. (b) The sphere is now placed in service and filled with gas until there is a volume change of $72 \times 10^{-6}\ m^3$. Determine the pressure exerted by the gas on the walls of the sphere. (c) To what value can the gas pressure be increased before failure occurs according to the maximum principal stress theory of elastic failure? For the material of the sphere $E = 200\ GN/m^2$, $v = 0.3$ and the yield stress σ_y, in simple

tension = 280 MN/m^2.

Solution

(a) Bulk modulus $K = \dfrac{\text{volumetric stress}}{\text{volumetric strain}}$

Now volumetric stress = pressure $p = 3\ MN/m^2$

and volumetric strain = change in volume ÷ original volume

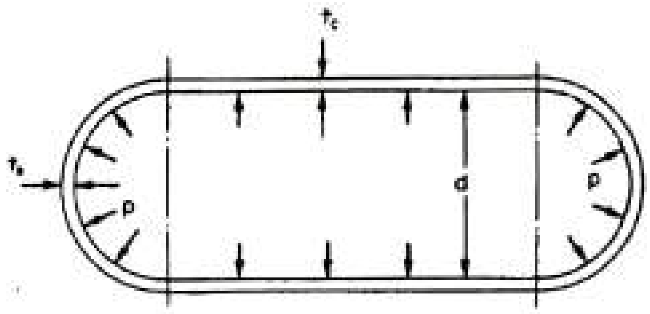
i.e. $$K = \frac{p}{\delta V/V}$$

∴ change in volume of water $= \dfrac{pV}{K} = \dfrac{3 \times 10^6}{2.1 \times 10^9} \times \dfrac{4\pi}{3}(0.5)^3$

$$= \mathbf{0.748 \times 10^{-3}\ m^3}$$

(b) From eqn. (9.9) the change in volume is given by

Cylindrical vessel with hemispherical ends:



Consider now the vessel as shown in which the wall thickness of the cylindrical and hemispherical portions may be different (this is sometimes necessary since the hoop stress in the cylinder is twice that in a sphere of the same radius and wall thickness). For the purpose of the calculation the internal diameter of both portions is assumed equal. From the preceding sections the following formulae are known to apply.

(a) For the cylindrical portion:

$$\text{hoop or circumferential stress} = \sigma_{H_c} = \frac{pd}{2t_c}$$

$$\text{longitudinal stress} = \sigma_{L_c} = \frac{pd}{4t_c}$$

$$\text{hoop or circumferential strain} = \frac{1}{E}[\sigma_{H_c} - \nu\sigma_{L_c}] = \frac{pd}{4t_c E}[2-\nu]$$

(b) For the hemispherical ends:

$$\text{hoop stress} = \sigma_{H_s} = \frac{pd}{4t_s}$$

$$\text{hoop strain} = \frac{1}{E}[\sigma_{H_s} - \nu\sigma_{H_s}] = \frac{pd}{4t_s E}[1-\nu]$$

Thus equating the two strains in order that there shall be no distortion of the junction

$$\frac{pd}{4t_c E}[2-\nu] = \frac{pd}{4t_s E}[1-\nu] \qquad \text{i.e.} \qquad \frac{t_s}{t_c} = \frac{(1-\nu)}{(2-\nu)}$$

$$\delta V = \frac{3pd}{4tE}(1-\nu)V$$

$\therefore$
$$72 \times 10^{-6} = \frac{3p \times 1 \times \frac{4}{3}\pi(0.5)^3(1-0.3)}{4 \times 6 \times 10^{-3} \times 200 \times 10^9}$$

$\therefore$
$$p = \frac{72 \times 10^{-6} \times 4 \times 6 \times 200 \times 10^6 \times 3}{3 \times 4\pi(0.5)^3 \times 0.7}$$

$$= 314 \times 10^3 \text{ N/m}^2 = \mathbf{314\ kN/m^2}$$

(c) The maximum stress set up in the sphere will be the hoop stress,

i.e.
$$\sigma_1 = \sigma_H = \frac{pd}{4t}$$

Now, according to the maximum principal stress theory failure will occur when the maximum principal stress equals the value of the yield stress of a specimen subjected to simple tension,

i.e. when
$$\sigma_1 = \sigma_y = 280 \text{ MN/m}^2$$

Thus
$$280 \times 10^6 = \frac{pd}{4t}$$

$$p = \frac{280 \times 10^6 \times 4 \times 6 \times 10^{-3}}{1}$$

$$= 6.72 \times 10^6 \text{ N/m}^2 = \mathbf{6.7\ MN/m^2}$$

The sphere would therefore yield at a pressure of 6.7 MN/m^2.

