



LECTURE NOTES IN

# Fundamentals of Digital Communications

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«أنت تعلم ما تعلمه من غير أن تعلمه في المدرسة»

"Education is what remains after you have forgotten  
what you have learned in school." Albert Einstein

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## Course Outline

Week	Revised Syllabus (Last Edition)
1	The Sampling Theory (Shannon's Theorem), Nyquist Sampling Rate & Shannon's Theorem
2	NRZ
3	New Codes in Communications & Error-Correction, CRC & ECC
4	Binary PCM, Differential PCM, Modulation, Noise in PCM, Adaptive PCM
5	QAM, PSK, Frequency and Guard Tones, PCM Mod, PCM Demodulation
6	Channel Capacity, Multi-level Modulation, Coding, PCM
7	Modulation of Error in Modulation
8	Sampling & Problems
9	Introduction to Digital Modulation, QAM (Modulation)
10	QPSK (Modulation)
11	QPSK (Modulation)
12	Modulation Through, Frequency, Power, and Phase, Spectral Efficiency, Modulation
13	Binary Modulation, QPSK (Modulation)
14	QPSK (Modulation)

### References

- 1. Introduction to Communications Systems, Forster & G. G. G. G.
- 2. Introduction to Systems

### Bibliography

- (1) Modern Digital and Analog Communications Systems, B. P. P. P.
- (2) Introduction to Systems as Introduced in Digital and Analog Systems of Communications, B. P. P.
- (3) Communications Systems Engineering, 2nd ed., P. P. P. P.
- (4) Digital and Analog Communications Systems, L. M. M.

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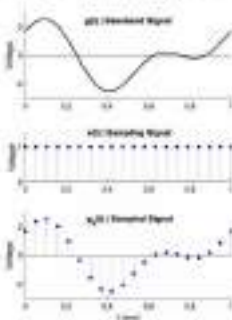
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## Part 1 SAMPLING & PULSE MODULATION

## 1.1 INTRODUCTION

Representations of data and mathematical functions are frequently displayed as continuous signals, even though a finite number of discrete points were used to construct the graph. If these points or samples have sufficiently short spacing, one obtains a smooth curve drawn through these elements. It can be said that a continuous curve is adequately specified by its sample points alone.

Sampling, therefore, enables it to be possible to transmit messages in the form of pulse modulation rather than as continuous signals. Usually, the pulses are quite short compared to the time between them.



Pulse modulation offers two main advantages over continuous-wave modulation:

- (1) The transmitted pulses can be received and then their contents obtained without great loss continuously.
- (2) The more extended between pulses can be filled with signals coming from other signals (TDM).

However, the main disadvantage is the requirement for large transmission bandwidth compared to the original message bandwidth.

## 1.2 SAMPLING THEOREM

### 1.2.1 Ideal Sampling

Ideal sampling\* is done by multiplying the baseband signal  $g(t)$  by the impulse signal  $\delta(t)$ .

$$g_s(t) = g(t) \cdot \delta(t)$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

The Fourier transform is  $s(f)$  is

$$s(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - n/T) \quad \text{where } T = \frac{1}{f_s}$$

Let's now evaluate  $G_s(f)$ , the Fourier transform of the sampled signal  $g_s(t)$ . It

$$G_s(f) = \mathcal{F}\{g_s(t)\} = \mathcal{F}\{g(t) \cdot \delta(t)\}$$

Multiplcation in the time domain is convolution in the frequency domain:

$$G_s(f) = G(f) * S(f) = G(f) * \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - n/T)$$

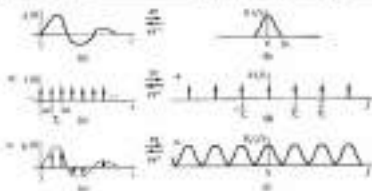
When  $\delta$  is used for convolution, we can apply simple properties of convolution to shift  $G(f)$ .  
Hence the next step is  $G_s(f)$  becomes

$$G_s(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} G(f - n/T)$$

By applying the shifting property of the delta function:

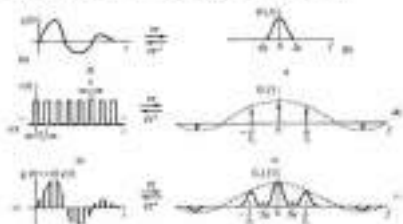
$$G_s(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} G(f - n/T) = \frac{1}{T} \left[ \begin{aligned} &G(f) + G(f - 1/T) + G(f - 2/T) + G(f - 3/T) + \dots \\ &+ G(f - 1/T) + G(f - 2/T) + G(f - 3/T) + \dots \end{aligned} \right]$$

\*This requires the periodic use of an ideal sampling clock to create continuous-time sampling impulses.



### 1.2.2 Ideal Sampling

The practical method of sampling is a naturally sampled signal which is produced by multiplying the bandpass with a carrier signal  $p(t)$  by the periodic pulse train  $s(t)$ .



Here, we also have  $x_s(t) = p(t)x(t) = p(t)x(t) \cdot s(t)$

Assume that  $s(t)$  is a periodic signal

$$s(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_s t}$$

Where  $C_k$  is the Fourier series coefficient  $C_k = \frac{1}{T} \text{rect}\left(\frac{k}{T}\right)$  for

$$x_d(t) = \mathcal{F}\left\{x(t) \sum_{k=-\infty}^{\infty} C_k e^{j2\pi k t}\right\} = \sum_{k=-\infty}^{\infty} C_k \mathcal{F}\{x(t) e^{j2\pi k t}\} = \sum_{k=-\infty}^{\infty} C_k X(f - k/T)$$

For every  $C_k \neq 0$  one copy of  $X(f)$  is added together, where in  $f^*$  was shifted by  $k/T$ , and multiplied by  $C_k$ .

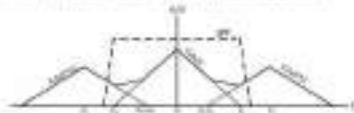
### 1.2.3 Reconstruction

To perform successful sampling, we keep  $f_s = 1/T_s \gg f_m$  (i.e.  $f_s \gg 2f_m$ ). This means the input  $x(t)$  covers a small section in  $C_k(f)$ . As a result,  $X(f)$  can be recovered exactly by simply passing  $C_k(f)$  through a LFF that passes only  $C_0(f)$ . Really, we introduce a part of  $1/T_s$  in the LFF to return the original signal.



### 1.2.4 Aliasing

When  $f_s = 2f_m$ , the copies are kept just barely, the perfect reconstruction becomes impossible. If this Nyquist criterion is not considered, the added half period overlaps the original spectrum. This results a new shape of the reconstructed spectrum filtered by LFF, and this undistortedly gives a signal different from  $x(t)$ . This problem is called "Aliasing".





In the following illustrations, we set  $\beta = 10$  for  $\mu(t) = 0.7$  and  $\beta = 1$  for  $\mu(t) = 0.7$  and  $\beta = 1$  for  $\mu(t) = 0.7$ .

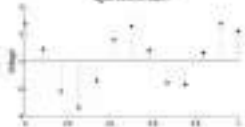
(a) Numerical Solution  $y(t)$  vs  $t$



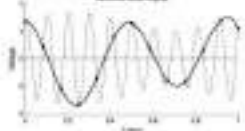
(b) Numerical Solution  $y(t)$  vs  $t$



(c) Numerical Solution



(d) Numerical Solution



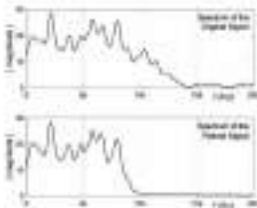
Lowest feasible "digital reconstruction" Sampling frequency $f_s$ (samples/sec)	highest feasible "analog reconstruction" analog frequency $f_a$ (Hz)
--	---

Two conditions are necessary to avoid this aliasing:

- (1) The input signal must be limited according to Nyquist conditions, i.e.,  $f_m \leq \frac{1}{2} f_s$ .
- (2) The sampling frequency must sufficiently greater than the maximum frequency component of the signal, i.e.,  $\frac{1}{2} \geq \frac{f_m}{f_s}$ . This condition is called "Nyquist limit".

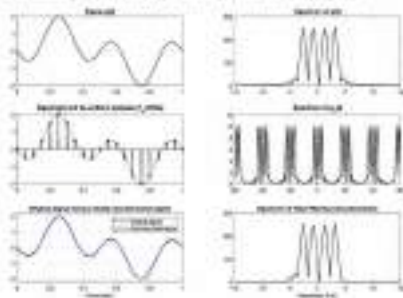


The illustration below shows a bandwidth signal whose spectrum extends up to 20kHz. If we want to perform Nyquist sampling at rate 40kHz, we must make an assumption that where cutoff frequency equals  $\frac{1}{2} \times 40\text{kHz}$ .



Another example from practical video transmission system. Although the average video signal bandwidth extends well up to 3MHz, most of the energy is concentrated in the range 300K–1.2MHz. For analogizing the bandwidth, 3MHz is sufficient. In video transmission, it is often more convenient to use a 3.58MHz LFF and then sampled at 60Hz. These are the standard values for television sampling.

In the following illustration, we set  $f_s = 1000$  for  $g(t) = \sin(2\pi f_s t)$  and  $h(t) = \sin(2\pi f_s t)$ .



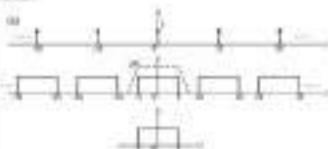
## Exercises

- The waveform  $x(t) = \sin(2\pi f_s t)$  is to be sampled periodically and then to be reconstructed from three sample values. (a) Plot the waveform illustrating time-aliasing between samples. (b) How many samples are needed to be used to reproduce  $x(t)$  exactly (if possible)?
- Draw the waveform (frequency) sampling rate for the signal  $x(t) = \sin(2\pi f_s t)$  where  $f_s$  is fixed to be 10, the signal  $y(t)$  is fixed to be 100,  $f_s$  is fixed to be 10, and  $f_s$  is fixed to be 10.
- A signal  $x(t)$  is fixed to be a sampling rate of 1000, 1000, 1000, 1000. If the sampled signal at the receiver is passed through an ideal LP with cutoff frequency

signal  $x(t)$ . Plot the spectrum of the signal at points A, B, C, and D, based on the following.



Assume:



We see that there is no aliasing:  $\omega_s/2 = \omega_N/2 = \omega_c$

Just make use (a) and (b) and proceed!

- (ii) Repeat the above example if the cutoff frequency of the filter is 10 Hz and  $\omega_c = 20\pi$  rad/s and the analog signal is  $x(t) = 4 \cos(20\pi t)$  and  $f_N = 10$  Hz.

Solution:  $x(t) = 4 \cos(20\pi t) = 2(e^{j20\pi t} + e^{-j20\pi t})$  and the Fourier transform can be written as:

$$X(\omega) = 2\pi \delta(\omega - 20\pi) + 2\pi \delta(\omega + 20\pi) = 2\pi \delta(\omega - 40) + 2\pi \delta(\omega + 40)$$

The spectrum of the ideally sampled signal is given by:

$$X_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) = 4\pi \delta(\omega)$$

So the spectrum of our sampled signal will be:

$$X_s(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \left[ \frac{4\pi}{2\pi} \delta(\omega - 40 - k40) + \frac{4\pi}{2\pi} \delta(\omega + 40 - k40) \right]$$



As shown above, there is no aliasing, and the cutoff frequency of the filter is suitable, and the output consists of the original signal only. We have no distortion. The output of the LFF is:

$$Y(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

Taking the inverse Fourier transform of this equation, we get:

$$y(t) = \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega = \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega$$

13. Suppose  $x(t)$  has the spectrum in the figure below, with  $\omega_c = 2000\pi$  and  $t = 100\pi$ . Sketch  $X_c(f)$  for  $f_c = 1000$ ,  $1000\pi$ ,  $1000\pi$  if ideal sampling is considered. Comment on whether or not the possible reconstruction of  $X_c(f)$  from  $X_c(f)$ .

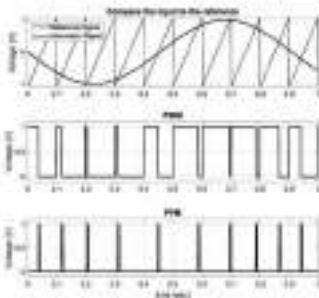


14. A signal  $x(t)$  is band limited to  $f_m$  and is sampled at a sampling rate of  $f_s$  Hz using ideal sampling (impulses). At another location, the sampled signal is filtered by an ideal low-pass filter (the passband is  $f_m$  Hz) and is reconstructed at  $f_s$  Hz. Express the form of the signal  $y(t)$ .

### 1.3 Pulse Modulation

Pulse modulation describes the process whereby the amplitude, the width or the position of individual pulses in a periodic pulse train are varied (i.e., modulated) in sympathy with the amplitude of a bandwidth information signal (i.e.).

