



Lecture Notes for the Class

# Engineering Probability and Statistics

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بلا إحصائيات، أنت مجرد شخص آخر لديه رأي.

*"Without data, you're just another person with an opinion."*

W. Edwards Deming

This print includes a collection of selected Materials from the books:

- Statistics for Engineers and Scientists, 4<sup>th</sup> edition, by Navidi
- Probability & Statistics for Engineers & Scientists, 9th edition, by Walpole et al.
- Schaum's Outline Probability and Statistics 9th edition, by Spiegel et al.

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# Introduction

## Statistical Engineering

An engineer is someone who solves problems of interest to society by the efficient application of scientific principles. Engineers accomplish this by either refining an existing product or process or by designing a new product or process that meets customer needs.

The field of statistical design deals with the collection, presentation, analysis, and use of data to make decisions, solve problems, and design products and processes.

**Variability** – measures characteristics of a system or phenomenon that produces the same results.

**Sources of variability:**

**Random variability:** for a fixed measurement,  $y$ , is constant, and  $y$  is random (stochastic)  $Y = \mu + \epsilon$ .

**Sample versus population:**



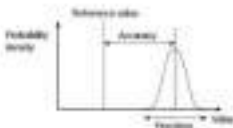
**Probability versus Statistics:** Probability deals with predicting the likelihood of future events, while statistics involves the analysis of the frequency of past events. Probability is generally a theoretical branch of mathematics, which studies the consequences of mathematical definitions. Statistics is generally an applied branch of mathematics, which aims to make sense of observations in the real world.

**Probability vs. Frequentism:** It is usually possible, at any point in time, to ask a "yes/no" question. If we could repeatedly pose this with the same conditions, the answer to the question is probabilistic.

Example: an event, "Today's temperature is below 10°C" is probabilistic.

# Accuracy vs Precision

Meaning	Meaning	Example
<b>Precision</b>	<ul style="list-style-type: none"> <li>Repeatability of being exact</li> <li>Accuracy without close the measured values to a desired value</li> </ul>	<ul style="list-style-type: none"> <li>Even precision is required to dig the pit hole accurately</li> <li>Precision leveling is used to design runway surfaces</li> </ul>
<b>Accuracy</b>	<ul style="list-style-type: none"> <li>Effect of bias/error on correct</li> <li>The ability to be something without making mistakes</li> <li>Accuracy is how close a measured value is to the actual (true) value</li> </ul>	<ul style="list-style-type: none"> <li>It's not good to be going with a car getting three off accuracy</li> </ul>



High Accuracy  
Low Precision



Low Accuracy  
High Precision



High Accuracy  
High Precision

- **Cherry picking** is the act of pointing to individual cases or data that seem to confirm a position while ignoring a significant number of related cases or data that may contradict that position. It is a kind of fallacy of selective attention, the more common example of which is the confirmation bias. Cherry picking may be committed intentionally or unintentionally. This refers to a more positive or particularity.

The focus is based on the perceived problem of the existing data, such as cherry. The pointer would be expected to only select the good and healthy fruits. In otherwise this only sets the selected fruit may thus wrongly conclude that more, or even all, of the fruit's fruit is in a likewise good condition. This can also give a false impression of the quality of the fruit (since it's only considered to be a representative sample).

Just as another cherry-picking is to include information that confirms a claim and then select only the data that supports that claim or ignores any contradiction of the data that could refute the claim. In other words, it's about selecting only data that confirms the claim.

- **Confirmation bias** is the tendency to search for, interpret, favor, and recall information in a way that confirms and is consistent with preexisting beliefs or hypotheses. It is a type of cognitive bias and a systematic error of selective reasoning. People display this bias when they gather or remember information selectively, or when they interpret it in a biased way. The effect is stronger for emotionally charged issues and for deeply entrenched beliefs. Confirmation bias is a variation of the more general tendency of perceptual.

People also tend to ignore or misinterpret evidence supporting their existing position. Based on such, the perceptual and memory process is considered to represent a confirmation bias. For example, a disagreement between two persons even though the different cases are representative for the same condition, belief perseverance (false beliefs persist after all evidence for them is shown to be false), the structural priming effect (a greater reliance on information associated with a word) and illusory correlation (false people likely perceive an association between non-associated situations).

- A **hasty generalization** is a fallacy in which a conclusion is not logically justified by sufficient or sufficient evidence. It's also called as *hasty generalization*. A *hasty generalization*, a *hasty generalization*, and *jumping to conclusions*.

Drawing a conclusion based on a small sample size, rather than making a conclusion that is made based on how with the typical or average situation.

- Example A: My father smoked like a pipe of cigarettes a day about 40 years ago but then he died with lung cancer. Therefore, smoking really can't be harmful for you.

Explanation: It is extremely unreasonable (and dangerous) to draw a personal conclusion about the health risks of smoking by the case study alone.

- Example B: Four out of five dentists recommended happy shiny dental amalgam based. Therefore, it must be great.

Explanation: It turns out that only five dentists were actually asked. When a random sampling of 100 dentists was polled, only 10% actually recommended the brand. The

One out of five said they were usually stopped tonight or yesterday morning. It just happened to be a bad night for me. I was going to sleep.

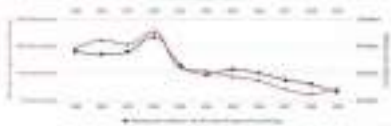
- **Sporadic Relationships:** the old adage says, "correlation does not imply causation," and because two events occur in tandem or tandem, this only proves that doesn't prove that they are meaningfully related to one another.

A sporadic correlation is a statistical relationship in which two or more events or variables are associated but not causally related. The further away from the present of a certain event, the more likely.

The example (see <http://www.oxfordjournals.org>) takes away through correlation in public data sets.

For example, the number of people who drowned by falling from a swimming pool increases with the number of times the word "Egg" appears in the text. Another example: the per capita consumption of chocolate correlates with the number of people who died by becoming tangled in their bedclothes. Age.

100 people will be killed by three thousand  
Deaths killed in collisions with railway track  
The number of people who died by falling from a swimming pool



## The Wisdom of Crowds

In 1907, Dr. Francis Galton asked 789 villagers to guess the weight of an ox. None of them got the right answer, but when Galton averaged their guesses, he arrived at a very precise estimate. This is a classic demonstration of the 'wisdom of the crowds', where groups of people pool their abilities to show collective intelligence.

Using statistics, the collective wisdom of a group will surpass that of even the most intelligent individual members.

Watch <https://www.youtube.com/watch?v=...>



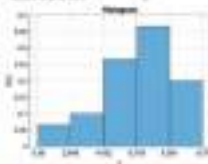


**Example 1** Give the histogram for the following data using 5 categories.

0.96, 4.95, 1.29, 6.12, 8.36, 1.46, 4.38, 8.28, 2.34, 6.81, 9.79, 3.41, 5.96, 6.88, 4.12, 6.43, 6.38, 6.96, 4.17, 1.27, 9.33, 6.34, 8.76, 6.66, 9.37, 7.04, 1.11, 6.79, 1.09, 3.11

**Solution:** First the number of points is  $N = 25$ , and the number of categories is  $k = 5$ , hence the range of each category is  $\Delta x = \frac{9.79 - 0.96}{5} = 1.79$ . So, we construct the table as

Category Index	Category	Frequency of $x$ $n_i$	Relative frequency $f(x) = \frac{n_i}{N}$
1	0.96 < $x$ < 2.75	3	0.1200
2	2.75 < $x$ < 4.54	3	0.1200
3	4.54 < $x$ < 6.33	8	0.3200
4	6.33 < $x$ < 8.12	10	0.4000
5	8.12 < $x$ < 9.91	6	0.2400



4 Note the difference between the notation  $x$  and  $\bar{x}$ .



As  $n$  increases, we can choose larger  $r$ . Thus, the shape of  $Y_{10}$  approaches to the "normal distribution".

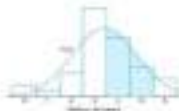
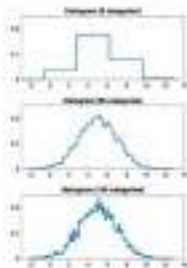
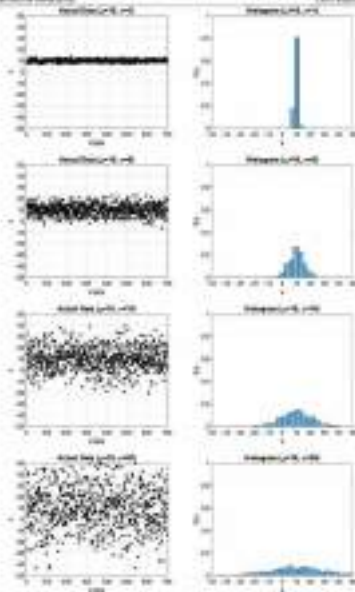


Figure 1.1: The Shape of Normal Distribution



Figure 1.2: The Shape of Data



# Probability

## Experiment

It is a process that results in an outcome that cannot be predicted in advance with certainty. Like tossing a coin, rolling a die, and measuring a voltage.

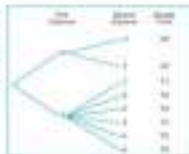
## Sample Space

For an experiment, it is the set of all possible outcomes of the experiment. For tossing a coin, we can say that  $S = \{H, T\}$  is the sample space. Consider the experiment of rolling a six-sided die. If we are interested in the number that shows on the top face, the sample space is  $S = \{1, 2, 3, 4, 5, 6\}$ . If we are interested only in whether the number is even or odd, the sample space is simply  $S = \{\text{even}, \text{odd}\}$ .

## Tree Diagrams

Example: An experiment consists of flipping a coin and then flipping it a second time if it lands heads. If it falls tails on the first flip, then a die is rolled once. To list the elements of the sample space, proceed the usual notation, we construct the tree diagram. By connecting all paths, realize that the sample space is:

$$S = \{HH, HT, TH, TT, T1, T2, T3, T4, T5, T6\}$$



## Statements

Some experiments have sample spaces with large or infinite number of outcomes. In such cases it is best to describe a set using a statement, which is expressed in the form  $A = \{x \in S \mid P(x)\}$ , where  $P$  is the predicate in  $P$  is a predicate. For example:

$$\bullet A = \{x \in S \mid 1 \leq x \leq 6\} = \{1, 2, 3, 4, 5, 6\}$$

$$\bullet A = \{x \in S \mid 3 \leq x \leq 4\} = \{3, 4\}$$

$$\bullet A = \{x \in S \mid 2 \leq x \leq 6 \text{ and } 3 \leq y \leq 6\} = \left\{ \begin{matrix} (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), \\ (3, 3), (3, 4), (3, 5), (3, 6), (4, 3), (4, 4), (4, 5), (4, 6), \\ (5, 3), (5, 4), (5, 5), (5, 6), (6, 3), (6, 4), (6, 5), (6, 6) \end{matrix} \right\}$$

Event:

It is a subset of a sample space.

**Example:** Consider the experiment of rolling two-sided die. The sample space for the die roll is  
 $S = \{1, 2, 3, 4, 5, 6\}$ . In the event  $A$  is the collection of the odd numbers only. Then  
 $A = \{1, 3, 5\}$  or  $\text{odd} = \{1, 3, 5\}$ .

## Complement

The complement of an event  $A$  with respect to  $S$  is the subset of all elements of  $S$  that are not in  $A$ . We denote the complement of  $A$  by the symbol  $A^c$ .

**Example:** Consider the sample space  $S = \{\text{black, red, blue, white, green, yellow}\}$ . Let the event  $A$  be  $A = \{\text{red, blue, yellow}\}$ . Then  $A^c = \{\text{black, white, green}\}$ .

## Containing Events

We often encounter events by combining simpler events. Because events are subsets of sample spaces, it is convenient to use the language of set or describe events composed in this way. We review the necessary notations.

- The **union** of two events  $A$  and  $B$ , denoted  $A \cup B$ , is the set of outcomes that belong either to  $A$ , or to  $B$ , or to both. In words,  $A \cup B$  means “ $A$  or  $B$ .” Then the event  $A \cup B$  occurs whenever either  $A$  or  $B$  occurs.
- The **intersection** of two events  $A$  and  $B$ , denoted  $A \cap B$ , is the set of outcomes that belong both to  $A$  and to  $B$ . In words,  $A \cap B$  means “ $A$  and  $B$ .” Then the event  $A \cap B$  occurs whenever both  $A$  and  $B$  occur.
- The **complement** of an event  $A$ , denoted  $A^c$ , is the set of outcomes that do not belong to  $A$ . In words,  $A^c$  means “not  $A$ .” Then the event  $A^c$  occurs whenever  $A$  does not occur.

Events can be graphically illustrated with Venn diagrams. Figure 1.1 illustrates the events  $A \cup B$ ,  $A \cap B$ , and  $A^c$ .



**FIGURE 1.1** Venn diagrams illustrating various events: (a)  $A \cup B$ , (b)  $A \cap B$ , (c)  $A^c$ .

## Mutually Exclusive Events

There are some events that cannot occur together. For example, it is impossible that a coin can come up both heads and tails, and it is impossible that a card pin can be both red and black. Events like this are said to be **mutually exclusive**.

### Definition

- The events  $A$  and  $B$  are said to be **mutually exclusive** if they have no outcomes in common.
- More generally, a collection of events  $A_1, A_2, \dots, A_n$  is said to be **mutually exclusive** if no two of them have any outcomes in common.

The Venn diagram in Figure 2.2 illustrates mutually exclusive events.



FIGURE 2.2 The events 1 and 2 are mutually exclusive.





## Counting Sample Points

If an operation can be performed in  $m$  ways, and if for each of these  $m$  ways a second operation can be performed in  $n$  ways, then the total number of ways to perform the two operations is  $mn$ .

### The Fundamental Principle of Counting

Suppose that operation  $\alpha$  can be performed in  $m$  ways, and if for each of these  $m$  ways a second operation,  $\beta$ , can be performed in  $n$  ways, then the total number of ways to perform the two operations is  $mn$ . For each of the  $m$  ways to perform the first operation, there are  $n$  ways to perform the second operation, and so the total number of ways to perform the two operations is  $n$  times the number of ways to perform the first operation, and so the total number of ways to perform the two operations is  $n$  times  $m$ .

## Example 2.1.1

When entering a password on a computer that can 1 letter of keyboard, 4 choices for the amount of memory, 3 choices of subunit, and 3 choices of picture, how many passwords are possible?

**Solution:**

The total number of ways to enter a computer is  $(1)(4)(3)(3) = 36$ .

**Example 2.1.2** How many possible passwords are there to the sample space when a pair of dice is thrown twice?

**Solution:** The first die can land face-up to any one of  $n_1 = 6$  ways. For each of these 6 ways, the second die can also land face-up to  $n_2 = 6$  ways. Therefore, the pair of dice can land to  $n_{\text{total}} = (6)(6) = 36$  possible ways. □

**Example 2.1.3** A diagram of a tree structure is often presented to show clearly a chain of further events, ordered, and conditional events, leading to results, decisions, and splits from these plans. In how many different ways can a house order one of these 'trees'?

**Solution:** Since  $n_1 = 4$  and  $n_2 = 3$ , a house can choose from

$$n_{\text{total}} = (4)(3) = 12 \text{ possible trees}$$



**Example:** If a 25-member club needs to elect a chair and a treasurer, how many different ways can they try to be elected?

**Solution:** For the chair position, there are 25 total possibilities. For each of these 25 possibilities, there are 24 possibilities to elect the treasurer. Using the multiplication rule, we obtain  $n_1 \cdot n_2 = 25 \cdot 24 = 480$  different ways. 4

### Permutations

A **permutation** is an ordered list of objects. For example, there are six permutations of the letters A, B, C: ABC, ACB, BAC, BCA, CAB, and CBA.

For any positive integer  $n$ ,  $n!$  is  $n!$  ( $n$  factorial) defined as follows:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots (2) \cdot (1)$$

The number of permutations of  $n$  objects is  $n!$ .

### Example

1.1

Five people stand in line at a movie theater. In how many different orders can they be arranged?

**Solution:**

The number of permutations of a collection of five people is  $5! = (5)(4)(3)(2)(1) = 120$ .

The number of permutations of  $n$  distinct objects taken  $r$  at a time is

$${}_nP_r = \frac{n!}{(n-r)!}$$

**Example:** In one year, three months (growth, heating, and service) will be given to a class of 25 graduate students in a master's department. If each student can receive at most one award, how many possible selections are there?

**Solution:** Since the awards are distinguishable, it is a permutation problem. The total number of sample points is

$${}_nP_1 = \frac{25!}{(25-1)!} = \frac{25!}{24!} = (25)(24)(23) \cdots (2)(1) = 25,000$$

**Example 10.1.1** A professor will choose one of the three students who are submitting a paper. The professor will choose one of the three students who are submitting a paper. The professor will choose one of the three students who are submitting a paper.

- (a) There are no solutions.
- (b) A will never win if he is present.

**Definition 10.1.1** The total number of students who are submitting a paper is

$$u_1 + u_2 + u_3 = 100 - 10 = 90.$$

- (c) Since A will never win if he is present, we have two situations: (i) A is not present, in which case all three students are submitting a paper; (ii) A is present, in which case only two students are submitting a paper. In the first case, the total number of students is  $u_1 + u_2 + u_3 = 100 - 10 = 90$ . In the second case, the total number of students is  $u_1 + u_2 + u_3 = 100 - 10 = 90$ .

## Example 10.1.2

Five different ways are available for the three students to submit a paper. There are three different ways for the three students to submit a paper. There are three different ways for the three students to submit a paper.

**Definition 10.1.1**

We use the fundamental principle of counting. There are three different ways for the three students to submit a paper. There are three different ways for the three students to submit a paper. There are three different ways for the three students to submit a paper.

### Combinations

In some cases, when choosing a set of objects from a larger set, we don't care about the ordering of the chosen objects; we are only interested in the choices.

The number of combinations of  $n$  objects chosen from a group of  $N$  objects is

$$\binom{N}{n} = \frac{N!}{n!(N-n)!} \quad (1.32)$$

### Example 2.11

It is common to have 30 people attend and 1 will be chosen as president or vice president of a group. The president or vice president is the order in which the people are chosen does not matter. How many different groups of two people can be chosen?

**Solution:**

Since the order of the two chosen people does not matter, we just calculate the number of combinations of 30 people from 30. This is

$$\begin{aligned} \binom{30}{2} &= \frac{30!}{2!28!} \\ &= \frac{(30 \cdot 29 \cdot 28!)(2 \cdot 1)(1!)}{(2 \cdot 1)(28!)} \\ &= 435 \text{ ways} \end{aligned}$$

**Exercise:** A group of 30 people will go to a class that has 100 computers. How many different groups of 10 people and 5 sports games can be chosen? How many ways are there that the students can get 2 lottery and 2 sports games?

**Solution:** The number of ways of selecting 2 computers from 10 is

$$\binom{10}{2} = \frac{10!}{2!8!} = 45$$

The number of ways of selecting 2 computers from 5 is

$$\binom{5}{2} = \frac{5!}{2!3!} = 10$$

Using the multiplication rule (Rule 4.1) with  $n_1 = 45$  and  $n_2 = 10$ , we have  $(45)(10) = 450$  ways. 4



## EXERCISES 12

**1.1** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.2** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.3** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.4** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.5** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.6** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.7** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.8** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.9** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.10** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.11** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.12** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.13** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.14** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.15** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.16** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.17** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.18** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.19** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.20** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

**1.21** A particle is projected vertically upwards with an initial speed of  $u$  m s<sup>-1</sup>. It reaches a maximum height of  $h$  m. Find  $u$  in terms of  $h$ .

## Probability of an Event

### Definition

Given any experiment with any possible

- The probability  $P(A)$  of an event occurring, denoted as  $A$ , occurs.
- $P(A)$  is the proportion of times that event  $A$  would occur in the long run, if the experiment were to be repeated over and over again.

Let  $S$  be a sample space containing  $N$  equally likely outcomes, and let  $A$  be an event containing  $k$  outcomes, then

$$P(A) = \frac{k}{N} \quad (1.1)$$

## Axioms of Probability

### The Axioms of Probability

1. For all  $A$  in the sample space, then  $0 \leq P(A) \leq 1$ .
2. For any event  $A$ ,  $P(\bar{A}) = 1 - P(A)$ .
3. If  $A$  and  $B$  are mutually exclusive events, then  $P(A \cup B) = P(A) + P(B)$ . More generally, if  $A_1, A_2, \dots$  are mutually exclusive events, then  $P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$ .

Let  $\bar{A}$  be  $A^c$ .

$$P(\bar{A}) = 1 - P(A) \quad (1.2)$$

Let  $\bar{A}$  denote the empty set, then

$$P(\bar{A}) = 0 \quad (1.3)$$

**Example 1.1** A coin is tossed twice. What is the probability that at least 1 head occurs?

**Solution:** The sample space for this experiment is

$$S = \{HH, HT, TH, TT\}$$

If the coin is balanced, each of these outcomes is equally likely to occur. Therefore, we assign a probability of  $1/4$  to each sample point. Thus, as in (1.1), as  $n(s) = 4$ , if  $A$  represents the event of at least 1 head occurring, then

$$A = \{HH, HT, TH\} \text{ and } P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

**Example:** A die is tossed in such a way that an even number is twice as likely to occur as an odd number. If  $E$  is the event that a number less than 4 occurs on a single toss of the die, find  $P(E)$ .

**Solution:** The sample space is  $S = \{1, 2, 3, 4, 5, 6\}$ . We assign a probability of  $x$  to each odd number and a probability of  $2x$  to each even number. Since the sum of the probabilities must be 1, we have  $3x = 1$  or  $x = 1/3$ . Hence, probabilities of 1/3 and 2/3 are assigned to each odd and even number, respectively. Therefore,

$$E = \{1, 2, 3\} \text{ and } P(E) = \frac{1}{3} + \frac{2}{3} + \frac{1}{3} = \frac{4}{3} \quad \square$$

**Example:** In the example above, let  $A$  be the event that an even number comes up and let  $B$  be the event that a number divisible by 3 occurs. Find  $P(A \cup B)$  and  $P(A \cap B)$ .

**Solution:** For the events  $A = \{2, 4, 6\}$  and  $B = \{3, 6\}$ , we have

$$A \cup B = \{2, 3, 4, 6\} \text{ and } A \cap B = \{6\}.$$

By assigning probabilities of 1/3 to each odd number and 2/3 to each even number, we have

$$P(A \cup B) = \frac{2}{3} + \frac{1}{3} + \frac{2}{3} + \frac{2}{3} = \frac{7}{3} \text{ and } P(A \cap B) = \frac{2}{3} \quad \square$$

### The Inclusion-Exclusion

Let  $A$  and  $B$  be any events. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (2.1)$$



If  $A$  and  $B$  are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B)$$

For three events  $A$ ,  $B$ , and  $C$ ,

$$\begin{aligned} P[A \cup B \cup C] &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C). \end{aligned}$$



**Example:** John is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed by his companies for them, he assumes that his probability of getting an offer from company A is 0.5, and his probability of getting an offer from company B is 0.30. If he believes that the probability that he will get offer from both companies is 0.2, what is the probability that he will get at least one offer from these two companies?

**Solution:** Using the addition rule, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.30 - 0.2 = 0.6. \quad \square$$

**Example:** What is the probability of getting a total of 7 or 11 when a pair of dice are tossed?

**Solution:** Let A be the event that 7 occurs and B the event that 11 occurs. Then, a total of 7 occurs for 6 of the 36 possible points, and a total of 11 occurs for only 2 of the possible points. Since all possible points are equally likely, we have  $P(A) = 1/6$  and  $P(B) = 1/18$ . The events A and B are mutually exclusive since a total of 7 and 11 cannot both occur on the same toss. Therefore,

$$P(A \cup B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}. \quad \square$$

This result could also have been obtained by counting the total number of points for the total AUB, namely 8, and writing

$$P(A \cup B) = \frac{8}{36} = \frac{2}{9} = \frac{2}{9}. \quad \square$$

If  $A_1, A_2, \dots, A_n$  is a partition of sample space S, then

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n) = P(S) = 1.$$

If A and A' are complementary events, then

$$P(A) + P(A') = 1.$$

**Example:** If the probabilities that an automobile mechanic will service A, B, C, D, E, F, G or H are, respectively, given as follows (in cents): 12, 8, 15, 9, 20, 5, 30, and 10, what is the probability that he will service at least 1 car on the next day of work?

**Solution:** Let E be the event that at least 1 car are serviced. Then,  $P(E) = 1 - P(E')$ , where E' is the event that none of the cars are serviced. Thus,

$$P(E') = 0.02 + 0.05 = 0.07.$$

It follows from Theorem 14 that

$$P(E) = 1 - 0.07 = 0.93. \quad \square$$

## EXERCISES

**2.88** Find the mean and variance of the following continuous random variables.

- The probability that an automobile will start within 5, 10, 15, or 20 minutes are given by the following set of probabilities: 0.10, 0.20, 0.30, 0.40, respectively.
- The probability that a left-side window is 10 ft, and the probability that it will not be 10 ft, are 0.20.
- The probabilities that a person will be in 1, 2, 3, or 4 years in a company are given by the following set of probabilities: 0.10, 0.20, 0.30, 0.40, respectively.

**2.89** Assuming that all elements of  $S$  in Exercise 2.8 are equal, find the probability of each event.

- The probability of event  $A$ .
- The probability of event  $B$ .
- The probability of event  $A \cup B$ .

**2.90** A box contains 100 marbles, of which 70 are red, 20 are blue, 5 are white, 10 are green, and 5 are yellow. The marbles are numbered 1 to 100. What is the chance that the first marble drawn is red? What is the chance that the first marble drawn is blue? What is the chance that the first marble drawn is white? What is the chance that the first marble drawn is green? What is the chance that the first marble drawn is yellow?

**2.91** A card is drawn from a standard deck of 52 cards. Find the probability that the card is a heart, a diamond, a spade, or a club. Find the probability that the card is a heart or a diamond. Find the probability that the card is a heart or a spade. Find the probability that the card is a heart or a club.

**2.92** A card is drawn from a standard deck of 52 cards. Find the probability that the card is a heart, a diamond, a spade, or a club. Find the probability that the card is a heart or a diamond. Find the probability that the card is a heart or a spade. Find the probability that the card is a heart or a club.

- It is a heart or a diamond.
- It is a heart or a spade.
- It is a heart or a club.

**2.93** A card is drawn from a standard deck of 52 cards. Find the probability that the card is a heart, a diamond, a spade, or a club. Find the probability that the card is a heart or a diamond. Find the probability that the card is a heart or a spade. Find the probability that the card is a heart or a club.

- It is a heart or a diamond.
- It is a heart or a spade.
- It is a heart or a club.

**2.94** A card is drawn from a standard deck of 52 cards. Find the probability that the card is a heart, a diamond, a spade, or a club. Find the probability that the card is a heart or a diamond. Find the probability that the card is a heart or a spade. Find the probability that the card is a heart or a club.

- It is a heart or a diamond.
- It is a heart or a spade.
- It is a heart or a club.

**2.95** Assuming that all elements of  $S$  in Exercise 2.8 are equal, find the probability of each event.

Event $A$	0.10
Event $B$	0.20
Event $C$	0.30
Event $D$	0.40
Event $E$	0.50

- What is the probability that a 10 is a 10?
- What is the probability that a 10 is a 10?
- What is the probability that a 10 is a 10?
- What is the probability that a 10 is a 10?

**2.96** Assuming that all elements of  $S$  in Exercise 2.8 are equal, find the probability of each event.

- What is the probability that a 10 is a 10?
- What is the probability that a 10 is a 10?
- What is the probability that a 10 is a 10?
- What is the probability that a 10 is a 10?

## Conditional Probability

A probability that is based on a part of a sample space is called a *conditional probability*. While the unconditional probability is based on the entire sample space.

Let  $A$  and  $B$  be events with  $P(B) > 0$ .

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

An additional situation occurs, requires that one sample space if in the presence of which is a well known situation. Suppose the requirements for a college degree. We shall categorize them as working or graduate and employment status. The data are given in Table 1.1.

Table 1.1: Employment of the labor force in a fixed time

	Employed	Unemployed	Total
Male	400	20	420
Female	120	200	320
Total	520	220	740

One of these individuals is to be selected at random. In a way throughout the process to gather the advantages of conditioning on the labor force. We shall be concerned with the following events:

- I: A man is chosen.
- E: The man chosen is employed.

Using the relevant sample space  $\mathcal{E}$ , we find that

$$P(E|I) = \frac{400}{420} = \frac{20}{21}$$

Let  $n(E)$  denote the number of elements in any set  $E$ . Using this notation, observed which has an equal chance of being selected, we can write

$$P(E|I) = \frac{n(E \cap I)}{n(I)} = \frac{n(E \cap I) \cdot n(\mathcal{E})}{n(I) \cdot n(\mathcal{E})} = \frac{P(E \cap I)}{P(I)}$$

where  $P(E \cap I)$  and  $P(I)$  are found from the original sample space  $\Omega$ . To verify this, simply note that

$$P(I) = \frac{420}{740} = \frac{21}{37} \quad \text{and} \quad P(E \cap I) = \frac{400}{740} = \frac{20}{37}$$

Thus,

$$P(E|I) = \frac{20/37}{21/37} = \frac{20}{21}$$

as before.

**Example:** The probability that a randomly selected light depends on time is  $P(D) = 0.03$ ; the probability that it arrives on time is  $P(A) = 0.95$ ; and the probability that it depends and arrives on time is  $P(A \cap D) = 0.78$ . Find the probability that a phone (a) arrives on time, given that it depends on time, and (b) depends on time, given that it has arrived on time.

**Solution:** Using Definition 2.10, we have the following:

- (a) The probability that a phone arrives on time, given that it depends on time,

$$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.03} = 0.03.$$

- (b) The probability that a phone depends on time, given that it has arrived on time, is

$$P(D|A) = \frac{P(A \cap D)}{P(A)} = \frac{0.78}{0.95} = 0.03.$$

4

## The Multiplicative Rule

If  $A$  and  $B$  are outcomes with  $P(B) > 0$ , then

$$P(A \cap B) = P(B|A)P(A) \quad (2.11)$$

**Example:** Suppose that we have a line bus containing 20 buses, of which 2 are defective. If 2 buses are selected at random and, without replacement, the bus is considered defective upon failing the test, what is the probability that both buses are defective?

**Solution:** We find that  $A$  is the event that the first bus is defective and  $B$  the event that the second bus is defective then we interpret  $A \cap B$  as the event that  $A$  occurs and then  $B$  occurs after  $A$  has occurred. The probability of first selecting a defective bus is  $1/10$ ; thus the probability of selecting a second defective bus from the remaining 9 is  $1/9$ . Hence,

$$P(A \cap B) = \left(\frac{1}{10}\right)\left(\frac{1}{9}\right) = \frac{1}{90}.$$

4

## Independent Events

Two events  $A$  and  $B$  are independent if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A).$$

assuming the occurrence of the conditional probabilities. Otherwise,  $A$  and  $B$  are dependent.

If  $A$  and  $B$  are independent events, then

$$P(A \cap B) = P(A)P(B) \quad (2.12)$$

### Example 2.22

A system contains two components, A and B. Both components contribute to the system's cost. The probability that component A fails is 0.001, and the probability that component B fails is 0.002. Assume the two components function independently. What is the probability that the system functions?

**Solution**

The probability that the system functions is the probability that both components function. Therefore

$$P(\text{system functions}) = P(A \text{ functions and } B \text{ functions})$$

Since the components function independently,

$$\begin{aligned} P(A \text{ functions and } B \text{ functions}) &= P(A \text{ functions})P(B \text{ functions}) \\ &= 1 - (1 - 0.999)(1 - 0.998) \\ &= 0.997 - 0.998 = 0.999 \\ &= 99.9\% \end{aligned}$$

### Example 2.23

At the microprocessor manufacturing plant, 10% are defective. Five microprocessors are chosen at random. Assume that failures are independent. What is the probability that they all work?

**Solution**

Let  $i = 1, \dots, 5$  and  $A_i$  denote the event that the  $i$ th microprocessor works. Then

$$\begin{aligned} P(\text{all function}) &= P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) \\ &= P(A_1)P(A_2)P(A_3)P(A_4)P(A_5) \\ &= 0.9^5 = 0.5905 \\ &= 59.05\% \end{aligned}$$

### Example 2.24

At Example 2.23, what is the probability that at least one of the microprocessors works?

