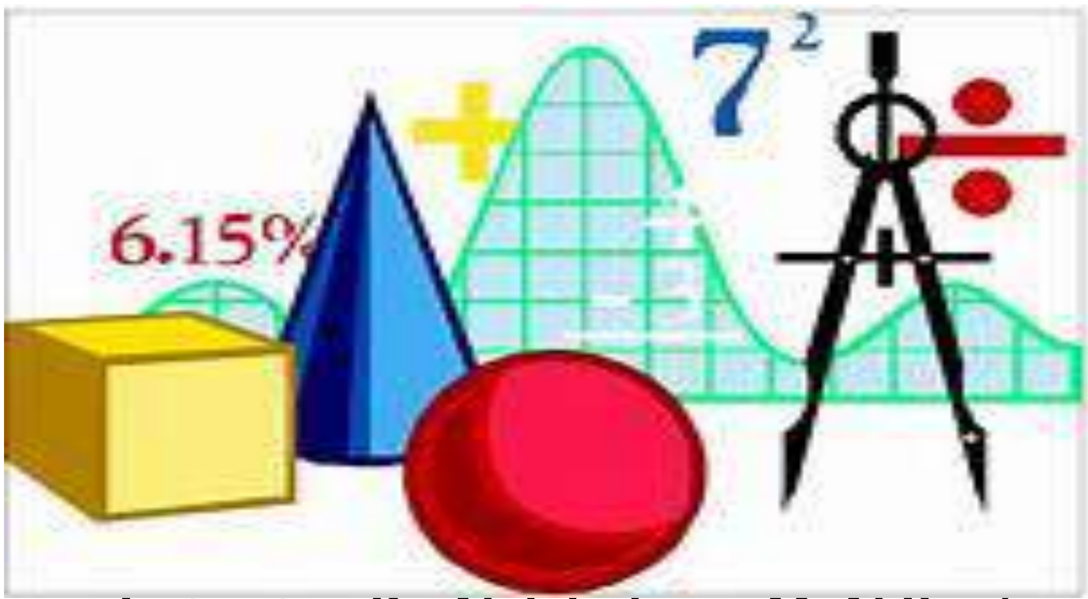
University of Anbar
College of Engineering
Mechanical Engineering Dept.

ME 2201 – Calculus III

2021-2022 First Semester



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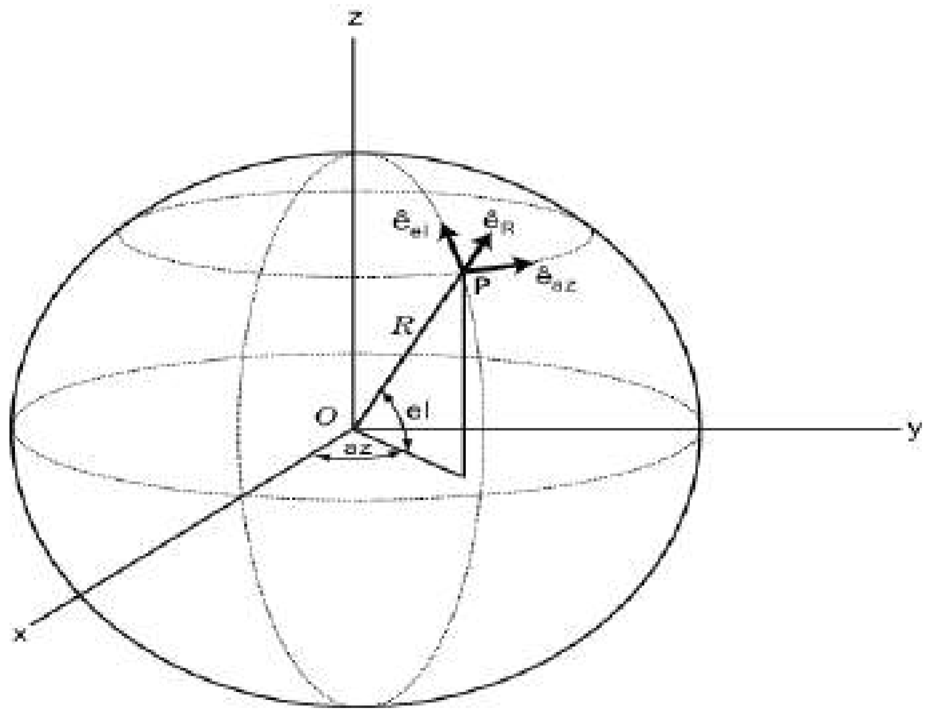
Chapter One