The pulses of PAM have	amplitude-variable, width-fixed,	position-fixed
The pulses of PWM have	amplitude-fixed, width-variable,	position-fixed
The pulses of PPM have	amplitude-fixed, width-fixed,	position-variable



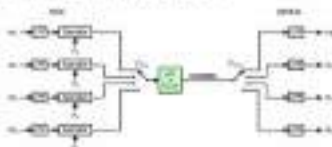
What modulation scheme could be used to sample TV frame numbers?

## 1.6 TIME DIVISION MULTIPLEXING (TDM)

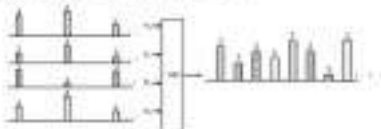
The use of clock under words in TDM, inserts sufficient spaces between samples for insertion of pulses other sampled signals. The result of combining several sampled signals in a defined sequence is called TDM.



The block diagram below demonstrates the TDM-MUX principle. In this system,  $n$  different signals are sampled at the same rate ( $f_s$  samples per second). The clock frequency  $f_{clk}$  must be fast enough to read the  $n$  samples without missing a sample from a signal (in this case  $f_{clk} = f_s$ ). The input signals are pre-filtered to prevent the aliasing. Sampling synchronization between MUX and DEMUX is critical for correct reception.



The diagram below illustrates the process of multiplexing.



In general, we need to understand the exact design for a TDM system. In the diagram below, let  $T_s$  be the time spacing between adjacent samples in a TDM input. If all input signals are sampled equally at  $T_s = 1/N T_c$ , then  $T_c = T_s/N$ , where  $N$  = number of multiplexed sampled input. The output TDM carrier frequency clock frequency is

$$f_c = \frac{1}{T_c} = \frac{N}{T_s} = N f_s$$

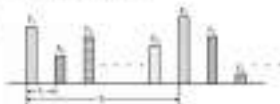


Figure 10.1 TDM

### Exercise

- 10 analog signals each is bandwidth 10 kHz are transmitted simultaneously over a channel via the TDM/TDM system. Compute the required: (i) sampling frequency, (ii) TDM clock frequency, (iii) TDM output transmission rate.
- Several bandwidth signals each bandwidth of 10 kHz can be sampled at the sampling frequency 30% over the Nyquist frequency then multiplexed through a channel with a bandwidth of 120 kHz, what would be the maximum number of signals that can be multiplexed?
- Determine the minimum transmission bandwidth required to transmit the following signals,  $x_1(t) = 20 \cos(2000\pi t)$ ,  $x_2(t) = 10 \cos(3000\pi t) \cos(2000\pi t)$ , and  $x_3(t) = 10 \cos(1500\pi t) + 15 \cos(4000\pi t)$  in a TDM/TDM system.

**Solution:** For TDM,  $f_s(t) = 20 \cos(2000\pi t)$

$x_2(t) = (10 \cos(3000\pi t) \cos(2000\pi t) = 5 \cos(1000\pi t) + 5 \cos(5000\pi t)$

$x_3(t) = 10 \cos(1500\pi t) + 15 \cos(4000\pi t)$

Minimum frequency 100 Hz, (1000 Hz) spectrum, and clock

$$f_c = 15 \times 100 = 1500 \text{ Hz} \rightarrow f_c = 15 \times 100 = 1500 \text{ Hz} \quad f_c = \frac{N}{T_s} = 1500 \text{ Hz}$$



### Transients and Guard Times

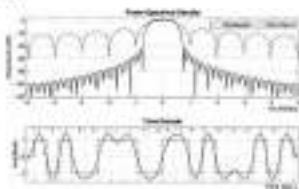
In addition to recording the two transients, ICDW equipment must record channel transients. If ICDW signal buffers have transients, if the transmission channel results in packet errors, both moving into the next frame state of the frame.



If we assume the transmission channel is like a 1<sup>st</sup> order LPF, the response to a rectangular pulse is an exponential decay. To reduce the packet loss, the transmitted pulses must be extended by  $t_p$ . The guard time  $t_g$  is the minimum pulse spacing so that the tail of the pulse decays to a value less than  $A_{th}$  by the time each pulse arrives.



### Pulse Shaping



## 1.3 PULSE CODE MODULATION (PCM)

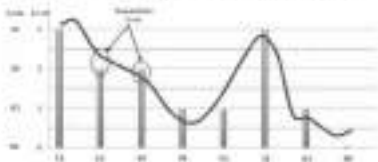
After sampling an analog signal, it is possible to code the samples into bits, such as binary. The transmission through binary alphabets is more efficient than the using numerals. If the value of a sample is sent using only two possible elements, the required bandwidth is small. If it is used for the decoder to reconstruct the waveform into two possible signals that represent the value of a continuous signal. This significantly helps in reducing the effects of distortion and additive noise.

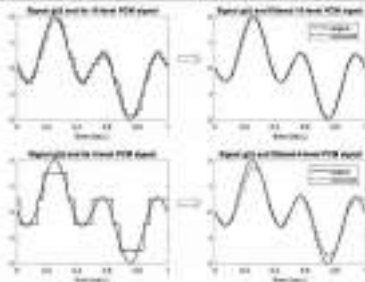
In PCM, the sample values are rounded to the values of certain levels. The rounding off operation is known as "quantization". These quantized samples are coded into binary number which is equivalent to the value of the quantizer level that has the least to the sample value.

### 1.3.1 Quantization

When an analog signal's amplitude is quantized, it becomes discrete in time only. It remains analog in amplitude since discrete values within the range are still present. This signal is said to be "quantized", when each point of the PCM signal is reduced in amplitude to certain value by means of quantization levels.

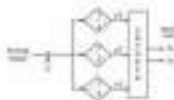
It is clear from the figure below, quantization error (noise) can be reduced by increasing the number of quantization levels (i.e. i.e. decreasing the value that is between the levels).





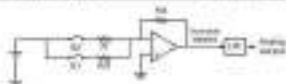
### 3.5.2 Encoding

The quantization samples are now to be converted into binary values. In Fig. 2.13.3a of the next chapter, quantization/variable intervals in the quantization table represent  $L \dots 0$  components. The following is a 2048 PCM code:

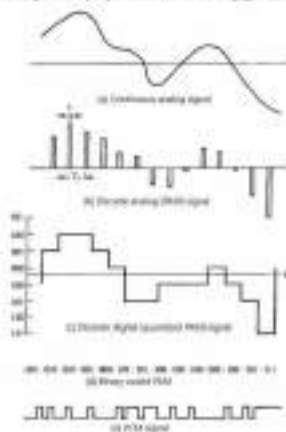


### 3.5.3 Decoding

The decoding results binary code by the conversion of the code. The decoder must provide the an. response back to a time location. This is done by associating each group of bits with the corresponding quantization level. The binary value of the quantization level is then converted by 127. In the example 3 for PCM codes, 61 & 52 are converted binary 01000001 and 01000010.



Die RCM funktioniert wie ein Differenzierer (DVR) an der Summation und liefert ein Ausgangssignal (DVR) an die Summation. Das folgende Diagramm zeigt die RCM.

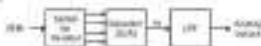


The complete block diagram of PBR systems:

**Encoder:**



**Decoder:**



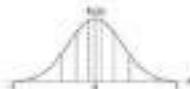
## 1.2.2 Non-Uniform Quantization

### Theoretical Basis Step

Uniform quantization assumes that the information signal has uniform PDF. In all quantization levels are used equally the more capacity it is not the case.

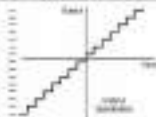
If the PDF of the information signal is not uniform (overflows become and bounded with zero), then we can optimize the locations of the quantization levels to obtain uniform quantization error (minimized).

An information signal  $x(n)$  has the signal probability function  $f_X(x)$ :



The shape of  $f_X(x)$  means  $\Delta(x) \neq 0$  most of the time. Therefore, we also use non-uniform quantization as indicated by the diagonal lines. The quantization levels are spaced here more or less after rule  $\Delta \propto 1/x$ . They are closer for large values of  $|x|$ , as large  $|x|$  occurs infrequently.

The diagonal lines show the uniform and the non-uniform distribution of the quantization levels.



Practically, one implementation is a difficult problem because of the knowledge of the signal. But, the problem is that the information signal has an unknown PMF so it is PMF-agnostic solutions. However, there are studies in some signals. For example, in the case of voice signals, the PMF shape of different speakers is usually similar, but the peaks have can vary widely between speakers, e.g. that is shouting and normal talking speakers. Therefore, the approach taken is to try to not to use waveform quantization but use lower compression (the compression).

### COMPRESSING INFORMATION (distortion)

This is the process of compressing the information signal prior to lower quantization of transmission. The compression is achieved via non-linear amplitude-to-amplitude conversion.



The receiver expands the reconstructed signal with the inverse characteristics to restore the original waveform.



In PCM with compressing system will be:

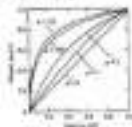


## Compressing Methods

There are two typical compressors that asymptotically and stably used for speech coding:

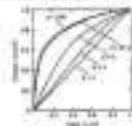
- (I) The *p*-law compressor (used in the G.723, G.726 and G.729). The parameter *p* controls the amount of compression and expansion. The standard compression rate *p* = 255 following a uniform quantizer with 128 levels (7 bit per sample).

$$y(x) = \frac{\ln(1 + ax^p)}{\ln(1 + a)}$$



- (II) The *μ*-law compressor (used in most commercial A-law based to the G.726).

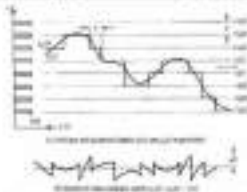
$$y(x) = \begin{cases} \frac{255x}{1 + \ln 2} & \text{for } 0 \leq x \leq \frac{1}{2} \\ \frac{1 + \ln(2/x)}{2 + \ln 2} & \text{for } \frac{1}{2} < x \leq 1 \end{cases}$$



## 4.2.2 Signal-to-Quantization-Noise Ratio

It is important to consider the quantization noise in the overall system quality. To evaluate  $SNR_q$  of a uniformly quantized signal, it is suitable to make the following assumptions:

- (1) Linear quantization (i.e. equal distances between quantization levels).
- (2) Zero mean signal (i.e. symmetrical PDF around the 0 level).
- (3) Uniform signal PDF (i.e. all input levels equally likely).



Let  $L$  be the number of the levels of the quantizer,  $I$  be the number of bits per a PCM word ( $L = 2^I$ ) and  $Q$  be the quantization level of the quantizer. The quantization interval is defined:

$$\Delta = \frac{2\Delta}{L}$$

The error of the quantized level is given by:

$$e(t) = \sum_{k=1}^L \frac{1}{L} \left( x(t) - \frac{k\Delta}{2} \right)$$

The mean square error of the quantization is:

$$\begin{aligned} \sigma_e^2 &= \int_{-\infty}^{\infty} x(t) x(t) dt - \frac{1}{L} \left[ \int_{-\infty}^{\infty} x(t) \left( -\frac{\Delta}{2} \right) dt + \int_{-\infty}^{\infty} x(t) \left( +\frac{\Delta}{2} \right) dt + \dots \right] \\ &= \frac{1}{L} \left( \frac{\Delta^2}{4} (1^2 + 3^2 + 5^2 + \dots + (L-1)^2) - \frac{1}{L} \left( \frac{\Delta^2}{4} \frac{L(L-1)(L+1)}{6} \right) \right) \\ &= \frac{\Delta^2}{12} (L^2 - 1) \end{aligned}$$

Forcing the quantization error (i.e., the difference between the unquantized and quantized signal) to  $\pm \Delta/2$ , then the PDF of  $x_q$  is:

$$p(x_q) = \begin{cases} \frac{1}{\Delta} & -\frac{\Delta}{2} \leq x_q \leq \frac{\Delta}{2} \\ 0 & \text{elsewhere} \end{cases}$$

The mean square quantization error (noise) is:

$$\frac{\Delta^2}{12}$$



$$\bar{Q} = \int_{-\infty}^{\infty} q(x) f(x) dx = \frac{1}{12}$$

Therefore the average  $Q_{A,B}$  is given

$$Q_{A,B} = \sqrt{2} \bar{Q} = \sqrt{2} = 1$$

Since the peak signal level is  $\sqrt{2}$  the peak  $Q_{A,B}$  is given

$$Q_{A,B} = \frac{(1.4142)^2}{1} = 2$$

So the average is 1 in addition to  $Q_{A,B} = 1.4142 = 1$ .

Where  $\alpha = 1.77$  for the peak  $Q_{A,B}$ , and  $\alpha = 1$  for the average  $Q_{A,B}$ . This equation is called the full rate and is plotted out that additional full improvement in the  $Q_{A,B}$  is obtained as  $\alpha$  is added to the full rate.