**Solution**

We can solve this by using the probability rule that

$$P(\text{at least one works}) = 1 - P(\text{all are defective})$$

Now, using 0.1 denotes the probability that a microprocessor is defective,

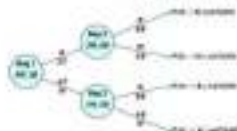
$$\begin{aligned} P(\text{all are defective}) &= P(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c \cap A_5^c) \\ &= P(A_1^c)P(A_2^c)P(A_3^c)P(A_4^c)P(A_5^c) \\ &= 0.1^5 \\ &= 0.0001 \end{aligned}$$

Therefore the probability that at least one works is  $1 - 0.0001 = 0.9999$ .

**Example:** One bag contains 1 white ball and 2 black balls, and a second bag contains 2 white balls and 1 black ball. One ball is drawn from the first bag and placed, unseen, in the second bag. What is the probability that a ball now drawn from the second bag is blue?

**Solution:** Let  $B_1$ ,  $B_2$ , and  $W_1$  represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1. Of course, neither in the case of the mutually exclusive events  $B_1 \cap B_2$  and  $W_1 \cap B_2$ . The various possibilities and their probabilities are illustrated in Figure 1.8. Thus

$$\begin{aligned} P(B_1 \cap B_2) + P(W_1 \cap B_2) &= P(B_1 \cap B_2) + P(W_1 \cap B_2) \\ &= P(B_1)P(B_2|B_1) + P(W_1)P(B_2|W_1) \\ &= \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{2}{3} \end{aligned}$$



## Reliability Analysis

### Example

1.20

Figure 1.9 illustrates a component, A, and B, connected in series as shown in the following diagram:



The system will function only if both components are live. The probability that A functions is given by  $P(A) = 0.95$ , and the probability that B functions is given by  $P(B) = 0.90$ . Assume that A and B function independently. Compute the probability that the system functions.

#### Solution

Since the system will function only if both components function, it follows that

$$\begin{aligned} \text{Proposed Solution: } P(A \cap B) &= P(A)P(B) \text{ by the assumption of independence} \\ &= 0.95 \times 0.90 \\ &= 0.855 \end{aligned}$$

# Example 2.28

A communication system consists of two components A and B connected in parallel as shown in the following diagram.



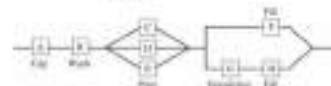
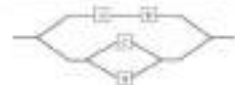
The system will operate if either A or B functions. The probability that A functions is 0.95 and the probability that B functions is 0.92. Assume A and B function independently. What is the probability that the system will function?

**Solution:**

Since the system will function as long as either of the two components functions, it follows that

$$\begin{aligned}
 P(\text{system functions}) &= P(A \cup B) \\
 &= P(A) + P(B) - P(A \cap B) \\
 &= P(A) + P(B) - P(A)P(B) \\
 &\quad \text{by the assumption of independence} \\
 &= 0.95 + 0.92 - (0.95)(0.92) \\
 &= 0.989
 \end{aligned}$$

The reliability of more complex systems can often be determined by decomposing the system into a number of subsystems, each of which is either a component or a connected series or parallel.







**Exercises on the Law of Total Probability  
or Bayes' Theorem**

**2.28** Suppose the lifespan of an electronic device is as given in Figure 2.10. What is the probability that the device works? Assume the components all work properly.



Figure 2.10: Diagram for Exercise 2.28

**Exercises on the Law of Total Probability  
or Bayes' Theorem**

**2.29** Consider the circuit in Figure 2.11. Assume the components all work properly.

(a) What is the probability that the circuit works?

(b) Assume that the circuit works. What is the probability that the component 2 is not working?

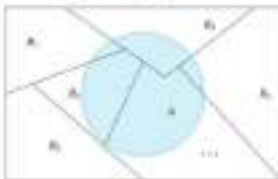


Figure 2.11: Diagram for Exercise 2.29

**The Law of Total Probability**

If the events  $A_1, A_2, \dots, A_k$  constitute a partition of the sample space  $\Omega$  such that  $P(A_i) > 0$  for  $i = 1, 2, \dots, k$ , then for any event  $B$  of  $\Omega$ ,

$$P(B) = \sum_{i=1}^k P(B|A_i)P(A_i) = \sum_{i=1}^k P(A_i)P(B|A_i).$$



**Example:** In a certain assembly plant, three machines,  $B_1$ ,  $B_2$ , and  $B_3$ , make 25%, 35%, and 40% respectively of the products. It is known from past experience that 2%, 3%, and 4% of the products made by each machine respectively are defective. Now suppose that a finished product is randomly selected. What is the probability that it is defective?

**Solution:** Consider the following events:

- $A$ : the product is defective,
- $B_1$ : the product is made by machine  $B_1$ ,
- $B_2$ : the product is made by machine  $B_2$ ,
- $B_3$ : the product is made by machine  $B_3$ .

Applying the rule of elimination, we can write

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3).$$

$$P(A|B_1)P(B_1) = (0.02)(0.25) = 0.005,$$

$$P(A|B_2)P(B_2) = (0.03)(0.35) = 0.0105,$$

$$P(A|B_3)P(B_3) = (0.04)(0.40) = 0.016.$$

and hence

$$P(A) = 0.005 + 0.0105 + 0.016 = 0.0315.$$

□

## Bayes' Rule

If  $A$  and  $B$  are two events, we have seen that in our cases  $P(A|B) \neq P(A|A)$ . Bayes' rule provides a formula that allows us to calculate one of the conditional probabilities if we know the other one. To be more specific, assume that we know  $P(A|B)$  and we wish to calculate  $P(B|A)$ . That is with the notion of conditional probability (Equation 2.14)

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Now use Equation (2.14) to substitute  $P(B) = P(A)P(B|A)$  for  $P(B)$ :

$$P(A|B) = \frac{P(A \cap B)P(A)}{P(A)} \quad (2.24)$$

Equation (2.24) is exactly Bayes' rule. When Bayes' rule is written, the expression  $P(A)$  in the denominator is usually replaced with the more complicated expression derived from the law of total probability. Specifically, since the events  $A$  and  $A^c$  are mutually exclusive and exhaustive, the law of total probability shows that

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) \quad (2.25)$$

**(Bayes' Rule)** If the events  $A_1, A_2, \dots, A_k$  constitute a partition of the sample space  $S$  such that  $P(A_i) \neq 0$  for  $i = 1, 2, \dots, k$ , then for any event  $B$  in  $S$  such that  $P(B) \neq 0$ ,

$$P(A_i|B) = \frac{P(B \cap A_i)}{\sum_{j=1}^k P(B \cap A_j)} = \frac{P(A_i|B)P(B)}{\sum_{j=1}^k P(A_j|B)P(B)} \quad \text{for } i = 1, 2, \dots, k.$$

### Revision Examples

**Ex. 1.** A batch of items contains items that contain 1 red ball, 4 white balls, and 5 blue balls. Assume the probability that a bag will attract a particular colour bag is equal to weight.

(a) **Method 1**

Let  $A$  be the event that a bag will attract a colour bag, and let  $B$  be the event that a bag will attract a red ball. Then

$$P(A) = \frac{\text{sum of weights of all bags}}{\text{sum of weights of all bags}} = 1 + 4 + 5 = 10, \quad P(B) = 1.$$

**Method 2**

Let sample space consist of 10 bags. Then if we assign equal probability  $1/10$  to each sample point, we see that  $P(B) = 1/10 = 1/10$ , since there are 1 sample point corresponding to 'red ball'.

(b)  $P(B) = 1/10 + 1/10 + 1/10 = 3/10$

(c)  $P(B) = 1/10 + 1/10 + 1/10 = 3/10$

(d)  $P(\text{red ball}) = P(B) = 1/10 + 1/10 + 1/10 = 3/10$  by part (c)

(e) **Method 1**

$$P(\text{red ball}) = P(B|A) = \frac{\text{sum of weights of red balls}}{\text{sum of weights of all bags}} = \frac{1}{10 + 4 + 5} = \frac{1}{10}.$$

We can also be checked using the results given in (a) to (c).

**Method 2**

$$P(B|A) = P(B) = P(B) = 1/10 + 1/10 + 1/10 = 3/10.$$

**Method 3**

Since events  $A$  and  $B$  are mutually exclusive, it follows from the part (c) that

$$P(B|A) = P(B) = P(B) = 1/10 + 1/10 + 1/10 = 3/10.$$

**Ex. 3.** Suppose it is found twice that the probability of getting a 0, 1, or 2 on the top of a 3-sided die is the same as the probability of getting a 3, 4, or 5 on the top of the same die.

Find the probability that the die is biased, and find the probability that the die is unbiased, given that it is found twice that the probability of getting a 0, 1, or 2 on the top of a 3-sided die is the same as the probability of getting a 3, 4, or 5 on the top of the same die.

**Solution:**

$$P(X_1 = 0, X_2 = 0) + P(X_1 = 0, X_2 = 1) + P(X_1 = 0, X_2 = 2) = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{9}$$

We have assumed the fairness of the die, so the result of the second throw is independent of the first so that  $P(X_1 = 0, X_2 = 0) = P(X_1 = 0)P(X_2 = 0)$ . Since we have found that  $P(X_1 = 0, X_2 = 0) = \frac{1}{9}$ , we can find the probability that the die is biased, and the probability that the die is unbiased, given that it is found twice that the probability of getting a 0, 1, or 2 on the top of a 3-sided die is the same as the probability of getting a 3, 4, or 5 on the top of the same die.

**Solution:**

Since we have found that the probability of getting a 0, 1, or 2 on the top of a 3-sided die is the same as the probability of getting a 3, 4, or 5 on the top of the same die, we can find the probability that the die is biased, and the probability that the die is unbiased, given that it is found twice that the probability of getting a 0, 1, or 2 on the top of a 3-sided die is the same as the probability of getting a 3, 4, or 5 on the top of the same die.

Since we have found that the probability of getting a 0, 1, or 2 on the top of a 3-sided die is the same as the probability of getting a 3, 4, or 5 on the top of the same die, we can find the probability that the die is biased, and the probability that the die is unbiased, given that it is found twice that the probability of getting a 0, 1, or 2 on the top of a 3-sided die is the same as the probability of getting a 3, 4, or 5 on the top of the same die.

$$P(X_1 = 0, X_2 = 0) = \frac{1}{9}$$

We have assumed the fairness of the die, so the result of the second throw is independent of the first so that  $P(X_1 = 0, X_2 = 0) = P(X_1 = 0)P(X_2 = 0)$ .

$$P(X_1 = 0, X_2 = 0) = \frac{1}{9} = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = P(X_1 = 0)P(X_2 = 0)$$

2.6. Find the probability of selecting a 7 or 11 taking into account events of a particular size.

The sample space for each roll of the dice is shown in Fig. 2.9. For example, (1, 2) means that 1 comes up on the first die and 2 on the second. Since the first and the second dice are 10 independent, we assume probability  $\frac{1}{10}$  for each.

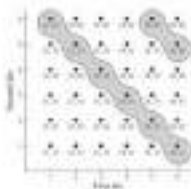


Fig. 2.9

It can be seen that event 7 or 11 (denoted by the two outcomes in Fig. 2.9) that is given as indicated, we have  $P(A) = 6/10 = 3/5$ . It follows that the probability of a 7 or 11 is given by

$$P(A) = 6/10 = 3/5 = \frac{1}{5} + \frac{1}{5}.$$

Using rules (1) and (2) to find the probability of the other two events the probability of a 7 or 11 is given by

$$P(A|B) = 1/5 + P(A|B) = \left[ \frac{1}{5} + \frac{1}{5} \right] = \frac{2}{5}.$$

Using the fact that the events are independent

1.11. These balls are then successively transferred to the box of Redden (7) until the probability that they are white is the same in all three, and then if each ball is (a) replaced in its own box, (b) interchanged.

Let  $B_1$  = event "ball in the draw"  $B_2$  = event "white on second draw"  $B_3$  = event "they are distributed" the region  $P(B_1 \cap B_2 \cap B_3)$ .

(a) If each ball is replaced then the events are independent and

$$\begin{aligned} P(B_1 \cap B_2 \cap B_3) &= P(B_1)P(B_2|B_1)P(B_3|B_1 \cap B_2) \\ &= P(B_1)P(B_2)P(B_3) \\ &= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{27} \end{aligned}$$

(b) If each ball is interchanged then the events are dependent and

$$\begin{aligned} P(B_1 \cap B_2 \cap B_3) &= P(B_1)P(B_2|B_1)P(B_3|B_1 \cap B_2) \\ &= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{27} \end{aligned}$$

1.12. Find the probability of not turning up a business transaction of a white die.

Let  $B_1$  = event "businessman" and  $B_2$  = event "it is a good year" then

$$B_1 \cap B_2 = \text{event "Was the man a businessman and it was a good year?"}$$

and the region  $P(B_1 \cap B_2)$ .

#### Method 1

Events  $B_1$  and  $B_2$  are not mutually exclusive; therefore are independent (from the table below)

$$\begin{aligned} P(B_1 \cap B_2) &= P(B_1) \times P(B_2) = P(B_1 \cap B_2) \\ &= P(B_1) \times P(B_2) = P(B_1)P(B_2) \\ &= \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \end{aligned}$$

#### Method 2

Then  $P(\text{businessman is a businessman}) = P(\text{it is a good year}) = 1$   
 $P(\text{businessman is a businessman}) = 1 = P(\text{it is a good year})$

$$\begin{aligned} &= 1 = P(\text{businessman is a businessman and it is a good year}) \\ &= 1 = P(B_1 \cap B_2) = 1 = P(B_1)P(B_2) \\ &= 1 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \end{aligned}$$

#### Method 3

From number of equally likely cases in which both events occur  $1/4 = 1/4 = 1/4$

also  
 Number of ways in which  $B_1$  occurs and  $B_2$  does not  
 Number of ways in which  $B_2$  occurs and  $B_1$  does not  
 Number of ways in which both  $B_1$  and  $B_2$  occur = 1

Then the number of ways in which both events of the events  $B_1$  and  $B_2$  occur =  $1/4 = 1/4 = 1/4$ . Therefore  $P(B_1 \cap B_2) = 1/4$ .

- 1.13. One bag contains 3 white balls and 1 black ball; another contains 1 white ball and 1 black ball. If one bag is chosen at random and a ball is drawn, find the probability that the ball is white, the ball is black, it is white and not in the 1st bag.

Let  $W_1$  = event that ball from the bag 1 is white;  $W_2$  = event that ball from bag 2 is white.

$$(a) P(W_1) = P(W_1 \cap W_1) + P(W_1 \cap W_2) = P(W_1 \cap W_1) = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \frac{3}{16}$$

$$(b) P(W_2) = P(W_2 \cap W_1) + P(W_2 \cap W_2) = P(W_2 \cap W_2) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16}$$

(c) The required probability is

$$1 - P(W_1) - P(W_2) = 1 - \frac{3}{16} - \frac{1}{16} = \frac{11}{16}$$

- 1.14. Box 1 contains 1 red and 2 blue marbles while Box 2 contains 1 red and 4 blue marbles. A ball is chosen at random. If the colour of the ball, a marble is chosen from Box 1. If a marble with 2 marbles is chosen from Box 2, find the probability that a red marble is chosen.

Let  $R$  denote the event 'a red marble is chosen' while  $B$  and  $B_2$  denote the event that Box 1 and Box 2 are chosen respectively. Then  $R$  and  $B$  are mutually exclusive events. Also  $B$  and  $B_2$  are not mutually exclusive events. So we can use the addition of probabilities with  $n = R$ ,  $A_1 = B$ ,  $A_2 = B_2$ . Therefore, the probability is choosing a red marble is

$$P(R) = P(R \cap B) + P(R \cap B_2) = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{5}\right) = \frac{8}{45}$$

- 1.15. At least how many ways can 7 different coloured balls be arranged in a row?

We can arrange the 7 marbles in 7! permutations. The first position can be occupied by any one of 7 marbles, i.e., there are 7 ways of filling the first position. When this has been done, there are 6 ways of filling the second position. Therefore, we have 7 ways of filling the first position, 6 ways of filling the second position, and finally with 5 ways of filling the last position. Therefore,

$$\text{Number of arrangements of 7 marbles in a row} = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7!$$

In general,

$$\text{Number of arrangements of } n \text{ different objects in a row} = n \times (n-1) \times (n-2) \times \dots \times 1 = n!$$

This is the called the number of a permutation of  $n$  different objects and is denoted by  $n!$ .

1.24. In how many ways can 10 people be seated in a circular table if only 6 seats are available?

The first person can choose any one of 10 seats, and then the last three must sit in 3 seats sitting to the immediate left of the first person, and 3 seats to the right of the first one. Therefore,

$$\text{Number of arrangements of 10 people when 6 seats are available} = 10 \times 3 \times 2 \times 1 = 120.$$

3-point.

Consider the arrangements of 6 different objects where one item =  $6! = 720$ ,  $5! = 120$ ,  $4! = 24$ ,  $3! = 6$ .

Which are called the number of permutations of 6 different objects taken 1, 2, 3, 4, 5, and 6 items by  ${}_6P_1$ . Then there are  $\dots + {}_6P_5 + {}_6P_6$  ways. Therefore 1.25

1.25. Find  ${}_6P_1 + {}_6P_2 + {}_6P_3 + {}_6P_4 + {}_6P_5 + {}_6P_6$ .

$$\text{Ans } {}_6P_1 = 6 \times 1 = 6 = 6! \quad {}_6P_2 = 6 \times 5 = 30 = 2 \times 3! \quad {}_6P_3 = 6 \times 5 \times 4 = 120 = 3! \times 4! \quad {}_6P_4 = 6 \times 5 \times 4 \times 3 = 360 = 4! \times 3! \quad {}_6P_5 = 6 \times 5 \times 4 \times 3 \times 2 = 720 = 5! \times 2! \quad {}_6P_6 = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 = 6!$$

1.26. If a computer can send 1 byte and 4 points that has no further benefit using the transmission, then how many different arrangements are possible?

The sequence is fixed by  $P_1$  byte and the order by  $P_2$  byte. Then arrangements of the sequence can be considered with each arrangement of the bytes. Hence

$$\text{Number of arrangements} = {}_4P_1 \times {}_4P_2 = (4 \times 3) \times (4 \times 3) = 288$$

1.27. How many 4-digit numbers can be formed with the 10 digits 1, 2, 3, ..., 9 if no repetition is allowed (the repetitions are not allowed for the first digit must be zero and repetitions are not allowed)?

(a) The first digit can be any one of 9 (since 0 is not allowed). The second, third and fourth digit can be any one of 10. Thus  $9 \times 10 \times 10 \times 10 = 9000$  numbers can be formed.

(b) The first digit can be any one of 9 (since zero is not)

The second digit can be any one of 9 (since for the first one the first digit)

The third digit can be any one of 8 (since the first one and the first digit)

The fourth digit can be any one of 7 (since the first one and the first digit)

$$\text{Thus } 9 \times 9 \times 8 \times 7 = 4536 \text{ numbers can be formed}$$

**Another method**

The first digit can be any one of 9, and the remaining three can be chosen in  ${}_3P_3$  ways. Thus  $9 \times 3! = 9 \times 6 = 54 = 4536$  numbers can be formed.

1.28. The first digit can be any one of 9, and the second is 0 or 1, and the third is 0 or 1. Thus  $9 \times 2 \times 2 = 36$  numbers can be formed.

**Another method**

The first digit can be chosen in 9 ways and the next two digits in  ${}_2P_2$  ways. Thus  $9 \times {}_2P_2 = 9 \times 2 \times 1 = 36$  numbers can be formed.





(ii) From (i) above  $\Rightarrow \sum_{i=1}^n \frac{1}{2^i} = \frac{1}{2}$  then

$$\text{the last 1 is added } \Rightarrow 1 = \frac{1}{2} + \frac{1}{2}$$

(iii) For the second column  $\Rightarrow \sum_{i=1}^n \frac{1}{2^i} \frac{dA_i}{dt} = \frac{1}{2}$

(iv) From above column, then, then  $\Rightarrow \frac{1}{2}$  of each value is added

$$\Rightarrow \sum_{i=1}^n \left( \frac{1}{2^i} \right) = \frac{1}{2} + \frac{1}{2} = 1$$

**Another method**

$$\begin{aligned} P(X_1 + X_2 \leq 1) &= P(X_1 \leq 1 - X_2) \\ &= \int_0^1 \left( \int_0^{1-x} \frac{1}{2} dx \right) \frac{1}{2} dx \\ &= \frac{1}{4} \end{aligned}$$

- Q12. A box contains 1 red and 9 white marbles. Two marbles are drawn successively from the box without replacement and it is noted that the second is a white. What is the probability that the first is also white?

**Solution 1**

If  $W_1$ ,  $W_2$  respectively "white on 1st draw" "white on 2nd draw" respectively, we are looking for  $P(W_1 | W_2)$ . This is given by

$$P(W_1 | W_2) = \frac{P(W_1 \cap W_2)}{P(W_2)} = \frac{1 \times 9 \times 8}{2 \times 9} = \frac{4}{5}$$

**Solution 2**

Since the second marble is white there are only 9 marbles left in the box. For the first marble the probability is  $\frac{1}{10}$ .

- Q13. The probability that a husband and wife will be alive 20 years from now are given by 0.8 and 0.6, respectively. Find the probability that in 20 years at least one of them (i) will have died (ii) will be alive.

Sol 1. If  $H$  and  $W$  denote that the husband and wife respectively, will survive in 20 years. Then  $P(H) = 0.8$ ,  $P(W) = 0.6$ . We suppose that  $H$  and  $W$  are independent events. Then we can find the probability

(i) that both will die  $\Rightarrow P(H^c \cap W^c) = P(H^c)P(W^c) = (1 - 0.8)(1 - 0.6)$

(ii) that both will be alive  $\Rightarrow P(H \cap W) = P(H)P(W) = (0.8)(0.6) = 0.48$

(iii) that both will die or both will be alive  $\Rightarrow 1 - P(H \cap W) = 1 - 0.48 = 0.52$

# Random Variables

## Concept of a Random Variable

It is often important to allocate a numerical description to the outcome. For example, the sample space giving a detailed description of each possible outcome where three electronic components are tested may be written  $S = \{BBB, BBB, BNB, BNB, NBB, NBN, BNB, BNB, NBN, NBN, NBN, NBN\}$  where  $B$  denotes non-defective and  $N$  denotes defective. One is naturally concerned with the number of defectives that occur. Thus each point in the sample space will be assigned a numerical value of 0, 1, 2, or 3. These numerical values of test results (random quantities) determined by the outcome of the experiment. Thus each possible value assumed by the random variable  $X$ , the number of defective items when three electronic components are tested. We shall use a capital letter  $X$  to denote a random variable and its corresponding small letter  $x$  for this space for assumed values.

A random variable assigns a numerical value to each outcome in a sample space.

**Example:** Three balls are drawn, by successive without replacement, from an urn containing 2 red balls and 3 black balls. The possible outcomes and the value  $x$  of the random variable  $X$ , where  $X$  is the number of red balls, are

Sample Space	$x$
RRB	2
RBR	2
RBB	1
RRB	1
RRB	0

## Discrete Random Variables

A random variable is **discrete** if it possible values from values are discrete. This does not mean the possible values are arranged in order, there is a gap between each value and the next one. The set of possible values may be infinite. For example, the set of all integers and the set of all positive integers are both discrete sets.

## Discrete Probability Distributions

A discrete random variable possesses each of its values with a certain probability. In deriving a probability mass function, the variable  $X$  representing the number of heads, assume the value 2 with probability 0.08, since 2 of the 8 equally likely sample points result in two heads and only 128.

$x$	0	1	2	3
$P(X)$	1/8	3/8	3/8	1/8

The **probability mass function** of a discrete random variable  $X$  is the function  $p(x) = P(X = x)$ . The probability mass function is sometimes called the **probability distribution**.

The set of ordered pairs  $\{(x, f(x))\}$  is a **probability function**, **probability mass function**, or **probability distribution** of the discrete random variable  $X$  if, for each possible outcome  $x$ ,

- i.  $f(x) \geq 0$ ,
- ii.  $\sum_x f(x) = 1$ ,
- iii.  $P(X = x) = f(x)$ .

### The Cumulative Distribution Function of a Discrete Random Variable

The probability mass function specifies the probability that a random variable is equal to a given value. A function called the **cumulative distribution function** specifies the probability that a random variable is less than or equal to a given value. The cumulative distribution function of the random variable  $X$  is the function  $F(x) = P(X \leq x)$ .

The **cumulative distribution function**  $F(x)$  of a discrete random variable  $X$  with probability distribution  $f(x)$  is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t) \quad \text{for } -\infty < x < \infty.$$

**Example:** Find the cumulative distribution function of the random variable  $X$  described in the following table. Verify that  $F(1) = \frac{1}{2}$ .

$x$	0	1	2	3	4	5
$f(x)$	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{10}$





Figure 15.1: Histogram of the number of successes in 10 trials



Figure 15.2: Continuous probability for 10 trials

### Computing Probabilities with the PDF

The function  $f(x)$  is a **probability density function** (pdf) for the continuous random variable  $X$ , defined over the set of real numbers,  $\mathbb{R}$ .

1.  $f(x) \geq 0$ , for all  $x \in \mathbb{R}$ .
2.  $\int_{-\infty}^{\infty} f(x) dx = 1$ .
3.  $P(a < X < b) = \int_a^b f(x) dx$ .



Let  $X$  be a continuous random variable with probability density function  $f(x)$ .  
 Let  $a$  and  $b$  be any two numbers, with  $a < b$ . Then

$$\Pr(a \leq X \leq b) = \Pr(a \leq X < b) = \Pr(a < X \leq b) = \Pr(a < X < b) = \int_a^b f(x) dx$$

In addition,

$$\Pr(X \leq a) = \Pr(X < a) = \int_{-\infty}^a f(x) dx \quad (12.12)$$

$$\Pr(X \geq a) = \Pr(X > a) = \int_a^{\infty} f(x) dx \quad (12.13)$$

**Example:** Suppose that the temperature in degrees Celsius,  $T$ , in a controlled laboratory experiment is a continuous random variable having the probability density function

$$f(t) = \begin{cases} \frac{1}{2}, & -2 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

- (a) Verify that  $f(t)$  is a density function.  
 (b) Find  $\Pr(-1 \leq T \leq 1)$ .

**Solution:** We use Definition 10.

- (a) Obviously,  $f(t) \geq 0$ . To verify condition 2 in Definition 10, we note

$$\int_{-\infty}^{\infty} f(t) dt = \int_{-2}^2 \frac{1}{2} dt = \frac{1}{2} \left[ t \right]_{-2}^2 = \frac{1}{2} (2 - (-2)) = 1$$

- (b) Using formula 2 in Definition 10, we obtain

$$\Pr(-1 \leq T \leq 1) = \int_{-1}^1 \frac{1}{2} dt = \frac{1}{2} \left[ t \right]_{-1}^1 = \frac{1}{2} (1 - (-1)) = 1$$

# **Example** **2.41**

A hole is drilled into the end of a component, as shown in Figure 2.41(a). The depth of the hole is equal to the difference between the radius of the hole and the radius of the shaft. An observation reveals it takes the following distribution. The probability density function of  $X$  is

$$f(x) = \begin{cases} 1.2(1-x)^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Calculate the probability that the hole is less than the original shaft diameter (i.e.  $x < 0$ ).

**Solution**

Figure 2.41(b) shows the probability density function of  $X$ . The function is only defined for  $0 \leq x \leq 1$  and for  $x < 0$  and for  $x > 1$ . We observe that the probability that the observation is greater than 1 unit. The probability that the observation is greater than 1 unit is  $P(X > 1)$ . The value of  $x$  is greater than 1 unit is the probability density function of the original hole.



**FIGURE 2.41** (a) Diagram of the probability density function of  $X$ , the observation of the hole diameter is equal to  $P(X < 0)$ .

This area is given by

$$\begin{aligned} P(X < 0) &= \int_{-\infty}^0 f(x) dx \\ &= \int_{-\infty}^0 1.2(1-x)^2 dx \\ &= 1.2 \left( -\frac{1}{3} \right) \Big|_{-\infty}^0 \end{aligned}$$



# The Cumulative Distribution Function of a Continuous Random Variable

The cumulative distribution function, denoted  $F(x)$  of a continuous random variable  $X$  with density function  $f(x)$  is

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt.$$

As an immediate consequence of Definition 3.1, one can write the two results

$$P(a < X < b) = F(b) - F(a) \text{ and } f(x) = \frac{dF(x)}{dx},$$

if the derivative exists.

**Example 1** Use the density function, find  $F(x)$ , and use it to compute  $P(0 < X \leq 1)$

$$f(x) = \begin{cases} \frac{1}{2}x, & -1 \leq x \leq 2, \\ 0, & \text{otherwise} \end{cases}$$

**Solution:** For  $-1 \leq x \leq 2$ ,

$$F(x) = \int_{-\infty}^x f(t) dt = \int_{-1}^x \frac{t}{2} dt = \left[ \frac{t^2}{4} \right]_{-1}^x = \frac{x^2 + 1}{4}.$$

Therefore,

$$F(x) = \begin{cases} 0, & x < -1, \\ \frac{x^2 + 1}{4}, & -1 \leq x \leq 2, \\ 1, & x > 2. \end{cases}$$

$$P(0 < X \leq 1) = F(1) - F(0) = \frac{1}{4} - \frac{1}{4} = \frac{1}{4}.$$

**Example:** The Operation 4 of Example 20.10 puts gas pistons into fuel and generally continues until it reaches the 100th piston. Call this number  $N$ . The RWE has determined that the density function of the winning piston is

$$f(x) = \begin{cases} \frac{1}{20} & 0 \leq x \leq 20 \\ \frac{3x-20}{400} & \text{otherwise} \end{cases}$$

Find  $P(N \leq 50)$  to determine the probability that the winning fuel piston that the RWE is performing exceeds 5.

**Solution:** For  $0 \leq x \leq 20$ ,

$$F(x) = \int_{-\infty}^x \frac{1}{20} dx = \frac{1}{20}x = \frac{50}{20} = \frac{5}{2}$$



Figure 20.6: Cumulative distribution function for Example 20.10.

Thus, 
$$F(x) = \begin{cases} \frac{1}{20}x & 0 \leq x \leq 20 \\ \frac{3x-20}{400} & 20 < x \leq 50 \\ 1 & x > 50 \end{cases}$$

To determine the probability that the winning fuel is less than the performance fuel estimate 5, we have

$$P(N \leq 50) = F(50) = \frac{3}{2} - \frac{1}{4} = \frac{5}{4} \quad \square$$

# Example 4.2.1

Verify in Example 4.1 that the cumulative distribution function is as given in (4.2.1).

**Solution**

The probability density function of  $X$  is given by  $f(x) = 6x(1-x)^5$ ,  $0 \leq x \leq 1$ ,  $f(x) = 0$  if  $x < 0$  or  $x > 1$ , and  $\int_{-\infty}^{\infty} f(x) dx = 1$ . The cumulative distribution function is given by  $F(x) = \int_{-\infty}^x f(t) dt$ . Since  $f(x)$  is defined separately on three intervals, we use the composition of the cumulative distribution function three separate times.

If  $x < 0$ ,

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(t) dt \\ &= \int_{-\infty}^x 0 dt \\ &= 0 \end{aligned}$$

If  $0 \leq x \leq 1$ ,

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(t) dt \\ &= \int_{-\infty}^0 f(t) dt + \int_0^x f(t) dt \\ &= \int_{-\infty}^0 0 dt + \int_0^x 6t(1-t)^5 dt \\ &= 0 + 12 \left( 1 - \frac{t^2}{2} \right) \Big|_0^x \\ &= 12 \left( 1 - \frac{x^2}{2} \right) \end{aligned}$$

If  $x > 1$ ,

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(t) dt \\ &= \int_{-\infty}^0 f(t) dt + \int_0^1 f(t) dt + \int_1^x f(t) dt \\ &= \int_{-\infty}^0 0 dt + \int_0^1 6t(1-t)^5 dt + \int_1^x 0 dt \\ &= 0 + 12 \left( 1 - \frac{t^2}{2} \right) \Big|_0^1 + 0 \\ &= 2 + 0 \\ &= 1 \end{aligned}$$

Example

$$f(x) = \begin{cases} 1.2\left(1 - \frac{x}{10}\right) & 0 \leq x \leq 10 \\ 0 & 10 < x \leq 15 \end{cases}$$

A graph of  $f(x)$  is presented here:



### Example

2.4.2

Refer to Example 2.4.1. Suppose we wish to determine the maximum value of the function  $f(x)$  on the interval  $[0, 15]$ .