With the normally accepted value of Poisson's ratio for general steel work of 0.3, the thickness ratio becomes :

$$\frac{t_s}{t_c} = \frac{0.7}{1.7}$$

i.e. the thickness of the cylinder walls must be approximately 2.4 times that of the hemispherical ends for no distortion of the junction to occur.

Example: A cylinder has an internal diameter of 230 mm, has walls 5 mm thick and is 1 m long. It is found to change in internal volume by 12.0 x m^3 when filled with a liquid at a pressure p . If E = 200GN/m^2 and γ= 0.25, and assuming rigid end plates, determine:

(a) the values of hoop and longitudinal stresses;

(c) the necessary change in pressure p to produce a further increase in internal volume of (longitudinal) are assumed; 15 %. The liquid may be assumed incompressible.

1. Determine the change in volume of a thin cylinder of original volume 65.5 x 10- m3 and length 1.3 m if its wall thickness is 6 mm and the internal pressure **14** bar (1.4 MN/m^2). For the ylinder material E = 210 GN/m^2; $\nu = 0.3$. ans:17.5 x 10-6m^3.]

2. What must be the wall thickness of a thin spherical vessel of diameter 1 m if it is to withstand an internal pressure of 70 bar (7 MN/m^2) and the hoop stresses are limited to 270 MN/m^2

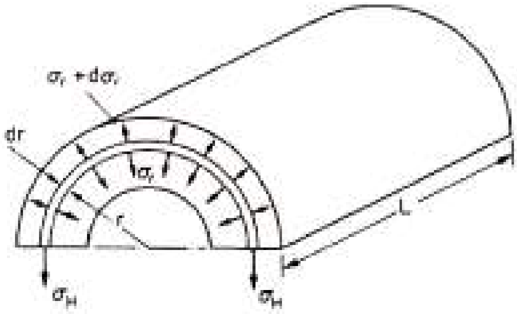
3. A steel cylinder 1 m long, of 150mm internal diameter and plate thickness 5mm, is subjected to an internal pressure of 70bar (7 MN/m^2); the increase in volume owing to the pressure is 16.8 x m3. Find the values of Poisson's ratio and the modulus of rigidity. Assume E = 210GN/m^2.

4. A spherical vessel of 1.7m diameter is made from 12mm thick plate, and it is to be subject4 to a hydraulic test. Determine the additional volume of water which it is necessary to pump into the vessel, which the vessel is initially just filled with water, in order to raise the pressure to the proof pressure of 116 bar (1 1.6 MN/m^2). The bulk modulus of water is 2.9 GN/m^2. For the material of the vessel , E = 200 GN/m^2, v = 0.3. ans:26.14 x10-6 m^3

Thick cylinders:

Lame theory:

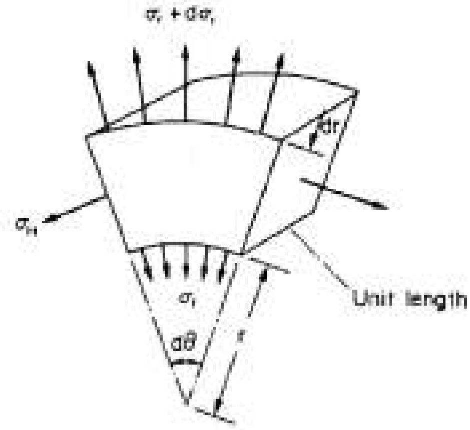
Consider the thick cylinder as shown. The stresses acting on an element of unit length at radius rare as shown in Fig. the radial stress increasing from σ_r to $\sigma_r + d\sigma_r$ over the element thickness dr (all stresses are assumed tensile),



For radial equilibrium of the element:

$$\sum F_r = \quad (\sigma_r + d\sigma_r)(r + dr)d\theta \times 1 - \sigma_r \times rd\theta \times 1 = 2\sigma_H \times dr \times 1 \times \sin\frac{d\theta}{2}$$

For small angles: $\sin\frac{d\theta}{2} = \frac{d\theta}{2}$ radian



Therefore, neglecting second-order small quantities,

$$rd\sigma_r + \sigma_r dr = \sigma_H dr \qquad \sigma_r + r\frac{d\sigma_r}{dr} = \sigma_H$$

Or: $$\sigma_H - \sigma_r = r\frac{d\sigma_r}{dr} \qquad (1)$$

Assuming now that plane sections remain plane, i.e. the longitudinal strain .zL is constant across the wall of the cylinder:

$$\varepsilon_L = \frac{1}{E}[\sigma_L - v\sigma_r - v\sigma_H \quad = \frac{1}{E}[\sigma_L - v(\sigma_r + \sigma_H)] = \text{constant}$$