## EXAMPLES

- (1) Consider the voltage of a quantized signal source over 12V. This signal is sampled and quantized through a 1 bit per level quantizer. What is the quantization error for the samples whose voltages are 1.6V, 2V, -0.2V and -0V?

Solution:  $L = 12V$  per sample at  $L = 12V$  in the quantizer. The maximum error voltage is  $E_{max} = 4V$ . The minimum error voltage is  $E_{min} = -5V$ .

Then  $E_{max} = E_{min} = 5V$ . The error rate is  $\alpha = \frac{L}{E_{max}} = 0.233$ .

The quantized voltage level is  $E_{max} = 4V$  and  $E_{min} = -5V$ .

The voltage of a quantization level is the average of the two adjacent quantized voltages. So the following table is constructed as below:

For the quantized voltage level 12V and 12V the quantizer output is 12V. For the sample 1.6V the error is  $E_{max} = 1.6V - 12V = -10.4V$ .

For  $E_{max} = 1.6V \rightarrow E_{min} = 1.6V - 12V = -10.4V$ .

For  $E_{min} = 2V \rightarrow E_{max} = 2V - 12V = -10V$  and  $E_{min} = 2V - 12V = -10V$ .

For  $E_{min} = -0.2V \rightarrow E_{max} = -0.2V - 12V = -12.2V$  and  $E_{min} = -0.2V - 12V = -12.2V$ .

For  $E_{min} = -0V \rightarrow E_{max} = -0V - 12V = -12V$  and  $E_{min} = -0V - 12V = -12V$ .

Then maximum  $E_{max} = 12V$  and  $E_{min} = -12V$ .

Normalized Pulse(s)	Quantized Pulse(s)	Quantized Number
0	0.000	0
0.25	0.001	1
0.5	0.002	2
0.75	0.003	3
1.0	0.004	4
1.25	0.005	5
1.5	0.006	6
1.75	0.007	7
2.0	0.008	8

- (ii) Repeat example (i) for  $L = 4$  levels. Compare and Discuss.
- (iii) Consider the output of a bandpass signal source with the range  $-1V$  to  $1V$ . This signal is sampled and quantized through an 8 levels quantizer. What is the quantization noise levels for each of the samples  $0.2V$ ,  $-0.8V$ ,  $1V$  and  $0.9V$ ?
- (iv) For PCM system,  $FSB_p$  is to be held to minimum of 3000. Determine the values of required quantizing levels  $L$  and find the corresponding value,  $FSB_q$ .
- Solution:** 12.14 dB
- (v) A binary channel with maximum allowable rate is 3000 bps is available for PCM used transmission. Find the appropriate values of the sampling frequency and quantization levels (assume noise signals are  $\pm 1.248V$ ).

**Solution:**  $f_s \leq 3000/2$  and  $L \geq 2V_p / \Delta = 2(1.248) / (3000/3200) = 3.276 \approx 4$   
 $L = 4 \Rightarrow 16$  levels  $f_s = F_{max} = 3000/5 = 600$  Hz

- (14) An analog signal with a bandwidth of 5 kHz is to be converted into binary PCM and transmitted over a channel. The peak signal to quantization ratio of the system output is given as least 40 dB. [a] If we assume that  $B_{\text{ch}} = 8$  and then there is no RL, what will be the worst length and the number of quantization steps needed? [b] When will be the quantization for part (a)? When will be the channel bandwidth required if we design the pulse signal per unit?

$$\begin{aligned} \text{Solution: (a) } (S/N)_q &= 1.5 \times 10^4 = (1.5 \log 4) = 6.02 \rightarrow (1) = 30 \text{ bits, and } (1) = 111 \\ \text{Length: } 5 \text{ kHz} \times 15 &= 75 \text{ kHz} = 75 \text{ Mbps. } (1) = 0.5 = \frac{1}{2} \rightarrow 11.0000 \end{aligned}$$

- (15) A digital recording system samples each of two voice channels equally with a 5000 Hz as the sampling rate per second. [a] Determine the output rate bits per second (bps) needed. [b] The bit stream of digital data is grouped by the addition of error-correcting, check, compression, digital and several bit codes. These additional bits represent 50% of the original. Then what is the output bit rate of the system? [c] If we would like to store an hour's worth of recording data, determine the memory per required. [d] For a compression, a high quality analogue bitstream may require 5000 bytes, 2 minutes per page, 1000 bytes per volume, 8 words per line, a letter per word and 7 bits per letter. All are average. Determine the memory size required to store it.

$$\begin{aligned} \text{Solution: (a) } (S/N)_q &= 1.5 \times 10^4 = 108 \text{ dB. } (1) = 108 \text{ dB} = 10 \times 10 \text{ dB} = 10 \times 10 \text{ dB} = 10 \times 10 \text{ dB} \\ \text{And including the overhead bits rate: } 2 \times 1.41 \text{ MHz} &= 2.82 \text{ MHz} \\ (1) = 1.01 \text{ MHz} &= 1.01 \text{ MHz} = 1.01 \text{ MHz} = 1.01 \text{ MHz} = 1.01 \text{ MHz} \\ \text{So, it can indicate the overhead bits for the final rate are more } 10\% \text{ of each hour} \\ \text{from the total of the hour of system.} \end{aligned}$$

- (16) The audio signal "bandwidth" is 500–5000 Hz, is sampled at the sampling rate 8000 samples per second. If the input signal average 100 dB, [a] what is the number of levels  $L$  needed and the corresponding code length? [b] Estimate the required system bandwidth.

$$\text{Approximated } 0.14 \text{ per sample. (1) } 100 \text{ dB}$$

- (ii) A sinusoidal message signal is transmitted by binary PCM without compression. If the SNR is required to be at least 37dB, determine the minimum value of  $L$  required. Determine the bit rate allowed with this  $L$ .
- (iii) A television signal (video and audio) has a bandwidth of 4.5MHz. This signal is sampled, quantised and binary coded to obtain a PCM signal. (a) Determine the sampling rate if the signal is to be sampled at a rate 20% above the Nyquist rate (ii) If the samples are quantised over 12dB levels, determine the number of binary codes required to encode each sample (c) Determine the binary value rate (b/s) of the binary-coded signal, and the minimum bandwidth required to transmit this signal.
- (iv) The original peak-to-peak 100-dB  $\mu$ V PCM waveform has 30dB. This does not peak 100 to 120dB. It was decided to increase the SNR to the desired value by increasing the number of quantisation levels  $L$ . Find the fractional increase in the transmission bandwidth required by this increase (a).

### 4.5.4 PCM Multiplexing

The output PCM signal rate  $R_{PCM} = 2f_s L$  (b/s), where:  $L$  = number of multiplexed signals,  $f_s$  = number of samples/sec and  $L$  = number of levels.





Lowest binary signal transmission	simplest transmission
Differential signaling (Manchester coding)	simple, convenient, efficient

- (2) What is the channel bandwidth required to transmit a 10-Mbit/s signal consisting of 24 equal signals each with 4000 baud/s? How many are needed and 25% guard band?

$$\text{Bitrate } f = \log_2 L = \text{Data rate (characters/s)} \times L \text{ in bits/character}$$

$$f = 24 \times 4000 = 96 \text{ kbps}$$

- (3) Two binary signals, each of bandwidth 344, are to be transmitted simultaneously by binary PCM. The maximum allowable error in sample amplitude is 0.2% of the peak signal amplitude. The signal must be sampled at least 25% over the Nyquist rate. Sampling and quantizing requires an additional 0.7% extra bits. Determine the minimum possible data rate that must be transmitted, and the minimum bandwidth required to transmit this signal.

- (4) Twenty-three analog signals each with a bandwidth of 3400, are sampled at 36 kHz and multiplexed together with a synchronization channel (8000) into a 720 kHz signal. Calculate the minimum frequency, the number of channels, and the channel bandwidth required for the channel.

- (5) Several high-density audio channels having  $f_s = 44.1$  kHz are to be transmitted via binary PCM with  $L = 12$ . The entire bandwidth of PCM signals can be multiplexed in a 1.44-MHz channel. How calculate the corresponding bandwidth efficiency?

### Quality Factor Required Bits

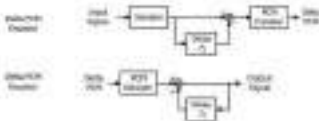
In addition to the channel effects, PCM performance depends primarily on the quantization error. To make the reconstructed signal like the original bandpass signal, we must reduce the quantizing error by increasing  $L$ . How we know that increasing the number of quantization levels requires more bits per sample to be transmitted. Large  $L$  is not perfect for encoded channels and due to some channel's limitation. So, usually the non-uniform quantization, several techniques are used to reduce the quantization error as the same number of bits  $L$ .

## 1.6 BANDWIDTH REDUCTION TECHNIQUES

The channel bandwidth is limited, and it is a valuable resource. A frequent objective of the communication engineer is to transmit the maximum information rate via the narrowest possible bandwidth. This is especially true for radio communication in which radio spectrum is a scarce and therefore valuable resource. The following techniques are used to transmit the same coding rate by using fewer bits per sample.

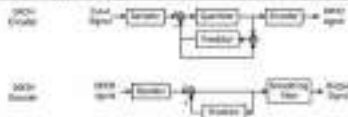
### 1.6.1 Delta PCM

Because the samples of most of the bandpass signals are highly correlated, it is possible to transmit the information about the changes between samples instead of sending the sample values themselves. A simple way for such systems is the Delta PCM. This method transmits the difference between adjacent samples through one word. This difference is significantly less than the actual sample values, hence it is coded using fewer binary symbols per word than the conventional PCM. However, Delta PCM systems cannot accommodate rapidly varying bandpass signals.



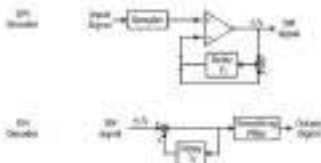
### 1.6.2 Differential PCM

Since neighboring samples of the many voice signals are highly correlated, Differential PCM (DPCM) uses an algorithm to predict future values. Such algorithms monitor the trend of the bandpass samples and use some models to predict the value of the incoming samples. These DPCM work until the actual value becomes available for examination and measure the difference to the already predicted value. The difference signal represents the difference signal's unpredictable part. By this means, DPCM reduces the redundancy in signal and stores the information to be transmitted using fewer symbols, less spectrum, and power loss.



## 2.7 Delta Modulation (DM)

If the quantizer of the DPCM system is constrained over its (i.e., the one level only) and the predictor is one sample delay then the resulting scheme is called DM. The information signal is represented by a stepped waveform. The resolution of this results is depends on  $\Delta$  &  $T$ , where:

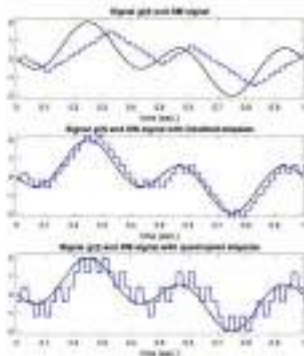




Not obvious from this figure alone that:

- 1) The system input is available (over the  $t$  horizon) up to reach the required level of the output.
- 2) If the system is more complex, its recommended structure (even with multiple inputs) may not allow some inputs which cannot be used at the horizon.
- 3) The input error (or best prediction) occurs when the input changes too rapidly for the sampled measurements follow.

The underlying problems are directly related to a dilemma about how to choose the time step (or  $\Delta t$ ). A small step size ( $\Delta t$ ) is desirable for accuracy, but the choice ( $\Delta t = 1/t_s$ ) should be large to avoid aliasing, which is not recommended. Namely, as  $\Delta t$  gets smaller, more small details of the continuous signal will be lost. This dilemma is illustrated in the figure below:



However, we can calculate the optimum values for these parameters that minimize the step-overhead problem. An estimate of the side-effect condition for DM can be obtained just easily by assuming modulation, but the input be  $x(t) = 1 \cos(\omega_{\text{mod}} t)$ , so that

$$\text{and } \left| \frac{dy(t)}{dt} \right| = 2\pi f_{\text{mod}} = \text{maximum derivative by input}$$

Then the maximum rate of rise is  $\frac{1}{T} = \Delta f_{\text{mod}}$ . Also

$$\Delta f_{\text{mod}} \geq 2\pi f_{\text{mod}} \Rightarrow \frac{1}{T} \geq \frac{2\pi f_{\text{mod}}}{T} \Rightarrow T < \frac{1}{2\pi f_{\text{mod}}}$$

The above  $f_{\text{mod}}$  condition may be applied to avoid channel signals by setting  $f_{\text{mod}}$  to the highest frequency component, and  $x = \max_p |x|$ .