**Solution.**

Let  $M$  denote the maximum value. We need to find  $f(x) = M$ . This requires a simple, if not trivial, argument. For  $x \in [0, 10]$ , the function is linear and increasing, so the maximum value is  $f(10) = 1.2$ . For  $x \in [10, 15]$ , the function is constant, so the maximum value is  $f(x) = 1.2$ .





## Probability and Statistics: Topics for EXAM 2B: Engineering

### 12. Randomness and Probability

**12.20** Measurements of a quantity sometimes are known within a tolerance, such as  $\pm 1\%$  tolerance. These are known as random or measurement errors. Just as one measurement is a single shot of one point like observation, suppose the measurement is just a shot of one point equally distributed by the known tolerance.

$$f(x) = \frac{1}{2\epsilon} (1 - \frac{|x|}{\epsilon}), \quad -\epsilon \leq x \leq \epsilon, \\ \text{elsewhere}$$

- What is the tolerance  $\epsilon$  for a random variable  $x$  that has  $f(x)$ ?
- What is the probability that a measurement is in some interval before time  $t$ ?
- For what particular measurements of  $x$  is the probability of the magnitude of the error just equal to half of  $\epsilon$ ? What is the probability that this occurs?

**12.21** Based on extensive testing, it is determined by the manufacturer of a lighting fixture that the time  $t$  for a single fixture of major brand to require service is given by the probability density function

$$f(t) = \frac{1}{2} e^{-t/2}, \quad t \geq 0, \\ \text{elsewhere}$$

- What is the average number of fixtures to be replaced a day and if it is desired to replace a major brand fixture for each hour, how many of the  $Q$  fixtures (assuming  $Q \gg 1$ )?
- What is the probability that a major brand fixture will be replaced after?

**12.22** The probability of the number  $N$  of occurrences of a particular company that is subjected to a communication interference within a certain time interval is given by the following discrete probability distribution. What is the probability of  $N$  occurrences is given by

$$f(N) = \frac{1}{2} (1 - \frac{N}{2}), \quad N \geq 0, \\ \text{elsewhere}$$

- Verify that this distribution is a valid discrete distribution.
- What is the probability that a company which is subject to communication interference will be subject to an interference on an interference on each particular company?
- What is the probability that a company which is subject to communication interference will be subject to an interference on an interference on each particular company?

## Engineering Probability and Statistics

### Random Variables

**12.23** Suppose a random variable  $x$  is given by the following probability density function. After obtaining a single value of  $x$ , find the probability that the value of  $x$  is greater than the value of  $x$  obtained from a single observation. What is the probability that the value of  $x$  is greater than the value of  $x$  obtained from a single observation?

$$f(x) = \frac{1}{2} (1 - \frac{|x|}{2}), \quad -2 \leq x \leq 2, \\ \text{elsewhere}$$

- What is the value of  $x$  that makes the value of  $x$  greater than the value of  $x$  obtained from a single observation?
- What is the probability that the value of  $x$  is greater than the value of  $x$  obtained from a single observation?
- What is the probability that the value of  $x$  is greater than the value of  $x$  obtained from a single observation?

**12.24** After obtaining a random variable  $x$  from a random variable  $x$  that is given by the probability density function

$$f(x) = \frac{1}{2} (1 - \frac{|x|}{2}), \quad -2 \leq x \leq 2, \\ \text{elsewhere}$$

- What is the probability that  $x$  is greater than the value of  $x$  obtained from a single observation?
- What is the probability that  $x$  is greater than the value of  $x$  obtained from a single observation?

## Joint Probability Distributions

When two or more random variables are associated with each item in a population, the random variables are said to be **jointly distributed**.

### Jointly Discrete Random Variables

If  $X$  and  $Y$  are two discrete random variables, the probability distribution for their simultaneous occurrence can be represented by a function with values  $f(x, y)$  for any pair of values  $(x, y)$  within the range of the random variables  $X$  and  $Y$ . It is customary to refer to this function as the **joint probability distribution** of  $X$  and  $Y$ .

Below is the discrete case:

$$f(x, y) = P(X = x, Y = y)$$

that is, the value  $f(x, y)$  gives the probability that outcomes  $x$  and  $y$  occur at the same time.

The function  $f(x, y)$  is a **joint probability distribution** or **probability mass function** of the discrete random variables  $X$  and  $Y$  if

1.  $f(x, y) \geq 0$  for all  $(x, y)$ .
2.  $\sum_x \sum_y f(x, y) = 1$ .
3.  $P(X = x, Y = y) = f(x, y)$ .

For any region  $R$  in the  $xy$ -plane,  $P\{(X, Y) \in R\} = \sum_x \sum_y f(x, y)$ .

### Example 12.51

Find the probability that  $X < Y$  and  $Y < X$ .

$x$	$y$	$f(x, y)$
1	1	0.1
1	2	0.2
1	3	0.3
2	1	0.2
2	2	0.3
2	3	0.2
3	1	0.1
3	2	0.2
3	3	0.3

**Solution:**

$$\begin{aligned}
 P(X < Y) &= P(1 < 2) + P(1 < 3) + P(2 < 3) = 0.2 + 0.3 + 0.2 \\
 &= 0.7 \\
 P(Y < X) &= P(2 < 1) + P(3 < 1) + P(3 < 2) = 0.2 + 0.1 + 0.2 \\
 &= 0.5
 \end{aligned}$$



# Example 2.33

Find the probabilities that  $X = 1$  or  $2$ .

*Solution*

$$\begin{aligned} P(X = 1) &= P(X = 1 | Y = 1)P(Y = 1) + P(X = 1 | Y = 2)P(Y = 2) \\ &= P(X = 1 | Y = 1) \times \frac{1}{2} + P(X = 1 | Y = 2) \times \frac{1}{2} \\ &= 0.05 + 0.15 = 0.20 \\ &= 0.20 \end{aligned}$$

It is possible to write the joint probability mass function  $P(x, y)$  as follows:

$x$	$y$	
	1	2
1	0.05	0.15
2	0.15	0.25
3	0.20	0.35

**Example:** A joint discrete probability function  $P(x, y)$  is expressed below, find  $P(1 + y \leq 3)$ .

$$P(x, y) = \begin{cases} \frac{1}{20} \binom{2}{x} \binom{2}{y} \left( \frac{1}{2} \right)^{x+y} & \text{for } x = 0, 1, 2; y = 0, 1, 2 \text{ and } 0 \leq x + y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

*Solution:* First we construct the table for the joint discrete probability distribution:

$P(x, y)$		$y$		
		0	1	2
$x$	0	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{1}{20}$
	1	$\frac{2}{20}$	$\frac{2}{20}$	0
	2	$\frac{1}{20}$	0	0

$$\begin{aligned} P(X + Y \leq 3) &= P(0, 0) + P(0, 1) + P(1, 0) \\ &= \frac{1}{20} + \frac{2}{20} + \frac{2}{20} = \frac{5}{10} \end{aligned}$$

# Jointly Continuous Random Variables

If  $X$  and  $Y$  are jointly continuous random variables, with joint probability density function  $f(x, y)$ , and  $a < u < v < b$ , then

$$P(a < X < b \text{ and } c < Y < d) = \int_a^b \int_c^d f(x, y) dy dx$$

The joint probability density function has the following properties:

$$f(x, y) \geq 0 \text{ for all } x \text{ and } y$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = 1$$

$$P(a < X < b \text{ and } c < Y < d) = \int_a^b \int_c^d f(x, y) dy dx$$





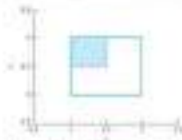
Assume that the joint probability density function of the distances from the centre of a city to the two nearest bus stops is given by

$$f(x, y) = \begin{cases} 24 - 4x - 4y & 0 \leq x \leq 3 \text{ and } 0 \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that a randomly chosen location has a distance to a bus stop of 1.5 km and is less than 2 km from the other bus stop.

#### Solution

We have defined  $P(X \leq 1.5 \text{ and } Y \leq 2) = P$ . The figure illustrates the region where the joint density is positive. The shaded rectangle illustrates the region where  $Y \leq 2$ ,  $X \leq 1.5$  and  $0 \leq x \leq 3$  and  $0 \leq y \leq 3$ , i.e. the probability that is required.



We express the joint probability density function with the indicated region

$$\begin{aligned} P(X \leq 1.5 \text{ and } Y \leq 2) &= \int_0^{1.5} \int_0^2 (24 - 4x - 4y) \, dy \, dx \\ &= \int_0^{1.5} \left[ 24y - 4xy - 2y^2 \right]_0^2 \, dx \\ &= \int_0^{1.5} \left( 48 - 8x - 8 \right) \, dx \\ &= \frac{1}{2} \end{aligned}$$

**Example:** A joint continuous probability function  $f(x, y)$  is expressed below. It results to be a valid probability distribution function for the given boundaries. Find  $P(X > Y > X)$ , where  $d = \{(x, y) | 0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\}$ .

$$f(x, y) = \begin{cases} \frac{1}{2}(2x + 3y) & \text{for } 0 < x < \frac{1}{2}, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

**Solution:** (a) The integration of  $f(x, y)$  over the whole region is

$$\begin{aligned} \int_0^1 \int_0^{\frac{1}{2}} f(x, y) dx dy &= \int_0^1 \int_0^{\frac{1}{2}} \frac{1}{2}(2x + 3y) dx dy \\ &= \int_0^1 \left( \frac{x^2}{2} + \frac{3xy}{2} \right) \Big|_0^{\frac{1}{2}} dy \\ &= \int_0^1 \left( \frac{1}{8} + \frac{3y}{4} \right) dy = \left( \frac{y}{8} + \frac{3y^2}{8} \right) \Big|_0^1 = \frac{1}{8} + \frac{3}{8} = 1 \end{aligned}$$

(b) To calculate the probability, we use

$$\begin{aligned} P(X > Y > X) &= \iint_d f(x, y) dx dy = \iint_d \frac{1}{2}(2x + 3y) dx dy \\ &= \int_{\frac{1}{4}}^{\frac{1}{2}} \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{2}(2x + 3y) dx dy \\ &= \int_{\frac{1}{4}}^{\frac{1}{2}} \left( \frac{x^2}{2} + \frac{3xy}{2} \right) \Big|_{\frac{1}{4}}^{\frac{1}{2}} dy = \int_{\frac{1}{4}}^{\frac{1}{2}} \left( \frac{1}{8} + \frac{3y}{4} \right) dy \\ &= \left( \frac{y}{8} + \frac{3y^2}{8} \right) \Big|_{\frac{1}{4}}^{\frac{1}{2}} \\ &= \frac{1}{8} \left[ \left( \frac{1}{2} + \frac{3}{4} \right) - \left( \frac{1}{4} + \frac{3}{16} \right) \right] = \frac{31}{160} \end{aligned}$$

# Marginal Distributions

The **marginal distributions** of  $X$  alone and of  $Y$  alone are

$$f_X(x) = p(x) = \sum_y p(x, y) \quad \text{and} \quad f_Y(y) = p(y) = \sum_x p(x, y)$$

for the discrete case, and

$$f_X(x) = p(x) = \int_{-\infty}^{\infty} p(x, y) dy \quad \text{and} \quad f_Y(y) = p(y) = \int_{-\infty}^{\infty} p(x, y) dx$$

for the continuous case.

The two marginal distributions are functions of the values of  $p(x, y)$  and  $p(x, y)$  are just the marginal cases of the respective systems and hence when the values of  $(x, y)$  are displayed in a rectangular table.

**Example:** compute the marginal distributions for the following distribution table.

		Y			
		-2	-1	0	
X	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	
	1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	
	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	
Marginal Totals		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1

**Solution:** For the random variable  $X$ , we see that

$$p(0) = f_X(0) = P(X=0) = P(0, -2) + P(0, -1) + P(0, 0) = \frac{1}{12} + \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$$

$$p(1) = f_X(1) = P(X=1) = P(1, -2) + P(1, -1) + P(1, 0) = \frac{1}{6} + \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

and

$$p(2) = f_X(2) = P(X=2) = P(2, -2) + P(2, -1) + P(2, 0) = \frac{1}{12} + \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$$

which are just the values shown in Table. In a similar manner we could show that the values of  $f_Y(y)$  are given by the row totals. In addition, these marginal distributions may be written as follows:

X	0	1	2
$f_X(x)$	$\frac{5}{12}$	$\frac{5}{6}$	$\frac{5}{12}$

Y	-2	-1	0
$f_Y(y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

**Example:** Find  $g(x)$  and  $g(y)$  for the joint density function

$$f(x, y) = \begin{cases} \frac{2}{9}(2x + 3y), & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

**Solution:** By definition,

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 \frac{2}{9}(2x + 3y) dy = \left( \frac{2xy}{1} + \frac{3y^2}{6} \right) \bigg|_{y=0}^{y=1} = \frac{4x+3}{3}$$

for  $0 \leq x \leq 1$ , and  $g(x) = 0$  otherwise. Similarly,

$$g(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^1 \frac{2}{9}(2x + 3y) dx = \frac{2x^2 + 3xy}{2} \bigg|_{x=0}^{x=1} = \frac{2y+3}{2}$$

for  $0 \leq y \leq 1$ , and  $g(y) = 0$  otherwise.

## Example 2.30

**Let  $X$  and  $Y$  have the bivariate probability density function of the bivariate  $T$  of Example 2.29. Find the marginal probability density function of the bivariate  $T$  of the bivariate  $T$ .**

**Solution:**

Find the marginal probability density function of  $X$  by  $g(x)$  and the marginal probability density function of  $Y$  by  $h(y)$ . Then

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} f(x, y) dy \\ &= \int_0^1 \frac{2}{9}(2x + 3y) dy \\ &= \frac{2}{9} \left( 2x + \frac{3}{2} \right) \quad \text{for } 0 \leq x \leq 1 \end{aligned}$$

and

$$\begin{aligned} h(y) &= \int_{-\infty}^{\infty} f(x, y) dx \\ &= \int_0^1 \frac{2}{9}(2x + 3y) dx \\ &= \frac{2}{9} \left( 2 + \frac{3}{2} \right) \quad \text{for } 0 \leq y \leq 1 \end{aligned}$$

# Example 1.10

Let  $f(x, y) = x^2 + y^2$  be a function of two variables. Suppose that  $f$  is defined on the region  $R$  in the  $xy$ -plane, where  $R$  is the region bounded by the circle  $x^2 + y^2 = 1$  and the line  $y = x$ . The line  $y = x$  and the circle  $x^2 + y^2 = 1$  intersect at the points  $(1, 1)$  and  $(-1, -1)$ . The region  $R$  is shown in the figure below. The points  $(1, 1)$  and  $(-1, -1)$  are the points where the circle and the line intersect.

$$f(x, y) = \begin{cases} x^2 + y^2 & \text{if } (x, y) \in R \\ 0 & \text{if } (x, y) \notin R \end{cases}$$

Find  $\iint_R f(x, y) \, dA$  and  $\iint_R f(x, y) \, dV$ .

## Solution

The region  $R$  is the region shown in Figure 1.10. The region  $R$  is the region in the  $xy$ -plane bounded by the circle  $x^2 + y^2 = 1$  and the line  $y = x$ . The region  $R$  is the region in the  $xy$ -plane bounded by the circle  $x^2 + y^2 = 1$  and the line  $y = x$ .

$$\begin{aligned} \iint_R f(x, y) \, dA &= \iint_R (x^2 + y^2) \, dA \\ &= \iint_R (x^2 + y^2) \, dA \\ &= \iint_R (x^2 + y^2) \, dA \\ &= \iint_R (x^2 + y^2) \, dA \\ &= \iint_R (x^2 + y^2) \, dA \end{aligned}$$



FIGURE 1.10 The region  $R$  is the region in the  $xy$ -plane bounded by the circle  $x^2 + y^2 = 1$  and the line  $y = x$ . The region  $R$  is the region in the  $xy$ -plane bounded by the circle  $x^2 + y^2 = 1$  and the line  $y = x$ .



Work on Example 1.56. Find the integral density of  $f$  and of  $F$ .

**Solution**

To compute (1) is the integral density of  $f$ , we fix  $x$  and integrate the function  $f$  along the vertical line through  $x$ , as shown in Figure 1.57. The integrand is the weight  $w(y)$  and the limits of integration are  $y = 0$  to  $y = x$  if  $x \leq 1$ .

$$\begin{aligned} f(x) &= \int_0^x w(y) dy \\ &= 4xy^2 \bigg|_0^x \\ &= 4x^3 \quad \text{if } 0 \leq x \leq 1 \end{aligned}$$



**Figure 1.57** The integrand  $w(y)$  is constantly increasing in  $y$  along the vertical line through  $x$ .

To compute (2) is the integral density of  $F$ , we fix  $x$  and integrate the function  $F$  along the horizontal line through  $x$ , as shown in Figure 1.58 (page 158). The integrand is the weight  $w(y)$  and the limits of integration are  $y = 0$  to  $y = x$  if  $x \leq 1$ .

$$\begin{aligned} F(x) &= \int_0^x w(y) dy \\ &= 4xy^2 \bigg|_0^x \\ &= 4x^3 \quad \text{if } 0 \leq x \leq 1 \end{aligned}$$





FIGURE 1.7 The area of Region 1 is equal to the area of Region 2, and so the total area is 1.

Now that 
$$\int_{-\infty}^{\infty} g(x) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = 1,$$

and

$$\begin{aligned} P(a < X < b) &= P(a < X < b, -\infty < Y < \infty) \\ &= \int_a^b \int_{-\infty}^{\infty} f(x, y) dy dx = \int_a^b g(x) dx. \end{aligned}$$

### Conditional Probability Distribution

Let  $X$  and  $Y$  be two random variables, discrete or continuous. The **conditional distribution** of the random variable  $Y$  given that  $X = x$  is

$$f_Y(y) = \frac{f(x, y)}{g(x)}, \text{ provided } g(x) > 0.$$

Similarly, the conditional distribution of  $X$  given that  $Y = y$  is

$$f_X(x) = \frac{f(x, y)}{h(y)}, \text{ provided } h(y) > 0.$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) > 0,$$

where  $A$  and  $B$  are now the events defined by  $X = x$  and  $Y = y$ , respectively, that

$$P(Y = y | X = x) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{f(x, y)}{g(x)}, \text{ provided } g(x) > 0,$$

where  $X$  and  $Y$  are discrete random variables.

If we wish to find the probability that the discrete random variable  $X$  falls between  $a$  and  $b$  when it is known that the discrete variable  $Y = y$ , we evaluate

$$P(a < X < b | Y = y) = \sum_{x=a+1}^b J(x, y)$$

where the summation extends over all values of  $X$  between  $a$  and  $b$ . When  $X$  and  $Y$  are continuous, we evaluate

$$P(a < X < b | Y = y) = \int_a^b f(x|y) dx.$$

**Example:** Find the conditional distribution of  $X$ , given that  $Y = 1$ . And use it to determine  $P(X < 0 | Y = 1)$ .

$P(x, y)$		$y$		
		0	1	2
$x$	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
	1	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
	2	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
	3	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

**Solution:** We need to find  $J(x, y)$  when  $y = 1$ . From the table, we find that

$$H(1) = \sum_x J(x, 1) = \frac{1}{4} = \frac{2}{4} = 1 - \frac{2}{4}$$

So,

$$J(x, 1) = \frac{J(x, 1)}{H(1)} = \left(\frac{1}{2}\right) J(x, 1), \quad x = 0, 1, 2, 3$$

Therefore,

$$J(0, 1) = \left(\frac{1}{2}\right) J(0, 1) = \left(\frac{1}{2}\right) \left(\frac{1}{8}\right) = \frac{1}{16}, \quad J(1, 1) = \left(\frac{1}{2}\right) J(1, 1) = \left(\frac{1}{2}\right) \left(\frac{1}{8}\right) = \frac{1}{16}$$

$$J(2, 1) = \left(\frac{1}{2}\right) J(2, 1) = \left(\frac{1}{2}\right) \left(\frac{1}{8}\right) = 0$$

and the conditional distribution of  $X$ , given that  $Y = 1$ , is

$$J(x|1) = \left\{ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right\}$$

Finally,

$$P(X < 0 | Y = 1) = J(0, 1) = \frac{1}{16}$$

**Example 12.1** The joint density for the random variables  $(X, Y)$ , where  $X$  is the total temperature change and  $Y$  is the proportion of specimens with that temperature change, is given by

$$f(x, y) = \begin{cases} 4xy^2, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

- (a) Find the marginal densities  $g(x)$ ,  $h(y)$ , and the conditional density  $f(y|x)$ .  
 (b) Find the probability that the specimens differ more than half of the total observations, given that the temperature is measured by 0.25 units.

**Solution:** (a) By definition,

$$\begin{aligned} g(x) &= \int_0^1 f(x, y) dy = \int_0^1 4xy^2 dy \\ &= \frac{4}{3}x \left[ y^3 \right]_0^1 = \frac{4}{3}x, \quad 0 < x < 1 \\ h(y) &= \int_0^1 f(x, y) dx = \int_0^1 4xy^2 dx = 4y^2 \left[ \frac{x^2}{2} \right]_0^1 = 2y^2, \quad 0 < y < 1 \end{aligned}$$

So

$$f(y|x) = \frac{f(x, y)}{g(x)} = \frac{4xy^2}{\frac{4}{3}x(1-x)} = \frac{3y^2}{1-x}, \quad 0 < y < 1, 0 < x < 1$$

(b) Therefore

$$P\left(Y > \frac{1}{2} \mid X = 0.25\right) = \int_{1/2}^1 f(y|x=0.25) dy = \int_{1/2}^1 \frac{3y^2}{1-0.25} dy = \frac{6}{5}$$

**Exercise 1** Given the joint density function

$$f(x, y) = \begin{cases} \frac{1}{8}(1+2xy), & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise,} \end{cases}$$

find  $\mu_X$ ,  $\mu_Y$ ,  $\sigma_X^2$ , and evaluate  $P\left\{\frac{1}{2} \leq X \leq \frac{3}{4}, \frac{1}{2} \leq Y \leq \frac{3}{4}\right\}$ .

**Solution:** By definition of the marginal density, for  $0 \leq x \leq 1$ ,

$$\begin{aligned} g(x) &= \int_0^1 f(x, y) dy = \int_0^1 \frac{1}{8}(1+2xy) dy \\ &= \left(\frac{y}{8} + \frac{xy^2}{4}\right) \Big|_0^1 = \frac{1}{8} \end{aligned}$$

and for  $0 \leq y \leq 1$ ,

$$\begin{aligned} h(y) &= \int_0^1 f(x, y) dx = \int_0^1 \frac{1}{8}(1+2xy) dx \\ &= \left(\frac{x^2}{8} + \frac{2x^2y}{8}\right) \Big|_0^1 = \frac{1+2y^2}{8} \end{aligned}$$

Therefore, using the conditional density definition, for  $0 \leq x \leq 1$ ,

$$f(x|y) = \frac{f(x, y)}{h(y)} = \frac{\frac{1}{8}(1+2xy)}{\frac{1+2y^2}{8}} = \frac{x}{1+2y^2}$$

and

$$P\left\{\frac{1}{2} \leq Y \leq \frac{3}{4}, \frac{1}{2} \leq X \leq \frac{3}{4}\right\} = \int_{1/2}^{3/4} \int_{1/2}^{3/4} \frac{x}{1+2y^2} dx dy = \frac{7}{48}$$

$P(Y, X|y)$  is very close to that of  $y$

# Example 3.38

Table 3.1 shows the joint probability mass function for length  $X$  and width  $Y$  of a fish. Compute the conditional probability mass function  $p_{Y|X}(y|1)$ .

**Solution**

The possible values for  $Y$  are 10 and 15 cm. From Table 3.1,  $P(Y = 10 \text{ and } X = 10) = 0.10$ , and  $P(X = 10) = 0.30$ . Therefore,

$$\begin{aligned} p_{Y|X}(y|1) &= P(Y = y | X = 10) \\ &= \frac{P(Y = y \text{ and } X = 10)}{P(X = 10)} \\ &= \frac{0.10}{0.30} \\ &= 0.33 \end{aligned}$$

We can also find  $P(Y = 15 | X = 10)$  using the complement rule. Since we know that  $p_{Y|X}(y|1) + 1 - p_{Y|X}(y|1) = 1$ , since  $y = 10$  and  $y = 15$  are the only two possible values for  $Y$ . Therefore,  $p_{Y|X}(15|10) = 1 - 0.33$ . The conditional probability mass function of  $Y$  given  $X = 10$  is therefore  $p_{Y|X}(10|10) = 0.33$ ,  $p_{Y|X}(15|10) = 0.67$  and  $p_{Y|X}(y|10) = 0$  for any value of  $y$  other than 10 or 15.

**TABLE 3.1** Joint and marginal masses  
 of the length  $X$  and width  $Y$  of a fish

$x$	$y$		Sum
	10	15	
10	0.10	0.20	0.30
15	0.20	0.20	0.40
Sum	0.30	0.40	



**Working Example 10.1** The joint probability density function for the known and unknown variables  $X$  and  $Y$  is given by  $f(x, y) = \frac{1}{4}(x + y)$  for  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ . Find the conditional probability density function of  $Y$  given  $X = 1/2$ . Find the probability that the total distance is less than 1 km and verify that the distance is 1 km.

**Solution**

For example 10.1 we are given the joint probability density function

$$f(x, y) = \frac{1}{4}(x + y) \quad \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1$$

The conditional probability density function of  $Y$  given  $X = x/2$  is

$$f(y|x=1/2) = \frac{f(1/2, y)}{f_X(1/2)}$$

$$= \begin{cases} \frac{\frac{1}{4}(\frac{1}{2} + y)}{\frac{1}{4}(\frac{1}{2} + \frac{1}{2})} & \text{for } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1/2 + y}{1} & \text{for } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The probability that the total distance is less than 1 km and is less than 1 km is given by the integral  $\int_0^1 \int_0^{1-x} f(x, y) dx dy$ . This is because the region  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$  is the region  $x + y \leq 1$ .

$$P(X + Y < 1) = \int_0^1 \int_0^{1-x} f(x, y) dx dy$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{4}(x + y) dx dy$$

$$= \frac{1}{8}$$

## Independent Random Variables

If  $f(x, y)$  does not depend on  $y$ , then  $f(x, y) = g(x)$  and  $f(x, y) = g(x)h(y)$ .

Let  $X$  and  $Y$  be two random variables, discrete or continuous, with joint probability distribution  $f(x, y)$  and marginal distributions  $g(x)$  and  $h(y)$ , respectively. The random variables  $X$  and  $Y$  are said to be **statistically independent** if and only if

$$f(x, y) = g(x)h(y)$$

for all  $(x, y)$  within their range.

**Example:** Show that two random variables  $X$  and  $Y$  are not necessarily independent.

	$Y(x, y)$	$x$		
		0	1	2
$x$	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
	1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
	2	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

**Proof:** Let us consider the joint pmf  $J(x, y)$ . From Table we find the three probabilities  $J(0, 1)$ ,  $J(0, 2)$  and  $J(1, 2)$  as:

$$\begin{aligned} J(0, 1) &= \frac{1}{12}, \\ J(0, 2) &= \sum_{x=0}^2 J(x, 2) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{12}, \\ J(1, 2) &= \sum_{x=1}^2 J(x, 2) = \frac{1}{12} + \frac{1}{12} + 0 = \frac{2}{12}. \end{aligned}$$

Clearly,

$$J(0, 1) \neq J(0)J(1),$$

and therefore  $X$  and  $Y$  are not statistically independent. □

### Example 2.4.3

The joint probability mass function of the length  $X$  and thickness  $Y$  (in cm) of a piece of glass is given in Table 2.4.3. Are  $X$  and  $Y$  independent?

**Solution:**

We now check to see if  $P(X = 1 \text{ and } Y = 1) = P(X = 1)P(Y = 1)$  or, equivalently, if  $0.1 = 0.2 \times 0.5$ . We find by adding  $z = (2, 1) = (2)$

$P(X = 1) = 0.2$  and  $P(Y = 1) = 0.5$  so  $P(X = 1 \text{ and } Y = 1) = 0.2 \neq 0.1 = P(X = 1)P(Y = 1)$ .  
 Concluding that the variables are not independent. We  $P(X = 1) = 0.2$  and  $P(Y = 1) = 0.5$  so  $P(X = 1 \text{ and } Y = 1) = 0.1$   
 is not equal to  $0.2 \times 0.5$ . Therefore,  $X$  and  $Y$  are not independent.

### Example 2.4.4

**Example 2.4.4:** The joint probability density function of the thickness  $X$  and hole diameter  $Y$  of a randomly chosen washer is given in Table 2.4.4. Are  $X$  and  $Y$  independent?

**Solution:**

**Check:** The sum of  $f(x, y)$  is 1. Therefore,  $X$  and  $Y$  are independent.

# EXERCISES 10

**1.1** Let  $X$  and  $Y$  be independent random variables with the following probability distributions:

- (i)  $P(X = 1) = 1/2, P(X = 2) = 1/2$
- (ii)  $P(Y = 1) = 1/3, P(Y = 2) = 2/3$

**1.2** Let  $X$  and  $Y$  be independent random variables with the following probability distributions:

- (i)  $P(X = 1) = 1/2, P(X = 2) = 1/2$
- (ii)  $P(Y = 1) = 1/3, P(Y = 2) = 2/3$

Find

- (i)  $P(X + Y = 3)$
- (ii)  $P(X + Y = 4)$
- (iii)  $P(X + Y = 5)$
- (iv)  $P(X + Y = 6)$

**1.3** Let  $X$  and  $Y$  be independent random variables with the following probability distributions:

- (i)  $P(X = 1) = 1/2, P(X = 2) = 1/2$
- (ii)  $P(Y = 1) = 1/3, P(Y = 2) = 2/3$

- (i) Find the probability distribution of  $X + Y$ .
- (ii) Find the probability distribution of  $X - Y$ .
- (iii) Find the probability distribution of  $X/Y$ .

**1.4** Let  $X$  and  $Y$  be independent random variables with the following probability distributions:

- (i)  $P(X = 1) = 1/2, P(X = 2) = 1/2$
- (ii)  $P(Y = 1) = 1/3, P(Y = 2) = 2/3$

- (i) Find  $P(X + Y = 3)$

**1.5** Let  $X$  and  $Y$  be independent random variables with the following probability distributions:

- (i)  $P(X = 1) = 1/2, P(X = 2) = 1/2$

**1.6** Let  $X$  and  $Y$  be independent random variables with the following probability distributions:

- (i)  $P(X = 1) = 1/2, P(X = 2) = 1/2$
- (ii)  $P(Y = 1) = 1/3, P(Y = 2) = 2/3$

Find  $P(X + Y = 3)$

**1.7** Let  $X$  and  $Y$  be independent random variables with the following probability distributions:

- (i)  $P(X = 1) = 1/2, P(X = 2) = 1/2$
- (ii)  $P(Y = 1) = 1/3, P(Y = 2) = 2/3$

Find

- (i)  $P(X + Y = 3)$
- (ii)  $P(X + Y = 4)$

**1.8** Let  $X$  and  $Y$  be independent random variables with the following probability distributions:

- (i)  $P(X = 1) = 1/2, P(X = 2) = 1/2$
- (ii)  $P(Y = 1) = 1/3, P(Y = 2) = 2/3$

- (i) Find  $P(X + Y = 3)$
- (ii) Find  $P(X + Y = 4)$
- (iii) Find  $P(X + Y = 5)$

**1.9** Let  $X$  and  $Y$  be independent random variables with the following probability distributions:

- (i)  $P(X = 1) = 1/2, P(X = 2) = 1/2$
- (ii)  $P(Y = 1) = 1/3, P(Y = 2) = 2/3$

Find  $P(X + Y = 3)$

- (i) Find the probability distribution of  $X + Y$ .
- (ii) Find the probability distribution of  $X - Y$ .



# 10.1. Independence

**10.1** The events  $A$  and  $B$  are independent if and only if  $P(A \cap B) = P(A)P(B)$ . If  $A$  and  $B$  are independent, then  $P(A|B) = P(A)$  and  $P(B|A) = P(B)$ . If  $A$  and  $B$  are not independent, then  $P(A|B) \neq P(A)$  and  $P(B|A) \neq P(B)$ .

$$P(A \cap B) = \begin{cases} P(A)P(B) & \text{if } A \text{ and } B \text{ are independent} \\ P(A)P(B) & \text{otherwise} \end{cases}$$

(a) Determine if  $A$  and  $B$  are independent.