It is also assumed that the longitudinal stress σ_L is constant across the cylinder walls at points remote from the ends

$$\sigma_r + \sigma_H = \text{constant} = 2A \text{ (say)} \qquad (2)$$

Substituting in (1) for σ_r: $$2A - \sigma_r - \sigma_r = r\frac{d\sigma_r}{dr}$$

Multiplying through by r and rearranging, $2\sigma_r r + r^2 \frac{d\sigma_r}{dr} - 2Ar = 0$

$$\frac{d}{dr}(\sigma_r r^2 - Ar^2) = 0$$

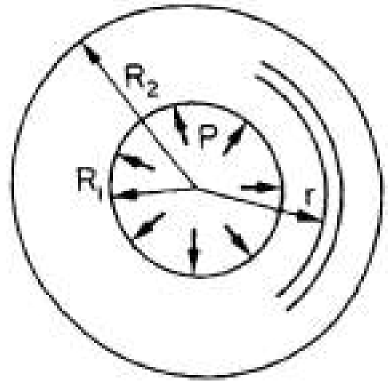
Therefore, integrating, $\sigma_r r^2 - Ar^2 = \text{constant} = -B$ (say) $\quad \sigma_r = A - \frac{B}{r^2}$

And from equation (2) : $\sigma_H = A + \frac{B}{r^2}$

Thick cylinder - internal pressure only:

Consider now the thick cylinder as shown subjected to an internal pressure P, the external pressure being zero:

At $r = R_1 \quad \sigma_r = -P$ and at $r = R_2 \quad \sigma_r = 0$



The internal pressure is considered as a negative radial stress since it will produce a radial compression (i.e. thinning) of the cylinder walls and the normal stress convention takes compression as negative.

Substituting the above conditions in radial stress equation:

$$-P = A - \frac{B}{R_1^2} \qquad \text{And} \qquad 0 = A - \frac{B}{R_2^2}$$

$$A = \frac{PR_1^2}{(R_2^2 - R_1^2)} \quad \text{and} \quad B = \frac{PR_1^2 R_2^2}{(R_2^2 - R_1^2)}$$

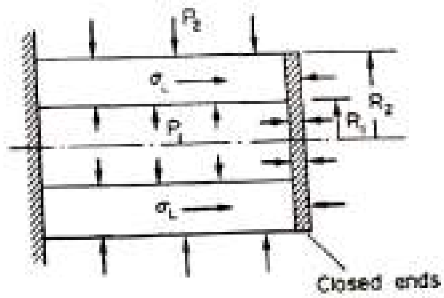
$$\text{radial stress } \sigma_r = A - \frac{B}{r^2} = \frac{PR_1^2}{(R_2^2 - R_1^2)}\left[1 - \frac{R_2^2}{r^2}\right]$$

where k is the diameter ratio $D_2/D_1 = R_2/R_1$

and $\qquad$ hoop stress $\sigma_H = \frac{PR_1^2}{(R_2^2 - R_1^2)}\left[1 + \frac{R_2^2}{r^2}\right]$

$$= \frac{PR_1^2}{(R_2^2 - R_1^2)}\left[\frac{r^2 + R_2^2}{r^2}\right] = P\left[\frac{(R_2/r)^2 + 1}{k^2 - 1}\right]$$

Longitudinal stress: Consider now the cross-section of a thick cylinder with closed ends subjected to an internal pressure P' and an external pressure P_2,



For horizontal equilibrium:

$$P_1 \times \pi R_1^2 - P_2 \times \pi R_2^2 = \sigma_L \times \pi(R_2^2 - R_1^2)$$

where σ_L is the longitudinal stress set up in the cylinder walls,

$$\text{longitudinal stress } \sigma_L = \frac{P_1 R_1^2 - P_2 R_2^2}{R_2^2 - R_1^2}$$

$\sigma_L = A$

Change of cylinder dimensions: (a) change in diameter

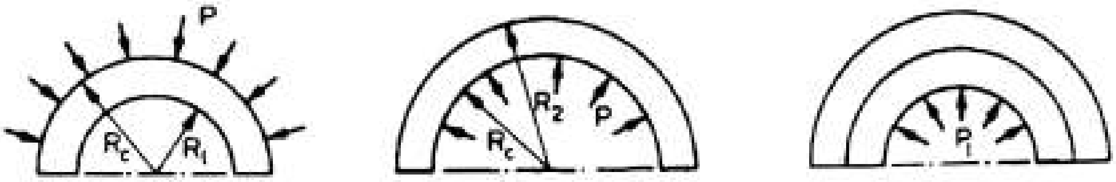
$$\varepsilon_H = \frac{1}{E}[\sigma_H - v\sigma_r - v\sigma_L]$$