Then the quantization error  $\sigma^2 = \int_{-\infty}^{\infty} \frac{f(x)}{2\pi} dx = \frac{\sigma^2}{12}$

and by choosing the error to be acceptable  $\epsilon$ , we get  $\sigma^2 = \frac{\epsilon^2}{12} = \frac{\sigma^2}{12}$

The corresponding value of the sub-carrier signal is

$$\begin{aligned} \beta &= \sqrt{\frac{12}{\epsilon^2}} = \left( \frac{12}{\epsilon^2} \right)^{\frac{1}{2}} = \frac{\sqrt{12}}{\epsilon} = \frac{1}{\epsilon} \sqrt{12} \\ &= 2\sqrt{3} \beta = \frac{\sqrt{12}}{\epsilon} \end{aligned}$$

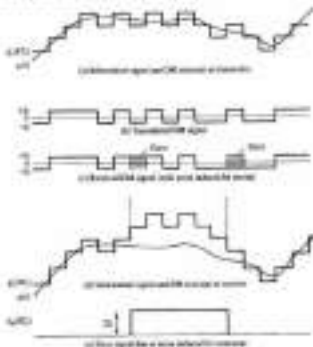
### 1.7.3 Adaptive Delta Modulation

We have seen that a high step size causes gross quantization errors, and a small step size results in sample-to-sample distortion. This means that a good choice for  $\Delta$  is a "medium" value, but in many cases, the performance of the low "medium" values is not satisfactory. An approach that can be used in these cases is to change the step size according to changes in the input. If the input tends to change rapidly the step size is chosen to be large (and vice versa) so the output can follow the input quickly without distortion.



## 4.7.2 Error in 30

If the 30 is not sufficiently high, then the 30 measure will occasionally have a positive error (i.e., it is instead  $\sim -2$  or less) even if the 30 is high. This is equivalent to the addition of an error of  $\sim 1$  to the measured signal at the 30 measure, or change in the 30 measure. This measure is used to estimate the error in the 30 measure, which is used to estimate the error in the 30 measure. This measure is used to estimate the error in the 30 measure, which is used to estimate the error in the 30 measure. This measure is used to estimate the error in the 30 measure, which is used to estimate the error in the 30 measure.



**Exercice 10**

- (1) On est à la recherche d'un signal d'entrée pour un système qui présente une caractéristique de transfert de la forme suivante avec  $k = 0,1$  :

$$G(s) = \frac{1}{s^2 + 2s + 2}$$

- (2) Quel est l'expression de la sortie pour un système à la fois d'ordre 1 et d'ordre 2 ?

- (3) Un système à la fois d'ordre 1 et d'ordre 2 est soumis à un signal d'entrée. On veut savoir si le signal d'entrée est un signal d'ordre 1 ou d'ordre 2. Quel est le signal d'entrée ?

- (4) Un système à la fois d'ordre 1 et d'ordre 2 est soumis à un signal d'entrée. On veut savoir si le signal d'entrée est un signal d'ordre 1 ou d'ordre 2. Quel est le signal d'entrée ?

$$\text{Solution : (1) On a une fonction de transfert de la forme } G(s) = \frac{k}{s^2 + 2s + 2}$$

$$k = 0,1 \Rightarrow G(s) = \frac{0,1}{s^2 + 2s + 2} = \frac{0,1}{(s + 1)^2 + 1}$$

$$G(s) = \frac{0,1}{(s + 1)^2 + 1} = \frac{0,1}{(s + 1)^2 + 1} = \frac{0,1}{(s + 1)^2 + 1}$$

$$\text{On a } G(s) = \frac{0,1}{(s + 1)^2 + 1} = \frac{0,1}{(s + 1)^2 + 1} = \frac{0,1}{(s + 1)^2 + 1}$$

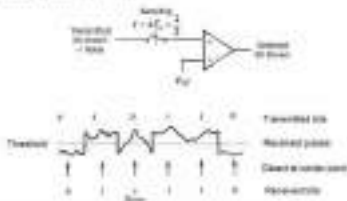
$$(2) \text{ On a } G(s) = \frac{0,1}{(s + 1)^2 + 1} = \frac{0,1}{(s + 1)^2 + 1} = \frac{0,1}{(s + 1)^2 + 1}$$

## 1.2 PROBABILITY OF ERROR AT RECEPTION

The detection of digital signals involves two problems:

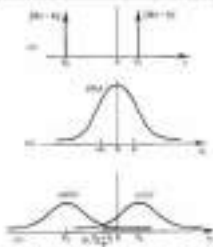
- (1) Reduction of each received voltage pulse (i.e. sampled) to a single numerical value, (not the quantisation).
- (2) Comparison of this value with a reference voltage (a decision level) which is itself not quantised.

Two bipolar binary symbols (0 and 1) are represented by two voltage levels (e.g.  $-V$  and  $+V$ ). In binary code as used in a simple example could be to set the reference  $V_{ref}$  halfway between the two voltage levels (i.e. at  $+V/2$ ).



In general, for bipolar digital signals the decision level is set to  $p = \frac{V_1 + V_2}{2}$ . For a simple construction of the symbols, we seek to determine the optimum threshold using more advanced mathematics.

Given the noise with a Gaussian PDF is common and statistically tractable, the bit error rate (BER) of a communication system is often modeled using Gaussian noise alone.



(a) The waveform of a binary communication system with antipodal signals (b) is used (c) The PDF of the received sample (d) is (e) The PDF of the received sample (f) is (g) The PDF of the received sample (h) is (i) The PDF of the received sample (j) is

Let the probability of sending symbol 0 (the voltage level  $V_0$ ) be

$$p(y_0) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_0 - \mu_0)^2}{2\sigma^2}}$$

And the PDF of sending symbol 1 (the voltage level  $V_1$ ) is

$$p(y_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_1 - \mu_1)^2}{2\sigma^2}}$$

Then, if the symbol 0 is transmitted, let  $P_{e0}$  be the probability of the received signal gives more than the threshold at the decision instant (i.e., even as voltage level  $V_1$ ), (shaded area under the curve  $p(y_0 | V_0)$  in Figure 1.2). The expression for the probability of error is

$$P_{e0} = \int_{V_1}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_0 - \mu_0)^2}{2\sigma^2}} dy_0$$

Also, if the digital symbol 1 is transmitted let  $P_{e1}$  be the probability that the received signal gives more than the threshold at the decision instant (i.e., even as voltage level  $V_1$ ).

So

$$r_{01} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}x^2} dx$$

It is clear from the symmetry of this problem that  $r_{01}$  is identical to  $r_{10}$ , and for equiprobable signals  $p_0(p_1) = p_1(p_0) = \frac{1}{2}$ , the error probability of error  $P_e$  will be:

$$P_e = p_0 P_{e|0} + p_1 P_{e|1} = \frac{1}{2} (P_{e|0} + P_{e|1}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$\text{using } x = \sqrt{2} \frac{1-Q}{2} \Rightarrow P_e = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x^2} dx$$

This integral cannot be evaluated analytically, but it can be given as a complementary error function, which is defined as:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-x^2} dx$$

$$\text{Then } P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

The advantage of using the Q function in the expression for  $P_e$  in this case is that the error can be accurately calculated.

For unipolar binary ( $P_0 = 1$ ,  $P_1 = 0 \Rightarrow p = \frac{1}{2}$ ), the average signal power is  $S = \frac{(1+0)}{2}$

$$\text{And the average noise power is } N = \sigma^2, \text{ so that } P_e = Q\left[\sqrt{\frac{S}{N}}\right]$$

For polar binary ( $P_0 = \frac{1}{2}$ ,  $P_1 = \frac{1}{2} \Rightarrow p = \frac{1}{2}$ ), the average signal power is  $S = \frac{(1+1)}{2}$

$$\text{And the average noise power is } N = \sigma^2, \text{ so that } P_e = Q\left[\sqrt{\frac{S}{N}}\right]$$

Therefore, the average (received) power for the unipolar signal must be twice that of the polar binary signal to achieve the same probability of error. For equiprobable signals, the polar binary signaling also has no advantage in the optimum receiver treatment. It is simply not as good as full, whereas the reason for the OQPSK binary signaling the threshold must be achieved is half the amplitude of the received signal.

## 2.9 CHANNEL CAPACITY

In any communication system, it is required to send data as fast as possible, that is the presence of noise and distortion along the channel, it is very hard to send across at the receiver. The Shannon-Hartley theorem of channel capacity states that the maximum rate of information transmission  $R_{max}$  over a channel of bandwidth  $B$  and the received signal to noise ratio is given by:

$$R_{max} = B \log_2 \left( 1 + \frac{S}{N} \right) \text{ bits/s}$$

Where  $C$  = channel capacity = the maximum rate at which information can be transmitted across the channel without error if it measured in bits per second (bps).

Let if the total operating communication rate:

- If  $R \leq C$ , it is possible to transfer data with small probability of error, even with noise.
- If  $R > C$ , error cannot be prevented regardless of the coding technique used.

If we would like to increase  $R_{max}$  in the above equation, we can increase  $B$  and/or the SNR.

- Larger SNR implies working at, for example, higher Tx powers or stronger antennas. In some cases, this is not that simple as above. But in general, as the SNR increased it would also increase without of loss the required  $B$  channel (however, you must note that, when  $R \rightarrow$  channel SNR  $\rightarrow \infty$  and hence  $R_{max} \rightarrow \infty$  regardless of  $B$  (if it possible)).
- As for increasing  $B$ , it implies changing the method of laying a license for radio bandwidth. In general, as  $B$  increased, it can allow faster changes in the information signal. Besides increasing  $B$ , there is also, when  $R \rightarrow \infty$ ,  $C$  that can approach it. The noise is expected to be white: the wider the bandwidth, the more the noise allowed in the system. This means, as  $B$  increases, SNR decreases as the same  $C$ .



### Channel Capacity

Suppose that the noise is white and PSD  $n_0/2$  [W/Hz], and assume the received signal power is fixed at a value  $P$  [W], the channel capacity would be  $C_{\text{max}} = B \log_2 \left( 1 + \frac{P}{n_0 B} \right)$ .

Now, when  $B \rightarrow \infty$  we get

$$C = \lim_{B \rightarrow \infty} B \log_2 \left( 1 + \frac{P}{n_0 B} \right) = \lim_{B \rightarrow \infty} \left[ \frac{P}{B} \log_2 \left( 1 + \frac{B}{P/n_0} \right) \right] = \frac{P}{n_0} \log_2 e$$

$$C = 1.44 \frac{P}{n_0}$$

This gives the maximum possible channel capacity as a function of the received signal power and the noise PSD. In a real system design, the channel capacity might be compared to this value to estimate whether additional resources in  $P$  is worthwhile.

### Example

- (1) Let  $S/N = 20$  dB at the receiver input. (a) What is the maximum possible rate of bandwidth data transmission without error? Assume the channel bandwidth is 20 kHz, 200 kHz, 1 MHz, 10 MHz, and 100 MHz. (b) Suppose (a) if the channel bandwidth is increased to the ranges indicated.

Solution:  $\frac{P}{n_0} = 10^{20/10} = 100 \text{ W/Hz}$  so  $C_{\text{max}} = C_{\text{max}}(B) = C_{\text{max}} \log_2 \left( 1 + \frac{P}{n_0 B} \right)$

At  $B_0 = 20 \text{ kHz}$ ,  $C = 250 \text{ kbps}$

At  $B_0 = 200 \text{ kHz}$ ,  $C = 2.5 \text{ Mbps}$

At  $B_0 = 1 \text{ MHz}$ ,  $C = 27.09 \text{ Mbps}$

At  $B_0 = 10 \text{ MHz}$ ,  $C = 262.26 \text{ Mbps}$

At  $B_0 = 100 \text{ MHz}$ ,  $C = 267.02 \text{ Mbps}$

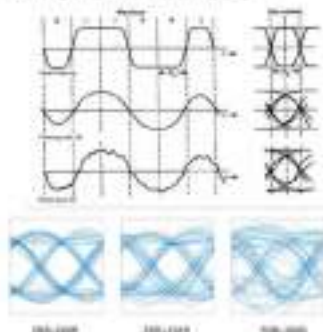
- (b) we can assume a VBF large number, which gives  $C = 267.02 \text{ Mbps}$ , but it is more practical and easier to use the approximation when  $B_0 \rightarrow \infty$

$C = 1.44 \frac{P}{n_0} = 267.12 \text{ Mbps}$  with less accuracy than

## 1.10 INTER-SAMPLE INTERFERENCE (ISI)

Previously, we discussed the essential problem (interference) caused by time dispersion. This results in spreading of time signals and overlapping, strong adjacent bit boundaries. The resulting is also known as ISI. It is caused not only by channel dispersion but also by multi-path effects.

To illustrate the ISI problem, we plotted the bit stream for just given:

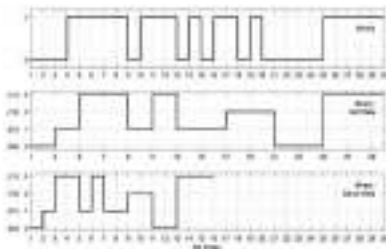


To reduce the ISI, we need:

- Consider the channel response for the transmitted rate.
- Frequency limitation of the transmitted signal to that.
- Reduce the effects of the multi-path channel (FIR).