(b) Find  $P(A|B)$  and  $P(B|A)$ .

**10.2** Let  $A$  denote the event that a person is a woman or a certain number of years old, and let  $B$  denote the event that a person is a certain number of years old. Let  $C$  denote the event that a person is a certain number of years old and a certain number of years old. Find the probability that  $A$  and  $B$  are independent.

	1	2	3	4
1	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.00

(a) Determine the marginal distribution of  $A$ .

(b) Determine the marginal distribution of  $B$ .

(c) Find  $P(A \cap B) = P(A)P(B)$ .

**10.3** Suppose that  $A$  and  $B$  have the following joint probability distribution:

	1	2	3
1	0.00	0.00	0.00
2	0.00	0.00	0.00
3	0.00	0.00	0.00

(a) Find the marginal distribution of  $A$ .

(b) Find the marginal distribution of  $B$ .

**10.4** Given the joint density function

$$f(x, y) = \begin{cases} 2xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

find  $P(X < 1/2 | Y < 1/2)$ .

**10.5** Suppose that  $A$  and  $B$  are independent events. Find  $P(A \cap B)$  and  $P(A|B)$ .

**10.6** Suppose that  $A$  and  $B$  are independent events. Find  $P(A \cap B)$  and  $P(A|B)$ .

**10.7** The joint density function of the random variables  $X$  and  $Y$  is

$$f(x, y) = \begin{cases} 2xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that  $A$  and  $B$  are independent.

(b) Find  $P(A \cap B) = P(A)P(B)$ .

# 10.2. Conditional Probability

**10.1** Let  $A$ ,  $B$ ,  $C$ , and  $D$  have the joint probability density function

$$f(x, y, z) = \begin{cases} 2xyz & 0 < x < 1, 0 < y < 1, 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find  $P(A)$ .

(b) Find  $P(B|A) = \frac{P(A \cap B)}{P(A)}$ .

**10.2** Suppose that the two random variables  $X$  and  $Y$  have the joint probability density function

**10.3** Suppose that the two random variables  $X$  and  $Y$  have the joint probability density function

**10.4** Suppose that the two random variables  $X$  and  $Y$  have the joint probability density function

$$f(x, y) = \begin{cases} 2xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal density function of  $X$ .

(b) Find  $P(X < 1/2 | Y < 1/2)$ .

**10.5** Suppose that the two random variables  $X$  and  $Y$  have the joint probability density function

$$f(x, y) = \begin{cases} 2xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal density function of  $X$ .

(b) Find  $P(X < 1/2 | Y < 1/2)$ .

**10.6** Suppose that the two random variables  $X$  and  $Y$  have the joint probability density function

$$f(x, y) = \begin{cases} 2xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find  $P(X < 1/2 | Y < 1/2)$ .

(b) Find  $P(Y < 1/2 | X < 1/2)$ .

11.27 Let the number of phone calls received in a neighborhood during a 15 minute interval be a discrete random variable  $X$  with probability function

$$p(x) = \frac{1}{2} \left( \frac{1}{2} \right)^x, \quad x = 0, 1, 2, \dots$$

- Calculate the probability that  $X$  equals 0, 1, 2, 3, 4, 5, and 6.
- Graph the probability mass function for  $X$  and label it.
- Calculate the cumulative distribution function for some values of  $X$ .

11.28 Consider the random variables  $X$  and  $Y$  with joint density function

$$f(x, y) = \begin{cases} 2x + y & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Find the marginal distributions of  $X$  and  $Y$ .
- Find  $P(X > 0.5, Y > 0.5)$ .

11.29 Consider the following joint probability density function of the random variables  $X$  and  $Y$ :

$$f(x, y) = \begin{cases} \frac{1}{2}xy & 1 \leq x \leq 2, 1 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

- Find  $f_X$ , the marginal density function of  $X$ , and  $f_Y$ .
- Are  $X$  and  $Y$  independent?
- Find  $P_X(Y > 1)$ .

11.30 The life span in hours of an electrical component is a random variable with probability density function

$$f(t) = \frac{1}{100} e^{-t/100}, \quad t \geq 0$$

- Calculate the probability density function.
- Calculate the probability that the life span exceeds 100 hours and label this value.

# Review Examples

## Discrete random variables and probability distributions

- 1.1. Suppose that a pair of fair dice are to be tossed and let the random variable  $X$  denote the sum of the points obtained. Find the probability distribution of  $X$ .

Answer

$x$	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36

- 1.2. (a) Find the probability distribution for the discrete random variable  $X$  defined by (1), and (b) graph the distribution.

(1) We have  $P(X = k) = P(X \leq k) - P(X \leq k-1)$  for  $k=1, 2, \dots, 10$ . From the results of Example 1.1, we find

$$P(X \leq k) = \begin{cases} 0 & \text{for } k \leq 0 \\ 1/36 & \text{for } k=1 \\ 2/36 & \text{for } k=2 \\ 3/36 & \text{for } k=3 \\ 4/36 & \text{for } k=4 \\ 5/36 & \text{for } k=5 \\ 6/36 & \text{for } k=6 \\ 7/36 & \text{for } k=7 \\ 8/36 & \text{for } k=8 \\ 9/36 & \text{for } k=9 \\ 10/36 & \text{for } k=10 \end{cases}$$

and, for Fig. 2(a),



Fig. 2(a)

## Continuous random variables and probability distributions

- 1.3. A random variable  $X$  has a density function  $f(x) = c \exp(-x^2)$ , where  $c$  is a constant. (a) Find the probability that  $X$  lies between 1 and 2.

(b) Show that  $\int_{-\infty}^{\infty} f(x) dx = 1$ , i.e.

$$\int_{-\infty}^{\infty} \exp(-x^2) dx = \sqrt{\pi}$$

where  $\pi = 3.14$

Let  $f$  be the probability density function corresponding to the density function of Problem 1(c).

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(u) du = \frac{1}{2} \int_{-\infty}^x e^{\frac{u}{2}} du = \frac{1}{2} \left[ 2e^{\frac{u}{2}} \right]_{-\infty}^x \\ &= \frac{1}{2} (2e^{\frac{x}{2}} - 2e^{-\frac{x}{2}} - 0) = \frac{1}{2} (e^{\frac{x}{2}} - e^{-\frac{x}{2}}) \\ &= \frac{1}{2} (e^{\frac{x}{2}} - e^{-\frac{x}{2}}) \end{aligned}$$

4.8. The characteristic function for a random variable  $X$  is

$$\phi(t) = \begin{cases} 1 - t^2 & -1 \leq t \leq 1 \\ 0 & t < -1 \text{ or } t > 1 \end{cases}$$

Find the density function, the probability that  $X > 1$ , and the probability that  $-1 < X < 1$ .

$$(a) \quad f(x) = \frac{d}{dx} \phi(x) = \begin{cases} -2x & -1 \leq x \leq 1 \\ 0 & x < -1 \text{ or } x > 1 \end{cases}$$

$$(b) \quad P(X > 1) = \int_1^{\infty} f(x) dx = \int_1^{\infty} 0 dx = 0$$

another method

the definition,  $P(X > 1) = P(X > 1 + \epsilon)$  for any  $\epsilon > 0$ .

$$P(X > 1 + \epsilon) = P(X > 1 + \epsilon + \epsilon) = 0$$

$$(c) \quad P(-1 < X < 1) = \int_{-1}^1 f(x) dx = \int_{-1}^1 -2x dx = \left[ -x^2 \right]_{-1}^1 = -1 - (-1) = 0$$

another method

$$\begin{aligned} P(-1 < X < 1) &= P(-1 < X < 1 + \epsilon) - P(1 < X < 1 + \epsilon) \\ &= P(X < 1 + \epsilon) - P(X < 1) \\ &= 0 - 0 = 0 \end{aligned}$$

#### Joint distributions and independent variables

4.9. The joint probability function of two discrete random variables  $X$  and  $Y$  is given by,  $1 \leq x \leq 2$ ,  $x \leq y \leq 2x$ , and  $x$  and  $y$  are non-negative integers such that  $P(X = x, Y = y) = \frac{1}{2^x}$  and  $P(X = 1) = 0$  otherwise.

(a) Find the value of the constant. (b) Find  $P(X = 1, Y = 1)$ .

(c) Find  $P(X = 1, Y = 1)$ .

4.10. The joint probability function of two discrete random variables  $X$  and  $Y$  is given by,  $1 \leq x \leq 2$ ,  $x \leq y \leq 2x$ , and  $x$  and  $y$  are non-negative integers such that  $P(X = x, Y = y) = \frac{1}{2^x}$  and  $P(X = 1) = 0$  otherwise.

Table 14

$X \backslash Y$	1	2	3	4	5	Total
1	0	0	1	0	0	1
2	0	0	0	1	0	1
3	0	0	0	0	1	1
Total	0	0	1	1	1	3



(a) From Table 14 we see that

$$P(Y = 3, X = 1) = 0 + \frac{1}{3}$$

(b) From Table 14 we see that

$$\begin{aligned} P(X = 1, Y = 3) &= \sum_{i=1}^3 \sum_{j=1}^3 P(X = i, Y = j) \\ &= 0 + 0 + 0 + 0 + 0 + 0 \\ &= 0 + \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \end{aligned}$$

as indicated by the joint distribution in the table.

28. Find the marginal probability functions for  $X$  and  $Y$  for the random variables of Problem 27.

(a) The marginal probability function for  $X$  is given by  $P(X = x) = \sum_{j=1}^3 P(X = x, Y = j)$  and can be obtained from the marginal distribution for  $X$  of Table 14. From this we see that

$$P(X = x) = P(X = 1) = \begin{cases} 0 & x = 1 \\ 0 & x = 2 \\ 1 & x = 3 \end{cases}$$

$$\text{Check: } \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

(b) The marginal probability function for  $Y$  is given by  $P(Y = y) = \sum_{i=1}^3 P(X = i, Y = y)$  and can be obtained from the marginal distribution for  $Y$  of Table 14. From this we see that

$$P(Y = y) = P(Y = 1) = \begin{cases} 0 & y = 1 \\ 0 & y = 2 \\ 1 & y = 3 \end{cases}$$

$$\text{Check: } \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

2.38 Show that the random variables  $X$  and  $Y$  of Problem 2.37 are dependent.

If the random variables  $X$  and  $Y$  are independent, then we must have, for all  $x$  and  $y$ ,

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

But we see from Problem 2.37 that

$$P(X = 1, Y = 1) = \frac{1}{12}, \quad P(X = 1, Y = 2) = \frac{1}{6}, \quad P(Y = 1) = \frac{1}{4}$$

so that

$$P(X = 1, Y = 1) \neq P(X = 1)P(Y = 1)$$

The result also follows from the fact that the joint probability function  $P(x, y)$  cannot be expressed as a function of  $x$  alone times a function of  $y$  alone.

2.44 The joint density function of two continuous random variables  $X$  and  $Y$  is

$$f(x, y) = \begin{cases} 24xy & 0 \leq x \leq 1, 0 \leq y \leq 1-x \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant.

(b) Find  $P(X \leq 0.5, Y \leq 0.5)$ .

(c) Find  $P(X \leq 0.5, 0.5 \leq Y \leq 1)$ .

(d) We now have the independence equal to 1, i.e.,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

Using the definition of  $f(x, y)$ , we do the integral for the value

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x, y) dx \right] dy \\ &= \int_{-\infty}^{\infty} \left[ \int_0^{1-y} 24xy dx \right] dy = \int_{-\infty}^{\infty} \left[ 12y(1-y)^2 - 0 \right] dy \\ &= \int_{-\infty}^{\infty} (12y - 24y^2) dy = 12y^2 - 8y^3 \Big|_{-\infty}^{\infty} = 20 \end{aligned}$$

Since this is not equal to 1, i.e.,

(a) Using the value  $f(x, y)$  from the previous problem

$$\begin{aligned} P_1 &= P(X \leq 0.5, Y \leq 0.5) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy \\ &= \frac{1}{24} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x, y) dx \right] dy = \frac{1}{24} \int_{-\infty}^{\infty} \left[ \int_0^{1-y} f(x, y) dx \right] dy \\ &= \frac{1}{24} \int_{-\infty}^{\infty} (12y - 24y^2) dy = \frac{1}{24} (12y^2 - 8y^3) \Big|_{-\infty}^{\infty} = \frac{1}{6} \end{aligned}$$

$$\begin{aligned}
 \text{so} \quad P(X \leq x, Y \leq y) &= \int_{-\infty}^x \int_{-\infty}^y \frac{1}{2\pi} e^{-\frac{1}{2}(u^2+v^2)} du dv \\
 &= \frac{1}{2\pi} \int_{-\infty}^x \left[ \int_{-\infty}^y e^{-\frac{1}{2}v^2} dv \right] dx = \frac{1}{2\pi} \int_{-\infty}^x \sqrt{2\pi} \Phi\left(\frac{y}{\sigma}\right) dx \\
 &= \frac{1}{2\pi} \int_{-\infty}^x \sqrt{2\pi} \Phi\left(\frac{y}{\sigma}\right) dx = \frac{1}{2\pi} \int_{-\infty}^x \sqrt{2\pi} \Phi\left(\frac{y}{\sigma}\right) dx
 \end{aligned}$$

2.11. Find the joint distribution function by the random variables  $X, Y$  of Problem 2.10.

From Problem 2.10 it is seen that the probability density function  $f(x, y)$  is the probability density function of a bivariate normal distribution with parameters  $\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho$ .

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right\}$$

using  $\mu_X = 0, \mu_Y = 0, \sigma_X = 1, \sigma_Y = 1, \rho = 0$  the joint distribution function is given by  $F(x, y) = P(X \leq x, Y \leq y)$ . The marginal distribution functions  $F_X(x)$  and  $F_Y(y)$  are determined by Problem 2.10 and by 2.9 that a bivariate normal distribution is given by

2.12. In Problem 2.11 find  $P(X < 1, Y < 1)$ .



Fig. 2.12

In Fig. 2.12 we have indicated the four regions  $R_1 = \{x < 0, y < 0\}$ ,  $R_2 = \{x > 0, y < 0\}$ ,  $R_3 = \{x < 0, y > 0\}$ , and  $R_4 = \{x > 0, y > 0\}$  with arrows pointing towards  $R_1$  and  $R_2$  at different intervals. The region probability is given by

$$P(R_1) = P(R_2) = P(R_3) = P(R_4) = \frac{1}{4}$$

where  $h$  is the height of the square root density  $h(x, y) = \sqrt{1 - x^2 - y^2}$  above the disk  $D$ . The probability is given by

$$\begin{aligned} \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2\pi} \sqrt{1-x^2-y^2} \, dy \, dx \\ &= \frac{1}{2\pi} \int_{-1}^1 \left[ \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2} \, dy \right] dx \\ &= \frac{1}{4\pi} \int_{-1}^1 \left[ \int_{-1}^1 \sqrt{1-x^2-y^2} \, dy \right] dx = \frac{1}{4\pi} \int_{-1}^1 \pi(1-x^2) \, dx = \frac{1}{4}. \end{aligned}$$



Fig. 1.10

### Conditional distributions

1.27. Find  $p(y|z)$  if  $X, Y, Z \sim N(0, 1)$  for the distributions of Problem 1.6.

(a) using the conditional distributions  $X|Y$  and  $Y|X$  and (b)

$$f(x, y) = \frac{f(x, y, z)}{\int_{-\infty}^{\infty} f(x, y, z) \, dz} = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

$$\text{and } f(y|z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$f(x|y) = \frac{f(x, y, z)}{\int_{-\infty}^{\infty} f(x, y, z) \, dx} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$(b) \quad f(y) = \int_{-\infty}^{\infty} f(x, y, z) \, dx = \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{3/2}} e^{-\frac{x^2+y^2+z^2}{2}} \, dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$



128. If  $X$  and  $Y$  have the joint density function

$$f(x, y) = \begin{cases} \frac{1}{8} + xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

find  $\text{Pr}(X < 0.5, Y < 0.5) = \text{Pr}(X < 0.5, Y < 0.5)$

Let  $X$  and  $Y$  be i.i.d.

$$f(x, y) = \begin{cases} \frac{1}{8} + xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let  $f(x, y) = \frac{1}{8} + xy$  for  $0 < x < 1, 0 < y < 1$   
 and  $f(x, y) = 0$  otherwise

The distribution of  $X$  and  $Y$  is not identical

(a)  $\text{Pr}(X < 0.5, Y < 0.5) = \int_0^{0.5} \int_0^{0.5} f(x, y) dx dy = \int_0^{0.5} \left[ \frac{1}{8}x + \frac{1}{2}xy^2 \right]_0^{0.5} dy = \frac{1}{16}$

129. The joint density function of the random variables  $X$  and  $Y$  is given by

$$f(x, y) = \begin{cases} 2xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (a) the marginal density of  $X$ , (b) the marginal density of  $Y$ , (c) the conditional density of  $X$ , (d) the conditional density of  $Y$ .

The region of interest for  $X$  is shown in the figure below.



Fig. 11.11

(a) To find the marginal density of  $X$ , let  $f_X(x)$  and integrate with respect to  $y$  (which is  $y$  is indicated by the vertical line in Fig. 11.11). The result is

$$f_X(x) = \int_0^1 2xy dy = x$$

for  $0 < x < 1$ . The distribution of  $X$  is uniform.

10. Find the marginal density of  $X$  by integrating with respect to  $y$  in (1) for  $x \in \mathbb{R}$ . Is this the same as the marginal density of  $X$  in (1)? The answer is **NO** ( $x = 1$ ).

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy = 0 \quad x \neq 0$$

The marginal density of  $X$  is  $f_X(x) = 0$ .

11. The conditional density function of  $X$  is, for  $0 < x < 1$ ,

$$f_{X|Y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \begin{cases} (2x - y) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

The conditional density function of  $Y$  is, for  $0 < y < 1$ ,

12. The conditional density function of  $X$  is, for  $0 < x < 1$ ,

$$f_{X|Y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \begin{cases} 2xy & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

The conditional density function of  $Y$  is, for  $0 < y < 1$ ,

$$\begin{aligned} \text{then} \quad f_{X,Y} &= [f_{X|Y}(x, y) \times f_Y(y)] = [2xy \times (1 - y)] \\ &= 2xy(1 - y) = \begin{cases} 2xy(1 - y) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \\ f_{X,Y} &= \begin{cases} 2xy(1 - y) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

### 1.28. Exercise: check the random variables defined in 1.27 are independent

In the independent of the  $X$  and  $Y$  in 1.27,  $f_{X,Y}(x, y) = f_X(x) \times f_Y(y)$ ,  $f_X(x) = 2x(1 - x)$ ,  $f_Y(y) = 2y(1 - y)$  for  $0 < x < 1$  and  $0 < y < 1$  are independent.

Instead of using the random variable  $f_{X,Y}(x, y) = f_X(x) \times f_Y(y)$  to check if the random variables are independent, you can check if the joint density function  $f_{X,Y}(x, y)$  is constant. This is not the case for the joint density function  $f_{X,Y}(x, y) = 2xy(1 - y)$  in 1.27, so the random variables are not independent.

### Worked example

- 1.29. Suppose that the random variables  $X$  and  $Y$  have a bivariate density given by

$$f_{X,Y}(x, y) = \begin{cases} 2xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the constant  $c$  for marginal density functions  $f_X(x)$  and  $f_Y(y)$  by the marginal density function  $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy = \int_0^1 2xy dy = x$ , and  $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx = \int_0^1 2xy dx = y$  for  $0 < x < 1$  and  $0 < y < 1$ .

13. The independence is given by

$$\begin{aligned} \int_0^1 \int_0^1 2xy dx dy &= \int_0^1 \left[ \int_0^1 2xy dx \right] dy \\ &= \int_0^1 \left[ xy - \frac{y^2}{2} \right]_0^1 dy = \frac{1}{2} \end{aligned}$$

Worked example 1.30. Let  $X$  and  $Y$  be

$$(b) \quad P(T < T + 3, T < 11) = \frac{1}{64} \int_{-1}^1 \int_{-1}^1 P(T) = 128(10) = \frac{13}{16}$$

$$(c) \quad P(T < 11) = \frac{1}{64} \int_{-1}^1 \int_{-1}^1 (2x + 3) dx = \frac{13}{16}$$

$$(d) \quad P(T < T + 11) = \int_{-1}^1 \int_{-1}^1 dx dy$$

where  $D$  is the shaded region of Fig. 1.18. Although the region is not, it is easier to see the fact that

$$P(T < T + 11) = 1 - P(T < 3 & 4) = 1 - \int_{-1}^1 \int_{-1}^1 dx dy$$

where  $D'$  is the cross-hatched region of Fig. 1.18. Therefore

$$P(T < T + 11) = \frac{1}{64} \int_{-1}^1 \int_{-1}^1 (2x + 3) dx = \frac{13}{16}$$

Thus,  $P(T < T + 4) = 13/16$ .

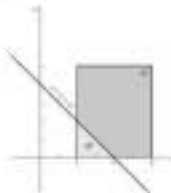


Fig. 1.18

- (g) The random variables are dependent since  $f(x, y) \neq f(x)f(y)$  or equivalently  $P(x, y) \neq P(x)P(y)$ .

# Mathematical Expectation

## Mean of a Random Variable

Let  $X$  be a random variable with probability distribution  $f(x)$ . The mean, or expected value, of  $X$  is

$$\mu = E(X) = \sum_{i=1}^n x_i f(x_i)$$

If  $X$  is discrete, we

$$\mu = E(X) = \sum_{i=1}^n x_i f(x_i)$$

If  $X$  is continuous,

### Example 2.24

A probability distribution is thought of as a function that maps the sample space of the random variable to the real numbers. A probability mass function is a function that maps the sample space of a discrete random variable to the real numbers. It has the following probability mass function:

$$f(x) = \begin{cases} \frac{1}{10} & \text{if } x = 1 \\ \frac{2}{10} & \text{if } x = 2 \\ \frac{3}{10} & \text{if } x = 3 \\ \frac{2}{10} & \text{if } x = 4 \\ \frac{1}{10} & \text{if } x = 5 \end{cases}$$

Find the mean  $\mu$ .

**Solution**

Using Equation (2.24) we compute

$$\mu = 1 \left( \frac{1}{10} \right) + 2 \left( \frac{2}{10} \right) + 3 \left( \frac{3}{10} \right) + 4 \left( \frac{2}{10} \right) + 5 \left( \frac{1}{10} \right) = 3$$



**FIGURE 2.24** The graph of the probability mass function of a discrete random variable.

Example: Find  $E(X)$  if  $f(x)$  is given by:

$$f(x) = \frac{(x+1)^2}{9}, \quad x = 0, 1, 2, 3$$

**Solution:**

Simple calculations yield:  $f(0) = 1/9$ ,  $f(1) = 4/9$ ,  $f(2) = 16/9$ , and  $f(3) = 16/9$ . Therefore,

$$E(X) = 0 \cdot \frac{1}{9} + 1 \cdot \frac{4}{9} + 2 \cdot \frac{16}{9} + 3 \cdot \frac{16}{9} = \frac{72}{9} = 8$$

Example: Let  $X$  be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \frac{10000}{x^2} & x > 100 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected life of this type of device.

**Solution:** Using Definition 4.3, we have

$$E(X) = \int_{100}^{\infty} x \cdot \frac{10000}{x^2} dx = \int_{100}^{\infty} \frac{10000}{x} dx = 100$$

Therefore, we can expect this type of device to last, on average, 100 hours.

Let  $X$  be a random variable with probability distribution  $f(x)$ . The expected value of the random variable  $g(X)$  is

$$E(g(X)) = E(g(X)) = \sum_i g(x_i) f(x_i)$$

If  $X$  is discrete, and

$$E(g(X)) = E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

If  $X$  is continuous,

**Example:** Suppose that the number of cars  $Y$  that pass through a one-way highway, 435 mi. and 225 mi. on any given Friday has the following probability distribution:

$Y$	1	2	3	4	5	6	7	8	9
$P(Y = y)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Let  $g(Y) = 4Y + 1$  represent the amount of money, in dollars, paid to the attendant for the attendant's expected earnings for this particular time period.

**Solution:**

$$\begin{aligned}
 E[g(Y)] &= E[4Y + 1] = \sum_{i=1}^9 (4y_i + 1)P(y_i) \\
 &= (7)\left(\frac{1}{8}\right) + (11)\left(\frac{1}{8}\right) + (15)\left(\frac{1}{8}\right) + (19)\left(\frac{1}{8}\right) \\
 &\quad + (23)\left(\frac{1}{8}\right) + (27)\left(\frac{1}{8}\right) + (31)\left(\frac{1}{8}\right) + (35)\left(\frac{1}{8}\right) + (39)\left(\frac{1}{8}\right) + (43)\left(\frac{1}{8}\right) \\
 &= 25.
 \end{aligned}$$

**Example:** Let  $X$  be a random variable with density function

$$f(x) = \begin{cases} \frac{2}{3} & -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of  $g(X) = 4X + 6$ .

**Solution:** By Theorem 1.1, we have

$$E[g(X)] = \int_{-\infty}^{\infty} \frac{(4x + 6)f(x)}{3} dx = \frac{1}{3} \int_{-1}^2 (4x^2 + 8x) dx = 6.$$

Let  $X$  and  $Y$  be random variables with joint probability distribution  $f(x, y)$ . The mean, or expected value, of the random variable  $g(X, Y)$  is

$$E[g(X, Y)] = E[g(X, Y)] = \sum_x \sum_y g(x, y)f(x, y)$$

If  $X$  and  $Y$  are discrete, and

$$E[g(X, Y)] = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f(x, y) dx dy$$

If  $X$  and  $Y$  are continuous,

**Exercise 1** Let  $X$  and  $Y$  be the random variables with joint probability distribution indicated in Table. Find the expected value of  $g(X, Y) = XY$ .

		$Y$			Row Totals
		0	1	2	
$X$	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
	1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
	2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
Column Totals		$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{3}$	1

**Solution:**

$$\begin{aligned}
 E(XY) &= \sum_{i=1}^n \sum_{j=1}^m xy_{ij}p_{ij}(x, y) \\
 &= (0)(0)\left(\frac{1}{12}\right) + (0)(1)\left(\frac{1}{6}\right) + (0)(2)\left(\frac{1}{6}\right) \\
 &\quad + (1)(0)\left(\frac{1}{6}\right) + (1)(1)\left(\frac{1}{3}\right) + (1)(2)\left(\frac{1}{3}\right) \\
 &\quad + (2)(0)\left(\frac{1}{6}\right) + (2)(1)\left(\frac{1}{3}\right) + (2)(2)\left(\frac{1}{3}\right) \\
 &= \frac{11}{3} \approx 3.67
 \end{aligned}$$

**Exercise 2** Find  $E(Y|X)$  for the density function

$$f(x, y) = \begin{cases} \frac{8xy^2}{9} & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

**Solution:** We have

$$E\left(\frac{Y}{X}\right) = \int_0^1 \int_0^1 \frac{E(Y|X)}{X} f(x, y) dy dx = \int_0^1 \frac{E(Y|X)}{X} dx = \frac{1}{2}$$

## Example 5.1.1

The displacement of a piston in an internal combustion engine is subject to the random force of the gases acting through the top of the piston of its rods. Let  $T$  represent the thickness of the cylinder head, in millimeters, and let  $V$  represent the length of the piston rods in millimeters. The displacement  $g$  for  $0 < t < T$  is, in mm,  $g = t + V$ . Let  $T$  and  $V$  be jointly distributed with joint probability density function

$$f(t, v) = \begin{cases} \frac{1}{9} & 0 \leq t \leq 1 \text{ and } 0 \leq v \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean of  $g$ .

**Solution:**

$$\begin{aligned}
 \mu &= \int_0^1 \int_0^2 \frac{t+v}{9} f(t, v) dv dt \\
 &= \int_0^1 \int_0^2 \frac{t+v}{9} dv dt \\
 &= \frac{11}{9} \approx 1.22
 \end{aligned}$$

The mean displacement is 1.22 mm and is approximately 0.05 in.

# EXERCISES 97

9.1. The probability distribution of the number of inspections per 20 square feet of electric wiring in a house is given by the following table (Exercise 9.14, Exercise 9.14).

$x$	0	1	2	3	4	5
$P(X=x)$	0.10	0.20	0.30	0.20	0.10	0.10

Find the average number of inspections per 20 square feet of this house.

9.2. The probability distribution of the number of days until a customer returns a product is given by the following table:

$x$	1	2	3	4	5	6	7	8	9	10
$P(X=x)$	0.10	0.15	0.20	0.25	0.20	0.15	0.10	0.05	0.05	0.05

Find the mean of  $X$ .

9.3. Find the value of the constant  $c$  such that the probability distribution of the number of calls received per hour is given by the following table:

$x$	0	1	2	3	4	5	6	7	8	9	10
$P(X=x)$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$

Find the value of  $c$ .

9.4. A company has a policy of giving a bonus to its employees if they work more than 100 hours in a month. Find the expected number of employees who will receive a bonus if the company has 100 employees.

9.5. A company has a policy of giving a bonus to its employees if they work more than 100 hours in a month. Find the expected number of employees who will receive a bonus if the company has 100 employees.

9.6. The probability of a customer returning a product is given by the following table (Exercise 9.2, Exercise 9.2).

$x$	1	2	3	4	5	6	7	8	9	10
$P(X=x)$	0.10	0.15	0.20	0.25	0.20	0.15	0.10	0.05	0.05	0.05

Find the expected number of days until a customer returns a product.

9.7. A company has a policy of giving a bonus to its employees if they work more than 100 hours in a month. Find the expected number of employees who will receive a bonus if the company has 100 employees.

9.8. A company has a policy of giving a bonus to its employees if they work more than 100 hours in a month. Find the expected number of employees who will receive a bonus if the company has 100 employees.

9.9. A company has a policy of giving a bonus to its employees if they work more than 100 hours in a month. Find the expected number of employees who will receive a bonus if the company has 100 employees.

$x$	0	1	2	3	4	5	6	7	8	9	10
$P(X=x)$	0.10	0.15	0.20	0.25	0.20	0.15	0.10	0.05	0.05	0.05	0.05

Find the mean of  $X$ .

9.10. The probability distribution of the number of calls received per hour is given by the following table:

$x$	0	1	2	3	4	5	6	7	8	9	10
$P(X=x)$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$

Find the constant value of  $c$ .

9.11. The probability of a customer returning a product is given by the following table (Exercise 9.2, Exercise 9.2).

$x$	1	2	3	4	5	6	7	8	9	10
$P(X=x)$	0.10	0.15	0.20	0.25	0.20	0.15	0.10	0.05	0.05	0.05

Find the expected number of days until a customer returns a product.

9.12. The probability of a customer returning a product is given by the following table (Exercise 9.2, Exercise 9.2).

$x$	0	1	2	3	4	5	6	7	8	9	10
$P(X=x)$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$

Find the constant value of  $c$ .

9.13. The probability of a customer returning a product is given by the following table (Exercise 9.2, Exercise 9.2).

$x$	0	1	2	3	4	5	6	7	8	9	10
$P(X=x)$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$	$c$



# **Chapter 10: Probability and Statistics**

## **10.1. Probability and Statistics**

**1.1** Suppose that the random variable  $X$  is uniformly distributed on a finite interval  $a, b$ . Find the probability density function  $f(x)$ .

$$f(x) = \frac{1}{b-a} \quad \text{if } a \leq x \leq b, \\ 0 \quad \text{otherwise.}$$

Find the expected value of  $X$ .

**1.2** Suppose that you are sampling  $n = 10$  of 1000 balls (blue, green, white) at one time. Find the probability that the number of blue balls is 3.

$$P(X=3) = \frac{1}{10!} \left( \frac{1}{10} \right)^3 \left( \frac{9}{10} \right)^7$$

Find the probability that the number of blue balls is 3.

**1.3** Let  $X$  be a random variable with the following probability distribution:

$$P(X=1) = \frac{1}{10}, P(X=2) = \frac{2}{10}, P(X=3) = \frac{3}{10}, P(X=4) = \frac{4}{10}$$

Find  $\mu_X$ ,  $\sigma_X$ ,  $\sigma_X^2$ ,  $\sigma_X^4$ ,  $\sigma_X^6$ ,  $\sigma_X^8$ .