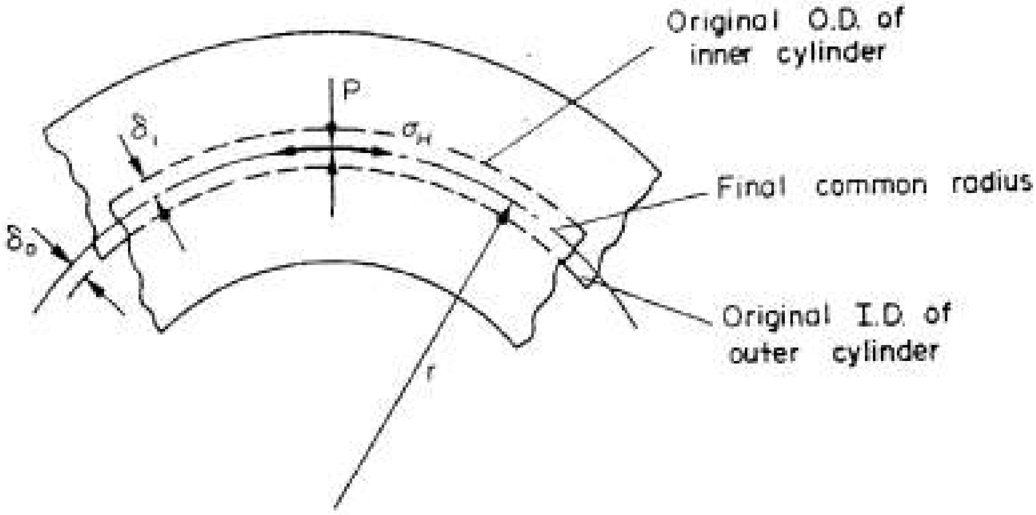
$$\Delta D = \frac{2r}{E}[\sigma_H - v\sigma_r - v\sigma_L]$$

(b) Change of length:

$$\Delta L = \frac{L}{E}[\sigma_L - \sigma v_r - v\sigma_H]$$

Compound cylinders:





(a) *shrinkage-internal cylinder:*

At $r = R_1$, $\sigma_r = 0$

At $r = R_c$, $\sigma_r = -p$ (compressive since it tends to reduce the wall thickness)

condition (b) *shrinkage – external cylinder:*

At $r = R_2$, $\sigma_r = 0$

At $r = R_c$, $\sigma_r = -p$

condition (c) *internal pressure – compound cylinder:*

At $r = R_2$, $\sigma_r = 0$

At $r = R_1$, $\sigma_r = -P_1$

Shrinkage or interference allowance:

since circumferential strain = diametral strain

$$\text{circumferential strain at radius } r \text{ on outer cylinder} = \frac{2\delta_o}{2r} = \frac{\delta_o}{r} = \varepsilon_{H_o}$$

$$\text{circumferential strain at radius } r \text{ on inner cylinder} = \frac{2\delta_i}{2r} = \frac{\delta_i}{r} = -\varepsilon_{H_i}$$

(negative since it is a *decrease* in diameter).

$$\text{Total interference or shrinkage} = \delta_o + \delta_i = r\varepsilon_{H_o} + r(-\varepsilon_{H_i})$$

$$= (\varepsilon_{H_o} - \varepsilon_{H_i})r$$

Now assuming open ends, i.e. $\sigma_L = 0$,

$$\varepsilon_{H_o} = \frac{\sigma_{H_o}}{E_1} - \frac{\nu_1}{E_1}(-p) \qquad \text{since } \sigma_{r_o} = -p$$

and

$$\varepsilon_{H_i} = \frac{\sigma_{H_i}}{E_2} - \frac{\nu_2}{E_2}(-p) \qquad \text{since } \sigma_{r_i} = -p$$

Therefore total interference or shrinkage allowance $= \left[\frac{1}{E_1}(\sigma_{H_o} + \nu_1 p) - \frac{1}{E_2}(\sigma_{H_i} + \nu_2 p)\right]r$

Generally, however, the tubes are of the same material.

$$\textbf{Shrinkage allowance} = \frac{r}{E}(\sigma_{H_o} - \sigma_{H_i})$$

Example1:

A thin cylinder 75 mm internal diameter, 250 mm long with walls 2.5 mm thick is subjected to an internal pressure of $7\ \mathrm{MN/m^2}$. Determine the change in internal diameter and the change in length.

If, in addition to the internal pressure, the cylinder is subjected to a torque of 200 N m, find the magnitude and nature of the principal stresses set up in the cylinder. $E = 200\ \mathrm{GN/m^2}$. $\nu = 0.3$.

Example 1: A thick cylinder of 100 mm internal radius and 150 mm external radius is subjected to an internal pressure of 60 MN/m^2 and an external pressure of 30 MN/m^2. Determine the hoop and radial stresses at the inside and outside of the cylinder together with the longitudinal stress if the cylinder is assumed to have closed ends.