Vectors



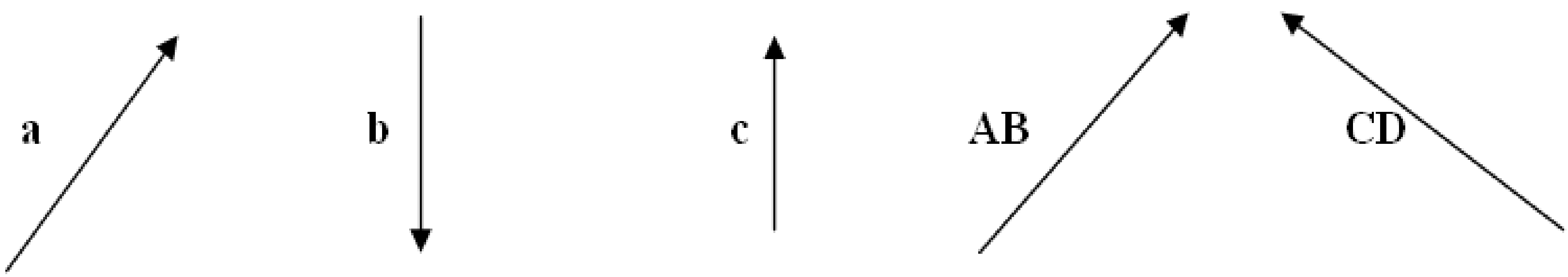
1.1 introduction

Some physical quantities describe only with there values such as temperature, area, length, mass, etc., theses quantities are called <u>scalars</u>.
other physical quantities are not enough to mention only their values, they need to mention also their direction, for example, force, velocity, acceleration , etc. these quantities are called <u>vectors.</u>

The vector usually represents by a directed line segment (arrow). The length is the magnitude of it and the direction of the arrow represents the direction of the vector.

The vector can be denoted by symbol

$$\overrightarrow{a}\ ,\ \overrightarrow{b}\ ,\ \overrightarrow{c}\ ,\ \overrightarrow{AB}\ ,\ \overrightarrow{CD}$$



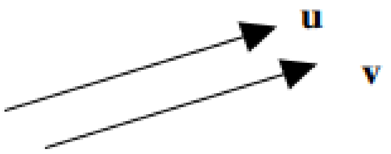
1.2 Some definitions of vectors

1.2.1 Magnitude of vector

The magnitude of a vector $\vec{a}$ written $|\vec{a}|$, is the length of its representative directed line segment.

1.2.2 Unit vector

A unit vector $\vec{u}$ is a vector of unit length, that is $|\vec{u}|$ =1.



1.2.3 Equal vectors

Two vectors u and v, which have the same length and same direction, are said to be equal vectors even though they have different initial points and different terminal points. If u and v are equal vectors we write u = v.

1.2.4 Zero vector

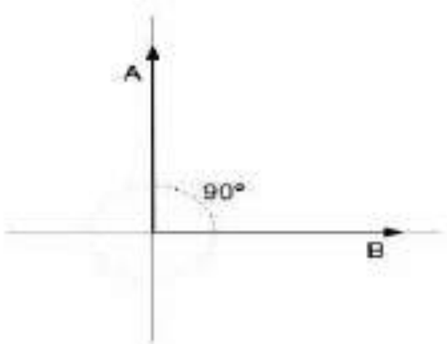
The zero vector, denoted 0, is the vector whose length is 0. Since a vector of length 0 does have any direction associated with it was shall agree that its direction is arbitrary; that is to say it can be assigned any direction we choose. The zero vector satisfies the property: v + 0 = 0 + v = v for every vector v

1.2.5 Negative vector

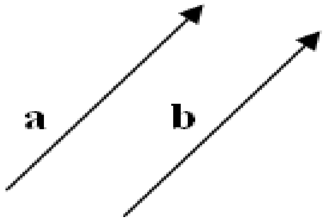
If u is a nonzero vector, we define the negative of u, denoted –u, to be the vector whose magnitude (or length) is the same as the magnitude (or length) of the vector u, but whose direction is opposite to that of u.

1.2.6 Orthogonal vector

Two vectors $\vec{A}$ and $\vec{B}$ are said to be orthogonal when the angle between them is 90 degree or one of then is a zero vector .



1. 3 Vectors algebra



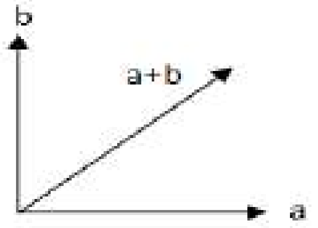
1. 3.1 Equality

$$\underset{a}{\rightarrow} = a_{1i} + a_{2j} \ \ and \ \ \underset{b}{\rightarrow} = b_{1i} + b_{2j}$$

$$a_1 = b_1 \ \& \ a_2 = b_2$$

These are two vectors are equal only if

Then $\underset{a}{\rightarrow} = \underset{b}{\rightarrow}$

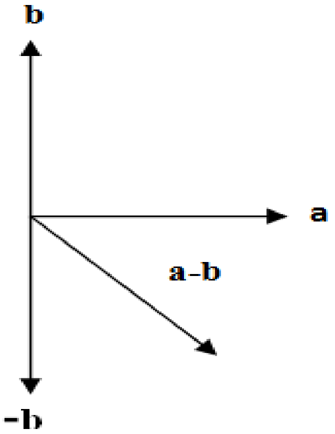


1.3.2 Addition and Subtraction

$$\underset{a}{\rightarrow} = a_{1i} + a_{2j} \ \ and \ \ \underset{b}{\rightarrow} = b_{1i} + b_{2j}$$

$$\underset{a}{\rightarrow} + \underset{b}{\rightarrow} = (a_1 + b_1)i + (a_2 + b_2)j$$

$$\underset{a}{\rightarrow} - \underset{b}{\rightarrow} = (a_1 - b_1)i + (a_2 - b_2)j$$