**1.4** Find the expected value of the random variable  $X$  if  $X$  has the probability distribution of Exercise 1.3.

**1.5** A group of students has a probability of 0.1 of passing the first exam, 0.2 of passing the second exam, 0.3 of passing the third exam, 0.4 of passing the fourth exam, 0.5 of passing the fifth exam, 0.6 of passing the sixth exam, 0.7 of passing the seventh exam, 0.8 of passing the eighth exam, 0.9 of passing the ninth exam, 1.0 of passing the tenth exam.

$$P(X=1) = \frac{1}{10}, P(X=2) = \frac{2}{10}, P(X=3) = \frac{3}{10}, P(X=4) = \frac{4}{10}, P(X=5) = \frac{5}{10}, P(X=6) = \frac{6}{10}, P(X=7) = \frac{7}{10}, P(X=8) = \frac{8}{10}, P(X=9) = \frac{9}{10}, P(X=10) = \frac{10}{10}$$

Find the expected value of the random variable  $X$  if  $X$  has the probability distribution of Exercise 1.5.

**1.6** A random variable  $X$  has the probability distribution of Exercise 1.5.

$$P(X=1) = \frac{1}{10}, P(X=2) = \frac{2}{10}, P(X=3) = \frac{3}{10}, P(X=4) = \frac{4}{10}, P(X=5) = \frac{5}{10}, P(X=6) = \frac{6}{10}, P(X=7) = \frac{7}{10}, P(X=8) = \frac{8}{10}, P(X=9) = \frac{9}{10}, P(X=10) = \frac{10}{10}$$

Find the expected value of  $X$  if  $X$  has the probability distribution of Exercise 1.6.

**1.7** A group of students has a probability of 0.1 of passing the first exam, 0.2 of passing the second exam, 0.3 of passing the third exam, 0.4 of passing the fourth exam, 0.5 of passing the fifth exam, 0.6 of passing the sixth exam, 0.7 of passing the seventh exam, 0.8 of passing the eighth exam, 0.9 of passing the ninth exam, 1.0 of passing the tenth exam.

# **Chapter 11: Probability and Statistics**

## **11.1. Probability and Statistics**

**1.1** Suppose that the random variable  $X$  is uniformly distributed on a finite interval  $a, b$ . Find the probability density function  $f(x)$ .

$$f(x) = \frac{1}{b-a} \quad \text{if } a \leq x \leq b, \\ 0 \quad \text{otherwise.}$$

Find the expected value of  $X$  if  $X$  has the probability density function  $f(x)$ .

**1.2** Suppose that  $X$  and  $Y$  have the following joint probability distribution:

	$Y=1$	$Y=2$	$Y=3$
$X=1$	0.1	0.2	0.3
$X=2$	0.2	0.3	0.4
$X=3$	0.3	0.4	0.5

Find the expected value of  $X$  if  $X$  has the probability distribution of Exercise 1.2.

Find the expected value of  $Y$  if  $Y$  has the probability distribution of Exercise 1.2.

**1.3** Suppose that  $X$  and  $Y$  have the following joint probability distribution:

	$Y=1$	$Y=2$	$Y=3$
$X=1$	0.1	0.2	0.3
$X=2$	0.2	0.3	0.4
$X=3$	0.3	0.4	0.5

Find the expected value of  $X$  if  $X$  has the probability distribution of Exercise 1.3.

Find the expected value of  $Y$  if  $Y$  has the probability distribution of Exercise 1.3.

$$P(X=1) = \frac{1}{10}, P(X=2) = \frac{2}{10}, P(X=3) = \frac{3}{10}, P(X=4) = \frac{4}{10}, P(X=5) = \frac{5}{10}, P(X=6) = \frac{6}{10}, P(X=7) = \frac{7}{10}, P(X=8) = \frac{8}{10}, P(X=9) = \frac{9}{10}, P(X=10) = \frac{10}{10}$$

Find the expected value of  $X$  if  $X$  has the probability distribution of Exercise 1.4.

**1.4** Suppose that  $X$  and  $Y$  have the following joint probability distribution:

# **Exercises involving type of discrete random variable** **4.18. Discrete random variable**

**4.18.** Consider the probability in Exercise 3.18 on page 55. The probability that the weight is more than 100 pounds is a certain fixed value.

$$P(X > 100) = \frac{1}{10} - P(75 < X < 100)$$

- Find the density function.
- Calculate the expected value or mean weight in pounds.
- Are you surprised or not surprised by (2)? Explain why or otherwise.

**4.19.** Exercise 3.19 on page 55 deals with an event that has the same probability distribution as

$$P(X = x) = \frac{1}{2} \left( \frac{1}{2} \right)^{x-1}, \quad x = 1, 2, \text{ etc.}$$

- Find the density function.
- Find the mean distribution.

**4.20.** In Exercise 3.21 on page 55, the distribution of the number of people in a family is given by

$$P(X = x) = \frac{1}{2} \left( \frac{1}{2} \right)^{x-1}, \quad x = 1, 2, \text{ etc.}$$

What is the expected value of the family size?

# **Exercises involving continuous random variable** **4.21. Continuous random variable**

**4.21.** Consider Exercise 3.22 on page 55.

- Why is the mean proportion of the budget allocated to maintenance and pollution control?
- What is the probability that a company spends no budget at all (zero dollars) on environmental pollution control? Is this probability possible? The expected mean value is 1/2.

**4.22.** In Exercise 3.23 on page 55, the distribution of the number of people in a family is given by

$$P(X = x) = \frac{1}{2} \left( \frac{1}{2} \right)^{x-1}, \quad x = 1, 2, \text{ etc.}$$

- Find the probability function.
- Find the expected number of people in a family.
- Find  $P(X > 1)$ .

## **Variance of Random Variables**

The mean or expected value of a random variable  $X$  is of great importance in statistics because it describes where the probability distribution is centered. It tells, however, this mean does not give an adequate description of the shape of the distribution. We also need to characterize the variability in the distribution. In Figure 4.1, we have the histograms of two discrete probability distributions that have the same mean,  $\mu = 2$ . They differ considerably in variability, or the dispersion of their observations about the mean.



Figure 4.1. Distributions with equal means and unequal dispersions.

Let  $X$  be a random variable with probability distribution  $f(x)$  and mean  $\mu$ . The variance of  $X$  is

$$\sigma^2 = E[(X - \mu)^2] = \sum_i (x_i - \mu)^2 f(x_i) \quad \text{if } X \text{ is discrete, and}$$

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx, \quad \text{if } X \text{ is continuous.}$$

The positive square root of the variance,  $\sigma$ , is called the **standard deviation** of  $X$ .

The quantity  $x - \mu$  in the definition above is relative deviation of a value  $x$  from  $\mu$ . Since the deviations are squared and then averaged,  $\sigma^2$  will be much smaller for a set of  $x$  values that are close to  $\mu$  than it will be for a set of values that vary considerably from  $\mu$ .

**Example 1.** Let the random variable  $N$  represent the number of automobiles that are used for official business purposes on any given working day. The probability distribution for company A (Figure 1.4(a)) is

$$\begin{array}{c|c} x & p(x) \\ \hline 0 & 0.1 \\ 1 & 0.3 \\ 2 & 0.4 \\ 3 & 0.2 \end{array}$$

and that for company B (Figure 1.4(b)) is

$$\begin{array}{c|c} x & p(x) \\ \hline 0 & 0.1 \\ 1 & 0.2 \\ 2 & 0.3 \\ 3 & 0.3 \\ 4 & 0.1 \end{array}$$

Show that the variance of the probability distribution for company B is greater than that for company A.

**Solution.** For company A, we find that

$$\mu_A = E(N) = (1)(0.1) + (2)(0.3) + (3)(0.4) + (0) = 2.0,$$

and then

$$\sigma_A^2 = \sum_{i=1}^n (x_i - \mu_A)^2 p(x_i) = (1 - 2)^2(0.1) + (2 - 2)^2(0.3) + (3 - 2)^2(0.4) + (0 - 2)^2(0.2) = 1.0.$$

For company B, we have

$$\mu_B = E(N) = (0)(0.1) + (1)(0.2) + (2)(0.3) + (3)(0.3) + (4)(0.1) = 2.0,$$

and then

$$\begin{aligned} \sigma_B^2 &= \sum_{i=1}^n (x_i - \mu_B)^2 p(x_i) \\ &= (0 - 2)^2(0.1) + (1 - 2)^2(0.2) + (2 - 2)^2(0.3) \\ &\quad + (3 - 2)^2(0.3) + (4 - 2)^2(0.1) = 1.6. \end{aligned}$$

Clearly, the variance of the number of automobiles that are used for official business purposes is greater for company B than for company A.  $\square$

The variance of a random variable  $N$  is

$$\sigma^2 = E(Y^2) - \mu^2.$$

**Example 1** Let the random variable  $X$  represent the number of defective parts for a machine when 5 parts are sampled from a production line and tested. The following is the probability distribution of  $X$ .

$$\begin{array}{c|c} x & P(X=x) \\ \hline 0 & \frac{1}{32} \\ 1 & \frac{5}{16} \\ 2 & \frac{15}{32} \\ 3 & \frac{5}{16} \\ 4 & \frac{1}{32} \end{array}$$

Using Theorem 6.2, calculate  $E(X)$ .

**Solution:** First, we compute

$$\mu = \sum_{i=1}^n x_i p_i = (0)\left(\frac{1}{32}\right) + (1)\left(\frac{5}{16}\right) + (2)\left(\frac{15}{32}\right) + (3)\left(\frac{5}{16}\right) + (4)\left(\frac{1}{32}\right) = 2.0625$$

So,

$$E(X) = (0)(\frac{1}{32}) + (1)(\frac{5}{16}) + (2)(\frac{15}{32}) + (3)(\frac{5}{16}) + (4)(\frac{1}{32}) = 2.0625$$

Therefore,

$$E(X^2) = (0)^2\left(\frac{1}{32}\right) + (1)^2\left(\frac{5}{16}\right) + (2)^2\left(\frac{15}{32}\right) + (3)^2\left(\frac{5}{16}\right) + (4)^2\left(\frac{1}{32}\right) = 4.9375 \quad \square$$

**Example 2** The weight (in grams) for a drinking water product is normally distributed. Then a test result of efficiency score is a continuous random variable. It having the probability density

$$f(x) = \begin{cases} \frac{1}{20}(x-1), & 1 \leq x \leq 21 \\ 0, & \text{otherwise} \end{cases}$$

Find the mean and variance of  $X$ .

**Solution:** Calculating  $E(X)$  and  $E(X^2)$ , we have

$$\mu = E(X) = \int_1^{21} x \cdot \frac{1}{20}(x-1) dx = \frac{1}{20} \int_1^{21} (x^2 - x) dx$$

and

$$E(X^2) = \int_1^{21} x^2 \cdot \frac{1}{20}(x-1) dx = \frac{1}{20} \int_1^{21} (x^3 - x^2) dx$$

Therefore,

$$\mu = \frac{1}{20} \left[ \frac{x^3}{3} - \frac{x^2}{2} \right]_1^{21} = \frac{1}{20} \left( \frac{9261}{3} - \frac{441}{2} \right) = 15.75 \quad \square$$

## Example 3.31

Find the variance and standard deviation for the random variable  $X$  specified in Example 3.30, assuming the number of items produced is unlimited.

**Solution:**

In Example 3.30, we computed the mean of  $X$  as  $\mu = 1.56$ . We compute the variance by using Equation (3.30)

$$\begin{aligned} E(X^2) &= (0)^2 \cdot 0.001 + (1)^2 \cdot 0.001 + (2)^2 \cdot 0.001 + (3)^2 \cdot 0.001 + (4)^2 \cdot 0.001 + (5)^2 \cdot 0.001 \\ &\quad + (6)^2 \cdot 0.001 + (7)^2 \cdot 0.001 + (8)^2 \cdot 0.001 + (9)^2 \cdot 0.001 + (10)^2 \cdot 0.001 \\ &\quad + (11)^2 \cdot 0.001 + (12)^2 \cdot 0.001 + (13)^2 \cdot 0.001 + (14)^2 \cdot 0.001 + (15)^2 \cdot 0.001 \\ &\quad + (16)^2 \cdot 0.001 + (17)^2 \cdot 0.001 + (18)^2 \cdot 0.001 + (19)^2 \cdot 0.001 + (20)^2 \cdot 0.001 \\ &= 1.56 \end{aligned}$$

The standard deviation is  $\sigma = \sqrt{E(X^2) - \mu^2} = 1.20$

### Example 3.18

Let the discrete random variable  $X$  have the probability mass function  $p_X(x)$  as shown in the table below. Find the mean, variance, and standard deviation of  $X$ .

**Solution**

By Example 3.17, the mean is  $\mu_X = E[X] = 1.2$ . The variance is

$$\begin{aligned}\sigma_X^2 &= E[(X - \mu_X)^2] = E[X^2 - 2\mu_X X + \mu_X^2] = E[X^2] - 2\mu_X E[X] + \mu_X^2 E[1] \\ &= E[X^2] - 2\mu_X^2 + \mu_X^2 = E[X^2] - \mu_X^2 \\ &= (0.16 + 0.36 + 0.64 + 1.00 + 1.44 + 1.80 + 2.25) - (1.2)^2 \\ &= 1.76\end{aligned}$$

### Example 3.19

A machine is considered to be in good condition if its lifetime is in the range  $(60, 100]$  h. An engineer studies this process. The probability mass function of the lifetime of the machine is  $p_X(x)$  for the probability that exactly one of them during the specified time is true, and the probability that neither of them during the specified time is false. Let  $X$  represent the number of machines in good condition. Find the probability mass function and the mean, variance, and standard deviation of  $X$ .

**Solution**

The probability mass function is  $P_X(0) = 0.1$ ,  $P_X(1) = 0.3$ ,  $P_X(2) = 0.4$ ,  $P_X(3) = 0.1$ , and  $P_X(4) = 0.1$  because a year has 4 h. The mean is

$$\begin{aligned}\mu_X &= (0)(0.1) + (1)(0.3) + (2)(0.4) + (3)(0.1) + (4)(0.1) \\ &= 1.9\end{aligned}$$

The variance is

$$\begin{aligned}\sigma_X^2 &= E[X^2] - \mu_X^2 = (0)(0.1) + (1)(0.3) + (4)(0.4) + (9)(0.1) + (16)(0.1) - (1.9)^2 \\ &= 0.89\end{aligned}$$

The standard deviation is  $\sigma_X = \sqrt{0.89} \approx 0.9433$ .

# Example 3.42

Water is pumped 1 m. Find the mean discharge rate (m³/s) and the standard deviation.

**Assumptions**

Using Equation (3.42) for mean discharge rate (m³/s).

$$\begin{aligned} \mu_Q &= \int_{-1}^1 Q f(Q) dQ \\ &= \int_{-1}^1 (1.2Q + 0.7) f(Q) dQ \\ &= 1.2 \left[ \frac{Q^2}{2} \right]_{-1}^1 + 0.7 \left[ \frac{Q}{1} \right]_{-1}^1 \\ &= 0.4 \text{ m}^3/\text{s} \end{aligned}$$

Using Equation (3.43) for standard deviation of  $Q$ , it is needed to use the standard formula Equation (3.43).

$$\begin{aligned} \sigma_Q^2 &= \int_{-1}^1 Q^2 f(Q) dQ - \mu_Q^2 \\ &= \int_{-1}^1 Q^2 (1.2Q + 0.7) f(Q) dQ - (0.4)^2 \\ &= 1.2 \left[ \frac{Q^3}{3} \right]_{-1}^1 + 0.7 \left[ \frac{Q^2}{2} \right]_{-1}^1 - (0.4)^2 \\ &= 0.04 \text{ m}^6/\text{s}^2 \end{aligned}$$

Let  $X$  be a random variable with probability distribution  $f(x)$ . The variance of the random variable  $g(X)$  is

$$\sigma_{g(X)}^2 = E[g(X)^2] - \mu_{g(X)}^2 = \sum_i g(x_i)^2 f(x_i) - \mu_{g(X)}^2$$

If  $X$  is discrete, and

$$\sigma_{g(X)}^2 = E[g(X)^2] - \mu_{g(X)}^2 = \int_{-\infty}^{\infty} g(x)^2 f(x) dx - \mu_{g(X)}^2$$

If  $X$  is continuous,

**Example 4.1** Calculate the variance of  $g(X) = 2X + 3$ , where  $X$  is a random variable with probability density function

$$f_X(x) = \begin{cases} \frac{8}{3} - \frac{1}{3}x & 0 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

**Solution:** First, we find the mean of the random variable  $2X + 3$ :

$$\begin{aligned} \mu_{2X+3} &= E[2X + 3] = \sum_{i=1}^n (2x_i + 3)f_X(x_i) = 8 \\ \sigma_{2X+3}^2 &= E[(2X + 3) - \mu_{2X+3}]^2 = E[(2X + 3 - 8)^2] \\ &= E[4X^2 - 12X + 9] = \sum_{i=1}^n 4x_i^2 - 12x_i + 9f_X(x_i) = 4. \end{aligned}$$

**Example 4.2** Let  $X$  be a random variable with density function

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the variance of the random variable  $g(X) = 4X + 5$ .

**Solution:** In Example 4.1, we found that  $\mu_{2X+3} = 8$ . Thus, using Theorem 4.2,

$$\begin{aligned} \sigma_{4X+5}^2 &= E[(4X + 5) - \mu_{4X+5}]^2 = E[(4X - 3)^2] \\ &= \int_{-1}^1 (x^2 - 3x + \frac{9}{4}) \, dx = \frac{1}{3} \int_{-1}^1 (3x^2 - 9x + 9) \, dx = \frac{16}{3}. \end{aligned}$$

## Covariance of Random Variables

Let  $X$  and  $Y$  be random variables with joint probability distribution  $f(x, y)$ . The covariance of  $X$  and  $Y$  is

$$c_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \sum_{i=1}^n \sum_{j=1}^m (x_i - \mu_X)(y_j - \mu_Y)f(x_i, y_j)$$

If  $X$  and  $Y$  are discrete, and

$$c_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y)f(x, y) \, dx \, dy$$

If  $X$  and  $Y$  are continuous,



The covariance of two random variables  $X$  and  $Y$  with means  $\mu_X$  and  $\mu_Y$ , respectively, is given by

$$\text{Cov}(X, Y) = \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)]$$

$$\text{Cov}(X, Y) = \begin{cases} \text{Positive} & \text{if } X \text{ and } Y \text{ increase together} \\ 0 & \text{if } X \text{ and } Y \text{ are independent} \\ \text{Negative} & \text{if } X \text{ and } Y \text{ decrease together} \end{cases}$$

**Example 4.6** The following is the joint probability distribution. Find the covariance of  $X$  and  $Y$ .

		$Y$		$P_{X,Y}$
$X$		0	1	
$X$	0	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$
	1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$
	2	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$
	3	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$
$P_X$		$\frac{3}{8}$	$\frac{3}{8}$	1

**Solution:** From Example 4.4, we see that  $E(XY) = \frac{5}{4}$ . Thus

$$\mu_X = \sum_{i=1}^n x_i p_i = 0 \left( \frac{3}{16} \right) + 1 \left( \frac{3}{8} \right) + 2 \left( \frac{3}{8} \right) + 3 \left( \frac{3}{16} \right) = \frac{3}{2}$$

and

$$\mu_Y = \sum_{j=1}^m y_j p_j = 0 \left( \frac{3}{8} \right) + 1 \left( \frac{3}{8} \right) + 2 \left( \frac{3}{8} \right) + 3 \left( \frac{3}{8} \right) = \frac{3}{2}$$

Hence,

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y = \frac{5}{4} - \left( \frac{3}{2} \right) \left( \frac{3}{2} \right) = -\frac{1}{4}$$

### Example 4.7

Consider Example 2.9. A random variable is having probability function given by the table below,  $x = 1, 2$ , and the law is  $x = 1$  (see Figure 2.13). If  $X, Y$  denote the random variable is a given only the value strictly  $x > 1$  and  $Y$  is given by

$$f(x, y) = \begin{cases} \frac{1}{4} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

**Solution:** (i) (ii)

# Example

We compute the mass of a plate  $D = \{(x, y) \mid 0 \leq x \leq y \leq 1\}$  with density function  $\delta(x, y)$ :

$$m = \int_0^1 \int_0^y \delta(x, y) dx dy$$

First, the plate density is positive, so the length density



To compute the length (in this region, not the value of  $\delta$ , as above), we compute the mass integral by integrating with respect to  $x$  along the vertical line through  $x$ . The limits of integration giving this line are  $x = 0$  to  $y = x$ . That is, we integrate with respect to  $y$  from  $y = x$  to  $y = 1$ , since all possible values of  $y$  in the region of integration in the case length are  $x \leq y \leq 1$ .

Then we

$$\begin{aligned} m &= \int_0^1 \int_x^1 \delta(x, y) dy dx \\ &= \int_0^1 \left( \int_x^1 \delta(x, y) dy \right) dx \\ &= \int_0^1 \frac{\delta(x, x)}{2} dx \\ &= \frac{1}{2} \end{aligned}$$

We find  $\mu_x$  and  $\mu_y$  by using the standard formulas presented in Example 1.10. These are

$$\begin{aligned} \mu_x &= \int_0^1 x \left( \int_x^1 \delta(x, y) dy \right) dx = \frac{1}{3} \\ \mu_y &= \int_0^1 \left( \int_x^1 y \delta(x, y) dy \right) dx = \frac{1}{3} \end{aligned}$$

We now compute  $u_x$  and  $u_y$ :

$$\begin{aligned}u_x &= \int_{-\infty}^{\infty} x \delta(x) dx \\&= \int_{-\infty}^{\infty} x^2 dx \\&= \frac{2}{3} \\u_y &= \int_{-\infty}^{\infty} y \delta(x) dx \\&= \int_{-\infty}^{\infty} (y^2 - x^2) dx \\&= \frac{2}{3}\end{aligned}$$

$$\text{Hence } \text{Curl}(\mathbf{u}, \mathbf{F}) = \frac{2}{3} - \left(\frac{2}{3}\right) \left(\frac{2}{3}\right) = \frac{2}{9} = 0.2222.$$

# Example 3.6.8

Quality-control checks are made randomly, thereby ensuring the number of surface flaws in a pipe joint. The inspector finds 0 or 1 or 2 flaws in the section of surface flaws due to surface degradation of the pipe joint in the testing material, and for 0, 1, or 2, the number of surface flaws due to inclusion of foreign particles in the fluid. The joint probability mass function for  $X$  and  $Y$  is given in the following table. The marginal probability mass functions are given below and in the margin of the table. Find the covariance of  $X$  and  $Y$ .

$x$	$y$	$f$		$g(y)$
		0	1	
0	0	0.05	0.15	0.20
1	0	0.05	0.15	0.20
2	0	0.05	0.15	0.20
$g(x)$	0.20	0.25	0.30	

## Solution

We will use the formula  $\text{Cov}(X, Y) = \mu_{xy} - \mu_x \mu_y$  (Equation (3.6)). First we compute  $\mu_{xy}$ :

$$\begin{aligned}\mu_{xy} &= \sum_{i=1}^n \sum_{j=1}^m x_i y_j f(x_i, y_j) \\ &= 0 \cdot 0 \cdot 0.20 + 0 \cdot 1 \cdot 0.15 + 0 \cdot 2 \cdot 0.15 + 1 \cdot 0 \cdot 0.15 \\ &\quad + 1 \cdot 1 \cdot 0.15 + 1 \cdot 2 \cdot 0.15 \\ &= 0.45 \quad \text{covariance of } X \text{ and } Y.\end{aligned}$$

Now we compute  $\mu_x$  and  $\mu_y$  and  $\mu_{xy}$ :

$$\mu_x = 0(0.20) + 1(0.25) + 2(0.30) = 1.15$$

$$\mu_y = 0(0.20) + 1(0.30) + 2(0.30) = 1.10$$

It follows that  $\text{Cov}(X, Y) = 0.45 - 1.15(1.10) = -0.715$ .

## Covariance Coefficient

Although the covariance between two random variables does provide information regarding the nature of the relationship, the magnitude of  $\sigma_{xy}$  does not indicate anything regarding the strength of the relationship, since  $\sigma_{xy}$  is not unit-free. Its magnitude will depend on the units used to measure both  $X$  and  $Y$ . There is a scale-free version of the covariance called the covariance coefficient that is used widely in statistics.

Let  $X$  and  $Y$  be jointly distributed random variables with standard deviations  $\sigma_X$  and  $\sigma_Y$ . The correlation between  $X$  and  $Y$  is denoted  $\rho_{XY}$  and is given by

$$\rho_{XY} = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y} \quad (2.76)$$

For any two random variables  $X$  and  $Y$ ,

$$-1 \leq \rho_{XY} \leq 1$$

Proof: Let

- $\rho_{XY}$  is the ratio of the sums of  $X$  and  $Y$ .

$$\rho_{XY} = 1 \quad \text{when } X \text{ and } Y \text{ are dependent}$$

$$0 < \rho_{XY} < 1$$

- Value meaning:  $\rho_{XY} = 0$  when  $X$  and  $Y$  are independent

$$0 > \rho_{XY} > -1$$

$$\rho_{XY} = -1 \quad \text{when } X \text{ and } Y \text{ are perfectly dependent}$$

**Example:** Find the correlation coefficient between  $X$  and  $Y$

		$Y$			
	$X$	0	1	2	$P_X$
$P_Y$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
	1	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
	2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
	$P_X$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1

**Solution:** Then

$$E(X^2) = P^2 \left( \frac{1}{11} \right) + P^2 \left( \frac{15}{20} \right) + P^2 \left( \frac{8}{30} \right) = \frac{25}{20}$$

and

$$E(Y^2) = P^2 \left( \frac{10}{30} \right) + P^2 \left( \frac{5}{2} \right) + P^2 \left( \frac{1}{30} \right) = \frac{1}{2}$$

so that

$$\sigma_X^2 = \frac{25}{20} - \left( \frac{11}{11} \right)^2 = \frac{9}{20} \quad \text{and} \quad \sigma_Y^2 = \frac{1}{2} - \left( \frac{1}{2} \right)^2 = \frac{1}{4}$$

Therefore, the correlation coefficient between  $X$  and  $Y$  is

$$\rho_{XY} = \frac{E(XY)}{\sigma_X \sigma_Y} = \frac{-4/20}{\sqrt{(9/20)(1/4)}} = -\frac{1}{\sqrt{3}}$$

### Example 3.68

**Solve Example 3.67. Find  $\mu_{X+Y}$ .**

**Solution**

In Example 3.67, we computed the CDF  $F_X(x)$  and PDF  $f_X(x)$  for  $X$  and  $f_Y(y)$  for  $Y$ . We now also compute  $\mu_X$  and  $\mu_Y$ . According to our the computed densities of  $X$  and of  $Y$ , which were presented in Example 3.67. Then we find that  $\mu_X = 1$  and  $\mu_Y = 1$ , and  $\mu_{X+Y} = 1 + 1 = 2$ . We obtain

$$\begin{aligned}\mu_X &= \sum_{i=1}^{\infty} x^i [h_X(x) - g_X(x)] \\ &= \sum_{i=1}^{\infty} x^i [x^i - \left(\frac{1}{2}\right)^i] \\ &= 0.6667 \\ \mu_Y &= \sum_{i=1}^{\infty} y^i [h_Y(y) - g_Y(y)] \\ &= \sum_{i=1}^{\infty} y^i [y^i - \left(\frac{1}{2}\right)^i] \\ &= 0.6667\end{aligned}$$

$$\mu_{X+Y} = \mu_X + \mu_Y = \frac{0.6667}{0.3333} + \frac{0.6667}{0.3333} = 2.000$$

### Example 3.78

**Solve Example 3.68. Find  $\mu_{X+Y}$ .**

**Solution**

In Example 3.68, we computed the CDF  $F_X(x) = 1 - (1/2)^x$ ,  $f_X(x) = (1/2)^x$ , and  $f_Y(y) = (1/2)^y$ . We now also compute  $\mu_X$  and  $\mu_Y$ . According to our the computed densities of  $X$  and of  $Y$ , which were presented in Example 3.68. We obtain

$$\begin{aligned}\mu_X &= \sum_{i=1}^{\infty} x^i [f_X(x) - g_X(x)] \\ &= \sum_{i=1}^{\infty} x^i [(1/2)^x - (1/2)^x] = 0.6667 \\ \mu_Y &= \sum_{i=1}^{\infty} y^i [f_Y(y) - g_Y(y)] \\ &= \sum_{i=1}^{\infty} y^i [(1/2)^y - (1/2)^y] = 0.6667\end{aligned}$$

It follows that

$$\mu_{X+Y} = \mu_X + \mu_Y = \frac{0.6667}{0.3333} + \frac{0.6667}{0.3333} = 2.000$$

For any random variable  $X$ ,  $\text{Cov}(X, X) = \sigma_X^2$  and  $\mu_{X+X} = 1$ .

- If  $\text{Corr}(X, Y) = \rho_{X,Y} = 0$ , then  $X$  and  $Y$  are said to be uncorrelated
- If  $X$  and  $Y$  are independent, then  $X$  and  $Y$  are uncorrelated
- It is mathematically possible for  $X$  and  $Y$  to be uncorrelated without being independent. This rarely occurs in practice.

# EXERCISES

- 4.11 Let  $X$  be a random variable with the following probability distribution:

$$\begin{array}{c|c} x & p(x) \\ \hline -1 & 1/10 \\ 0 & 3/10 \\ 1 & 4/10 \\ 2 & 2/10 \end{array}$$

Find the standard deviation of  $X$ .

- 4.12 The random variable  $X$ , representing the number of times you roll four of numbers one, has the following probability distribution:

$$\begin{array}{c|c} x & p(x) \\ \hline 0 & 1/16 \\ 1 & 1/8 \\ 2 & 3/16 \\ 3 & 1/4 \\ 4 & 3/16 \\ 5 & 1/8 \\ 6 & 1/16 \end{array}$$

Using Theorem 4.2 on page 113, find the variance of  $X$ .

- 4.13 Suppose that the probability mass function  $p(x)$  of a discrete random variable  $X$  is given by  $p(x) = \frac{1}{2^x}$  for  $x = 1, 2, 3, \dots$ . Find the variance of  $X$ .

- 4.14 A discrete random variable  $X$  has the probability mass function  $p(x)$  given by  $p(x) = \frac{1}{2^x}$  for  $x = 1, 2, 3, \dots$ . Find the variance of  $X$ .

- 4.15 The probability mass function of a discrete random variable  $X$  is given by  $p(x) = \frac{1}{2^x}$  for  $x = 1, 2, 3, \dots$ . Find the variance of  $X$ .

- 4.16 The probability mass function of a discrete random variable  $X$  is given by  $p(x) = \frac{1}{2^x}$  for  $x = 1, 2, 3, \dots$ . Find the variance of  $X$ .

- 4.17 Letting  $p(x)$  be the probability mass function of a discrete random variable  $X$ , find the variance of  $X$ .

- 4.18 Find the standard deviation of the random variable  $X$  with  $p(x) = \frac{1}{2^x}$  for  $x = 1, 2, 3, \dots$ .

- 4.19 Using the results of Exercise 4.18 on page 116, find the variance of  $p(X) = \frac{1}{2^X}$ , where  $X$  is a random variable having the density function given in Exercise 4.11 on page 113.

- 4.20 The length of time in seconds that an observer recorded observing for rainfall is a random variable  $X$  with

density function  $f(x) = \frac{1}{10}e^{-x/10}$ , where  $x$  is the number of seconds.

$$f(x) = \frac{1}{10}e^{-x/10}, \quad x \geq 0$$

Find the mean and variance of the random variable  $X$ .

- 4.21 Find the variance of the random variable  $X$  with  $f(x) = \frac{1}{10}e^{-x/10}$  on page 116.

- 4.22 Find the variance of the random variable  $X$  with  $f(x) = \frac{1}{10}e^{-x/10}$  on page 116.

- 4.23 Find the variance of the random variable  $X$  with  $f(x) = \frac{1}{10}e^{-x/10}$  on page 116.

- 4.24 Suppose the probability mass function  $p(x)$  of a discrete random variable  $X$  is given by  $p(x) = \frac{1}{2^x}$  for  $x = 1, 2, 3, \dots$ . Find the variance of  $X$ .

$$\begin{array}{c|c} x & p(x) \\ \hline 1 & 1/2 \\ 2 & 1/4 \\ 3 & 1/8 \\ 4 & 1/16 \end{array}$$

Find the variance and standard deviation of the random variable  $X$ .

- 4.25 The probability mass function of a discrete random variable  $X$  is given by  $p(x) = \frac{1}{2^x}$  for  $x = 1, 2, 3, \dots$ . Find the variance of  $X$ .

$$p(x) = \frac{1}{2^x}, \quad x = 1, 2, 3, \dots$$

Find the variance and standard deviation of  $X$ .