## 1.11 MULTI-LEVEL BINARY SIGNALING (M-ARY)

Binary (two-level) signaling means sending a single bit over the channel interval  $T_b$  (or at the bit rate  $1/T_b$  bits/sec). To increase the data transfer rate, it is possible to combine several bits at any instant to form a unique received one symbol per  $M$  data bits. We use the represented methods (e.g., between 0 and 1) where  $M = 2^n - 1$ . The plot below illustrates the M-ary representation.



In this example, we use 2 bits per symbol (since  $2^2 = 4$  levels). The information rate is still unchanged, but the symbol rate is reduced (as before we had 40). In other words, we can send twice the information rate at the same symbol rate (as in the first plot). Practically, one difficulty is to make the multi-level waveform signaling through wires and detecting it correctly. The receiver has to distinguish the received symbols according to their levels. Thus, the probability of error increases as  $M$  increases (e.g., per).

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Practical Engineering: Elements of Error	in the Transmission of Signals

#### Advantage

- A higher information transfer rate is possible for a given channel rate and a fixed bandwidth.

#### Disadvantages

- If any bandwidth signaling results in reduced signal-to-noise ratio when it is compared to the binary signaling.
- It increases the complexity of the receiver and transmitter.
- It imposes a greater requirement for frequency and/or reduced distortion in the TDM hardware and/or channels.

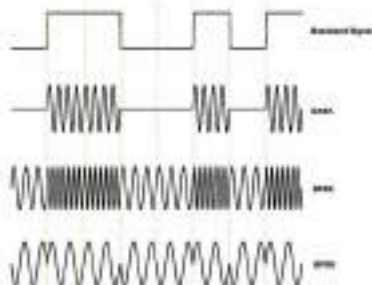
#### Exercise

1. A multilevel digital communication system transmits one of 16 possible levels over the channel every 5 bits. (a) What is the number of bits corresponding to each level? (b) What is the baud rate? (c) What is the bit rate?
2. An analog multiplex with  $f_m = 1 \text{ kHz}$  is to be transmitted as a  $\geq 200$  levels and transmitted via a binary PCM signal (during  $M = 2^N$ ). Find the minimum allowed values of  $N$ ,  $f_s$ , and the corresponding value of  $m$ . If the available transmission bandwidth is 10 kHz.

## Part 2 DIGITAL MODULATION

## 2.3 Binary Digital Modulation

Since pulse-modulated signals consist of "low" frequencies, they cannot be efficiently transmitted through a channel with bandpass characteristics. Hence, for communication systems employing band-pass channels it becomes advantageous to modulate a carrier signal with the digital data stream prior to transmission. Three basic forms of digital modulation corresponding to ASK, FSK & PSK are known as Amplitude Shift Keying (ASK), Frequency Shift Keying (FSK), and Phase Shift Keying (PSK).



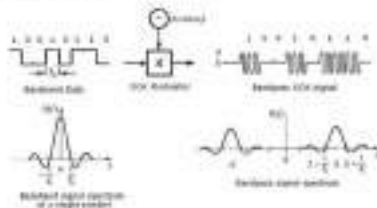
## 2.2 BINARY AMPLITUDE SHIFT KEYING (BASK)

In BASK, the amplitude of a high-frequency carrier is modulated between two values, ON-OFF Keying (OOK).

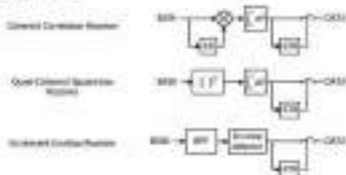
$$A(t) = \begin{cases} A \cos(\omega_c t) & \text{for logic 1} \\ 0 & \text{for logic 0} \end{cases}$$

The bandwidth of BASK signal =  $\omega_c$

### 2.2.1 Generation



### 2.2.2 Demodulation



Lowest Order Digital Communication	Open Questions
Discrete Subcarriers / Orthogonal Signals	in the Frequency Domain

### 4.2.3 Probability of Error in FSK

The received wave model based on two possibilities

$$r(t) = \begin{cases} \cos(\omega_c t) + n_c(t) & \text{for signal 1} \\ n_c(t) & \text{for signal 0} \end{cases}$$

Where  $n_c(t)$  is the noise added to the discrete signal.

At receiver's output, for  $p_1(t) = p_0(t) = \frac{1}{2}$ , the optimum decision threshold is set at  $E_b/2$ . ( $E_b$  = received signal energy for signal 1), when  $E_b = 0$ , the noise is Gaussian distributed with the probability of error is

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Where  $N_0$  = the average received signal power =  $\frac{E_b}{T_b}$

$E_b$  = average received signal energy =  $\frac{(E_s + E_i)}{2} = \frac{E_s}{2}$

$E_s$  = signal energy =  $\frac{1}{2}$ , and  $T_b$  is the interval of binary data.

$\beta$  = the signal to noise ratio =  $\frac{E_s}{N_0}$ ,  $\beta$  = the bandwidth of the discrete signal.

$$P_e = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

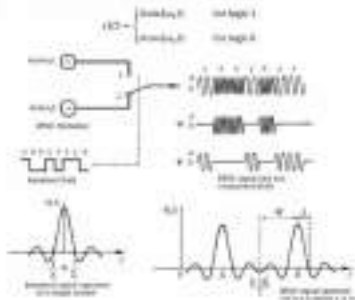
$$P_e = Q\left(\sqrt{\gamma_b \beta \frac{E}{P}}\right) \quad \text{where } \gamma_b \text{ is the SNR, } \beta = 1$$

$$P_e = \frac{1}{2} - \frac{1}{2} \exp\left(-\frac{\gamma_b E}{N_0}\right) \quad \text{multitone FSK}$$



## 1.1 Binary Frequency Shift Keying (BFSK)

In BFSK, the carrier frequency of the carrier is switched between two values in response to the binary code. We can consider the BFSK waveform as a composition of two DASK components of different carrier frequencies.



### 1.1.1 Derivation

We derive above a BFSK carrier frequency as:

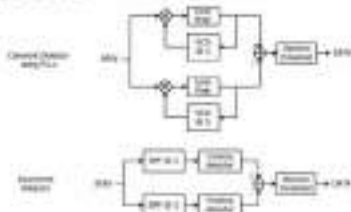
$$f_c = \frac{f_1 + f_2}{2}$$

Assuming BFSK frequency is dependent on:

$$f_c = \frac{f_1 + f_2}{2}$$

To define the bandwidth of a BFSK signal, we'll consider the bandwidth work at the first zero crossing point in the BFSK spectrum. So  $B = (2f_1 - 2f_2)$ , where  $f_1 = 1/T$  is the first zero of the bandwidth data points, which is assumed here to be the nominal bandwidth of the binary.

## 2.2.2 Noncoherent



## 2.2.3 Probability of Error in FSK

$$P_b = Q\left(\sqrt{\frac{E_{b,FSK}}{N_0}}\right)$$

$$\text{As } E_{b,FSK} = \frac{E_b}{2} \Rightarrow \frac{E_b}{2} = \frac{E_b}{2} \Rightarrow P_b = P_e$$

$$P_b = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$P_b = Q\left(\sqrt{\gamma_b \frac{E_b}{N_0}}\right) \quad \text{Coherent Detection: } \gamma_b = 2$$

$$P_b = \frac{1}{2} - \frac{1}{2} \exp\left(-\frac{\gamma_b E_b}{N_0}\right) \quad \text{Noncoherent Detection}$$

Remember that for the FSK results to show any difference!

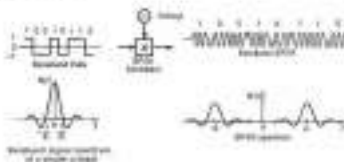
## 2.8 BINARY PHASE SHIFT KEYING (BPSK)

BPSK converts the baseband binary to passband by changing the carrier's phase in compliance with the transmitted digital data.

$$s(t) = \begin{cases} \cos(\omega_c t) & \text{for logic 1} \\ \cos(\omega_c t + \pi) & \text{for logic 0} \end{cases}$$

In BPSK,  $\phi = 180^\circ$

### 2.8.1 Generation



### 2.8.2 Demodulation

Coherent Demodulation Receiver



### 2.8.3 Probability of Error in BPSK

$$P_b = P\left(\sqrt{\frac{2E_b N_0}{\pi}}\right)$$

Since  $E_{avg} = E_b = E_s$ . The average signal rate is  $\gamma_s = E_b/N_0$ .

$$P_b = P\left(\sqrt{\frac{2\gamma_s}{\pi}}\right) \quad \text{and} \quad P_b = Q\left(\sqrt{2\gamma_s \frac{E_b}{N_0}}\right)$$

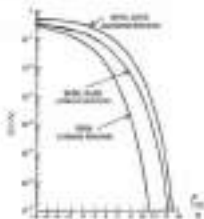
Thus, unlike BPSK is better than other the most schemes. (What is the modulation scheme?)

## 2.5 Comparison of Binary Keying Techniques

### 2.5.1 Through the Average Power

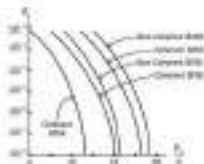
OSK & BPSK have equivalent  $E_b/N_0$  from the same  $\gamma$ , BPSK has better performance (lower  $\gamma$ )

As shown:



### 2.5.2 Through the Peak Power

Forget the lower  $E_b/N_0$  BPSK requires double the power of ASK but also requires half the power of BPSK, so all the energy of BPSK concentrated in one half the time (25% of ASK). As shown:



### 2.5.4 Through the Spectral Efficiency

We can define the spectral efficiency of the transmission as

$$\eta = \frac{R_b}{W} \text{ bits/s/Hz}$$

$M$  = number of levels =  $2^N$ , the total  $N$  =  $2, 4, 8, \dots$

$$R_b = \text{transmission rate (in bps)} = \frac{1}{T} = \frac{1}{T_{\text{sig}}}$$

$E$  = the required signal bandwidth of the transmitted signal

Then the throughput will be

	FSK	PSK	QPSK	QAM
Bits Rate (bps)	$\frac{1}{T}$	$\frac{1}{T}$	$\frac{1}{T}$	$\frac{1}{T}$
Bandwidth (Hz)	$\frac{1}{T}$	$\frac{1}{T}$	$(1+\frac{1}{2}) \frac{1}{T}$	$\frac{1}{T}$
Throughput (bits/s/Hz)	1	1	$\frac{1}{(1+\frac{1}{2})}$	1

### 2.5.5 Through Systems

FSK

- The transmission of FSK are very straightforward
- They have the advantage of transmitting no power at 0, such systems that were applications in short range wireless telemetry systems.
- The receiver for non-coherent FSK is easy to build. The difference in performance between coherent & non-coherent detection is highly compared to the increase in complexity required. Therefore coherent detection of FSK is generally uncommon.
- The detection threshold in the receiver must be adjusted with changes in the level of the received signal. Therefore, it requires an Automatic Gain Control (AGC).

PSK

- As contrast to FSK, the PSK systems operate continuously above a pre-determined threshold level, regardless of the received strength.
- A variation of PSK is digitally more complex than binary FSK.

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- Non-coherent FSK receiver is relatively easy to implement and popular choice for low to medium data communication rates.
- FSK requires more bandwidth than PSK & QPSK.

#### PSK

- The performance of PSK systems is superior to both ASK & FSK systems.
- PSK systems require transmitted power for a given  $P_r$ .
- Synchronous detection is required, and hence receiver systems are more difficult (therefore more expensive) to build.

The degree of which scheme of the digital modulation to be selected depends on the trade off between (performance, cost, bandwidth, etc.). In addition (propagation, distortion, fading, non-synchronous p.p., interference) non-linearities, etc.) may affect the choice.