1.3.3 Multiplication by a scalar

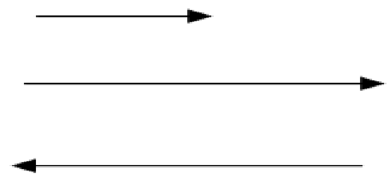
- If $\vec{a} = a_{1i} + a_{2j}$ and S is the scalar

$$s\vec{a} = (sa_1)_i + (sa_2)_j$$

$\vec{a}$

$2\vec{a}$

$-2\vec{a}$



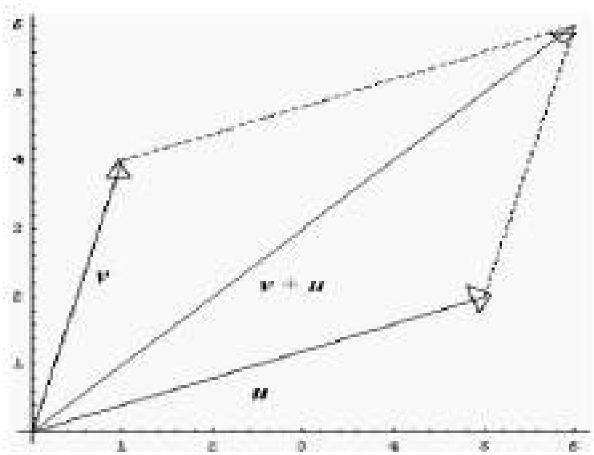
- If (s) is positive, the direction of vector $s\vec{a}$ at the same direction of vector $\vec{a}$
- If (s) is negative, the direction of vector $s\vec{a}$ at the opposite direction of vector $\vec{a}$

Example 1.1 if $\vec{u} = 5_i + 2_j$ and $\vec{v} = 1_i + 4_j$

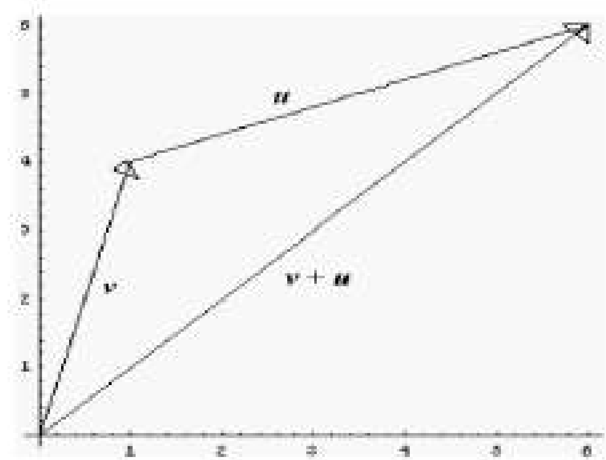
Find u+v

Solution

$$\overrightarrow{u+v} = 6_i + 6_j$$



or



1.4 unit vectors (i.j, &k)

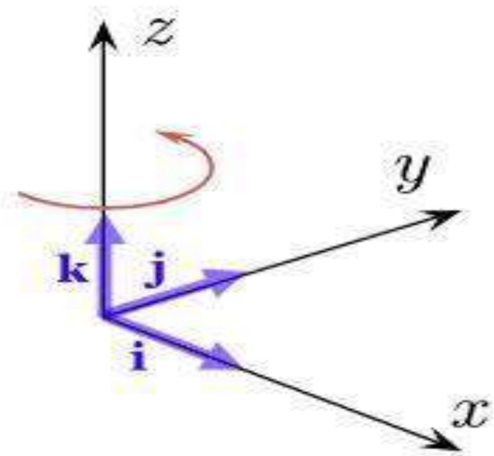
Let i, j and k are unit vectors

Where

(i) Is a unit vector in the positive x-axis direction.

(j) Is a unit vector in the positive y-axis direction

(k) Is a unit vector in the positive z-axis direction



That m $|i|=|j|=|k|=1$

And

i, j, and k are orthogonal

To find the unit vector for any Let $\underset{a}{\rightarrow} = 3_i + 4_j$

$$unit\ vector\ of\ a = \frac{a}{|a|} = \frac{3_i + 4_j}{\sqrt{3^2 + 4^2}} = \frac{3_i + 4_j}{5} = \frac{3}{5}i + \frac{4}{5}j$$

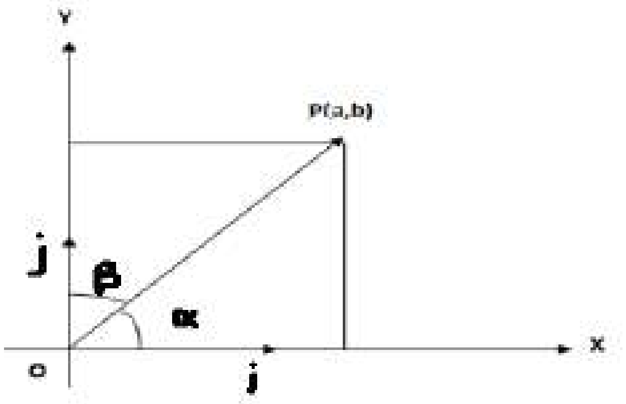
1.5 Vector in plane

If $\overrightarrow{A}$ is a vector from the origin (o) to the point P(a, b).