Solution:

at $r = 0.1$ m, $\sigma_r = -60$ MN/m^2 at $r = 0.15$ m, $\sigma_r = -30$ MN/m^2

$$-60 = A - 100B \qquad -30 = A - 44.5B$$

B = 0.54 and A = -6

$$\sigma_H = A + \frac{B}{r^2} = -6 + 0.54 \times 100 = \mathbf{48\ MN/m^2}$$

and at $r = 0.15$ m, $\sigma_H = -6 + 0.54 \times 44.5 = -6 + 24$

$$= \mathbf{18\ MN/m^2}$$

From eqn. (10.7) the longitudinal stress is given by

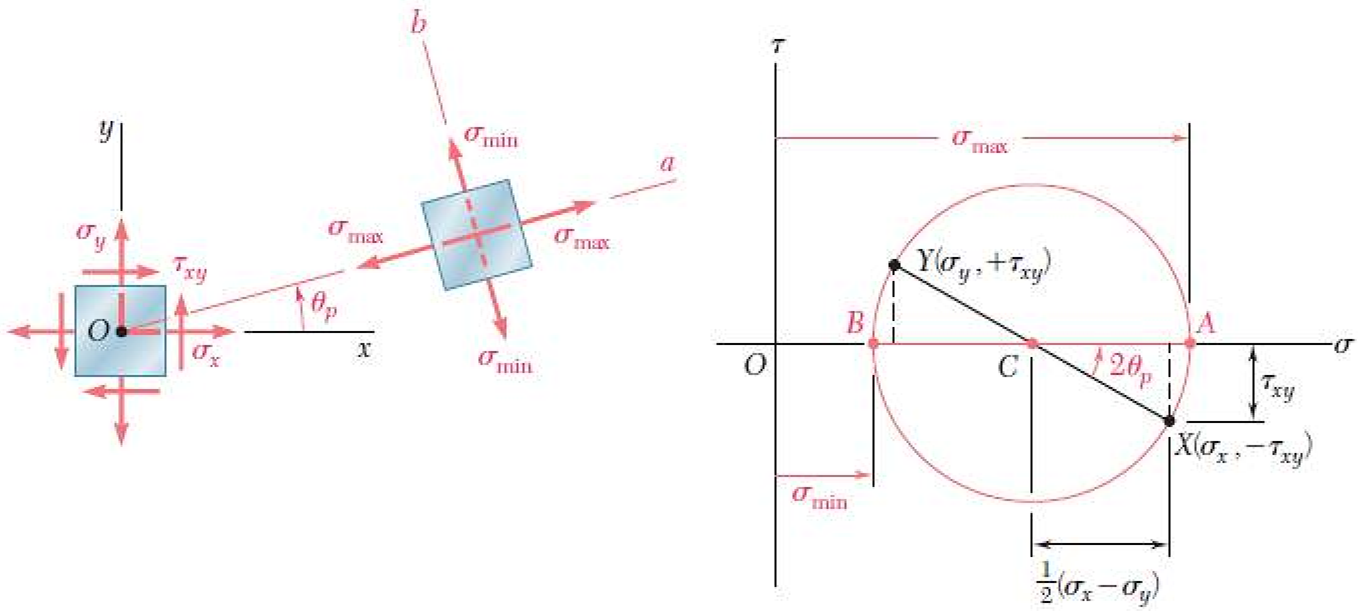
$$\sigma_L = \frac{P_1R_1^2 - P_2R_2^2}{(R_2^2 - R_1^2)} = \frac{(60 \times 0.1^2 - 30 \times 0.15^2)}{(0.15^2 - 0.1^2)}$$