- 4.26 Letting  $p(x)$  be the probability mass function of a discrete random variable  $X$ , find the variance of  $X$ .

$$p(x) = \frac{1}{2^x}, \quad x = 1, 2, 3, \dots$$

Find the variance and standard deviation of  $X$ .



# Linear Combinations of Random Variables

## The Random Variable

If  $X$  and  $Y$  are random variables, then

$$E(aX + b) = aE(X) + b$$

*Proof:* By the definition of expected value,

$$E(aX + b) = \int_{-\infty}^{\infty} (ax + b)f(x) dx = a \int_{-\infty}^{\infty} xf(x) dx + b \int_{-\infty}^{\infty} f(x) dx$$

The first integral on the right is  $E(X)$  and the second integral equals 1. Therefore we have

$$E(aX + b) = aE(X) + b$$

□

Setting  $a = 1$ , we see that  $E(X) = E(X)$ .

Setting  $b = 0$ , we see that  $E(aX) = aE(X)$ .

If  $X$  is a random variable, it will not necessarily be  $E(X) = aE(X) + b$ :

$$E(X) = aE(X) + b$$

$$(1-a)E(X) = b$$

**Example:** Find  $E(X)$  if  $P(X = i) = \frac{1}{6}$  for  $i = 1, 2, 3, 4, 5, 6$ .

*Solution:*

$$\text{Now } E(X) = 1 = E(X) + 0$$

$$\begin{aligned} 0 &= E(X) = \sum_{i=1}^6 i f(i) \\ &= (1) \left( \frac{1}{6} \right) + (2) \left( \frac{1}{6} \right) + (3) \left( \frac{1}{6} \right) + (4) \left( \frac{1}{6} \right) + (5) \left( \frac{1}{6} \right) + (6) \left( \frac{1}{6} \right) = \frac{21}{6} \end{aligned}$$

Therefore,

$$0 = 1 = (2) \left( \frac{1}{6} \right) + \frac{1}{6} = E(X)$$

□

**Example:** Let  $X$  be a random variable with density function

$$f(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of  $g(X) = 4X^2 + 3$ .

**Solution:**

$$E(4X^2 + 3) = 4E(X^2) + 3$$

So

$$E(X^2) = \int_{-1}^1 x^2 \left( \frac{1}{2} \right) dx = \int_{-1}^1 \frac{x^2}{2} dx = \frac{1}{3}$$

Therefore,

$$E(4X^2 + 3) = 4 \left( \frac{1}{3} \right) + 3 = \frac{14}{3}$$

4

## Example

2.10

A telephone number is composed of seven digits and is called a heptadigit number. Let  $X$  represent the sum of the digits of a heptadigit number. Assume that the probability density function of  $X$  is

$$f(x) = \begin{cases} \frac{1}{6} & 0 \leq x \leq 100 \\ 0 & \text{otherwise} \end{cases}$$

Let  $Y = 10X^2$  represent the area of the room. Find the expected  $Y$ .

**Solution:**

$$\begin{aligned} E(Y) &= \int_0^{100} \frac{10x^2}{6} f(x) dx \\ &= \int_0^{100} \frac{10x^2}{6} \left( \frac{1}{6} \right) dx \\ &= 100 \end{aligned}$$

So the area is 100 sq ft.

The expected value of the sum or difference of two or more functions of a random variable  $X$  is the sum or difference of the expected values of the functions. That is,

$$E(g(X) + h(X)) = E(g(X)) + E(h(X))$$

**Proof:** By definition,

$$\begin{aligned} E(g(X) + h(X)) &= \int_{-\infty}^{\infty} (g(x) + h(x)) f(x) dx \\ &= \int_{-\infty}^{\infty} g(x) f(x) dx + \int_{-\infty}^{\infty} h(x) f(x) dx \\ &= E(g(X)) + E(h(X)) \end{aligned}$$

4

**Example 10.1.1** Let  $X$  be a random variable with probability distribution as follows:

$$P(X) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Find the expected value of  $Y = (X - 1)^2$ .

**Solution:** We can write

$$E(Y) = E[(X - 1)^2] = E(X^2 - 2X + 1) = E(X^2) - 2E(X) + E(1).$$

$E(1) = 1$ , and by direct summation,

$$E(X) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} + 5 \cdot \frac{1}{4} = 3 \text{ and}$$

$$E(X^2) = 1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{4} + 3^2 \cdot \frac{1}{4} + 4^2 \cdot \frac{1}{4} + 5^2 \cdot \frac{1}{4} = 11.$$

Thus,

$$E(Y) = E[(X - 1)^2] = 11 - 2(3) + 1 = 6.$$

**Example 10.1.2** The weekly demand for a certain article is denoted by  $X$ , and is known that the demand  $X$  is a continuous random variable  $p(x) = kx^2 + 2 - 3x$  where  $X$  has the density function

$$p(x) = \begin{cases} 3x^2 - 4x + 2, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the expected value of the weekly demand for the item.

**Solution:** By Theorem 10.1.1, we write

$$E(X^2 + 2 - 3X) = E(X^2) + E(2) - 3E(X).$$

$E(X) = 0$ , and by direct integration,

$$E(X) = \int_0^2 x(3x^2 - 4x + 2) dx = \frac{2}{3} \text{ and } E(X^2) = \int_0^2 x^2(3x^2 - 4x + 2) dx = \frac{11}{15}.$$

Thus,

$$E(X^2 + 2 - 3X) = \frac{11}{15} + \frac{2}{3} - 3 \cdot \frac{2}{3} = \frac{1}{5}.$$

# Two Random Variables

For two random variables  $X$  and  $Y$ , let  $Z = aX + bY + c$ , where  $a, b$ , and  $c$  are constants.

$$f(Z) = af(X) + bf(Y) + c$$

$$\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\text{Cov}(X,Y)$$

If  $X$  and  $Y$  are independent,  $\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2$ .

The expected value of the sum or difference of two random functions of the random variables  $X$  and  $Y$  is the sum or difference of the expected values of the functions. That is,

$$E[g(X,Y)] = E[g(X,Y)] + E[h(X,Y)].$$

Let  $X$  and  $Y$  be two independent random variables. Then

$$E(XY) = E(X)E(Y).$$

**Proof:** By definition,  $E(X)$

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x,y) dx dy$$

Since  $X$  and  $Y$  are independent, we may write

$$f(x,y) = f_X(x)f_Y(y),$$

where  $f_X(x)$  and  $f_Y(y)$  are the marginal distributions of  $X$  and  $Y$ , respectively. Thus,

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_X(x)f_Y(y) dx dy = \int_{-\infty}^{\infty} x f_X(x) dx \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= E(X)E(Y). \end{aligned}$$

**Example 1.14**  $X$  and  $Y$  are independent random variables with the joint density function

$$f(x, y) = \begin{cases} \frac{1}{8}(x+y), & 0 < x < 2, 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

Show that  $E(XY) = E(X)E(Y)$ .

**Solution:** By definition,

$$E(XY) = \int_0^2 \int_0^2 \frac{1}{8}(x+y) dy dx = \frac{3}{4}, \quad E(X) = \frac{1}{2}, \quad \text{and} \quad E(Y) = \frac{3}{4}$$

Thus,

$$E(X)E(Y) = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) = \frac{3}{8} = E(XY)$$

### Example 1.15

Continuing Example 1.14, it is clear that the random variable  $XY$  forms a random process  $\{X(t)\}$  uniformly in the sense  $\{X(t)\}$  just then along the  $x$ -axis in the range. Find the expected variance of the distance travelled.

**Solution:**

The distance travelled in the case  $X = 2$ . The means of  $X$  and of  $Y$  were computed in Example 1.14. Thus the  $\mu_X = 0.5$  and  $\mu_Y = 0.75$  and  $\mu_{XY} = 0.375$ . We compute

$$\begin{aligned} \sigma_{XY}^2 &= E(XY)^2 - \mu_{XY}^2 \\ &= 0.396 - 0.141 \\ &= 0.255 \end{aligned}$$

We compute  $\sigma_{XY}^2$  from Equation (1.75) of Example 1.14 as compared to  $\sigma_{XY}^2 = 0.255$ . In Example 1.14 we computed  $\sigma_X^2 = 0.0833$  and  $\sigma_Y^2 = 0.0049$ . Therefore

$$\begin{aligned} \sigma_{XY}^2 &= \sigma_X^2 + \sigma_Y^2 + 2E(XY) \\ &= 0.0833 + 0.0049 + 2(0.375) \\ &= 0.8641 \end{aligned}$$

# Example 2.39

Assume  $X$  (expressed in kWh) has a normal distribution with mean  $\mu_X = 10$  and  $\sigma_X = 2$ . Assume  $Y$  (cost of the electricity) has a normal distribution with mean  $\mu_Y = 0.15$  and  $\sigma_Y = 0.02$ . Assume  $X$  and  $Y$  are independent. Find the mean and the standard deviation of  $Z = 0.005X + 0.02Y$ .

**Solution:**

$$\begin{aligned}\mu_Z &= \mu_{0.005X + 0.02Y} \\ &= 0.005\mu_X + 0.02\mu_Y \\ &= 0.005(10) + 0.02(0.15) \\ &= 0.007\end{aligned}$$

Since  $X$  and  $Y$  are independent, we can use Equation (2.37) to find the standard deviation:

$$\begin{aligned}\sigma_Z &= \sqrt{\sigma_{0.005X + 0.02Y}^2} \\ &= \sqrt{(0.005\sigma_X)^2 + (0.02\sigma_Y)^2} \\ &= \sqrt{(0.005(2))^2 + (0.02(0.02))^2} \\ &= 0.007\end{aligned}$$

Exercises 10.2

10.1. Assuming as in Exercise 10.1(a) that  $Z$  has the same joint density as the discrete random variable  $Z = (Z_1, \dots, Z_n)$ , where  $Z_i$  represents the number of votes for  $i$ th candidate.

10.2. Using Theorems 10.1 and Theorem 10.2, find the joint and marginal of the random variable  $Z = (Z_1, \dots, Z_n)$ , where  $Z_i$  has the probability distribution of Exercise 10.1 on page 430.

10.3. Suppose that the joint density of the random variables  $X$  and  $Y$  is given by  
 $f(x, y) = 2 - x - y$ ,  $0 \leq x, y \leq 1$   
 $f(x, y) = 0$  elsewhere.

10.4. Let  $X$  and  $Y$  be a bivariate random variable with the following joint probability distribution:

$$\begin{matrix} & Y=1 & Y=2 & Y=3 & Y=4 \\ X=1 & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \\ X=2 & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ X=3 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{matrix}$$

Find the joint and marginal densities, using Theorem 10.1 and Theorem 10.2.

10.5. A bivariate random variable  $Z$  has the joint density

$$f(x, y) = \begin{cases} 2 - x - y & 0 \leq x, y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the joint and marginal densities of  $Z$  and the marginal density of  $X$  and  $Y$ .

10.6. If a random variable  $Z$  is defined such that  
 $Z = (Z_1, \dots, Z_n)$  and  $Z_i = 1$  or  $0$

find  $f(x, y, z)$ .

10.7. Suppose that  $Z$  and  $Y$  are bivariate random variables having the joint probability distribution

$$\begin{matrix} & Y=1 & Y=2 & Y=3 \\ Z=1 & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ Z=2 & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{matrix}$$

Find

(a)  $f(Z, Y)$

(b)  $f(Y)$

10.8. If  $Z$  and  $Y$  are bivariate random variables with densities  $f_Z(z)$  and  $f_Y(y)$ , find the density of the random variable  $Z = (Z_1, \dots, Z_n)$ .

10.9. Suppose that the joint density of  $Z$  and  $Y$  is given by  $f_Z(z)$  and  $f_Y(y)$ .

10.10. If the joint density function of  $Z$  and  $Y$  is given by

$$f_Z(z) = \frac{1}{2} e^{-z/2}, \quad 0 \leq z < \infty, \quad f_Y(y) = \frac{1}{2} e^{-y/2}, \quad 0 \leq y < \infty$$

find the joint density function of  $Z$  and  $Y$ .

10.11. The joint probability density function of  $Z$  and  $Y$  is given by

$$f_Z(z) = \frac{1}{2} e^{-z/2}, \quad 0 \leq z < \infty, \quad f_Y(y) = \frac{1}{2} e^{-y/2}, \quad 0 \leq y < \infty$$

(a) Find the joint density function of  $Z$  and  $Y$  using Theorem 10.1.

(b) Find the joint density function of  $Z$  and  $Y$  using Theorem 10.2.

(c) Find the joint density function of  $Z$  and  $Y$  using Theorem 10.3.

10.12. The joint density function

$$f_Z(z) = \frac{1}{2} e^{-z/2}, \quad 0 \leq z < \infty, \quad f_Y(y) = \frac{1}{2} e^{-y/2}, \quad 0 \leq y < \infty$$

find the joint density function of  $Z$  and  $Y$ .

10.13. Find the joint density function of  $Z$  and  $Y$ .

10.14. Find the joint density function of  $Z$  and  $Y$ .

10.15. The joint density function of  $Z$  and  $Y$  is given by

$$f_Z(z) = \frac{1}{2} e^{-z/2}, \quad 0 \leq z < \infty, \quad f_Y(y) = \frac{1}{2} e^{-y/2}, \quad 0 \leq y < \infty$$

(a) Find the joint density function of  $Z$  and  $Y$ .

(b) Find the joint density function of  $Z$  and  $Y$ .

10.16. Find the joint density function of  $Z$  and  $Y$  using the joint probability density function

$$f_Z(z) = \frac{1}{2} e^{-z/2}, \quad 0 \leq z < \infty, \quad f_Y(y) = \frac{1}{2} e^{-y/2}, \quad 0 \leq y < \infty$$

1.1. Questions 1-4 (40 marks)

1.1.1. Suppose that  $f(x) = \frac{1}{2}x^2$  is a function of a particle. The type of motion concerned is a constant velocity with positive direction. Suppose

$$f(x) = \frac{1}{2}x^2 \quad x \geq 0$$

otherwise.

(a) Determine the mean length  $E(X)$  of this type of molecule concerning.

(b) Find the variance and standard deviation of  $X$ .

(c) Find  $E(X^2)$ .

1.1.2. Suppose that the function  $f(x) = \frac{1}{2}x^2$  is a particle. The motion is that the particle is at rest.

$$f(x) = \frac{1}{2}x^2 \quad x \geq 0$$

otherwise.

(a) Find the mean length of this molecule.

(b) Find  $E(X^2)$ .

(c) Find the variance and standard deviation of the random variable  $X$ .

1.1.3. Suppose the given function is

$$f(x) = \frac{1}{2}x^2 \quad x \geq 0, \quad x \leq 0$$

otherwise.

Determine the variance coefficient.



# Review Examples

8.1. A car manufacturer has 2000 units of 100, 200 and 300 cc engines. The total value of the units is \$1,000,000. How many units of each engine should be produced?

Let  $x$  be the number of units of 100 cc engine,  $y$  be the number of units of 200 cc engine, and  $z$  be the number of units of 300 cc engine. The constraints are:

$$x + y + z = 2000$$

$$100x + 200y + 300z = 1,000,000$$

$$x = 2000 - y - z$$

Substituting  $x = 2000 - y - z$  into the second equation, we get:

$$100(2000 - y - z) + 200y + 300z = 1,000,000$$

Table 8.1

Equation	$x$	$y$	$z$
(1) $x + y + z = 2000$	2000	1	1
(2) $100x + 200y + 300z = 1,000,000$	2000	100	200

8.2. Find the equation of the line of best fit for the data in Table 8.2.

Let  $x$  be the number of units of 100 cc engine,  $y$  be the number of units of 200 cc engine, and  $z$  be the number of units of 300 cc engine.

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$$

Using the method of least squares, we get:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$$

8.3. A continuous random variable  $X$  has probability density given by

$$f(x) = \begin{cases} 2x - x^2 & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Find  $E(X)$ ,  $E(X^2)$ , and  $E(X^3)$ .

$$E(X) = \int_0^2 xf(x)dx = \int_0^2 x(2x - x^2)dx = \int_0^2 (2x^2 - x^3)dx$$

$$= \left[ \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_0^2 = \frac{16}{3} - \frac{4}{1} = \frac{10}{3}$$

$$E(X^2) = \int_0^2 x^2f(x)dx = \int_0^2 x^2(2x - x^2)dx$$

$$= \int_0^2 (2x^3 - x^4)dx = \left[ \frac{1}{2}x^4 - \frac{1}{5}x^5 \right]_0^2 = \frac{16}{2} - \frac{32}{5} = \frac{48}{5}$$

1.10. The joint density function of two random variables  $X$  and  $Y$  is given by

$$f(x, y) = \begin{cases} \frac{1}{2}xy & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (a)  $E(X)$ , (b)  $E(Y)$ , (c)  $E(XY)$ , (d)  $E(X^2)$ , (e)  $E(Y^2)$ .

$$(a) \quad E(X) = \int_0^1 \int_0^1 x \cdot \frac{1}{2}xy \, dy \, dx = \int_0^1 \int_0^1 x^2 \left(\frac{y}{2}\right) dy \, dx = \frac{1}{6}$$

$$(b) \quad E(Y) = \int_0^1 \int_0^1 y \cdot \frac{1}{2}xy \, dy \, dx = \int_0^1 \int_0^1 y^2 \left(\frac{x}{2}\right) dx \, dy = \frac{1}{6}$$

$$(c) \quad E(XY) = \int_0^1 \int_0^1 xy \cdot \frac{1}{2}xy \, dy \, dx = \int_0^1 \int_0^1 x^2 y^2 \left(\frac{1}{2}\right) dy \, dx = \frac{1}{24}$$

$$(d) \quad E(X^2) = \int_0^1 \int_0^1 x^2 \cdot \frac{1}{2}xy \, dy \, dx = \int_0^1 \int_0^1 x^3 \left(\frac{y}{2}\right) dy \, dx = \frac{1}{12}$$

**Another method**

(1) Since  $X$  and  $Y$  are independent, we can, using part (a) and (b),

$$E(XY) = E(X)E(Y) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$$

(2) By Theorem 1.4 and 1.5, pages 36–37, together with (a) and (b),

$$E(X^2) = E(Y) + 2E(XY) + E(X) = 2\left(\frac{1}{6}\right) + \frac{1}{36} = \frac{1}{4}$$

# **Variance and standard deviation**

1.11. Find (a) the variance, (b) the standard deviation of the sum obtained in rolling eight of the dice.

(a) Referring to Problem 1.1, we have  $\phi(x) = \phi(2) = 1/12$ . Hence,

$$E(X^2) = E(X) + 2\left(\frac{1}{12}\right) + 2\left(\frac{1}{12}\right) + \cdots + 2\left(\frac{1}{12}\right) + \frac{1}{12}$$

Thus, by Theorem 1.4,

$$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{12} + \left(\frac{1}{12}\right)^2 = \frac{1}{12}$$

and, since  $X$  and  $Y$  are independent, Theorem 1.5 gives

$$V(X + Y) = V(X) + V(Y) = \frac{1}{6}$$

$$(b) \quad \sigma_{X+Y} = \sqrt{V(X+Y)} = \sqrt{\frac{1}{6}}$$

2.6. Find the volume of the solid generated by revolving the curve  $y = \sin x$  about the  $y$ -axis.

2.7. A solid is formed by revolving the curve  $y = \sin x$  about the  $y$ -axis. Find the volume.

$$\begin{aligned} V &= \pi \int_0^{\pi} (\sin x)^2 dx = \pi \int_0^{\pi} \frac{1 - \cos 2x}{2} dx \\ &= \frac{\pi}{2} \left[ x - \frac{\sin 2x}{2} \right]_0^{\pi} = \frac{\pi}{2} \left[ \pi - 0 \right] = \frac{\pi^2}{2} \end{aligned}$$

Answer verified

by Wolfram|Alpha

$$V = \pi \int_0^{\pi} (\sin x)^2 dx = \pi \int_0^{\pi} \frac{1 - \cos 2x}{2} dx = \frac{\pi}{2} \left[ x - \frac{\sin 2x}{2} \right]_0^{\pi} = \frac{\pi^2}{2}$$

2.8. Find the area of the region bounded by the curve  $y = \sin x$  and the  $y$ -axis.

2.9. Find the first four moments about the origin for a continuous random variable having density function

$$f(x) = \begin{cases} 2(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_1 = E(X) = \int_0^1 x \cdot 2(1-x) dx = \frac{1}{3}$$

$$\mu_2 = E(X^2) = \int_0^1 x^2 \cdot 2(1-x) dx = \frac{1}{6}$$

$$\mu_3 = E(X^3) = \int_0^1 x^3 \cdot 2(1-x) dx = \frac{1}{12}$$

$$\mu_4 = E(X^4) = \int_0^1 x^4 \cdot 2(1-x) dx = \frac{1}{30}$$

2.22. Find and simplify the ECF, the PDF, the CDF and the PDF of  $XY$  for a bivariate PDF, the CDF of  $Z$ , in terms of the random variables  $X$  and  $Y$  as defined in Problem 2.21.

$$\begin{aligned} \text{(a)} \quad E(X) &= \sum_i \sum_j x_{ij} p_{ij} = \sum_i \left[ \sum_j x_{ij} p_{ij} \right] \\ &= 0.0000 + 0.0000 + 0.0400 + 0.00 = \frac{2}{25} = \frac{4}{50} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad E(Y) &= \sum_i \sum_j y_{ij} p_{ij} = \sum_j \left[ \sum_i y_{ij} p_{ij} \right] \\ &= 0.0000 + 0.0000 + 0.0400 + 0.0000 + 0.00 = \frac{2}{25} = \frac{4}{50} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad E(XY) &= \sum_i \sum_j x_{ij} y_{ij} p_{ij} \\ &= 0.0000 + 0.0000 + 0.0400 + 0.0000 \\ &\quad + 0.0000 + 0.0000 + 0.0000 + 0.0000 \\ &\quad + 0.0000 + 0.0000 + 0.0000 + 0.0000 \\ &= 0.04 = \frac{2}{25} = \frac{4}{50} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \text{var}(X) &= \sum_i \sum_j x_{ij}^2 p_{ij} - \left[ \sum_i \sum_j x_{ij} p_{ij} \right]^2 \\ &= 0.0000 + 0.0000 + 0.0400 + 0.00 = \frac{2}{25} = \frac{4}{50} \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad \text{var}(Y) &= \sum_i \sum_j y_{ij}^2 p_{ij} - \left[ \sum_i \sum_j y_{ij} p_{ij} \right]^2 \\ &= 0.0000 + 0.0000 + 0.0400 + 0.0000 + 0.00 = \frac{2}{25} = \frac{4}{50} \end{aligned}$$

$$\text{(f)} \quad \text{cov}(X) = \text{var}(X) = \text{var}(Y) = \frac{4}{50} = \left( \frac{2}{5} \right)^2 = \frac{4}{25}$$

$$\text{(g)} \quad \text{cov}(Y) = \text{var}(Y) = \text{var}(X) = \frac{4}{50} = \left( \frac{2}{5} \right)^2 = \frac{4}{25}$$

$$\text{(h)} \quad \text{cov}(X, Y) = \text{cov}(Y, X) = \frac{4}{50} = \left( \frac{2}{5} \right) \left( \frac{2}{5} \right) = \frac{4}{25}$$

$$\text{(i)} \quad \rho = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X) \text{var}(Y)}} = \frac{\frac{4}{25}}{\sqrt{\frac{4}{25} \frac{4}{25}}} = \frac{\frac{4}{25}}{\frac{4}{25}} = 1.0000 \text{ (perfect)}$$

2.23. Write Problems 2.21 if the random variables  $X$  and  $Y$  as defined in Problem 2.21.

Using  $\rho = 1.0000$ , we have

$$\text{(a)} \quad \text{cov}(X) = \frac{1}{25} \int_{-1}^1 \int_{-1}^1 xy dx dy = \frac{1}{25} \int_{-1}^1 y^2 dy = \frac{2}{75}$$

$$\text{(b)} \quad \text{cov}(Y) = \frac{1}{25} \int_{-1}^1 \int_{-1}^1 xy dx dy = \frac{1}{25} \int_{-1}^1 x^2 dx = \frac{2}{75}$$

$$\text{(c)} \quad \text{cov}(X) = \frac{1}{25} \int_{-1}^1 \int_{-1}^1 xy dx dy = \frac{1}{25} \int_{-1}^1 y^2 dy = \frac{2}{75}$$



## The Percentiles of a Population

Let  $0 < p < 100$ . The  $p^{\text{th}}$  percentile of a population is the value  $x_p$  such that  $p\%$  of the population values are less than or equal to  $x_p$ . Thus if  $X$  is a continuous random variable with probability density function  $f(x)$ , the  $p^{\text{th}}$  percentile of  $X$  is the point  $x_p$  that solves the equation

$$P(X \leq x_p) = \int_{-\infty}^{x_p} f(x) dx = \frac{p}{100}$$



FIGURE 10.1 (a) Graph of a continuous random variable's density function,  $f(x)$ . (b) The probability that a continuous random variable is less than the  $p^{\text{th}}$  percentile.

## The Median of a Population

The median is a special case of a percentile. It is the 50th percentile.

Let  $X$  be a continuous random variable with probability density function  $f(x)$  and cumulative distribution function  $F(x)$ .

- The median of  $X$  is the point  $x_m$  that solves the equation  $P(X \leq x_m) = P(X \geq x_m) = \int_{-\infty}^{x_m} f(x) dx = 0.5$ .

## The Mode of a Population

The mode of a discrete random variable is the value which occurs most often or, in other words, has the greatest probability of occurring. The mode of a continuous random variable  $X$  is the value (or values) of  $X$  where the probability density function has a relative maximum.



# Example 1.37

A certain random variable has a probability density function given by: The first moment is known, in general, to be finite, with probability density function

$$f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Find the random variable's moments. Find the 95th percentile of the distribution.

**Solution**

The moment  $\mu_1$  is the solution to  $\int_{-\infty}^{\infty} x f(x) dx = \mu_1$ . We therefore have that

$$\begin{aligned} \int_{-\infty}^{\infty} x f(x) dx &= \mu_1 \\ \int_0^{\infty} x f(x) dx &= \mu_1 \\ \int_0^{\infty} x 2e^{-2x} dx &= \mu_1 \\ \int_0^{\infty} 2xe^{-2x} dx &= \mu_1 \\ -x e^{-2x} + \frac{1}{2} e^{-2x} &= \mu_1 \\ -x e^{-2x} + \frac{1}{2} e^{-2x} &= \mu_1 \end{aligned}$$

that if the above probability expression is the limit of  $f(x)$ , and both are positive.

Let  $\mu_1$  generally,  $\mu_2$  to be the solution to  $\int_{-\infty}^{\infty} x^2 f(x) dx = \mu_2$ . We proceed as before, substituting  $\mu_2$  for  $\mu_1$ , and let  $\mu_2 = \mu_2$ . We obtain

$$\begin{aligned} \int_{-\infty}^{\infty} x^2 f(x) dx &= \mu_2 \\ \int_0^{\infty} x^2 f(x) dx &= \mu_2 \\ \int_0^{\infty} x^2 2e^{-2x} dx &= \mu_2 \\ -x^2 e^{-2x} + 2x e^{-2x} - e^{-2x} &= \mu_2 \\ -x^2 e^{-2x} + 2x e^{-2x} - e^{-2x} &= \mu_2 \\ \mu_2 &= 0.50 \end{aligned}$$

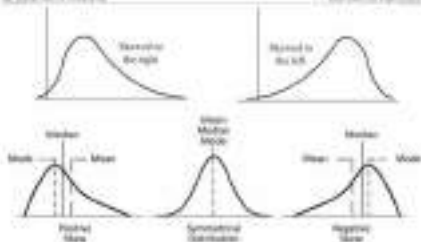
Each aspect of the above moment calculation, we have that  $\mu_1 = 0.5$  and  $\mu_2 = 0.50$ .

## Observations

When a distribution is not symmetric about any value, but instead has one of its tails longer than the other. If the longer tail occurs to the right, the distribution is said to be skewed to the right, while if the longer tail occurs to the left, it is said to be skewed to the left. Skewness describes how far a frequency distribution departs from being symmetric.

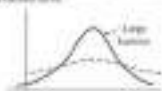
$$\alpha_1 = \frac{E(X - \mu)^3}{\sigma^3} = \frac{\mu_3}{\sigma^3}$$

The moment  $\alpha_1$  will be positive or negative depending on what far the distribution is skewing to the right or left, respectively. For a symmetric distribution,  $\alpha_1 = 0$ .



## KURTOSIS

Some times a distribution may have its values concentrated near the mean so that the distribution has a large peak or followed by the wild nature of the figure below. In other cases, the distribution may be relatively flat across the charted area.



Measurements of the degree of peak around the mean of a distribution are called coefficients of kurtosis or simply kurtosis.

$$k = \frac{E[(X - \mu)^4]}{\sigma^4} = \frac{\mu_4}{\sigma^4}$$

This is usually compared with the normal curve which has a coefficient of kurtosis equal to 3.





**Example:** The density function of a continuous random variable  $X$  is

$$f(x) = \begin{cases} 4x(2-x)^2/30 & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the mean; (b) Find the median; (c) Compute mode, median, and mean.

**Solution:**

(a) The mode is obtained by finding where the density (if it has a relative maximum). The relative maximum of (a) is found using the derivative as follows:

$$\frac{d}{dx} \left[ \frac{4x(2-x)^2}{30} \right] = \frac{32}{15}(2-x) = 0$$

Thus  $x = \sqrt{2} \approx 1.41$  appears, which is the required mode. Therefore the mode gives the maximum value for the density function,  $f(x)$ , at  $x = \sqrt{2}$ .

(b) The median is that value  $x$  for which  $F(x) = .5$  (i.e., find the  $x$  for  $F(x) = .5$ ).

$$F(x) = .5 = \frac{4}{30} \int_0^x (2t - t^2) dt = \frac{4}{30} \left( 2x - \frac{t^3}{3} \right)$$

Setting this equal to .52, we find that

$$3x^2 - 3x + 2 = 0$$

One of the

$$x = \frac{3 \pm \sqrt{3(4) - 4(2)(2)}}{2(3)} = \frac{3 \pm \sqrt{3(4) - 4(2)(2)}}{2(3)}$$

Therefore, the required median, which must be between 0 and 2, is given by

$$x = 1 + \frac{1}{3}\sqrt{3}$$

One of the  $x = 1 + \frac{1}{3}\sqrt{3}$  appears.

$$(c) \quad F(x) = \frac{4}{30} \int_0^x (2t - t^2) dt = \frac{4}{30} \left( 2x - \frac{t^3}{3} \right) = .50$$

which is algebraically solved for the median. The mode, median, and mean are shown in Fig. 1-4.

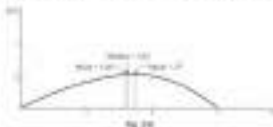


Fig. 1-4

### Example

Examine the 1st, 10th, 50, 150, and 750 percentile values for the distribution of the example above.

### Solution

From the example, we have

$$f(x) = c \cdot \left( \frac{10^x}{1} + \frac{10^x}{2} \right) = \frac{15c}{2} \cdot 10^x$$

(a) The 1st percentile is the value of  $x$  for which  $F(x) = 0.01$ . In the continuous case,  $F(x) = \int_0^x f(t) dt = 0.01$ . Using the method of Problem 3.18, we find  $x = 1.000$  approx.

(b) The 10th percentile is the value of  $x$  such that  $F(x) = 0.10$ , and we find  $x = 1.000$  approx.

(c) The 50th percentile is the value of  $x$  such that  $F(x) = 0.50$ , and we find  $x = 1.000$  approx.

### Homework

For each of the density functions of a continuous random variable  $X$  shown below, find (a) the mode, (b) the median, and (c) the mean.

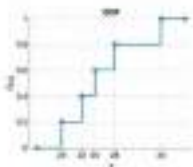
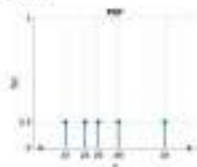
$$11. \quad f(x) = \begin{cases} \frac{125}{114} (7 - x^2) & -4 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$12. \quad f(x) = \begin{cases} (6\sqrt{3} + 6\sqrt{2} - 1) \sqrt{2} - 1 & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$13. \quad f(x) = \begin{cases} (1982 + \cos(x - \sqrt{2})) & -\pi \leq x \leq \pi \\ 0 & \text{otherwise} \end{cases}$$



Example: the discrete random variable  $X$  is uniformly distributed on  $\{1, 2, 3, 4, 5\}$ . Plot PMF and CDF.