$$\underset{A}{\rightarrow} = \underset{OP}{\rightarrow} = a_i + b_j$$

$$\left|\underset{A}{\rightarrow}\right| = \sqrt{a^2 + b^2}$$

$$\propto + \beta = \frac{\pi}{2}$$



Unit vector $= \underset{U}{\rightarrow} = \frac{\vec{A}}{\left|\underset{A}{\rightarrow}\right|} = \frac{a_i + b_j}{\left|\underset{A}{\rightarrow}\right|} = \frac{a}{\left|\underset{A}{\rightarrow}\right|} i + \frac{b}{\left|\underset{A}{\rightarrow}\right|} j$

$$\underset{U}{\rightarrow} = \cos \propto i + \cos\beta j$$

Where

$$\cos \propto = \frac{a}{\left|\underset{A}{\rightarrow}\right|}, \cos \beta = \frac{b}{\left|\underset{A}{\rightarrow}\right|}$$

Example 1.2

Find the direction and the length of $\underset{A}{\rightarrow} = 4i + 3j$

Solution

Length of $\underset{A}{\rightarrow} = \left|\underset{A}{\rightarrow}\right| = \sqrt{(4)^2 + (3)^2} = 5$

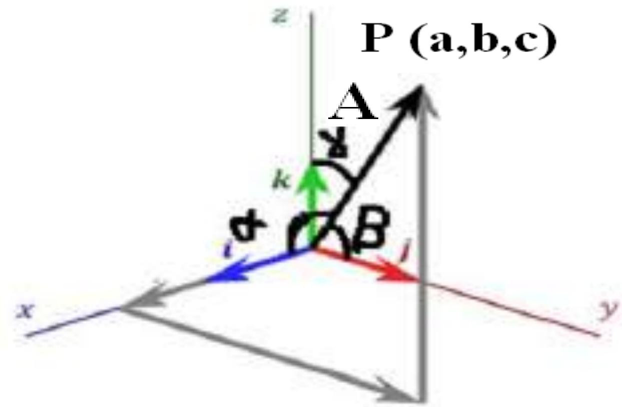
Unit vector of $\underset{A}{\rightarrow} = \frac{A'}{\left|\underset{A}{\rightarrow}\right|} = \frac{4}{5} i + \frac{3}{5} j$

$$\cos \alpha = \frac{4}{5} \rightarrow \alpha = \cos^{-1}\left(\frac{4}{5}\right) \rightarrow \alpha = 36.8^o$$

1.6 vector in space

- Suppose That A is a vector from the origin to a point P (a, b, c)

$$\underset{A}{\rightarrow} = \underset{OP}{\rightarrow} = ai + bj + ck \quad then \quad \left|\underset{A}{\rightarrow}\right| = \sqrt{a^2 + b^2 + c^2}$$



$$unit\ vector = \underset{u}{\rightarrow} = \frac{\vec{A}}{\left|\underset{A}{\rightarrow}\right|} = \frac{ai + bj + c}{\left|\underset{A}{\rightarrow}\right|} k$$

$$= \frac{a}{\left|\underset{A}{\rightarrow}\right|} i + \frac{b}{\left|\underset{A}{\rightarrow}\right|} j + \frac{c}{\left|\underset{A}{\rightarrow}\right|} k$$

$$\underset{u}{\rightarrow} = \cos \propto i + \cos \beta + \cos\gamma$$

$$\left|\underset{u}{\rightarrow}\right| = 1 = (\cos \propto)^2 + (\cos \beta)^2 + (\cos\gamma)^2$$

<u>Example 1.3</u>

Find the unit vector of vector $\underset{V}{\rightarrow} = 4i + 3j + 12k$

Solution

$$\left|\underset{V}{\rightarrow}\right| = \sqrt{(4)^2 + (3)^2 + (12)^2} = 13$$

and

$$\underset{u}{\rightarrow} = \frac{\vec{V}}{\left|\underset{V}{\rightarrow}\right|} = \frac{4}{13} i + \frac{3}{13} j + \frac{12}{13} k$$

<u>Example 1.4</u>

Find a vector 6 units long in the direction of vector $\underset{A}{\rightarrow} = 2i + 2j - k$

Solution $\underset{u}{\rightarrow} = \frac{\vec{A}}{\left|\underset{A}{\rightarrow}\right|} = 6 \frac{2i+2j-k}{\sqrt{(2)^2+(2)^2+(-1)^2}} = 4i + 4j - 2k$

$$vector = 6 \underset{u}{\rightarrow} = 4i + 4j - 2k$$

Example 1.5

Find a vector of length 2 units that makes angle 60 degree with x-axis and 30 degree with y-axis.