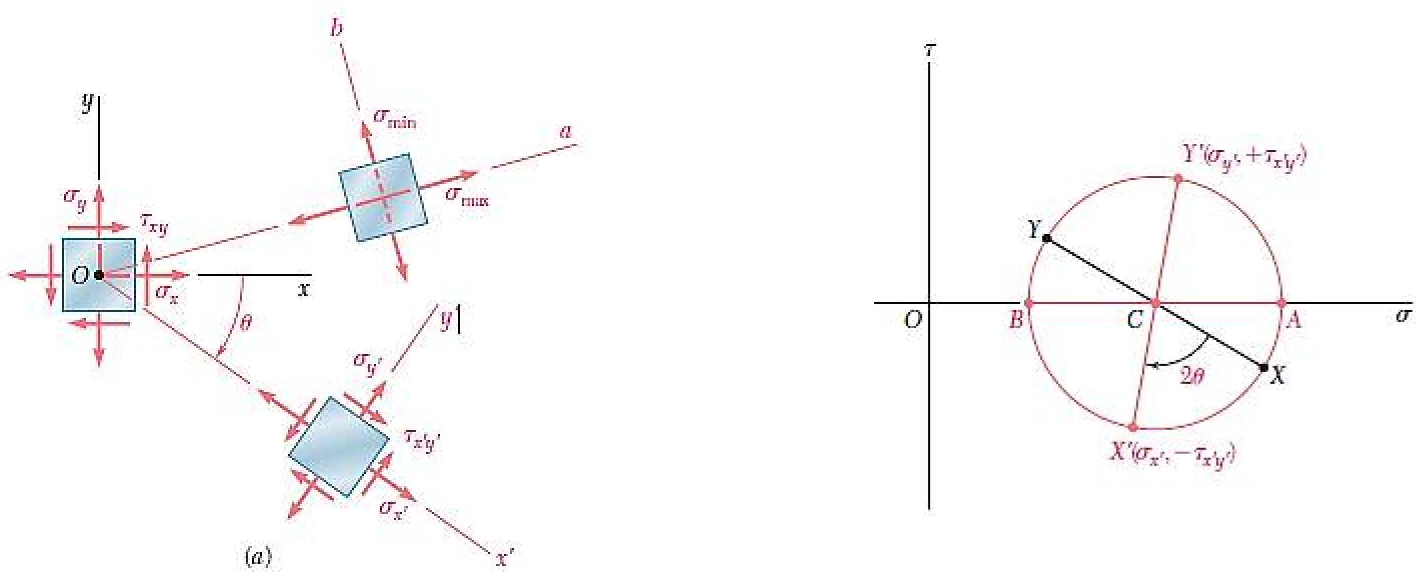
$$= \frac{10^2(60 - 30 \times 2.25)}{1.25 \times 10^2} = -\mathbf{6\ MN/m^2} \quad \text{i.e. compressive}$$

Example An external pressure of 10 MN/m^2 is applied to a thick cylinder of internal diameter 160 mm and external diameter 320 mm. If the maximum hoop stress permitted on the inside wall of the cylinder is limited to 30 MN/m^2, what maximum internal pressure can be applied assuming the cylinder has closed ends? What will be the change in outside diameter when this pressure is applied? E = 207 GN/m^2, v = 0.29.

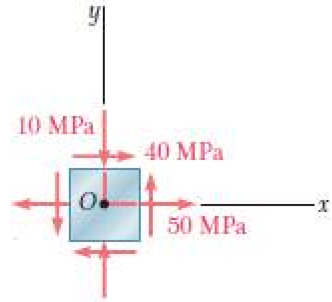
Combined stresses:

The circle used in the preceding section to derive some of the basic formulas relating to the transformation of plane stress was first introduced by the German engineer Otto Mohr (1835–1918) and is known as *Mohr's circle* for plane stress. This method is based on simple geometric considerations and does not require the use of specialized formulas. While originally designed for graphical solutions.



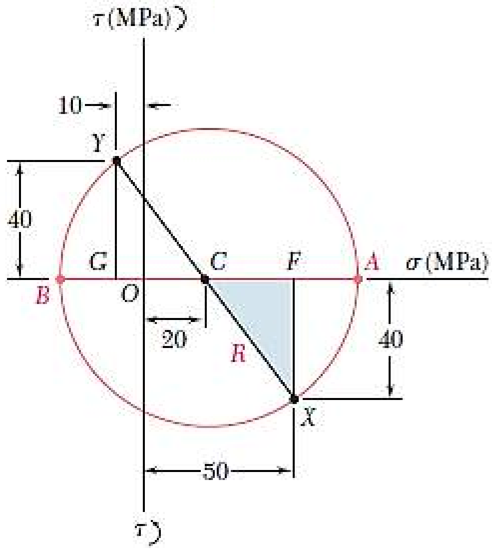


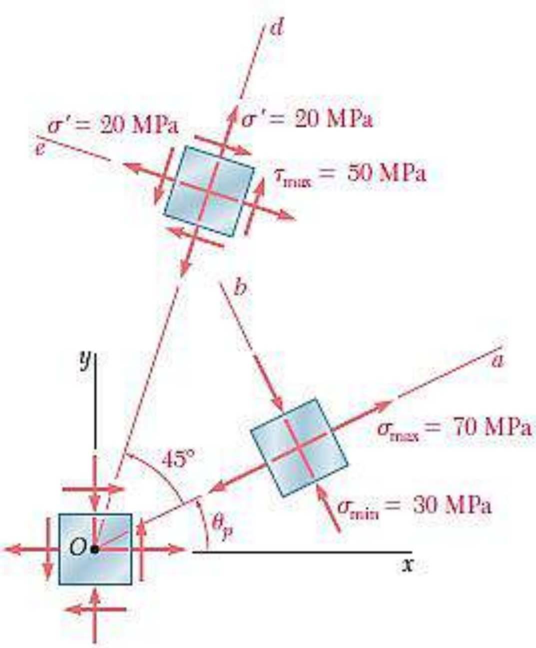
Example1: For the state of plane stress already considered as shown in figure, **(a)** Construct Mohr's circle, **(b)** Determine the principal stresses, **(c)** Determine the maximum shearing stress and the corresponding normal stress.