### Example 1

- (1) A 140Mbps (1.4 × 10<sup>8</sup> bits/sec) digital system uses pulse shaping to transmit its transmission to the radio frequency bandwidth. The received signal power is 0.04W and the associated noise power spectral density is 10<sup>-16</sup>W/Hz. Find the bit error rate (BER) expected in this system.

$$\begin{aligned} \text{Solution: Signal BW Bandwidth} &= \frac{1}{2} R_b = B = 70 \text{ MHz} \\ \therefore B &= 70 \times 10^6 = 7 \times 10^7 \text{ Hz} \\ R_b &= 2 \left( \frac{P}{N} \right) = 2 \left( \frac{0.04 \text{ W}}{10^{-16} \text{ W/Hz}} \right) = 80 \text{ MPT/s} = 8 \times 10^7 \\ \text{So the BER} &= R_b \times P_b = 74.2 \text{ Error/bit-sec-second} \end{aligned}$$

- (2) A rectangular pulse (NRZ) signal is transmitted over a channel, at the receiver the signal is corrupted because of 10<sup>-16</sup>W/Hz. The received signal BER is 10<sup>-5</sup> N/Hz. What minimum SNR can this system support at this transmitting a  $R_b = 10^6 \text{ s}^{-1}$ .

$$\begin{aligned} \text{Solution: } R_b &= 10^6 \text{ s}^{-1} \Rightarrow R_b \left( \frac{1}{2} \right) = 5 \times 10^5 \text{ s}^{-1} \\ \therefore R_b &= \frac{P}{N} = 501.55 \text{ Mbps} \end{aligned}$$

- (3) At BER (minimum) signal is detected by a receiver, the receiver is coded at the receiver input is a rectangular pulse with amplitude 100mV and duration of 10ms. The noise at this point is known to be white and Gaussian with an rms value of 100mV when measured by a rectangular pulse of 10ms duration.  $R_b$

$$\begin{aligned} \text{Solution:} \\ R_b &= R_{b_{min}} \times P_b = \left( \frac{100 \times 10^{-3}}{10 \times 10^{-3}} \right)^2 \times 10 \times 10^{-3} = 1 \times 10^{-4} \\ R_{b_{min}} &= \frac{R_b}{P_b} = \frac{1}{10} = 10 \times 10^{-4} \\ \therefore \frac{P}{N} &= \frac{10 \times 10^{-4}}{10} = \frac{10 \times 10^{-4}}{10 \times 10^{-3}} = 1.25 \times 10^{-2} \text{ N/Hz} \\ \text{So } P_b &= 2 \left( \frac{1}{2} \right) = 2 \left( \frac{1}{2} \right) = 2 \left( \frac{1}{2} \right) = 1.565 \times 10^{-4} \end{aligned}$$

- (14) A certain matched filter receives a signal as an input. The input signal is the sum of the signal  $s(t)$  and a rectangular pulse with amplitude 1 V and duration of time  $T$ . The sum at the input of the filter is  $s(t)$  and has a mean value of 11.10 V when measured at a noise bandwidth of 10 Hz. What is the probability of detection?

Answer:  $P_d = 7.69 \times 10^{-7}$

- (15) Find the constant  $K$  needed for a BPSK receiver to have  $P_b < 1.5 \times 10^{-5}$  if the following detectors are used: (a) coherent (b) non-coherent. Repeat for QPSK and 8PSK.

Solution: For (a)  $K = 10^{-5} / (2 \times 10^{-5}) = 0.5$

For (b)  $K = 10^{-5} / (2 \times 10^{-5}) = 0.5$

For (c)  $K = 10^{-5} / (2 \times 10^{-5}) = 0.5$

For (d)  $K = 10^{-5} / (2 \times 10^{-5}) = 0.5$

Compute and explain these results.

- (16) A certain 100 W signal is received with a peak voltage of 10 V and a current of 10 A. The signal is  $s(t) = 10 \cos(2\pi f t)$  V. Its derivative is  $s'(t) = 10 \sin(2\pi f t)$  V. What is the total average power? (a)  $P_s = P_{s'} = 10$  W. Repeat if  $P_s = 10$  W. Compute and explain.

Answer: 10 W

- (17) A given high frequency transmitter is peak power limited to 1.0 W. Both base and envelope are sinusoidal. The carrier is 1.0 MHz and the carrier wave is  $s(t) = 1.0 \cos(2\pi f t)$  V. Find the maximum value of (a)  $P_s$  (b)  $P_{s'}$ .

Solution:  $P_s = 1.0$  W,  $P_{s'} = 1.0$  W

For (c)  $P_s = 1.0$  W,  $P_{s'} = 1.0$  W





Course Section: Digital Communications Electrical Engineering 1 (Electronics 1) Notes	Open Problems to be Submitted: 20/04/2020
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Characterise the efficiency of this system (1) (i) As a complete system (ii) as a very large bandwidth is taken to the maximum transmission rate possible?

- (12) A digital transmission system consisting of two separate binary pulse signalling channels of 100 kbps is to be a transmitted over a bandwidth limited. For NRZ signalling this is limited by a bandwidth digital and pulse is overlapping after setting  $T_{10}$  and  $T_{11}$ , then (i) the magnitude spectrum and identify the value of first null-point bandwidth. Assume a carrier frequency of 1200 kHz.

- (13) Repeat the previous problem for the case of (a) NRZ signalling (b) NRZ signalling assuming the Mark frequency is 0.5 MHz and the space frequency is 0.5 MHz.

- (14) An ASK signal with non-coherent pulse carrier. Assume  $\eta_1$  and  $\eta_2$ . Assuming average probabilities are  $P_1 = P_2 = 0.5$ , give assume  $A_1 = 2 \text{ mV}$ ,  $A_2 = 0.5 \text{ V}$ ,  $\eta_1 = 1 \text{ ms}$ , and  $\eta_2 = 4 \text{ ms}$  is  $10^6 \text{ Hz}$ , calculate the optimum threshold voltage, (ii) find the probability of error for  $\eta_1$ .

$$\text{Solution: } \text{SNR} = \frac{A_1^2}{\eta_1} = \frac{A_2^2}{\eta_2} \quad A_1 = 0.5 \text{ V}, T_1 = 4 \text{ ms}, P_1 = 0.5 \left( \frac{P_2}{P_1} \right) =$$

- (15) NRZ and coherent NRZ systems use the same peak pulse amplitude and have the same values of  $A_1$  and  $\eta$ . How are the bit duration  $T_b$  related?

## 2.6 MODULATION TECHNIQUES WITH INCREASED SPECTRAL EFFICIENCY

In addition to the reliability and factors  $P_b$ , the efficient usage of the bandwidth is an important aspect to the design of digital communication systems.

As we know, the bandwidth is permeates the maximum information rate through the minimum possible bandwidth. We define this as the spectral efficiency  $\eta$ . From the  $\eta$  we get an equation, we get:

- a. Therefore it is by altering the transmitted signal prior to the transmission. But for in this method suffer:

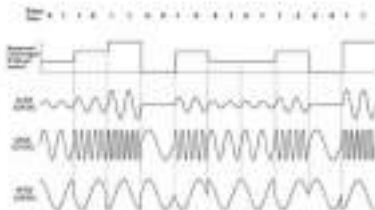
- 1- Increase the number of transmitted bits per symbol ( $R$ ), but we must consider the Shannon-Hartley limit of channel capacity. We avoid RS, QPSK and 16QAM for bandwidth efficient and QPSK is for low-power signals.
- 2- The final option is to increase  $M$  (i.e. increase the number of bits per a transmitted symbol). Practical systems currently use with  $M$  of 4, 16, 64, ... 1024.

The modulation per symbol ( $M$  symbols) required for the modulated digital modulation will refer to as 16QAM, 64QAM and 1024QAM.

Rate:

$P_b$  is the probability of bit errors.

$P_s$  is the probability of symbol errors.



In practice, only 16QAM and 64QAM are used, because:

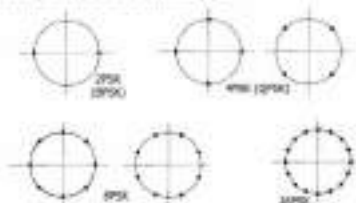
- 1- the increase of the M-QAM results in level ambiguity or distortion. The channel offers the low noise and distortion make the process of the detection very hard. Nevertheless, increasing  $M$  increases  $P_b$ , but it is still acceptable for small  $M$  through good channel.
- 2- as we studied before, the increase of  $M$  in 16QAM requires larger bandwidth to accommodate the modulated signal, which increases  $P_b$ .

### 4.4.1 M-Symbol Phase Shift Keying (MPSK)

MPSK implies the existence of the number of the allowed phase states fixed to  $M$  ( $M = 2^k$ ), i.e., with the carrier amplitude being constant, the phase is changing according to the input binary code of the designed phase states.

$$s(t) = A \cos(\omega_c t + \phi_k) \quad \text{where } k = 0, 1, 2, \dots, M-1 \quad \dots \quad \phi_k = \frac{2\pi k}{M} \quad \text{or} \quad \phi_k = \frac{(2Q+1)\pi}{M}$$

Under example, an MPSK is shown in the following figure.



### Quadrature Modulation

The phase and amplitude of the carrier is not given that determines the location on the constellation. The amplitude of the I and Q channels are derived from the rectangular coordinates of the carrier's amplitude and phase.



**Quadrature Modulated Signals (QPSK)**

Here, we study the coherent QPSK as an example of PSK. In QPSK, as with BPSK, the message conveyed by the transmitted signal is contained in the phase. The phase of the carrier takes on one of four equally representative, or



possible values, we may define the transmitted signal as

$$s(t) = \sqrt{2} \cos \left[ \omega_c t + \frac{(2i-1)\pi}{4} \right] \quad \text{where } i = 1, 2, 3, 4$$

$$= \sqrt{2} \cos \left[ \frac{(2i-1)\pi}{4} \right] \cos \omega_c t + \sqrt{2} \sin \left[ \frac{(2i-1)\pi}{4} \right] \sin \omega_c t$$

Equation (1) represents an M-PSK, we can make the following observations:

- There are two orthogonal basis functions, defined by a pair of quadrature carriers

$$\phi_1 = \cos \omega_c t \quad \text{and} \quad \phi_2 = \sin \omega_c t$$

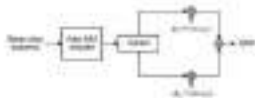
- There are four message points, and the transmitted signal vectors are defined by:

$$\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \cos \left[ \frac{(2i-1)\pi}{4} \right] \\ \sqrt{2} \sin \left[ \frac{(2i-1)\pi}{4} \right] \end{bmatrix} \quad \text{where } i = 1, 2, 3, 4$$

The elements of the signal vector have their values measured in the table below:

i	i-to-message input to bits	Phase of QPSK input to bits	Coordinates of Message (s1, s2)	
			s <sub>1</sub>	s <sub>2</sub>
1	00	$\pi/4$	+1	+1
2	01	$3\pi/4$	-1	+1
3	11	$5\pi/4$	-1	-1
4	10	$7\pi/4$	+1	-1

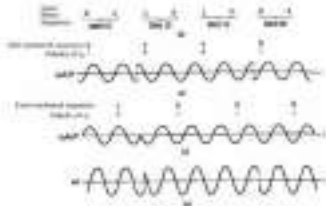
# Module 2



## Exercise 1



The figure below illustrates the implementation and results of a discrete-time Fourier transform (DTFT) system.



### Probability of Error in BPSK

As  $T$  increases with  $M$ , the probability of error would also be increased. So, the symbol probability of error is

$$P_s = 1 - Q\left(\sqrt{\frac{2E_b}{N_0}} \sin\left(\frac{\pi}{2M}\right)\right) \quad \text{or} \quad P_s = 1 - Q\left(\sqrt{\frac{2E_b}{N_0}} \sin\left(\frac{\pi}{2M}\right)\right)$$

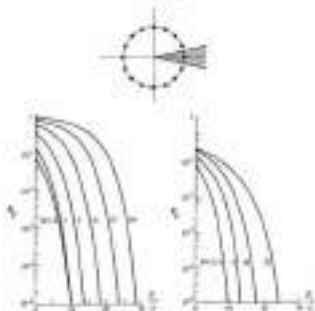
Using these formulas and for  $M = 4$  only, and if we use this we'll get twice the correct result of BPSK.

Now if we are given only to map binary symbols to phase states, the probability of error is

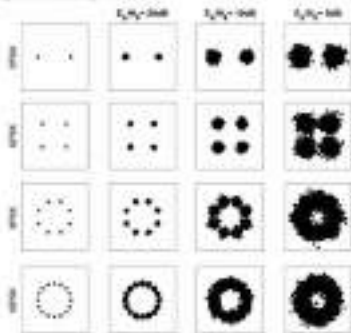
$$P_b = \frac{P_s}{\log_2 M}$$

To compare the performance of BPSK through  $M$ , we should express  $P_b$  in terms of  $P_b$  values

$$P_b = \frac{P_s}{\log_2 M}$$



For 12-PK, 12-MK, and higher  $N$  values, it will become so difficult (if not impossible) to distinguish between these plots:



### Bandwidth of MPSK Signals

The power spectral of MPSK signals is a main lobe bounded by well-defined spectral tails (i.e., frequencies at which the power spectral density is zero). Accordingly, the spectral width of the main lobe provides a simple and popular measure for the bandwidth (well-defined bandwidth).

So, the channel bandwidth required to pass MPSK signals (more precisely, the main spectral lobe of MPSK signals) is given by:

$$B = B_s = \frac{R_b}{E_s} B$$

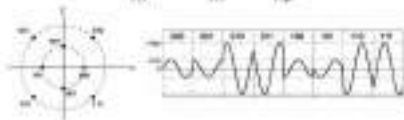
Where  $B_s$  is the signal rate (bauds per second),  $R_b$  is the binary data rate (bits per second).