$\alpha=60^o, \quad \beta=30,^o \quad \gamma=?$

Solution

$$(\cos \propto)^2+(\cos \beta)^2+(\cos\gamma\ )^{\,2}\ =1$$

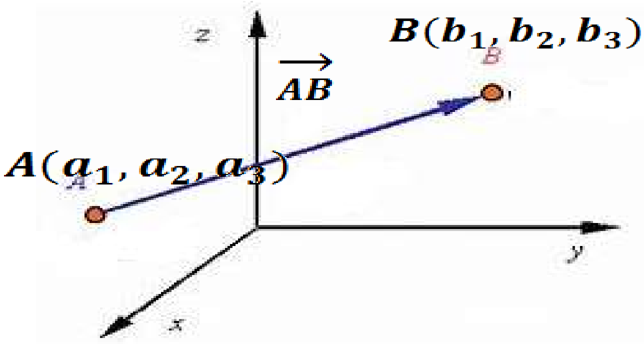
$$0.25+0.75+(\cos\gamma\ )^{\,2}\ =1$$

$$(\cos\gamma\ )^{\,2}\ =0, \qquad \cos\gamma=0, \ \ \gamma=90^o$$

$$\underset{u}{\rightarrow}=\cos\alpha i+\cos\beta j+\cos\gamma k=0.5\mathrm{i}+0.86\mathrm{j}$$

$$\underset{V}{\rightarrow}=2\left|\underset{V}{\rightarrow}\right|.\ \underset{u}{\rightarrow}=\ \ 2(0.5i+0.86j)\ =\ i+1.72j$$

1.7 vector between two points



Let $A(a_1,a_2,a_3)$ *and* $B(b_1,b_2,b_3)$

Its possible to find a vector between A&B

$$\underset{AB}{\rightarrow}=(b_1-a_1)i+(b_2-a_2)j+(b_3-a_3)k$$

Example 1.6

Find a vector and its unit vector from P1(1,0,1) to P2(3,2,0)

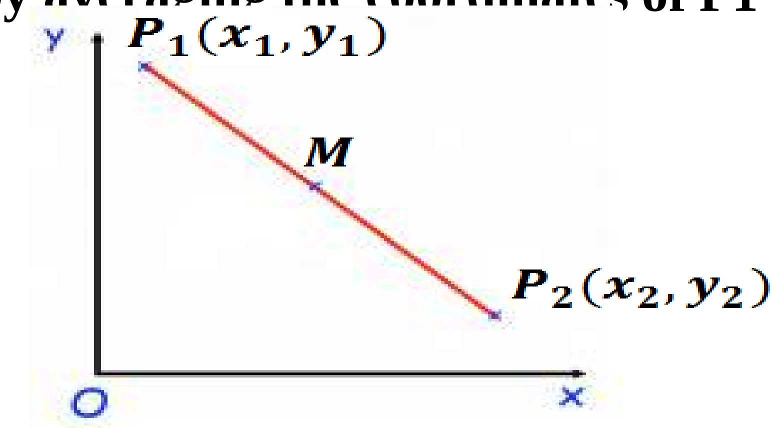
Solution

$$\underset{P_1P_2}{\longrightarrow}=(3-1)i+(2-0)j+(0-1)k$$

$$\left|\underset{P_1P_2}{\longrightarrow}\right|=\sqrt{(2)^2+(2)^2+(-1)^2}=3 \quad \text{and} \quad \underset{u}{\rightarrow}=\frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|}=\frac{2}{3}i+\frac{2}{3}j-\frac{1}{3}k$$

1.8 Mid point of line segments

The coordinates of the mid point M of the line segment joining two points P1(X1, Y1) and P2 (X2,y2) and found by averaging the coordinates of P1 and P2. That is,



$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$M = \left(\frac{3 + 7}{2}, \frac{-2 + 4}{2}\right)$$

$P_1(3, -2)$ *and* $P_2(7, 4)$

__Example 1.7__

Find the midpoint of the segment joining $A(2, -1)$, $B(-3, 2)$

Solution

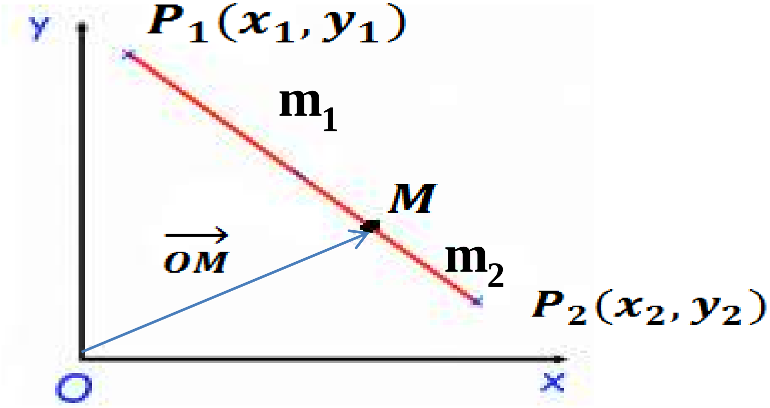
$$c = \left(\frac{2 - 3}{2}, \frac{-1 + 2}{2}\right) = \left(\frac{-1}{2}, \frac{1}{2}\right)$$

__Example 1.8__

Find the vector $\overrightarrow{oc}$ where C is the midpoint between

$$\overrightarrow{oc} = \left(-\frac{1}{2} - 0\right) i + \left(\frac{1}{2} - 0\right) j = -\frac{1}{2} i + \frac{1}{2} j$$

Note: The coordinates of a point which divides the line in the ratio $^{m_1}/_{m_2}$ as shown in the Fig.



$$M = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

Example 1.9

Find the vector $\overrightarrow{OM}$ where M is a point divides the line between $P_1(4, -2)\ and\ P_2(-8, 9)$ with a ratio $^3/_2$.