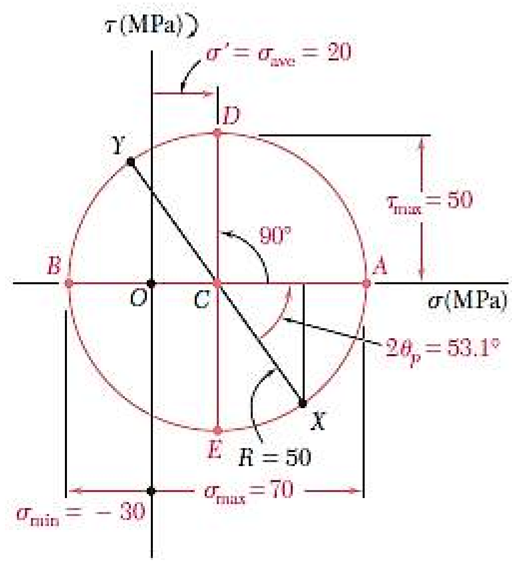


(a)

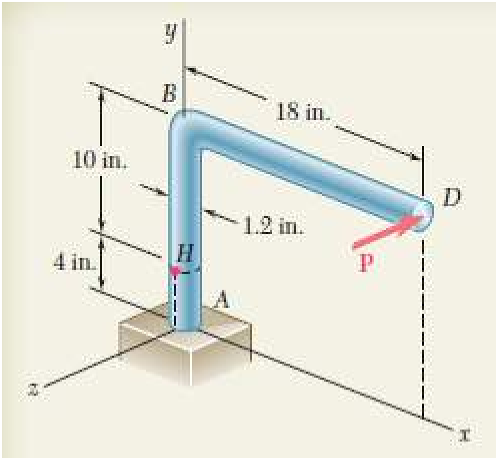
Solution:

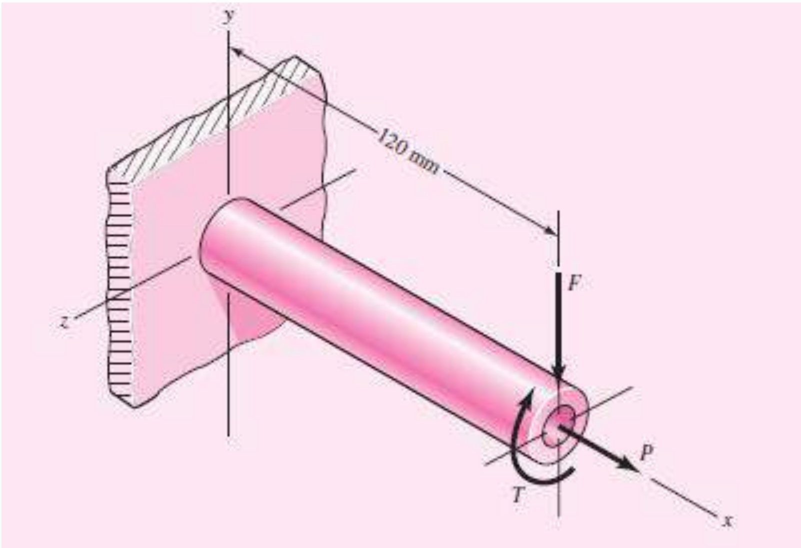




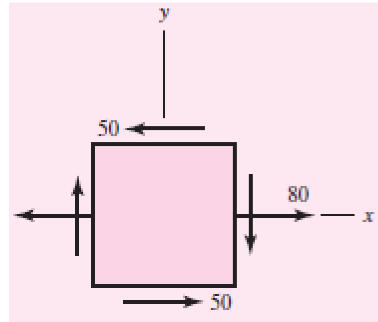


Example2: single horizontal force P of magnitude 150 lb is applied to end D of lever ABD. Knowing that portion AB of the lever has a diameter of 1.2 in., determine (a) the normal and shearing stresses on an element located at point H and having sides parallel to the x and y axes, (b) the principal planes and the principal stresses at point H.