### Continuous Uniform Distribution

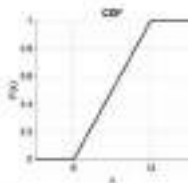
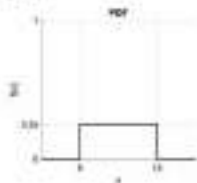
It is also called the rectangular distribution. It has constant probability on the interval  $[a, b]$ .

$$f(x) = \frac{1}{b-a} \quad a \leq x \leq b$$

$$f(x) = 0 \quad \text{otherwise}$$

$$F(x) = \int_{-\infty}^x f(t) dt = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

Example: the continuous random variable  $X$  is uniformly distributed on the range  $[0, 12]$ . Plot PMF and CDF.



$$E(X) = \int_0^1 \frac{1}{x} \cdot 4x \, dx = \frac{4}{2} \ln x \Big|_0^1 = \frac{4 \cdot 1}{2} = 2$$

$$E(X^2) = \int_0^1 x^2 \cdot 4x \, dx = \frac{4}{4} x^3 \Big|_0^1 = 1$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 1 - 2^2 = -3$$

**Example:** Suppose that a large conference room is a rectangular room that is divided into 10 rooms, each 10 feet long and 10 feet wide. Suppose that the room is divided into 10 rooms, each 10 feet long and 10 feet wide. Suppose that the room is divided into 10 rooms, each 10 feet long and 10 feet wide.

- What is the probability density function?
- What is the probability that any given conference room is at least 10 feet long?

**Solution:** (a) The appropriate density function for the probability distribution random variable  $X$  is the function

$$f(x) = \begin{cases} \frac{1}{10} & 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

$$(b) P(X \geq 10) = \int_{10}^{\infty} f(x) \, dx = 0$$

0

## The Bernoulli Discrete Distribution

Imagine an experiment that can result in one of two outcomes: success or failure. The probability of success is denoted by  $p$ . The probability of failure is denoted by  $1 - p$ . This is called a Bernoulli trial with success probability  $p$ . The number of successes in  $n$  trials is denoted by  $X$ . The number of successes in  $n$  trials is denoted by  $X$ . The number of successes in  $n$  trials is denoted by  $X$ .

For any Bernoulli trial, we define a random variable  $X$  as follows: if the experiment results in success, then  $X = 1$ ; otherwise  $X = 0$ . It follows that  $X$  is a discrete random variable, with probability mass function  $p(x)$  defined by:

$$p(0) = P(X = 0) = 1 - p$$

$$p(1) = P(X = 1) = p$$

$$p(x) = 0 \text{ for any value of } x \text{ other than } 0 \text{ or } 1$$



**FIGURE 11.1** (a) The distribution in problem 1. (b) The distribution in problem 2.

### Example 11.1

11.1

A coin is tossed twice. Let  $X$  be the number of heads. What is the distribution of  $X$ ?

**Solution**

Since  $X = 0$  when both coins are tails,  $X = 1$  when exactly one coin is heads, and  $X = 2$  when both coins are heads, the distribution of  $X$  is as follows:  $P(X = 0) = 1/4$ ,  $P(X = 1) = 1/2$ , and  $P(X = 2) = 1/4$ .

### Example 11.2

11.2

A coin is tossed twice. Let  $X$  be the number of heads. What is the distribution of  $X$ ?

**Solution**

The distribution of  $X$  is as follows:  $P(X = 0) = 1/4$ ,  $P(X = 1) = 1/2$ , and  $P(X = 2) = 1/4$ .

### Example 11.3

11.3

A coin is tossed twice. Let  $X$  be the number of heads. What is the distribution of  $X$ ?

**Solution**

The distribution of  $X$  is as follows:  $P(X = 0) = 1/4$ ,  $P(X = 1) = 1/2$ , and  $P(X = 2) = 1/4$ .

## Mean and Variance of a Bernoulli Random Variable

**11.4** — Bernoulli:  $X$  is a Bernoulli random variable.

$$P(X = 1) = p \quad (11.4.1)$$

$$P(X = 0) = 1 - p \quad (11.4.2)$$



Write a function `binomial(n, p, k)` such that

**Inputs:**

`n` is a non-negative integer, the number of trials; `p` is a number in  $[0, 1]$  (and `1-p` is a number in  $[0, 1]$ ); `k` is a non-negative integer

## The Binomial Discrete Distribution

If a total of  $n$  Bernoulli trials are conducted, and

- The trials are independent
- Each trial has the same success probability  $p$
- $X$  is the number of successes in  $n$  trials

then  $X$  has the binomial distribution with parameters  $n$  and  $p$ , denoted  $X \sim \text{Binomial}(n, p)$ .

## Probability Mass Function of a Binomial Random Variable

A Bernoulli trial succeeds or fails with probability  $p$  and a failure with probability  $q = 1 - p$ . Thus the probability mass function of the binomial random variable  $X$ , the number of successes in  $n$  independent trials, is

$$p(x) = P(X = x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2, \dots, n$$

$$p(x) = \sum_{i=0}^n \binom{n}{i} p^i q^{n-i}$$

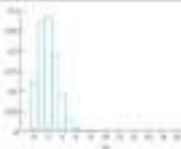
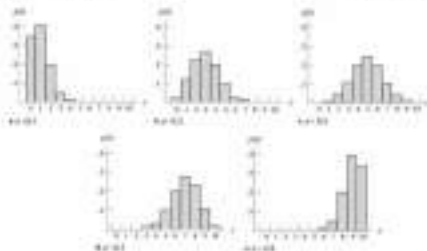


FIGURE 8.2 The Binomial PMF for  $n=10$  and  $p=0.5$  (left) and  $p=0.8$  (right).



**FIGURE 4.3**  
 Probability distributions of the sum of  $n$  independent Bernoulli trials, where  $n = 1, 2, 3, 4, 5$ .

(Source: Adapted from: Probability and Statistics for Engineers and Scientists, 9th ed., 2013.)



**Example 4.3** Find the probability mass function of the random variable  $X$  if  $X \sim \text{Binomial}(n, p)$ , where  $n = 10$ .

**Solution**

We are given that  $n = 10$  and  $p = 0.1$ . The probability mass function is

$$p(x) = \begin{cases} \frac{10!}{x!(10-x)!} (0.1)^x (0.9)^{10-x} & x = 0, 1, \dots, 10 \\ 0 & \text{otherwise} \end{cases}$$

$$p(4) = \frac{10!}{4!(10-4)!} (0.1)^4 (0.9)^{10-4} = 0.0073$$





A lot of binomial experiments. Find the probability that in some binomial experiments the number of successes is a binomial distribution and some binomial experiments the number of successes is a binomial distribution.

Suppose that the number of successes in a binomial experiment is a binomial distribution. Then the number of successes in a binomial experiment is a binomial distribution. We want to find  $P(X = k)$  for  $k = 0, 1, \dots, n$ .

$$\begin{aligned} P(X = k) &= \sum_{i=0}^k P(X = i) + \sum_{i=k+1}^n P(X = i) \\ &= \sum_{i=0}^k \binom{n}{i} p^i (1-p)^{n-i} + \sum_{i=k+1}^n \binom{n}{i} p^i (1-p)^{n-i} \\ &= \sum_{i=0}^k \binom{n}{i} p^i (1-p)^{n-i} + \sum_{i=k+1}^n \binom{n}{i} p^i (1-p)^{n-i} \\ &= 0.250 + 0.750 = 1.000 \\ &= 1.000 \end{aligned}$$

### Where Does the Name Binomial Come From?

The binomial distribution derives its name from the fact that the  $n+1$  terms in the binomial expansion of  $(p+q)^n$  correspond to the values of  $P(X=k)$  for  $k=0, 1, 2, \dots, n$ . That is,

$$\begin{aligned} (p+q)^n &= \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \dots + \binom{n}{n} p^n q^0 \\ &= P(X=0) + P(X=1) + P(X=2) + \dots + P(X=n) \end{aligned}$$

Since  $p+q=1$ , we see that

$$\sum_{k=0}^n P(X=k) = 1$$

a condition that must hold for any probability distribution.

Traditionally, we are interested in problems that are binomial, so that  $P(X=k)$  is  $\binom{n}{k} p^k (1-p)^{n-k}$ .

$$P(X=k) = \sum_{i=0}^n \binom{n}{i} p^i (1-p)^{n-i}$$

**Example 10.** The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are found to have contracted this disease, what is the probability that (a) at least 10 recover, (b) none of them recover, and (c) exactly 5 recover?

**Solution:** Let  $X$  be the number of people who recover.

$$(a) \quad P(X \geq 10) = 1 - P(X \leq 5) = 1 - \sum_{k=0}^5 \binom{15}{k} (0.4)^k (0.6)^{15-k} \\ = 0.9899$$

$$(b) \quad P(X = 0) = 0.6^{15} = \sum_{k=0}^0 \binom{15}{k} (0.4)^k (0.6)^{15-k} = \sum_{k=0}^0 \binom{15}{k} (0.4)^k (0.6)^{15-k} \\ = 0.0005 \approx 0.0005 = 0.05\%$$

$$(c) \quad P(X = 5) = \binom{15}{5} (0.4)^5 (0.6)^{10} = \sum_{k=5}^5 \binom{15}{k} (0.4)^k (0.6)^{15-k} = \sum_{k=5}^5 \binom{15}{k} (0.4)^k (0.6)^{15-k} \\ = 0.0007 = 0.07\% \approx 0.07\%$$

**Example 11.** A large chain retailer purchases a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective rate of the device is 0.5%.

(a) The supplier randomly picks 20 items from a shipment. What is the probability that there will be at least one defective item among these 20?

(b) Suppose that the retailer receives 20 shipments in a month and the supplier randomly sends 20 devices per shipment. What is the probability that there will be exactly 10 shipments each containing at least one defective device among the 20 that are received and tested from the shipment?

**Solution:** (a) Let  $X$  be the number of defective devices among the 20. Then  $X$  follows a  $N(20, 0.005)$  distribution. Thus

$$P(X \geq 1) = 1 - P(X = 0) = 1 - (0.995)^{20} \\ = 1 - (0.995)^{20} \approx 0.0976 \approx 9.76\%$$

(b) If, likewise, each shipment can either contain at least one defective item or not. Then, testing of each shipment can be treated as a Bernoulli trial with  $p = 0.0976$  from part (a), assuming before-hand from shipment to shipment and denoting by  $Y$  the number of shipments containing at least one defective item,  $Y$  follows a binomial distribution  $B(20, 0.0976)$ . Therefore

$$P(Y = 10) = \binom{20}{10} (0.0976)^{10} (0.9024)^{10} \approx 0.1061$$

**Question 10** It is suggested that an inspection system is 99% of all failing wells is a viable cost-effective. The next to question might take the true extent of the problem. It is determined that some testing is necessary. It is cost expensive to test all of the wells in the area, so 10 are randomly selected for testing.

(a) Using the binomial distribution, what is the probability that exactly 3 wells have the property, assuming that the proportion is correct?

(b) What is the probability that more than 3 wells are damaged?

**Solution:** (a) The number

$$X \sim \text{Bin}(10, 0.03) \Rightarrow \sum_{k=0}^3 \binom{10}{k} (0.03)^k (0.97)^{10-k} = 0.377061 + 0.377061 + 0.268242$$

(b) In this case,  $P(X \geq 4) = 1 - 0.999939 = 0.000061$

4

### The Mean and Variance of a Binomial Random Variable

If  $X \sim \text{Bin}(n, p)$ , then the mean and variance of  $X$  are given by

$$\mu_X = np \quad (4.3)$$

$$\sigma_X^2 = np(1-p) \quad (4.4)$$

<https://www.cambridge.org/9781107025816/9781107025816.008>

<https://www.cambridge.org/9781107025816/9781107025816.008>



# Review Examples

- Q1. Find the acceleration that is needed to accelerate from 0 to 100 km/h in 10 s and 1 hour.  
 (Note that 1 hour is 3600 seconds, not 1 s.)

## Solution 1

Let  $u$  denote speed and  $t$  denote time, and suppose that acceleration  $a(t)$  has a constant value  $a$  for the whole acceleration period from 0 to 100 km/h.

Then, speed after time  $t$  is  $at$  because  $a$  is constant, then we have  $100 \text{ km/h} = 100000 \text{ m/hour}$  in the complete acceleration period. Then we

$$\text{speed after } t \text{ hours is } at, \text{ for } 0 \leq t \leq 100$$

We could only have acceleration equal to 100 km/h, therefore

$$\text{at } 100 \text{ hours} = 100000 \Rightarrow a = \frac{1}{100}$$

$$\text{at } 100 \text{ min and } 1 \text{ hour} = 100000 \text{ m/hour} \Rightarrow 100$$

$$\Rightarrow 100000 = at = 100000a \Rightarrow a = \frac{1}{100} = \frac{1}{60} + \frac{1}{60} = \frac{1}{30}$$

$$\text{at } 100 \text{ km/hour}$$

$$\Rightarrow 100000 = at \text{ hours}$$

$$\Rightarrow 100000 = 100000a \Rightarrow a = 100000$$

$$\Rightarrow 100000 = at = 100000a \Rightarrow 100000 = 100000a \Rightarrow a = 100000$$

$$\Rightarrow 100000 = 100000a \Rightarrow 100000 = 100000a \Rightarrow 100000 = 100000a \Rightarrow a = 100000$$

Therefore

$$\text{The time } 100 \text{ km/h} = 100000 = 100000a \Rightarrow a = \frac{1}{100} = \frac{1}{60} + \frac{1}{60}$$

$$\text{at } 100 \text{ km/hour} = 100000 = 100000a \Rightarrow a = 100000$$

$$\Rightarrow 100000 = 100000a \Rightarrow a = 100000$$

$$\Rightarrow 100000 = 100000a \Rightarrow 100000 = 100000a \Rightarrow a = 100000$$

$$\Rightarrow 100000 = 100000a \Rightarrow 100000 = 100000a \Rightarrow a = 100000$$

$$\Rightarrow \frac{1}{100} = \frac{1}{60} + \frac{1}{60}$$

## Solution 2 using Newton's

$$\text{at } 100 \text{ km/hour} = \frac{100000}{100} = 1000$$

$$\text{at } 100 \text{ min and } 1 \text{ hour} = \frac{100000}{100} = 1000$$

$$\text{at } 100 \text{ km/hour} = 100000 = 100000a \Rightarrow a = 100000$$

$$\Rightarrow 100000 = 100000a \Rightarrow 100000 = 100000a \Rightarrow a = 100000$$

$$\Rightarrow 100000 = 100000a \Rightarrow 100000 = 100000a \Rightarrow a = 100000$$

Therefore

$$\text{The time } 100 \text{ km/h} = 100000 = 100000a \Rightarrow a = 100000$$

$$\Rightarrow 100000 = 100000a \Rightarrow a = 100000$$



- 4.5. Consider probability that in a family of  $n$  children there will be at least one boy (B) and at least one girl (G) and at least 1 girl, assume that the probability of a child being a boy is  $\frac{1}{2}$ .

$$\text{Let } P(B \text{ boy}) = \overbrace{(\frac{1}{2})^n}^{n \text{ children}} = \frac{1}{2^n} \quad \text{and } P(B \text{ boy}) = \overbrace{(\frac{1}{2})^n}^{n \text{ children}} = \frac{1}{2^n}$$

$$P(B \text{ boy}) = \overbrace{(\frac{1}{2})^n}^{n \text{ children}} = \frac{1}{2^n} \quad \text{and } P(B \text{ boy}) = \overbrace{(\frac{1}{2})^n}^{n \text{ children}} = \frac{1}{2^n}$$

Then

$$\begin{aligned} P(\text{at least 1 boy}) &= P(B \text{ boy}) + P(B \text{ boy}) + P(B \text{ boy}) + \dots + P(B \text{ boy}) \\ &= \frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{2^n} + \dots + \frac{1}{2^n} \end{aligned}$$

Another method

$$P(\text{at least 1 boy}) = 1 - P(\text{no boy}) = 1 - \overbrace{(\frac{1}{2})^n}^{n \text{ children}} = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

$$\text{Let } P(\text{at least 1 boy and 1 girl}) = 1 - P(\text{no boy}) - P(\text{no girl})$$

$$= 1 - \frac{1}{2^n} - \frac{1}{2^n} = \frac{2^n - 2}{2^n}$$

We could describe a similar problem by finding if for a random variable describing the number of boys in families with  $n$  children, then, for example, we have

$$P(X = 0) = P(X = 1) = P(X = 2) = \dots = P(X = n) = \frac{1}{2^n} \quad \text{and } P(X = 0) = \frac{1}{2^n}$$

- 4.6. One of 1000 families with 4 children each, how many would you expect to have (a) at least 1 boy and 1 girl or (b) at least 1 girl or (c) at least 1 boy?

Following definition 4.1, we get

$$\text{a) Expected number of families with at least 1 boy} = 1000 \left( \frac{2^n - 1}{2^n} \right) = 1000$$

$$\text{b) Expected number of families with at least 1 girl} = 1000 \left( \frac{2^n - 1}{2^n} \right) = 1000$$

$$\text{c) Let } P(\text{at least 1 boy}) = P(B \text{ boy}) + P(B \text{ boy})$$

$$= P(B \text{ boy}) + P(B \text{ boy}) = \frac{1}{2^n} + \frac{1}{2^n} = \frac{2}{2^n}$$

$$\text{Expected number of families with at least 1 girl} = 1000 \left( \frac{2}{2^n} \right) = 1000$$

$$\text{d) Expected number of families with at least 1 boy} = 1000 \left( \frac{2}{2^n} \right) = 1000$$





## The Normal Continuous Distribution

The normal distribution (also called the Gaussian distribution) is by far the most commonly used distribution in statistics. The distribution provides a good model for many, although not all, continuous populations.

The normal distribution is continuous rather than discrete. The mean of a normal random variable may have any value, and the variance may have any positive value. The probability density function of a normal random variable with mean  $\mu$  and variance  $\sigma^2$  is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad -\infty < x < \infty$$

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) dt$$

If  $Z \sim N(\mu, \sigma^2)$ , then the mean and variance of  $Z$  are given by

$$\begin{aligned} \mu_Z &= \mu \\ \sigma_Z^2 &= \sigma^2 \end{aligned}$$

Figure 3.4 presents a plot of the normal probability density function with mean  $\mu$  and standard deviation  $\sigma$ . The normal probability density function is sometimes called the normal curve, from the fact the normal curve is symmetric around  $\mu$ , so that  $\mu$  is the median as well as the mean. It is also the case that for any normal population:

- About 68% of the population is in the interval  $\mu \pm \sigma$ .
- About 95% of the population is in the interval  $\mu \pm 1.96\sigma$ .
- About 99.7% of the population is in the interval  $\mu \pm 2.58\sigma$ .

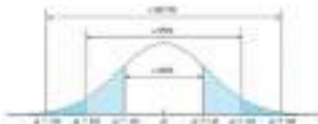
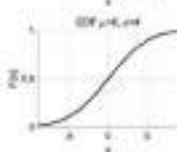
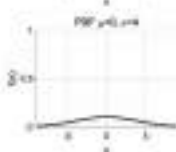
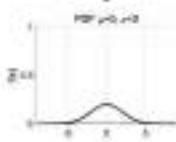
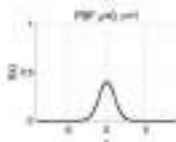
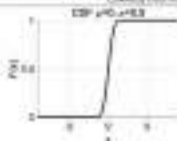
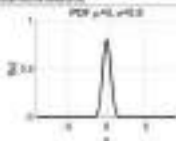
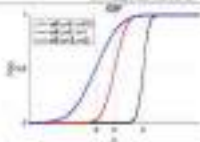
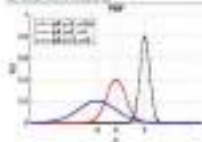


FIGURE 3.4 Probability density function of a normal random variable with mean  $\mu$  and standard deviation  $\sigma$ .





The following are some important properties of the normal curve:

1. The mode, which is the point on the horizontal axis where the curve is a maximum, occurs at  $x = \mu$ .
2. The curve is symmetric about a vertical axis through the mean  $\mu$ .
3. The area under the curve is divided into  $2 = 2$  parts by the vertical line through  $\mu$ ,  $-\infty < x < \mu < \infty$  and  $x > \mu$  and is equally spread between them.
4. The normal curve approaches the horizontal axis asymptotically as we proceed in either direction away from the mean.
5. The total area under the curve will cover the horizontal axis (equal to 1).

### Area under the Normal Curve

The curve of any continuous probability distribution is density function in one dimension so that the area under the curve bounded by the two ordinates  $x = x_1$  and  $x = x_2$  equals the probability that the random variable  $X$  assumes a value between  $x = x_1$  and  $x = x_2$ . Thus, for the normal curve in Figure 10.6,

$$P(x_1 < X < x_2) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{x_1}^{x_2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

is represented by the area of the shaded region.



exactly under the eye of a good statistician  
is somewhat different.

Suppose that the density  
function is

The area under the curve between any two values  $x_1$  and  $x_2$  depends on the values  $\mu$  and  $\sigma$ . This is evident in the following figure, where two lines shaded regions corresponding to  $P(x_1 < X < x_2)$  for two curves with different means and variances. Obviously, the two shaded regions are different in size; therefore, the probability associated with such distribution will be different for the two given values of  $X$ .



The difficulty associated in using integrals of normal density functions necessitates the tabulation of area under curves under the graph calculator. However, it would be tedious and cumbersome to set up separate tables for every conceivable value of  $\mu$  and  $\sigma$ .

Important: we convert  $x$  to the standard score  $z$  as

$$z = \frac{x - \mu}{\sigma}$$

Thus, if  $x$  is an item sampled from a normal population with mean  $\mu$  and variance  $\sigma^2$ , the standard unit equivalent of  $x$  is the number  $z$ . The number  $z$  is an item sampled from a normal population with mean 0 and standard deviation 1. This normal population is called the standard normal population.

## Example

4.3.3

Electricity bills sent to India's large cities have the known to thousands of us which are normally distributed with mean 10 and standard deviation 1.3. A particular bill is 15.5 thousands of rupees which find the value.

**Solution:**

Recognizing 15.5 to be an observation from a normal population with mean  $\mu = 10$  and standard deviation  $\sigma = 1.3$ . Therefore

$$z = \frac{15.5 - 10}{1.3} = 4.23$$

### Example 4.20

Refer to Example 4.19. The thickness of a metal sheet has a  $z$ -score of  $-1.1$ . Find the thickness of the sheet in the original units of the number of inches.

**Solution**

We use Equation (4.20), substituting  $z = -1.1$  for  $z$  and solving for  $x$ . We obtain

$$-1.1 = \frac{x - 1.75}{0.12}$$

Solving for  $x$  yields  $x = 1.12$ . The sheet is 1.12 inches thick.

Areas under the standard normal curve (Figure 4.1) have been extensively computed. A typical book table of area under normal curves (Table A.1) is given in Table A.1 (Appendix A). To find areas under a normal curve with a different mean and standard deviation convert to standard units and use Table A.1. Table A.2 provides areas to the left-hand tail of the curve for values of  $z$ .

### Example 4.21

Find the area under the normal curve to the left of  $z = 0.87$ .

**Solution**

From the  $z$ -table the area is 0.8090. See Figure 4.2.



FIGURE 4.2 Example 4.21

### Example 4.22

Find the area under the normal curve to the right of  $z = -1.34$ .

**Solution**

From the  $z$ -table the area to the left of  $z = 1.34$  is 0.9082. Therefore the area to the right is  $1 - 0.9082 = 0.0918$ . See Figure 4.3.



FIGURE 4.3 Example 4.22

# Example 6.43

For the area under the normal curve between  $z = 0.71$  and  $z = 1.28$ .

**Solution**

From the  $z$ -table the area to the left of  $z = 1.28$  is 0.8997. The area to the left of  $z = 0.71$  is 0.7611. The area between  $z = 0.71$  and  $z = 1.28$  is therefore  $0.8997 - 0.7611 = 0.1386$ . See Figure 6.7.



FIGURE 6.7 Solution to Example 6.43

# Example 6.44

What  $z$ -value corresponds to the 75th percentile of a normal curve? The 25th percentile? The median?

**Solution**

To answer this question, we use the  $z$ -table in reverse. We need to find the  $z$ -value for which 75% of the area of the curve is to the left. From the body of the table the proportion 0.75 is 0.7580 corresponding to a  $z$ -value of 0.67. Therefore the 75th percentile is approximately 0.67. By the symmetry of the curve the 25th percentile is  $z = -0.67$  after suitably looking up in the table itself. See Figure 6.8. The median is  $z = 0$ .



FIGURE 6.8 Solution to Example 6.44

# Example 3.43

A lifetime of a battery is a random variable that is normally distributed with mean 30 hours and standard deviation 3 hours. Find the probability that a randomly chosen battery will last between 22 and 32 hours.

**Solution**

Let  $X$  represent the lifetime of a randomly chosen battery. Then  $X \sim N(30, 3^2)$ . Figure 4.3 (page 436) presents the probability density function of the  $N(30, 3^2)$  population. The shaded area represents  $P(22 < X < 32)$ , the probability that a randomly chosen battery has a lifetime between 22 and 32 hours. To compute this area, we will use the  $z$ -table. First we convert values in the problem to  $z$ -scores. We have

$$z = \frac{22 - 30}{3} = -2.67 \quad \text{and} \quad z = \frac{32 - 30}{3} = 0.67$$

From the  $z$ -table, the area to the left of  $z = -2.67$  is 0.0038, and the area to the left of  $z = 0.67$  is 0.7486. The probability that a battery has a lifetime between 22 and 32 hours is  $0.7486 - 0.0038 = 0.7448 \approx 0.745$ .



FIGURE 4.3 Solution to Example 3.43

# Example 3.44

Find the 90th percentile of battery lifetimes.

**Solution**

From the  $z$ -table, the area to the left of 0.67 is 0.7486, corresponding to a  $z$ -score of 0.67. The population information was mean 30 and standard deviation 3. We then multiply the standard 29 standard deviations below the mean. We find this—by converting the  $z$ -score to a  $x$ -score—using Equation 3.43b:

$$x = 30 + (-0.67)(3)$$

Using the  $x$ -value  $x = 28.91$ . The 90th percentile of battery lifetimes is 28.91 hours. See Figure 4.3b.



FIGURE 4.3b Solution to Example 3.44

# Example 4.43

A generic specification for bearing shaft diameters are uniformly distributed with mean 1.000 cm and standard deviation 0.005 cm. Specifications call for the diameter to be in the range of 0.995 to 1.005 cm. What proportion of the shaft diameters will meet the specification?

**Solution**

Let  $X$  represent the diameter of a randomly chosen shaft bearing. Then  $X \sim \text{Uni}(\text{low} = 0.995, \text{high} = 1.005)$ . Figure 4.11 presents the probability density function of the first order statistic,  $X_{(1)}$ , for the shaft diameters. We find that  $P(0.995 \leq X_{(1)} \leq 1.005)$  is the proportion of bearings that meet the specification.

We compute the values in Table 4.11.

$$c = c_1 = \frac{1.005 - 0.995}{0.005} = 2, \quad c = c_2 = \frac{0.995 - 0.995}{0.005} = 0.00$$

The area to the left of  $c = c_1 = 2$  is 0.0000. The area to the left of  $c = c_2 = 0.00$  is 0.7500. The area between  $c = 0.00$  and  $c = 2$  is  $0.7500 - 0.0000 = 0.7500$ , or 75.00%, approximately 3/4 of the diameters will meet the specification.



FIGURE 4.11 Uniform Example 4.43





# Example 6.49

As in Example 6.47 and 6.48, assume that the ground that is excavated to lay the main drainage is only 2 ft wide. In what situation the smallest drainage is needed so that 95% of the drainage will cover the aquifer zone?

## Solution

The aquifer zone is at a 1.25-ft depth. We want to know how wide the main drainage and needed 95% of the population is that having drainage. By Figure 6.17, this means that the 2.5% of the area to the left of  $z = -1.96$ . This means that the 2.5% of the area to the right of  $z = 1.96$  also belong to the population of the area. In other words, the length of the main drainage is 2.4 ft, for a center of  $-1.96$ , while the upper limit of 1.96 has a center of 1.96. (Note that the area to the right of 1.96 is 0.025.)

$$z = \frac{1.25 - 1.96}{\sigma}$$

Substitute in problem 6.488 to get



FIGURE 6.17 Finding a  $z$ -value such that  $z = -1.96$  has approximately 95% of the population within between  $z$  and  $z$ .

- Example:** Given a standard normal distribution, find the area under the curve that lies
- (a) to the left of  $z = 1.64$  and
  - (b) between  $z = -1.65$  and  $z = 1.65$ .

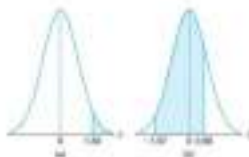


Figure 4.9 Areas for Example 4.2

**Solution:** See Figure 4.9 for the specific areas.

- (a) The area to Figure 4.9(a) to the right of  $z = 1.64$  is equal to 1 minus the area to Table 4.2 to the left of  $z = 1.64$ , namely  $1 - 0.9505 = 0.0495$ .
- (b) The area to Figure 4.9(b) between  $z = -1.65$  and  $z = 1.65$  is equal to the area to the left of  $z = 1.65$  minus the area to the left of  $z = -1.65$ . Using Table 4.2 we find the shaded area is  $(0.9511) - (0.0494) = 0.9017$ . ■

See page 17 for sample problems and solutions and/or assignments/assignments/assignments...

**Example 6.2.1** Suppose standard normal distribution. Find the value of  $z$  such that

(a)  $P(Z \leq z) = 0.9049$  and

(b)  $P(Z \leq z) = 0.3949$  or  $0.4051$ .

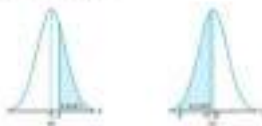


Figure 6.16. Areas for Example 6.2.1

**Solution:** The following graph illustrates what we already know in Figure 6.16:

(a) In Figure 6.16(a), we see that the  $z$ -value having an area of 0.9049 to the right must have a symmetric area of 0.0951 to the left. From Table A.2 it follows that  $z = 1.3$ .

(b) From Table A.2 we note that the  $z$ -value to the left of  $-1.3$  is equal to 0.0951. In Figure 6.16(b), we see that the  $z$ -value corresponding to 0.4051 is the same as the  $z$ -value for which  $0.5000 - 0.0951 = 0.4049$ . Hence, from Table A.2 we have  $z = 1.3$ . 4

**Example 6.2.2** Suppose random variable  $Z$  having a normal distribution with  $\mu = 10$  and  $\sigma = 2$ . Find the probability that  $Z$  assumes a value between 8 and 12.



Figure 6.17. Area for Example 6.2.2

**Solution:** The  $z$ -values corresponding to  $x_1 = 8$  and  $x_2 = 12$  are

$$z_1 = \frac{8 - 10}{2} = -1 \text{ and } z_2 = \frac{12 - 10}{2} = 1.$$

Therefore,

$$P(Z \leq 2) = 95\% = P(Z \leq 0) + P(0 \leq Z \leq 2),$$

so  $1 - 0.05 = 0.95$  is equal to the area to the right of the random variable  $Z = 0$  plus the area to the left of  $Z = 2$ . Using Table A.3, we have

$$\begin{aligned} P(Z \leq 2) = 0.95 &= P(Z \leq 0) + P(0 \leq Z \leq 2) = P(Z \leq 0) + (P(Z \leq 2) - P(Z \leq 0)) \\ &= 0.5000 + 0.4495 = 0.9495. \end{aligned} \quad \blacktriangle$$

**EXAMPLE 4.11** Using Table A, we have found probabilities that  $Z \leq 1.95$  and  $Z \geq 2.00$ . But the probability that  $Z$  is between 1.95 and 2.00?

**Solution:** The correct probability calculation with the density area under the curve is shown in Figure 4.12. To find  $P(1.95 \leq Z \leq 2.00)$ , we need to calculate the area under the normal curve to the right of  $Z = 1.95$ . This can be done by consulting Table A.3 for the corresponding  $z$  value, obtaining the area to the left of  $z$  from Table A.3 and then subtracting the area from 1. We find that

$$1 - \frac{0.97 - 0.50}{2} = 0.23.$$

Thus,

$$P(1.95 \leq Z \leq 2.00) = P(Z \geq 1.95) - 1 + P(Z \leq 2.00) = 1 - 0.9700 + 0.9775. \quad \blacktriangle$$



Figure 4.11. Area to the right of  $z$ .