Solution $\quad ^{m_1}/_{m_2} = {}^3/_2$

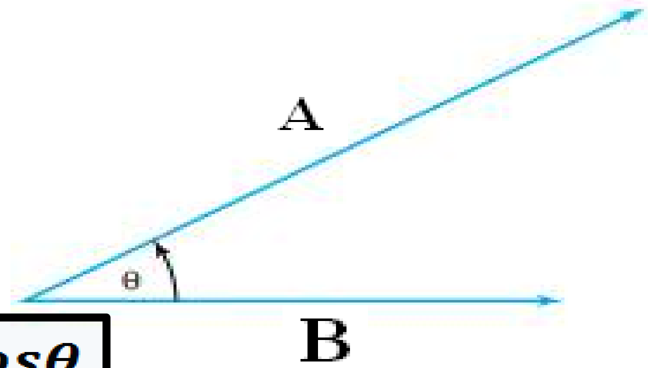
$$M = \left(\frac{3(-8) + 2(4)}{5}, \frac{3(9) + 2(-2)}{5}\right)$$

$$M = \left(\frac{-16}{5}, \frac{23}{5}\right) \quad \overrightarrow{OM} = \frac{-16}{5}i, + \frac{23}{5}j$$

1.9 The Dot Product (Scalar Product)

A product of two vectors A and B can be formed in such a way that the result is a scalar. The result is written a · b and called the dot product of a and b. The names scalar product and inner product are also used in place of the term dot product.

As shown in the Fig. where dot or scalar product $0 \le \theta \le \pi$. Then the dot product of a and b is defined as the number.



$$\underset{A}{\rightarrow} = a_{1i} + a_{2j} + a_{3k}$$

$$\underset{B}{\rightarrow} = b_{1i} + b_{2j} + b_{3k}$$

$$\underset{A}{\rightarrow} . \underset{B}{\rightarrow} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \boxed{\left|\underset{A}{\rightarrow}\right| \left|\underset{B}{\rightarrow}\right| cos\theta}$$

1.9.1 Properties of the dot product

- A· B = B · A
- λA · μB = μA · λB = λμA · B
- A · (B + C) = A · B + A· C
- A · (λB + μC) = λA · B + μA · C

- **To find the angle between two vectors**

$$\cos\theta = \frac{A.B}{|A||B|} \qquad 0 \le \theta \le \pi$$

- **If the two vectors are parallel**

$$A.B = |A||B| \quad and\, A.A = |A|^{2}$$

- **if the two vectors are orthogonal**

$$A.B = 0$$

Also

$$i.i = j.j = k.k = 1$$
$$i.j = j.i = i.k = k.i = j.k = k.j = 0$$

Example 1.10

Find A · B and the angle between the vectors a and b, given that
A = i + 2j + 3k and B = 2i − j − 2k.

Solution

$$cos\theta = \frac{A.B}{|A||B|} \quad ,$$

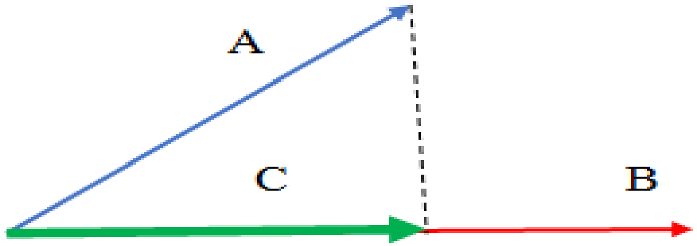
$$|A| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{14}\, and\, |B| = \sqrt{(2)^2 + (-1)^2 + (-2)^2} = 3$$

$$A.B = (1.2) + (2.(-1)) + (3.(-2)) = -6 \quad \theta = cos^{-1}\left(\frac{-6}{\sqrt{14}.3}\right) = 122.3^o$$

1.9.1 The projection of a vector onto the line of another vector

The projection of vector a onto the line of vector b is a scalar, and it is the projecting a vector onto a line signed length of the geometrical projection of vector a onto a line parallel to b, with the sign positive for 0 ≤ θ < π/2 and negative for π/2 < θ ≤ π. This is illustrated in Fig below.

$$\underset{C}{\rightarrow} = Proj_{\underset{B}{\rightarrow}} \overrightarrow{A} = \frac{\overrightarrow{A}.\overrightarrow{B}}{|B|^2} \cdot \underset{B}{\rightarrow}$$



1.11 Example

Find the vector projection of $\underset{A}{\rightarrow} = i + j + k \quad on \quad \underset{B}{\rightarrow} = 2i + 2j$ and then find the scalar component of vector A in the direction of vector B.