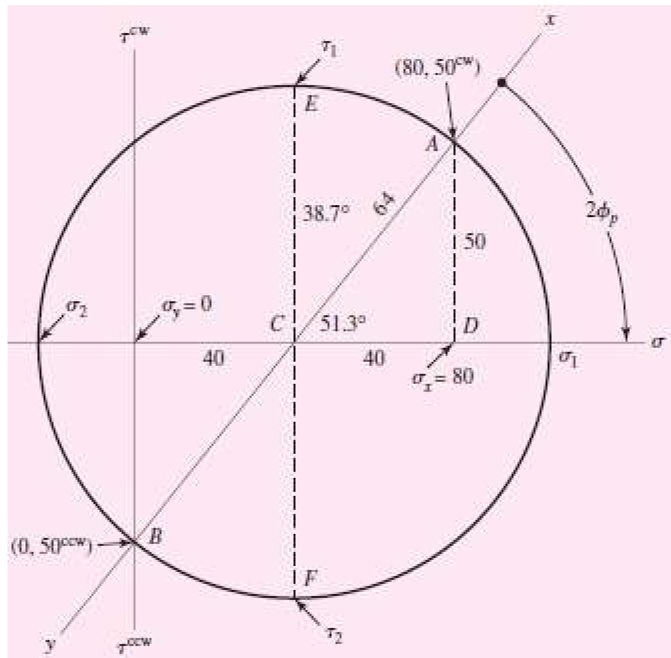


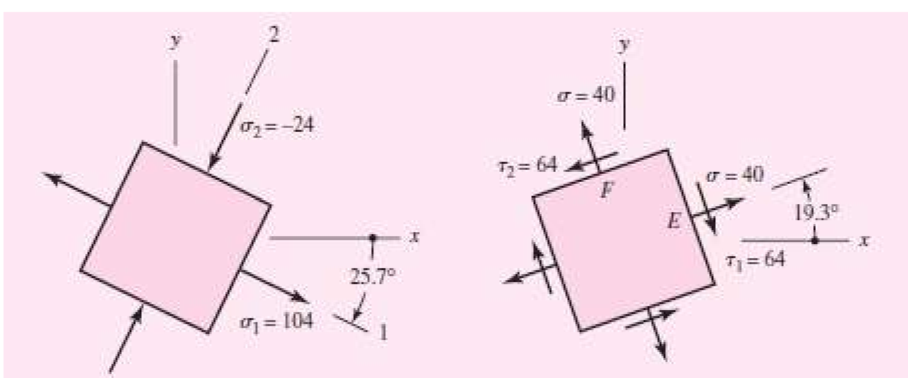


Example3: A stress element has σ_x = 80 MPa and τ_{xy} = 50 MPa cw, as shown in Figure. Using Mohr's circle, find the principal stresses and directions, and show these on a stress element correctly aligned with respect to the xy coordinates. Draw another stress element to show τ_1 and τ_2, find the corresponding normal stresses, and label the drawing completely.



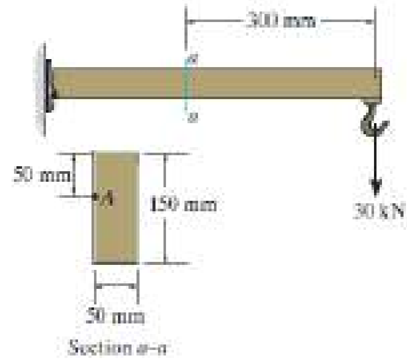
Solution:



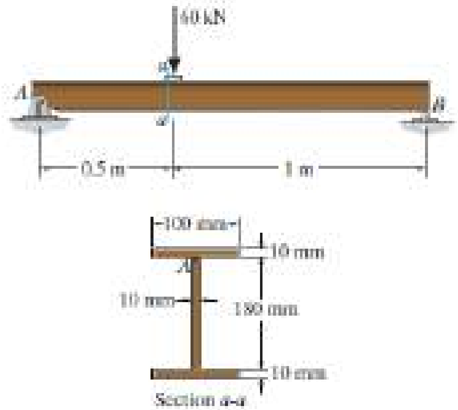


H.w:

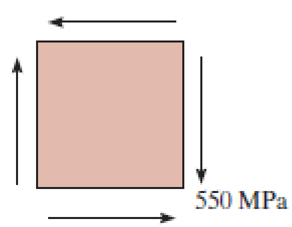
1. Determine the principal stress developed at point A on the cross section of the beam at section a–a.



2. Determine the maximum in-plane shear stress developed at point A on the cross section of the beam at section a–a, which is located just to the left of the 60-kN force. Point A is just below the flange.



3. Determine the equivalent state of stress if an element is oriented 25° counterclockwise from the element shown.



4. Determine the principal stress, the maximum in-plane shear stress, and average normal stress. Specify the orientation of the element in each case.