According to Chebyshev's theorem (see Section 12), the probability that a random variable assumes a value within 2 standard deviations of the mean is at least 0.75. If the random variable has a normal distribution, the  $z$  values corresponding to  $z_1 = -2$  and  $z_2 = 2$  are the  $z$  values we are looking for.

$$z_1 = \frac{0 - 2\sigma - \mu}{\sigma} = -2 \text{ and } z_2 = \frac{0 + 2\sigma - \mu}{\sigma} = 2$$

Thus,

$$\begin{aligned} P(-2\sigma \leq Z \leq 2\sigma) &= P(-2 \leq Z \leq 2) = P(Z \leq 2) - P(Z \leq -2) \\ &= 0.9775 - 0.0225 = 0.9550, \end{aligned}$$

which is a much stronger statement than that given by Chebyshev's theorem.

## Using the Normal Curve to Measure

Sometimes, we are required to find the value of  $x$  corresponding to a specified probability that falls between values listed in Table A.4 (see Example 4.6). For convenience, we will assume a standard normal value corresponding to the tabular probability that is more direct in the statistical calculation.

The preceding two examples were solved by going left (from a value of  $x$  to a value of  $z$ ) but inverting the desired area. In Example 4.6, the area to the right of  $x$  is given with a formula for the probability that the  $x$  value and then determine  $x$  by inverting the formula:

$$x = \frac{\sigma}{\sigma} z \quad \text{so given } z = -0.85 \text{ and } \sigma = 10$$

**Example 4.7** Assume a normal distribution with  $\mu = 90$  and  $\sigma = 10$ . Find the value of  $x$  that has (a) 90% of the area to the left and (b) 90% of the area to the right.



Figure 4.7 Answer for Example 4.7

**Solution:** (a) The area of 0.90 to the left of the desired  $x$  value is shown in Figure 4.7(a). We require a  $z$  value that leaves an area of 0.10 to the left. From Table A.4 we find  $P(Z < -0.85) = 0.10$ , so the desired  $z$  value is  $-0.85$ . Thus,

$$x = -90(-0.85) = 90 + 0.85(10)$$

(b) In Figure 4.7(b), we desire an area equal to 0.10 to the right of the desired  $x$  value. That is, we require a  $z$  value that leaves 0.10 of the area to the right and results in an area of 0.90 to the left. Again, from Table A.4, we find  $P(Z < 0.85) = 0.90$ , so the desired  $z$  value is 0.85 and

$$x = 90 + 0.85(10) = 90 + 8.5 = 98.5$$

## 6.4 Applications of the Normal Distribution

Some of the many problems for which the normal distribution is appropriate are treated in the following examples. The use of the normal curve to approximate binomial probabilities is considered in Section 6.5.

**Example 6.11** A certain type of storage battery (type AA) averages 300 hours when it is used. Assuming that the battery life is normally distributed, find the probability that a given battery will last less than 320 hours.

**Solution:** First we must set a standard scale as in Figure 6.14, showing the given distribution of battery lives and the desired area. To find  $P(X < 320)$ , we need to evaluate the area under the normal curve to the left of 320. This is accomplished by finding the area to the left of the corresponding  $z$ -score. Hence, we find that

$$z = \frac{320 - \mu}{\sigma} = \frac{20}{100} = 0.2$$

and then, using Table A.1, we have

$$P(X < 320) = P(Z < 0.2) = 0.5793$$



Figure 6.14 Area for Example 6.11



Figure 6.15 Area for Example 6.4

**Example 6.4** An electrical firm manufactures light bulbs that have a life, being measured, that is normally distributed with a mean of 300 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 270 and 330 hours.

**Solution:** The distribution of light bulb life is illustrated in Figure 6.15. The  $z$ -score corresponding to  $x = 270$  and  $x = 330$  is

$$z_1 = \frac{270 - 300}{40} = -0.75 \text{ and } z_2 = \frac{330 - 300}{40} = 0.75$$

Hence

$$\begin{aligned} P(270 < X < 330) &= P_{-0.75}^{0.75} = P(Z < 0.75) - P(Z < -0.75) \\ &= 0.7733 - 0.2267 = 0.5466 \end{aligned}$$

**Example 6.6** In an industrial process, the lifetime of a bulb having a normal distribution is known. The mean and standard deviation for the lifetime is 300 and 40, res.

specification is that specified tolerance limits (these specifications will be accepted) is 4 times that of the process standard deviation of a ball bearing (a normal distribution with mean  $\mu = 1.0$  and standard deviation  $\sigma = 0.001$ ). For tolerance limits more restrictive than ball bearings will be accepted?

**Solution:** The distribution of diameters is illustrated by Figure 6.16. The values corresponding to the specification limits are  $x_1 = 1.00$  and  $x_2 = 1.04$ . The corresponding z values are

$$z_1 = \frac{1.00 - 1.0}{0.001} = 0 \quad \text{and} \quad z_2 = \frac{1.04 - 1.0}{0.001} = 4.0$$

Then

$$P(1.00 \leq Y \leq 1.04) = P(0 \leq Z \leq 4.0) = 0.9999$$

Thus, 99.99% of the balls will be acceptable. This is a property of the normal distribution that we find often.

$$P(Z \leq 4.0) = P(Z \leq 0.0) + P(0.0 < Z \leq 4.0) = 0.5000 + 0.4999 = 0.9999$$

As a result, it is anticipated that, on average, 99.99% of manufactured ball bearings will be acceptable. 4



Figure 6.16: Area for Example 6.6

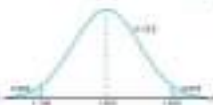


Figure 6.17: The distribution for Example 6.6

**Example 6.10** Engineers are used to report all measurements. In 1980, a certain dimension of an engine was specified as  $1.00 \pm 0.1$  in. Assume that this measurement is normally distributed with mean 1.00 and standard deviation 0.1. Determine the value of  $x$  such that a hypothesis test (with 95% of  $x$  in measurements)

**Solution:** From Table 4.1 we find that

$$P(Z \leq 1.96) = P(Z \leq 1.96) = 0.98$$

Then, since

$$1.96 = \frac{1.00 + x - 1.00}{0.1}$$

then we find  $x$  to be

$$x = 0.1(1.96) = 0.196$$

As a result, if the specification is done as Figure 6.17,



**EXAMPLE 4.1.2** A certain machine makes physical resistors during a seven-minute interval of 45 minutes each, each hour of business. Assuming that the resistance follows a normal distribution and that its standard deviation is 10 ohms, what percentage of resistors will have a resistance exceeding 55 ohms?

**Solution:** A percentage is found by subtracting the value required by (20), then the entire frequency for the normal is equal to the probability of a value falling to the mean, or mean half the area to the right of  $x = 45$  in Figure 4.14. This can be done by computing  $z = 45 - 45$  by substituting  $z$  into the following area to the left of  $z$  from Table A.1, and then subtracting 0.5 from the  $z$  value.

$$z = \frac{45 - 45}{10} = 0$$

Therefore,

$$P(x > 55) = P(x > 45 + 10) = 1 - P(x < 45 + 10) = 1 - 0.5000 = 0.5000$$

Thus, 50.00% of the resistors will have a resistance exceeding 45 ohms. A



Figure 4.14 Area to the right of 45.



Figure 4.15 Area to the right of 55.

**EXAMPLE 4.1.3** Find the percentage of resistors exceeding 45 ohms by Example 4.1.2 assuming it is normal to the mean value.

**Solution:** This problem differs from that in Example 4.1.1 in that we now begin a distribution of 45 ohms to all resistors whose resistance we measure from (20) and use mean 45A. We are probably representing a discrete distribution for values of a resistance versus the ohms. The required area is the region shaded to the right of 45 in Figure 4.14. We use Equation

$$z = \frac{45 - 45}{10} = 0$$

Thus,

$$P(x > 45) = P(x > 45) = 1 - P(x < 45) = 1 - 0.5000 = 0.5000$$

Therefore, 50.00% of the resistors exceed 45 ohms when measured to the nearest ohm. The difference  $0.5000 - 0.0000 = 0.5000$  between the areas under both of Example 4.1.2 represents all those resistors whose greater than 45 and less than 45.4 ohms are now being grouped to 45 ohms. A

**Example 4.10.1** The average grade for a course is 74, and the standard deviation is 7. If 25% of the class is given As, and the grades are scored as follows: A = 90, B = 80, C = 70, D = 60, and F = 50, what are the grades possible for A?

**Solution:** It helps to graph the normal curve of probability, find the  $z$ -value, and then determine  $x$  from the formula  $x = \mu + z\sigma$ . The area of 0.25, corresponding to a deviation of standard deviation, is shown in Figure 6.10. We require a  $z$ -value that leaves 0.25 of the area to the right side, hence, an area of 0.75 to the left. From Table A.1,  $P(Z \leq z) = 0.75$  for the  $z$ -score  $z = 0.67$ , so the desired  $x$ -value is 79.69.

$$x = (25)(144) + 74 = 81.69.$$

Therefore, the lowest A is 80 and the highest B is 74.

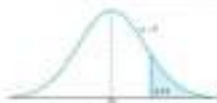


Figure 6.10: Area for Example 4.10.1



Figure 6.11: Area for Example 4.10.1

**Example 4.10.2** Refer to Example 4.10.1 and find the  $z$ -value desired.

**Solution:** The probability, written  $P_z$ , is the portion that corresponds to the area to the left, as shown in Figure 6.11. From Table A.1 we find  $P(Z \leq 0.67) = 0.75$  is the  $z$ -value desired to obtain 0.75. Thus  $z = 0.67(144) = 74.28$ . Hence,  $74 = 74.28$ . This is 0.28% of the grade on 74 or less.

## Exercises

6.1. Given a continuous uniform distribution, what are

(a)  $a = 10$  and  $b = 20$ ;

(b)  $a = 10$  and  $b = 20$ ;

(c)  $a = 10$  and  $b = 20$ ;

6.2. Suppose  $X$  follows a continuous uniform distribution from 1 to 7. Determine the mean, standard deviation, and  $P(2 < X < 4)$ .

6.3. The daily amount of coffee, in liters, dispensed by a machine located in an airport lobby is a random

variable. It follows a continuous uniform distribution with  $a = 7$  and  $b = 15$ . Find the probability that on a given day the amount of coffee dispensed by the machine will be

(a) no more than 10 liters;

(b) more than 7.5 liters but less than 10 liters;

(c) at least 10.5 liters.

6.4. It has been determined that the amount of coffee, in liters, dispensed by a machine located in an airport lobby is a random variable with a continuous uniform distribution.

6.10 When a coin is tossed, what is the probability that the outcome will occur  $k$  times? Justify.

6.11 When a coin is tossed, what is the probability that the outcome will be between  $k$  and  $k+1$  times?

6.12 When a standard normal distribution, find the area under the curve that

- (a) is to the left of  $z = -1.00$ ;
- (b) is to the right of  $z = 1.00$ ;
- (c) lies between  $z = -0.50$  and  $z = 0.50$ ;
- (d) is to the left of  $z = -1.50$ ;
- (e) is to the right of  $z = 1.50$ ;
- (f) lies between  $z = -0.50$  and  $z = 1.50$ .

6.13 Find the value of  $z$  if the area under a standard normal curve

- (a) is to the right of  $z$  is 0.4000;
- (b) is to the left of  $z$  is 0.5000;
- (c) lies between  $z$  and 0 is 0.2500;
- (d) lies between  $z$  and  $z$ , with  $z > 0$  is 0.5000.

6.14 When a standard normal distribution, find the value of  $k$  such that

- (a)  $P(Z > k) = 0.1000$ ;
- (b)  $P(Z < k) = 0.9000$ ;
- (c)  $P(Z < 0) = 0.5000$ ;
- (d)  $P(Z < 0) = 0.5000$ .

6.15 When a normal distribution with  $\mu = 10$  and  $\sigma = 5$ , find

- (a) the standard normal  $z$ -score to the right of  $x = 15$ ;
- (b) the standard normal  $z$ -score to the left of  $x = 10$ ;
- (c) the standard normal  $z$ -score between  $x = 10$  and  $x = 15$ ;
- (d) the value of  $x$  that has 90% of the normal curve area to the left;
- (e) the  $z$ -score value  $z$  that contains the middle 90% of the normal distribution.

6.16 When the normally distributed variable  $Z$  has mean 0 and standard deviation 1, find

- (a)  $P(Z < 1)$ ;
- (b) the value of  $k$  such that  $P(Z < k) = 0.9000$ ;
- (c) the value of  $k$  such that  $P(Z < k) = 0.5000$ ;
- (d)  $P(Z < 0) = 0.5000$ .

6.17 A certain machine is supposed to fill 16-ounce soft-drink cans with 16.00 ounces per can. If the amount of liquid in a certain soft-drink can is a standard normal distribution, find

- (a) what fraction of the soft-drink cans will have the soft-drink
- (b) what is the probability that a soft-drink can will have less than 16.00 ounces?
- (c) how many soft-drink cans will probably contain 16.00 ounces per can for the next 1000 cans?
- (d) how many soft-drink cans will contain 16.00 ounces or less?

6.18 The mean of a normal distribution is found to be 10.00 and the standard deviation is found to be 2.00. Assuming that the variable is normally distributed, what percentage of the total are

- (a) larger than 12.00?
- (b) between 10.00 and 12.00?
- (c) between 10.00 and 12.00?

6.19 A certain variable is said to have a normal distribution if the mean of the variable is 10.00 and the standard deviation is 2.00. Assuming that the variable is normally distributed, what is the probability that a given value will be

- (a) more than 12.00?
- (b) less than 12.00?
- (c) between 10.00 and 12.00?

6.20 The standard normal distribution of a given data is normally distributed with a mean of 10.00 and a standard deviation of 2.00. Assuming

- (a) What percentage of the data will have a value less than 12.00?
- (b) What is the probability that a given data will have a value greater than 12.00?
- (c) What is the value of  $z$  such that the area under the normal curve to the left of  $z$  is 0.9000?

6.21 A normal distribution has a mean of 10.00 and a standard deviation of 2.00. Assuming that the variable is normally distributed, what is the probability that a given value will be

- (a) more than 12.00?
- (b) less than 12.00?
- (c) between 10.00 and 12.00?

9.17 If the average life expectancy is 72.5 years, explain in terms of the data from 9.16 why most 60s is the value for the probability that someone will die.

9.18 Find the length of each column which are half the length 100 of the first.

9.19 Find the width of each row if the first 10 rows will take up half 1.00 hour.

9.20 In the November 1990 issue of *Scientific American*, Figure 1 is a study of the life span of people who are 60 years old in 1990. The study was done by using a sample of 1000 people who are 60 years old in 1990. The study found that the probability of a person dying in the next 10 years is 0.10.

9.21 What percentage of the people who died in the study are 60 years old in 1990?

9.22 What percentage of the people who died in the study are 60 years old in 1990?

9.23 The average life expectancy of people who are 60 years old in 1990 is 12.5 years. The study found that the probability of a person dying in the next 10 years is 0.10. Explain in terms of the data from 9.20 why the average life expectancy is 12.5 years.

9.24 The number of 1000 people who are 60 years old in 1990 is 1000. The study found that the probability of a person dying in the next 10 years is 0.10. Explain in terms of the data from 9.20 why the average life expectancy is 12.5 years.

9.25 How many 1000 people?

9.26 How many 1000 people?

9.27 How many 1000 people?

9.28 How many 1000 people?

9.29 A study was conducted in 1990 to find out how many people are 60 years old in 1990. The study found that the probability of a person dying in the next 10 years is 0.10.

9.30 The probability of the number of people who are 60 years old in 1990 is 1000.

9.31 The number 100 of the number of people who are 60 years old in 1990 is 1000.

9.32 The number of people who are 60 years old in 1990 is 1000. The study found that the probability of a person dying in the next 10 years is 0.10. Explain in terms of the data from 9.20 why the average life expectancy is 12.5 years.

9.33 How many 1000 people?

9.34 How many 1000 people?

9.35 How many 1000 people?

9.36 The number of people who are 60 years old in 1990 is 1000. The study found that the probability of a person dying in the next 10 years is 0.10. Explain in terms of the data from 9.20 why the average life expectancy is 12.5 years.

9.37 The number of people who are 60 years old in 1990 is 1000. The study found that the probability of a person dying in the next 10 years is 0.10. Explain in terms of the data from 9.20 why the average life expectancy is 12.5 years.

9.38 The number of people who are 60 years old in 1990 is 1000. The study found that the probability of a person dying in the next 10 years is 0.10. Explain in terms of the data from 9.20 why the average life expectancy is 12.5 years.

9.39 The number of people who are 60 years old in 1990 is 1000. The study found that the probability of a person dying in the next 10 years is 0.10. Explain in terms of the data from 9.20 why the average life expectancy is 12.5 years.

9.40 The number of people who are 60 years old in 1990 is 1000. The study found that the probability of a person dying in the next 10 years is 0.10. Explain in terms of the data from 9.20 why the average life expectancy is 12.5 years.

# Review Examples

- 4.33 Find the area under the standard normal curve above (a) Fig. 4.3 (a) between  $z = 0$  and  $z = 1.2$ , (b) below  $z = -0.44$  and  $z = 0$ , (c) between  $z = -0.44$  and  $z = 0.6$ , (d) between  $z = 0.6$  and  $z = 1.46$ , (e) to the right of  $z = -1.28$ .

(a) Using the table in Appendix C, please show the relevant entries in Appendix C.2 to explain that desired area is shaded in Fig. 4.3 (a). The result of 24% of the area under the curve is the probability for the interval found in Fig. 4.3 (a). 3 lines.

$$P(0 < Z < 1.2) = \frac{1}{\sqrt{2\pi}} \int_0^{1.2} e^{-t^2/2} dt = 0.2420$$



Fig. 4.3

(b) Suppose now  $z$  is distributed  $z = 0$  and  $z = -0.44$  the probability. The first standard normal curve below shaded is with area corresponding to the interval right of shaded rectangle.

The result of 33% of the area under the curve is the probability for the interval  $z = 0$  and  $z = -0.44$  of Fig. 4.3 (b) shaded.

$$\begin{aligned} P(-0.44 < Z < 0) &= \frac{1}{\sqrt{2\pi}} \int_{-0.44}^0 e^{-t^2/2} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{0.44} e^{-t^2/2} dt = 0.1679 \end{aligned}$$



Fig. 4.4



Fig. 4.5

- (c) Suppose now  $z$  is distributed  $z = -0.44$  and  $z = 0.6$   
 $z = -0.44$  and  $z = 0.6$   
 $z = -0.44$  and  $z = 0.6$   
 $z = -0.44$  and  $z = 0.6$   
 $z = -0.44$  and  $z = 0.6$

The area under the curve is the probability for the interval  $z = -0.44$  and  $z = 0.6$  of Fig. 4.5 (b) shaded.

$$\begin{aligned} P(-0.44 < Z < 0.6) &= \frac{1}{\sqrt{2\pi}} \int_{-0.44}^{0.6} e^{-t^2/2} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-0.44}^0 e^{-t^2/2} dt + \frac{1}{\sqrt{2\pi}} \int_0^{0.6} e^{-t^2/2} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{0.44} e^{-t^2/2} dt + \frac{1}{\sqrt{2\pi}} \int_0^{0.6} e^{-t^2/2} dt = 0.1679 + 0.2244 \\ &= 0.3923 \end{aligned}$$



but if a more precise decision would be given then the two values  $-1.7$  and  $1.7$  (which are 43% from 0) must be applied:

Table 1: The application to the right of  $-1.7$  (Fig. 4.41)

area between  $-1.7$  and  $z$  = Area between  $-1.7$  and 0

+ area between 0 and  $z$

$0.957 - 0.0437 = \text{area between } 0 \text{ and } z$

Therefore area between  $0$  and  $z$  is  $0.9063 = 0.9063 \times 0.5$  (from 0 to  $z$  and  $z$  to  $0$ )  $z = 1.35$



Fig. 4.41



Fig. 4.42

From 4.1,  $z$  is approximately the tail of  $N(0,1)$  (Fig. 4.11).

$$\begin{aligned} \text{Area between } z \text{ and } -z &= \text{area between } z \text{ and } 0 \\ &= \text{area between } -z \text{ and } 0 \\ \Phi(z) - \Phi(-z) &= \text{area between } z \text{ and } 0 = 0.4772 \end{aligned}$$

Thus the area between  $z$  and  $z$  is  $0.9544$  or  $95.44\%$  and  $z = 1.96$  by using these approximations, or  $z$  is slightly less than  $1.96$ .

- 4.10. The mean weight of 500 male students is 68 kg and the standard deviation is 10 kg, assuming that the weights are normally distributed, find how many students weighing between 55 and 85 kg, the more than 95% is.

- 4.11. The mean marks in an example of 200 students per class is 60 marks and the standard deviation is 10 marks. The papers for which these students are awarded class A equivalent to marks in the interval of 50 to 65 marks, otherwise they are given no equivalent to class A. Assuming the percentage of eligible students produced to the students, assuming the distribution of marks is normally distributed.

$$0.95 = \text{area between } -z \text{ and } z = 0.9544 \text{ (Table 4.1)}$$

$$0.95 = \text{area between } -z \text{ and } z = 0.9544 \text{ (Table 4.1)}$$

Required standard deviation

$$\Rightarrow \text{area under normal curve between } z = -1.96 \text{ and } z = 1.96$$

$$\Rightarrow \text{area between } -z \text{ and } z = 0.9544 \text{ (Table 4.1)}$$

$$\Rightarrow 0.9544 = 0.9544 \text{ or } 95\%$$

Therefore the percentage of eligible students is  $95\% = 75\% = 135$  (Fig. 4.12).



Fig. 4.12

Now that it is found that the mean is 60 and the standard deviation is 10, assuming the marks of 200 students are normally distributed, find how many students are awarded class A equivalent to marks in the interval of 50 to 65 marks, otherwise they are given no equivalent to class A.



# Data Sampling

## Populations and Samples

A **population** consists of the totality of the observations with which we are concerned. In other words, it is all the possible outcomes from an experiment or a system. Population can be finite or infinite.

In the field of **statistical inference**, statisticians are interested in deriving an conclusion concerning a population when it is impossible or impractical to observe the entire set of observations that make up the population. For example, in attempting to determine the average length of life of a certain brand of light bulbs, it would be impossible to test all such bulbs if we are to have any left to sell. Therefore, we must depend on a subset of observations from the population to help us make inferences concerning that same population. This brings us to consider the nature of sampling.

A **sample** is a subset of a population. We must choose samples that are representative of the population to ensure our inferences from the sample to the population are to be valid. There are many ways to choose a sample by selecting the most convenient members of the population. Such a nonrandom way leads to errors known as **nonrandom samples**. Any sampling procedure that produces inferences that consistently overestimate or consistently underestimate some characteristic of the population is said to be **biased**. To eliminate any possibility of bias in the sampling procedure, it is desirable to have a random sample in the sense that the observations are made independently and at random.

Let  $X_1, X_2, \dots, X_n$  be a independent random variables, each having the same probability distribution  $f(x)$ . Define  $Y_1, Y_2, \dots, Y_n$  to be a **random sample** of size  $n$  from the population  $f(x)$  and write its joint probability distribution as

$$f(y_1, y_2, \dots, y_n) = f(y_1)f(y_2)\dots f(y_n).$$

Any function of the random variables constituting a random sample is called a **statistic**.

## The Sample Mean, Median, and Mode

(i) *Sample mean:*

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

Note that the statistic  $\bar{X}$  assumes the value  $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$  when  $X_i$  assumes the value  $x_i$ .  $\bar{X}_2$  assumes the value  $x_2$  and so forth. The term *sample mean* is applied to both the statistic  $\bar{X}$  and its computed value  $\bar{x}$ .

(ii) *Sample median:*

$$\bar{x} = \begin{cases} x_{(n+1)/2} & \text{if } n \text{ is odd,} \\ \frac{1}{2}(x_{(n/2)} + x_{(n/2+1)}) & \text{if } n \text{ is even.} \end{cases}$$

The *sample median* is also a location measure that shows the middle value of the sample. Examples for both the sample mean and the sample median can be found in Section 3.3. The *sample mode* is defined as follows:

(i) The *sample mode* is the value of the sample that occurs most often:

## The Sample Variance and Standard Deviation

*Sample variance:*

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2. \quad (3.1.1)$$

If  $X^2$  is the variance of a random sample of size  $n$ , we may write

$$s^2 = \frac{1}{n-1} \cdot \mathbb{E} \left[ \sum_{i=1}^n (X_i - \bar{X})^2 \right]$$

The *sample standard deviation* is  $s = \sqrt{s^2}$ .

**Example 1:** A comparison of coffee prices at a randomly selected grocery store in San Diego showed increases from the previous month of 12, 15, 17, and 20 cents for a 4-pound bag. Find the variance of the random variable of price increases.

**Solution:** Calculating the sample mean, we get

$$\bar{x} = \frac{12 + 15 + 17 + 20}{4} = 16 \text{ cents.}$$

Then, we

$$\begin{aligned} s^2 &= \frac{1}{n} \sum_{i=1}^n x_i^2 - 16^2 = \frac{(12)^2 + (15)^2 + (17)^2 + (20)^2}{4} - 16^2 \\ &= \frac{1 + 4^2 + 1 + 3^2 + 1 + 1^2 + 4^2}{4} = \frac{30}{4}. \end{aligned}$$

**Example 2:** Find the variance of the data 5, 4, 5, 5, 6, and 7, representing the number of times caught by a random sample of 6 fishermen on June 15, 1996, at Little Altona.

**Solution:** We find that  $\sum_{i=1}^n x_i^2 = 171$ ,  $\sum_{i=1}^n x_i = 33$ , and  $n = 6$ . Hence,

$$s^2 = \frac{1}{36(36)} [6(171) - (33)^2] = \frac{11}{6}.$$

Thus, the sample standard deviation is  $s = \sqrt{11/6} \approx 1.37$  and the sample range is  $7 - 4 = 3$ .



# The Central Limit Theorem

The first important sampling distribution to be considered is that of the mean  $\bar{X}$ . Suppose that a random sample of  $n$  observations is taken from a normal population with mean  $\mu$  and variance  $\sigma^2$ . Each observation  $X_i$ ,  $i = 1, 2, \dots, n$ , of the random sample will then have the same normal distribution as the population from which it is sampled. Hence, by the reproductive property of the normal distribution established in Theorem 7.11, we conclude that

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + \dots + X_n)$$

has a normal distribution with mean

$$\mu_{\bar{X}} = \frac{1}{n}(\underbrace{\mu + \mu + \dots + \mu}_{n \text{ terms}}) = \mu \text{ and variance } \sigma_{\bar{X}}^2 = \frac{1}{n^2}(\underbrace{\sigma^2 + \sigma^2 + \dots + \sigma^2}_{n \text{ times}}) = \frac{\sigma^2}{n}.$$

If we are sampling from a population with unknown distribution, other than  $n$  infinite, the sampling distribution of  $\bar{X}$  will still be approximately normal with mean  $\mu$  and variance  $\sigma^2/n$ , provided that the sample size is large. This amazing result is an immediate consequence of the following theorem, called the Central Limit Theorem.

**Central Limit Theorem** If  $\bar{X}$  is the mean of a random sample of size  $n$  values from a population with mean  $\mu$  and finite variance  $\sigma^2$ , then the limiting form of the distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

as  $n \rightarrow \infty$  is the standard normal distribution  $N(0, 1)$ .

The normal approximation for  $\bar{X}$  will generally be good if  $n \geq 30$ , provided the population distribution is not terribly skewed. If  $n < 30$ , the approximation is good only if the population is not too different from a normal distribution; such as noted above, if the population is known to be normal, the sampling distribution of  $\bar{X}$  will follow a normal distribution exactly, no matter how small the size of the sample.

The sample size  $n = 30$  is a guideline to use for the Central Limit Theorem. However, as the statement of the theorem implies, the presumption of normality in the distribution of  $\bar{X}$  becomes more accurate as  $n$  grows larger. In fact, Figure 8.1 that you have just observed, shows how the distribution of  $\bar{X}$  becomes closer to normal as  $n$  grows larger, beginning with the clearly nonnormal distribution of an individual observation ( $n = 1$ ). It also illustrates that the mean of  $\bar{X}$  remains  $\mu$  for any sample size and the variance of  $\bar{X}$  gets smaller as  $n$  increases.

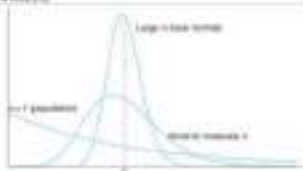


Figure 4.1: Illustration of the Central Limit Theorem: summation of  $X_i$  for  $n = 1$ , moderate  $n$ , and large  $n$ .

**Example:** An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed, with mean equal to 400 hours and a standard deviation of 60 hours. Find the probability that a random sample of 36 bulbs will have an average life of less than 375 hours.

**Solution:** The sampling distribution of  $\bar{X}$  will be approximately normal, with  $\mu_{\bar{X}} = 400$  and  $\sigma_{\bar{X}} = 60/\sqrt{36} = 10$ . The desired probability is given by the area of the shaded region in Figure 4.2.



Figure 4.2: Area for Example 4.1.

Corresponding to  $Z = 775$ , we find that

$$z = \frac{775 - 400}{10} = -3.75$$

and therefore

$$P(\bar{X} < 375) = P(Z < -3.75) = 0.0001$$

## APPENDIX A

[illegible]





# APPENDIX B

Area under the Standard Normal Curve from  $z$  to  $\infty$

$$P(Z > z)$$



$z$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4601	.4561	.4521	.4481	.4441	.4401	.4361	.4321	.4281	.4241
0.2	.4201	.4161	.4121	.4081	.4041	.4001	.3961	.3921	.3881	.3841
0.3	.3801	.3761	.3721	.3681	.3641	.3601	.3561	.3521	.3481	.3441
0.4	.3401	.3361	.3321	.3281	.3241	.3201	.3161	.3121	.3081	.3041
0.5	.3001	.2961	.2921	.2881	.2841	.2801	.2761	.2721	.2681	.2641
0.6	.2601	.2561	.2521	.2481	.2441	.2401	.2361	.2321	.2281	.2241
0.7	.2201	.2161	.2121	.2081	.2041	.2001	.1961	.1921	.1881	.1841
0.8	.1801	.1761	.1721	.1681	.1641	.1601	.1561	.1521	.1481	.1441
0.9	.1401	.1361	.1321	.1281	.1241	.1201	.1161	.1121	.1081	.1041
1.0	.1001	.0961	.0921	.0881	.0841	.0801	.0761	.0721	.0681	.0641
1.1	.0601	.0561	.0521	.0481	.0441	.0401	.0361	.0321	.0281	.0241
1.2	.0201	.0161	.0121	.0081	.0041	.0001	.0000	.0000	.0000	.0000
1.3	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
1.4	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
1.5	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
1.6	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
1.7	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
1.8	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
1.9	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2.1	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2.2	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2.3	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2.4	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2.5	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2.6	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2.7	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2.8	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
2.9	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
3.0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

[illegible]

# APPENDIX C



Areas under the Standard Normal Curve from 0 to  $z$

$z$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5518	.5558	.5598	.5638	.5677	.5717	.5757
0.2	.5797	.5837	.5877	.5917	.5957	.5997	.6037	.6077	.6117	.6157
0.3	.6197	.6237	.6277	.6317	.6357	.6397	.6437	.6477	.6517	.6557
0.4	.6597	.6637	.6677	.6717	.6757	.6797	.6837	.6877	.6917	.6957
0.5	.6997	.7037	.7077	.7117	.7157	.7197	.7237	.7277	.7317	.7357
0.6	.7397	.7437	.7477	.7517	.7557	.7597	.7637	.7677	.7717	.7757
0.7	.7797	.7837	.7877	.7917	.7957	.7997	.8037	.8077	.8117	.8157
0.8	.8197	.8237	.8277	.8317	.8357	.8397	.8437	.8477	.8517	.8557
0.9	.8597	.8637	.8677	.8717	.8757	.8797	.8837	.8877	.8917	.8957
1.0	.8997	.9037	.9077	.9117	.9157	.9197	.9237	.9277	.9317	.9357
1.1	.9397	.9437	.9477	.9517	.9557	.9597	.9637	.9677	.9717	.9757
1.2	.9797	.9837	.9877	.9917	.9957	.9997	.9997	.9997	.9997	.9997
1.3	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
1.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
1.5	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
1.6	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
1.7	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
1.8	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
1.9	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
2.0	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
2.1	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
2.2	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
2.3	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
2.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
2.5	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
2.6	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
2.7	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
2.8	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
2.9	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997
3.0	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997