Solution

Let C is the vector projection

$$\underset{C}{\rightarrow} = Proj_{\underset{B}{\rightarrow}} \overrightarrow{A} = \frac{\overrightarrow{A}.\overrightarrow{B}}{|B|^2} \cdot \underset{B}{\rightarrow}$$

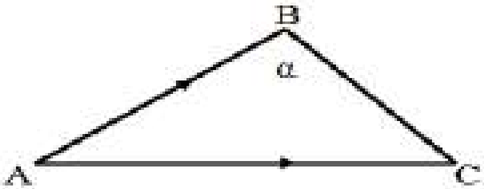
$$\underset{A}{\rightarrow} . \underset{B}{\rightarrow} = 2 + 2 + 0 = 4 \qquad \left|\underset{B}{\rightarrow}\right|^2 = \left|\underset{B}{\rightarrow}\right| . \left|\underset{B}{\rightarrow}\right| = 4 + 4 = 8$$

$$\underset{C}{\rightarrow} = \frac{4}{8}(2i + 2j) = i + j$$

Scalar component $\quad \left|\underset{C}{\rightarrow}\right| = \frac{\overrightarrow{A}.\overrightarrow{B}}{\left|\underset{B}{\rightarrow}\right|} = \frac{4}{2\sqrt{2}} = \sqrt{2}$

- **Example 1.12**

Given a triangle Δ ABC whose vertices are A(1,-1,0), B(-2,3,1) and C(0,1,-2), Find 1- the projection of vector AB onto Vector AC.
2- The angle α = ▶ABC



Solution

1- $\underrightarrow{AB} = (-2-1)i + (3+1)j + (1-0)k = -3i + 4j + k$

$\underrightarrow{AC} = (0-1)i + (1+1)j + (-2-0)k = -i + 2j - 2k$

$\underrightarrow{AB} . \underrightarrow{AC} = (-3)(-1) + (4)(2) + (4)(-2) = 3 + 8 - 2 = 9$

$\left|\underrightarrow{AC}\right| = \sqrt{(-1)^2 + (2)^2 + (-2)^2} = 3$

$Proj\ {}_{\underrightarrow{AC}}^{\underrightarrow{AB}} = \dfrac{\underrightarrow{AB} . \underrightarrow{AC}}{|AB|^2}\ \underrightarrow{AC} = \dfrac{9}{9}\ (-i + 2j - 2k) = -i + 2j - 2k$

$\underrightarrow{BA} . \underrightarrow{BC} = \left|\underrightarrow{BA}\right| \left|\underrightarrow{BC}\right| \cos\alpha \quad \rightarrow \quad \cos\alpha = \dfrac{\underrightarrow{BA} . \underrightarrow{BC}}{\left|\underrightarrow{BA}\right| \left|\underrightarrow{BC}\right|}$

$\underrightarrow{BA.} = 3i - 4j - k \text{ and } \underrightarrow{BA.} = 2i - 2j - 3k \qquad \underrightarrow{BA} . \underrightarrow{BC} = 6 + 8 + 3 = 17$

$\left|\underrightarrow{BA.}\right| = \sqrt{(3)^2 + (-4)^2 + (-1)^2} = \sqrt{26} \qquad \left|\underrightarrow{BC}\right| = \sqrt{(2)^2 + (-2)^2 + (-3)^2} = \sqrt{17}$

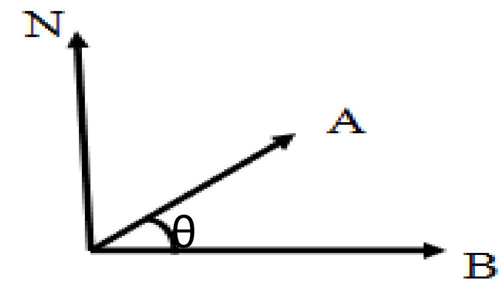
2-

$$\alpha = cos^{-1}\left(\frac{17}{\sqrt{26}\sqrt{17}}\right) = 36^o$$

1.10 Cross product (vector product)

A product of two vectors A and B can be defined in such a way that the result is a vector. The result is written A×B and called the cross product of A and A. The name vector product is also used in place of the term cross product.

Let



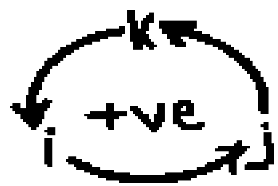
$\vec{A} = a_{1i} + a_{2j} + a_{3k}$ $\qquad$ $\vec{B} = b_{1i} + b_{2j} + b_{3k}$

$\vec{A} \times \vec{B} = \left|\vec{A}\right| \left|\vec{B}\right| \sin\theta \, \vec{N}$

Where N is a unit vector perpendicular on both vectors A and B.