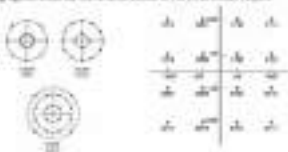
### 3.6.2 Hybrid Amplitude/Phase Modulation (QAM)

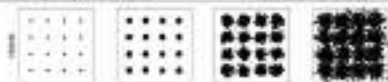
Also called Quadrature Amplitude Modulation. Here, it is possible to transmit amplitude as well as phase information using an improved distribution of quadrature code phase angles. The following Quadrature code (QAM)

Bit Combination	Phase Shift (deg.)	Amplitude
000	0	Low
001	90	Low
010	45	High
011	135	High
100	225	Low
101	315	Low
110	315	High
111	45	High



The following figure illustrates the distribution of 16PSK in complex (QAM)





#### 44-QAM Constellation

By adding more levels to the I and Q channels, higher data rates can be achieved

- The higher the number of levels, the more often there will be a difference in the levels
- 64-QAM uses 8 levels in the I dimension and 8 levels in the Q dimension for a total of 64 unique symbols
- 256-QAM uses 16 levels in the I dimension and 16 levels in the Q dimension for a total of 256 unique symbols

#### 64-QAM Constellation



#### 256-QAM Constellation



Probability of Error is  $10^{-5}$

$$\eta \approx 4 \left( 1 - \frac{1}{4M} \sqrt{\frac{E_{avg}}{4(M-1)}} \right)$$

This figure shows that for the 64-QAM is much better than the 16-QAM for the same SNR



- (2) An FD tree system is to operate with  $10^7$  PAM symbols over a 120-MHz channel. The channel is required to carry at 500 kbps. What minimum SNR is required to transmit signals with  $b_1$  as minimum  $10^{-3}$ ?

**Solution:** For FD tree least rate is  $R_1 = 1/2 \times R = 120$  symbols per second. Therefore  $R_1$  is 120 symbols per second, but the input of channel is  $R$ , rate  $R_c = 500$  kbps, so since  $R_c = m \times R_1$ , where  $m = \log_2 B$ .

$$\text{so } \log_2 B = 7.3 \rightarrow m = 7.3 \text{ bits per symbol, } B = 158 \text{ symbols/sec}$$

$$B = R_1 \times m = 9.24 \times 10^6 \text{ and using } B = 1/4 \left( \sqrt{1.2 \times \frac{R_c}{B}} + \cos \frac{1}{2} \right) \text{ then}$$

$$R_1 = \frac{R_c}{2} = \frac{500000}{2} = 250 \times 10^3 \text{ symbols/sec and } R_1 = \frac{1}{2} \times 9.24 \times 10^6 = 4.62 \times 10^6$$

$$R_c B = 500000 \times 10^3 \times 1.28 \times 10^6 = 1.28 \times 10^{12} \text{ Then}$$

$$B = 10^6 = 1 + 4 \left( \sqrt{1.28 \times \frac{R_c}{B}} + \cos \frac{1}{2} \right)$$

$$B = 10^6 = 4 \left( 0.32793 + \sqrt{\frac{R_c}{B}} \right)$$

$$= \frac{1}{2} + \left( \frac{1.28 \times 10^{12}}{B} \right)^{1/2} = \left( \frac{R_c}{B} \right)^{1/2} = 0.32793 + 0.32793$$

- (3) An MFSK transmitting system is used to transmit data at a rate 10 kbps over a 10-MHz channel. (a) Find the minimum number of tones levels of the MFSK that should be used, and the corresponding bandwidth. (b) Calculate the BER of this system in the presence of additive white Gaussian noise with uncorrelated power spectral density  $1.25 \times 10^{-10}$  W/Hz. Assume Gray coding a transmitter average power of 100 W and required bit error rate of  $10^{-5}$ .

**Solution:** (a)  $R_1 = R_c$ , so  $\log_2 B = 1.34 \rightarrow 5.48$  bits per symbol,  $B = 54$  symbols per second  $\rightarrow R_1 = \frac{R_c}{2} = 50$  symbols per second.

$$(b) \text{ so } 40 \times R_1 = R_c = (m \times B = 112 \times 54) \text{ MHz } \rightarrow B = 38.18 \log_2 B$$

$$R_1 = 1/2 \left( \sqrt{1.28 \times \frac{R_c}{B}} + \cos \frac{1}{2} \right) = 0.32793 + 0.32793 = 0.65586 \times 10^6 \therefore R_1 = 3.2793 \times 10^5$$

$$\text{BER} = 0.5/B = 0.77 \text{ bits per second}$$

- (1) A 120 kbit/s data link operates at 10 MHz. What is the underlying channel rate in the channel? And what is the capacity of the link?

**Solution:** We know  $C = \log_2 M = 7$  bits per symbol. We need to find

$$B_1 = \frac{C}{T} = 10 \text{ MHz}$$

The minimum signal width occupied by 120 kba channels  $B_2 = 100 \text{ kHz}$

- (2) An OFDM system is used for transmitting data at a rate 100 Mbps over a channel with bandwidth extending from 500 kHz to 700 kHz. (a) Over the link exactly the required information rate is the given channel? If not, find the minimum number of M-ary levels at the input that should be used (b) Find the minimum bits per tone in a carrier rate in the link channel and the corresponding symbol rate (c) the required SNR is 10 dB.

**Solution:** (a)  $B_2 = 100 \text{ kHz}$  - therefore, we must have  $B_1 \geq 100 \text{ kHz}$  (Channel 1)

$\therefore B_1 = 100 \text{ kHz}$  Symbols are present in the BWH can be used as  $B_1 = B_2$

(c), we must use the OFDM system  $C = \log_2 M = 7$  bits per symbol

As  $B_2 = 100 \text{ kHz} - B_1 < 100 \text{ kHz} \rightarrow \text{we } \geq 2.1 \text{ kHz} \rightarrow \text{we } = 1 \text{ kHz per symbol. BWH}$

$$\therefore B_2 = \frac{C}{T} = 1.10 \text{ kHz}$$

$$(SNR = 10 \log_{10} 10 = 10 \text{ dB}) = 10 \log_{10} (1 + 10) = 21.04 \text{ dB}$$

$$\text{we } \geq \frac{100 \text{ Mbps}}{100 \text{ kHz}} = 1.00 \rightarrow 1 \text{ kHz per symbol}$$

$$\therefore B_2 = \frac{C}{T} = 1.10 \text{ kHz} \text{ (SNR Channel Bandwidth } = 1.10)$$

- (3) It is required to transmit 100 Mbps with  $B_2 \leq 10^6 \text{ Hz}$ . Three possible schemes are considered (a) 16QAM (b) 16-ary PSK (c) 16-ary PPM. What is the minimum number of the frequency bandwidth and the signal power required if the channel input is Gaussian?

$$\text{Solution: (a) } B \geq B_1 = B_2 = 100 \text{ kHz} \quad C = \frac{C}{T} = 100 \text{ kHz}$$

$$(M = 16, B_1 = \frac{C}{T} = 100 \text{ kHz}, C = \frac{C}{T} = 1.10 \text{ kHz}) \rightarrow \frac{C}{T} = 1.10 \text{ kHz}$$

$$(1.10 \text{ kHz}, B_2 = \frac{C}{T} = 1.10 \text{ kHz}, C = \frac{C}{T} = 1.10 \text{ kHz}) \rightarrow 1.10 \text{ kHz}$$

- (7) Show the following for BPSK: (a) BER with a maximum average transmit rate (BER) of 10<sup>-5</sup> (b) minimum required efficiency (c) probability of symbol error of the system under the condition of minimum required efficiency (approximate) (d) the Gray coded probability of bit error.

Solution: (a)  $\bar{r} = \frac{1}{2} \log_2 M$  BPSK, for minimum (BER) efficiency  $\bar{r} = 0.5$ ,  $M = 2$  bits per second per Hz.

(b)  $\bar{r} = \frac{1}{2} \log_2 M = 0.5$ ,  $M = 2$  and for minimum (BER)  $\bar{r} = 0.5$ , we get

$$\bar{r} = 0.5 \left( 1 - \frac{1}{20} \right) \left( 1 + \frac{10 \log_{10} 2}{10 \log_{10} 2} \right) \approx 1.479 \times 10^{-5}$$

$$\bar{r} \approx 1.479 \times 10^{-5} = 1.479 \times 10^{-5}$$

- (8) A BPSK system is to be transmitted via QPSK. (a) What is the signal rate? (b) What is the minimum required channel bandwidth? (c) What is the symbol error probability if received average power is 12 dB and  $\sigma = 1$  dB/Hz? (d) What is signal error rate?

Solution: (a) 14.4 kbps per sec. (b) 1.44 kHz. (c)  $1.44 \times 10^{-5}$ . (d) 1.44 kHz per sec per sec.

- (9) Determine the probability of error per symbol for a BPSK QPSK system where the received signal at the receiver input is 1 dBm, average additive white Gaussian noise and the system bandwidth is 120 kHz.

- (10) Show that two BPSK systems can operate simultaneously over the same channel by using orthogonal carriers (sine and cosine). Draw a block diagram for the transmitters and receivers. What is the overall aggregate data rate for the orthogonal carrier modulation system as a function of the channel and bandwidth? How does the aggregate data rate of the system compare to the data rate for a spreader-free communication system for two channels and their bandwidths the TDMA data rate (BPSK) are they?

- (11) What is the minimum base channel bandwidth of a BPSK signal carrying 1000 bps?

- (12) A baseband signal bandwidth is 54 kHz is sampled at 20% over Nyquist rate and coded using PCM system having 256 levels quantizer. The resulting PCM sample frequency is assumed to be sufficiently close to maximum channel data rate specified with ITU-T. The bandwidth of this channel was not less 70% of the operating binary rate (data rate) estimated by a limited access rate value (rate of 152 + 2.5% VHz per km, calculate the required minimum transmit power of the received signal for other parameters: the operating MFS is not more than 122 error bits per second (for  $E_b/N_0 \geq 1.7$ ).

$$\begin{aligned} \text{Solution: } B_s &= (1 + 0.2) \cdot B = 2.54 + 0.2 = 307 = 307 \text{ kHz} \\ B_s &= 57600_s = 480 \cdot 12000_s; f_s B_s = 1.7 + B_s = 493031 \text{ k symbols per second} \\ m &= \frac{B_s}{f_s} = 2.1730 = 11.06 \text{ per symbol}; \rightarrow R_s = \frac{B_s}{m} = 25.225 \text{ symbols per second} \\ B_c &= \frac{f_s}{m} = 1.860 \times 10^{-3} \text{ s} = 2 + 2 = 2.2 = 2.097 \text{ Hz} \\ \sigma_B &= 2.8 \left( \sqrt{\frac{B_s}{f_s}} + \sigma_{\frac{B_s}{f_s}} \right) = 2.8 \sqrt{2.1730} = 4.275 \text{ dB} \\ E &= \left( \frac{2.097}{1.860} \right)^2 = 1.277 \text{ dBW} / 2 \text{ s} = 1.433 \text{ dBm} \end{aligned}$$

- (13) Assume a channel has the following specifications:  $\gamma = 10^{-7.5} \text{ W/Hz}$ , Losses=0.05 per km under bandwidth= 12.5 kHz, it is assumed that a power of 5.0 W is not less received located at 2.0 km away from the transmitter. Using the BPSK scheme select the binary data rate of 10 to estimate the quantity of signal to be received that  $1.1 \times 10^{-7.5} \text{ W/Hz}$  will be the spectral efficiency of this channel?

$$\text{Answer: selecting on the given parameters, we get } \sigma = 22.065 \text{ per symbol W/Hz} \\ \text{have to set as multiplier } k \text{ at } 3, \text{ for the curves of } E_b \text{ show increasing } E_b \text{ is the same} \\ \text{and subtract the } E_b, \text{ the amount change } \sigma = 1.065 \text{ dB} = 5 \text{ levels per symbol}$$

- (14) An MFSK modulating system is used to transmit data at a rate 100 kbps over 50 k channel, assume the presence of additive white Gaussian noise with associated power spectral density  $1.01 \times 10^{-7.5} \text{ W/Hz}$ . Given coding a transmitter average power of 400 W, and expected bit error rate of 1000, calculate the minimum number of binary levels of the MFSK that should be used, and the corresponding bandwidth. (2) Compute the BER of this system.

$$\text{Answer: (1) } M = 64 \text{ levels per symbol (2) } \text{BER} = 1.012 \text{ error bits per second}$$

- (124) Given the input bits are 000110100101. Sketch the signal of a BPSK, QPSK, DPSK and 4QAM modulation.
- (125) A central user is having two-bit available bandwidth of 120kHz. Find the maximum data rate that can be supported in this line to produce a reliable communication of 120kHz of QPSK signaling with a single carrier. Is real?
- (126) Assume that  $R_b = 1$  Mbps for a binary data stream, compare the advantages and disadvantages of using each of these signaling methods.
- (127) Sketch the binary signal modulation system if 1 bit is binary value bit is (a) on most complex, (b) lowest peak power requirement for a given bit rate.
- (128) If an MPSK system is used to send binary data at a rate 700 kbps over a 120 kHz channel. Calculate the required 2% ratio of maximum frequency with 34 error rate maximum from (4.7).
- (129) An MPSK system is used to transmit through a 100 kHz channel. If the minimum data rate required is 10 kbps, what is the lowest SNR for all 3 processes to keep an  $k$  data transfer with probability of error bits not more than  $1.5 \times 10^{-3}$ ?
- (130) Assume a channel has the following specifications:  $\gamma = 10^{-24}$  W/Hz, 1 megawatt power source. Bandwidth of 120 kHz. A transmitted data power of 120 W is to be sent over a distance of 10 km away from the transmitter. Using the MPSK scheme, what should be the minimum value of  $M$  to maintain probability of symbol error no more than  $1.5 \times 10^{-3}$ ?
- (131) It is required to transmit 100 kbps with  $P_b \leq 10^{-3}$  using the binary PSK scheme at noise PSD of  $10^{-11}$  W/Hz. Determine the transmission bandwidth and the signal power required at the receiver input.



# Part 3 APPENDIX: Q-FUNCTION

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$$

$$Q(0) = 0.5$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{1}{2} e^{-x^2/2} \right) \quad \text{for } x > 0$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$$

$$Q(0) = 0.5$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{1}{2} e^{-x^2/2} \right)$$

x	Q	Q	Q	Q	Q	Q	Q	Q	Q	Q	Q
0	0.5000	0.4999	0.4998	0.4997	0.4996	0.4995	0.4994	0.4993	0.4992	0.4991	0.4990
0.1	0.4602	0.4599	0.4596	0.4593	0.4590	0.4587	0.4584	0.4581	0.4578	0.4575	0.4572
0.2	0.4207	0.4203	0.4200	0.4197	0.4194	0.4191	0.4188	0.4185	0.4182	0.4179	0.4176
0.3	0.3809	0.3805	0.3802	0.3799	0.3796	0.3793	0.3790	0.3787	0.3784	0.3781	0.3778
0.4	0.3438	0.3434	0.3431	0.3428	0.3425	0.3422	0.3419	0.3416	0.3413	0.3410	0.3407
0.5	0.3085	0.3081	0.3078	0.3075	0.3072	0.3069	0.3066	0.3063	0.3060	0.3057	0.3054
0.6	0.2743	0.2739	0.2736	0.2733	0.2730	0.2727	0.2724	0.2721	0.2718	0.2715	0.2712
0.7	0.2420	0.2416	0.2413	0.2410	0.2407	0.2404	0.2401	0.2398	0.2395	0.2392	0.2389
0.8	0.2107	0.2103	0.2100	0.2097	0.2094	0.2091	0.2088	0.2085	0.2082	0.2079	0.2076
0.9	0.1809	0.1805	0.1802	0.1799	0.1796	0.1793	0.1790	0.1787	0.1784	0.1781	0.1778
1.0	0.1524	0.1520	0.1517	0.1514	0.1511	0.1508	0.1505	0.1502	0.1499	0.1496	0.1493
1.1	0.1353	0.1349	0.1346	0.1343	0.1340	0.1337	0.1334	0.1331	0.1328	0.1325	0.1322
1.2	0.1193	0.1189	0.1186	0.1183	0.1180	0.1177	0.1174	0.1171	0.1168	0.1165	0.1162
1.3	0.1044	0.1040	0.1037	0.1034	0.1031	0.1028	0.1025	0.1022	0.1019	0.1016	0.1013
1.4	0.0904	0.0900	0.0897	0.0894	0.0891	0.0888	0.0885	0.0882	0.0879	0.0876	0.0873
1.5	0.0802	0.0799	0.0796	0.0793	0.0790	0.0787	0.0784	0.0781	0.0778	0.0775	0.0772
1.6	0.0714	0.0711	0.0708	0.0705	0.0702	0.0699	0.0696	0.0693	0.0690	0.0687	0.0684
1.7	0.0629	0.0626	0.0623	0.0620	0.0617	0.0614	0.0611	0.0608	0.0605	0.0602	0.0599
1.8	0.0568	0.0565	0.0562	0.0559	0.0556	0.0553	0.0550	0.0547	0.0544	0.0541	0.0538
1.9	0.0530	0.0527	0.0524	0.0521	0.0518	0.0515	0.0512	0.0509	0.0506	0.0503	0.0500
2.0	0.0497	0.0494	0.0491	0.0488	0.0485	0.0482	0.0479	0.0476	0.0473	0.0470	0.0467
2.1	0.0469	0.0466	0.0463	0.0460	0.0457	0.0454	0.0451	0.0448	0.0445	0.0442	0.0439
2.2	0.0450	0.0447	0.0444	0.0441	0.0438	0.0435	0.0432	0.0429	0.0426	0.0423	0.0420
2.3	0.0432	0.0429	0.0426	0.0423	0.0420	0.0417	0.0414	0.0411	0.0408	0.0405	0.0402
2.4	0.0415	0.0412	0.0409	0.0406	0.0403	0.0400	0.0397	0.0394	0.0391	0.0388	0.0385
2.5	0.0408	0.0405	0.0402	0.0399	0.0396	0.0393	0.0390	0.0387	0.0384	0.0381	0.0378
2.6	0.0401	0.0398	0.0395	0.0392	0.0389	0.0386	0.0383	0.0380	0.0377	0.0374	0.0371
2.7	0.0394	0.0391	0.0388	0.0385	0.0382	0.0379	0.0376	0.0373	0.0370	0.0367	0.0364
2.8	0.0387	0.0384	0.0381	0.0378	0.0375	0.0372	0.0369	0.0366	0.0363	0.0360	0.0357
2.9	0.0380	0.0377	0.0374	0.0371	0.0368	0.0365	0.0362	0.0359	0.0356	0.0353	0.0350
3.0	0.0374	0.0371	0.0368	0.0365	0.0362	0.0359	0.0356	0.0353	0.0350	0.0347	0.0344

$\alpha$	$\beta$	$\gamma$	$\delta$	$\epsilon$	$\zeta$	$\eta$	$\theta$	$\iota$	$\kappa$	$\lambda$
1	1.11	1.12	1.13	1.14	1.15	1.16	1.17	1.18	1.19	1.20
21	1.21	1.22	1.23	1.24	1.25	1.26	1.27	1.28	1.29	1.30
31	1.31	1.32	1.33	1.34	1.35	1.36	1.37	1.38	1.39	1.40
41	1.41	1.42	1.43	1.44	1.45	1.46	1.47	1.48	1.49	1.50
51	1.51	1.52	1.53	1.54	1.55	1.56	1.57	1.58	1.59	1.60
61	1.61	1.62	1.63	1.64	1.65	1.66	1.67	1.68	1.69	1.70
71	1.71	1.72	1.73	1.74	1.75	1.76	1.77	1.78	1.79	1.80
81	1.81	1.82	1.83	1.84	1.85	1.86	1.87	1.88	1.89	1.90
91	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99	2.00
101	2.01	2.02	2.03	2.04	2.05	2.06	2.07	2.08	2.09	2.10
111	2.11	2.12	2.13	2.14	2.15	2.16	2.17	2.18	2.19	2.20
121	2.21	2.22	2.23	2.24	2.25	2.26	2.27	2.28	2.29	2.30
131	2.31	2.32	2.33	2.34	2.35	2.36	2.37	2.38	2.39	2.40
141	2.41	2.42	2.43	2.44	2.45	2.46	2.47	2.48	2.49	2.50
151	2.51	2.52	2.53	2.54	2.55	2.56	2.57	2.58	2.59	2.60
161	2.61	2.62	2.63	2.64	2.65	2.66	2.67	2.68	2.69	2.70
171	2.71	2.72	2.73	2.74	2.75	2.76	2.77	2.78	2.79	2.80
181	2.81	2.82	2.83	2.84	2.85	2.86	2.87	2.88	2.89	2.90
191	2.91	2.92	2.93	2.94	2.95	2.96	2.97	2.98	2.99	3.00
201	3.01	3.02	3.03	3.04	3.05	3.06	3.07	3.08	3.09	3.10
211	3.11	3.12	3.13	3.14	3.15	3.16	3.17	3.18	3.19	3.20
221	3.21	3.22	3.23	3.24	3.25	3.26	3.27	3.28	3.29	3.30
231	3.31	3.32	3.33	3.34	3.35	3.36	3.37	3.38	3.39	3.40
241	3.41	3.42	3.43	3.44	3.45	3.46	3.47	3.48	3.49	3.50
251	3.51	3.52	3.53	3.54	3.55	3.56	3.57	3.58	3.59	3.60
261	3.61	3.62	3.63	3.64	3.65	3.66	3.67	3.68	3.69	3.70
271	3.71	3.72	3.73	3.74	3.75	3.76	3.77	3.78	3.79	3.80
281	3.81	3.82	3.83	3.84	3.85	3.86	3.87	3.88	3.89	3.90
291	3.91	3.92	3.93	3.94	3.95	3.96	3.97	3.98	3.99	4.00
301	4.01	4.02	4.03	4.04	4.05	4.06	4.07	4.08	4.09	4.10
311	4.11	4.12	4.13	4.14	4.15	4.16	4.17	4.18	4.19	4.20
321	4.21	4.22	4.23	4.24	4.25	4.26	4.27	4.28	4.29	4.30
331	4.31	4.32	4.33	4.34	4.35	4.36	4.37	4.38	4.39	4.40
341	4.41	4.42	4.43	4.44	4.45	4.46	4.47	4.48	4.49	4.50
351	4.51	4.52	4.53	4.54	4.55	4.56	4.57	4.58	4.59	4.60
361	4.61	4.62	4.63	4.64	4.65	4.66	4.67	4.68	4.69	4.70
371	4.71	4.72	4.73	4.74	4.75	4.76	4.77	4.78	4.79	4.80
381	4.81	4.82	4.83	4.84	4.85	4.86	4.87	4.88	4.89	4.90
391	4.91	4.92	4.93	4.94	4.95	4.96	4.97	4.98	4.99	5.00
401	5.01	5.02	5.03	5.04	5.05	5.06	5.07	5.08	5.09	5.10
411	5.11	5.12	5.13	5.14	5.15	5.16	5.17	5.18	5.19	5.20
421	5.21	5.22	5.23	5.24	5.25	5.26	5.27	5.28	5.29	5.30
431	5.31	5.32	5.33	5.34	5.35	5.36	5.37	5.38	5.39	5.40
441	5.41	5.42	5.43	5.44	5.45	5.46	5.47	5.48	5.49	5.50
451	5.51	5.52	5.53	5.54	5.55	5.56	5.57	5.58	5.59	5.60
461	5.61	5.62	5.63	5.64	5.65	5.66	5.67	5.68	5.69	5.70
471	5.71	5.72	5.73	5.74	5.75	5.76	5.77	5.78	5.79	5.80
481	5.81	5.82	5.83	5.84	5.85	5.86	5.87	5.88	5.89	5.90
491	5.91	5.92	5.93	5.94	5.95	5.96	5.97	5.98	5.99	6.00
501	6.01	6.02	6.03	6.04	6.05	6.06	6.07	6.08	6.09	6.10
511	6.11	6.12	6.13	6.14	6.15	6.16	6.17	6.18	6.19	6.20
521	6.21	6.22	6.23	6.24	6.25	6.26	6.27	6.28	6.29	6.30
531	6.31	6.32	6.33	6.34	6.35	6.36	6.37	6.38	6.39	6.40
541	6.41	6.42	6.43	6.44	6.45	6.46	6.47	6.48	6.49	6.50
551	6.51	6.52	6.53	6.54	6.55	6.56	6.57	6.58	6.59	6.60
561	6.61	6.62	6.63	6.64	6.65	6.66	6.67	6.68	6.69	6.70
571	6.71	6.72	6.73	6.74	6.75	6.76	6.77	6.78	6.79	6.80
581	6.81	6.82	6.83	6.84	6.85	6.86	6.87	6.88	6.89	6.90
591	6.91	6.92	6.93	6.94	6.95	6.96	6.97	6.98	6.99	7.00
601	7.01	7.02	7.03	7.04	7.05	7.06	7.07	7.08	7.09	7.10

The  $\alpha$  of Equation can be represented as

$$\alpha = \frac{1}{2} \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right)$$

For  $v = 0.5c$ ,  $\alpha = 0.06066$ ,  $\alpha = 0.06066$ , and  $\alpha = 0.06066$ .

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