1.10.1 Properties

- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
- $\vec{A} \times \vec{B} = 0$ **if A and B vectors are parallel**
- $i \times i = j \times j = k \times k = 0$
- $i \times j = k, \qquad j \times k = i, \qquad k \times i = j$
- $j \times i = -k, \qquad i \times k = -j, \qquad k \times j = -i$



$(s\vec{A}) \times (t\vec{B}) = (st)(\vec{A} \times \vec{B})$, $\quad$ *s&t are scalar*

1.10.1 Determinants

1.10.1.1 $2 \times 2\ determinat$

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

For example $\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = (3 \times 5) - (-2 \times 4) = 23$

$$= a_1(b_2 c_3 - b_3 c_2) - a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1$$

1.10.1.2 $3 \times 3\ determinat$

For example

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 3 \begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} + (-5) \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}$$

$$= 3(4 \times 2 - (-4) \times 3) + 2(1 \times 2 - 0) - 5(1 \times 3 - 0) = 49$$

Assume $\underset{A}{\rightarrow} = a_{1i} + a_{2j} + a_{3k}$ and $\underset{B}{\rightarrow} = b_{1i} + b_{2j} + b_{3k}$

Then

$$\underset{A}{\rightarrow}\times\underset{B}{\rightarrow} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$$= (a_2 b_3 - a_3 b_2)i - (a_1 b_3 - a_3 b_1)j + (a_1 b_2 - a_2 b_1)k$$

Example 1.13

Let $\underset{A}{\rightarrow} = i + 2j - 2k$ and $\underset{B}{\rightarrow} = 3i + k$ find $\underset{A}{\rightarrow}\times\underset{B}{\rightarrow}$ & $\underset{B}{\rightarrow}\times\underset{A}{\rightarrow}$

Solution

$$\underset{A}{\rightarrow}\times\underset{B}{\rightarrow} = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} i - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} k = 2i - 7j - 6k$$

$$\underset{B}{\rightarrow}\times\underset{A}{\rightarrow} = \begin{vmatrix} i & j & k \\ 3 & 0 & 1 \\ 1 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 2 & -2 \end{vmatrix} i - \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} j + \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} k = -2i + 7j + 6k$$

$$\underset{A}{\rightarrow}\times\underset{B}{\rightarrow} = -\underset{B}{\rightarrow}\times\underset{A}{\rightarrow} \quad \textit{Proved}\,!$$

Example 1.14

If $\underset{A}{\rightarrow} = 2i - 3j + k \;\; and \;\; \underset{B}{\rightarrow} = -i + 2j - 3k$ **, find a vector of length 2 units perpendicular on both** $\underset{A}{\rightarrow} \; and \; \underset{B}{\rightarrow}$

Solution

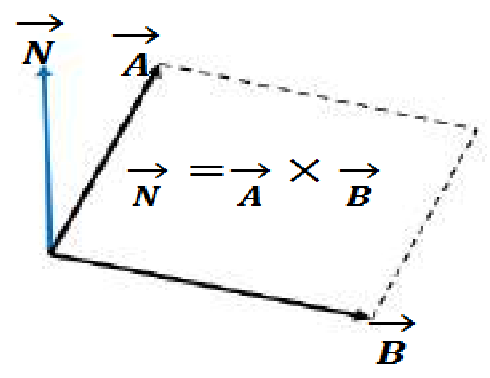
Assume vector C is the perpendicular vector on both vectors A&B.

$$\underset{C}{\rightarrow} = \underset{A}{\rightarrow} \times \underset{B}{\rightarrow} = \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 2 & -3 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} j + \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} k = 7i + 5j + k$$

$Unit\ vector\ of\ \underset{C}{\rightarrow} = \dfrac{7i + 5j + k}{\sqrt{(7)^2 + (5)^2 + (1)^2}} = \dfrac{1}{5\sqrt{3}}(7i + 5j + k)$**=1 unit length**

The new vector is 2 units length

$$= \frac{7i + 5j + k}{\sqrt{(7)^2 + (5)^2 + (1)^2}} = \frac{2}{5\sqrt{3}}(7i + 5j + k)$$



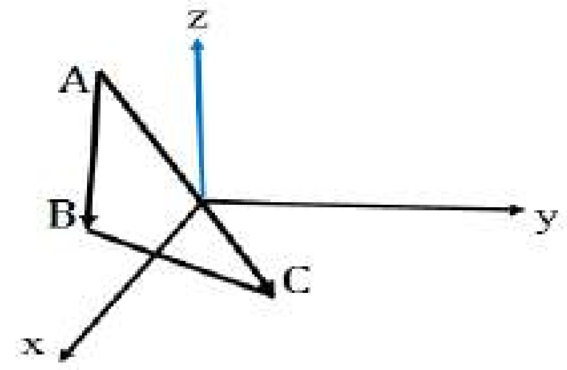
1.10.2 Area of Parallelogram

If $\left| \underset{N}{\rightarrow} \right| = Area\ of\ parallogram$

Then $Area\ of\ triangle = \dfrac{1}{2}(Area\ of\ parallogram)$

Example 1.15

Find the area of a triangle Δ ABC whose vertices are A(1,-1,3), B(2,0,1) and C(-1,2,-3) by using vector methods



Solution

$$Area\ of\ triangle = \frac{1}{2}\left|\underset{AB}{\to} \times \underset{AC}{\to}\right|$$

$$\underset{AB}{\to} = i + j - 2k \quad and \quad \underset{AC}{\to} = -2i + 3j - 6k$$

$$\underset{AB}{\to} \times \underset{AC}{\to} = \begin{vmatrix} i & j & k \\ 1 & 1 & -2 \\ -2 & 3 & -6 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} i - \begin{vmatrix} 1 & -2 \\ -2 & -6 \end{vmatrix} j + \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} k = 0i + 2j + 5k$$

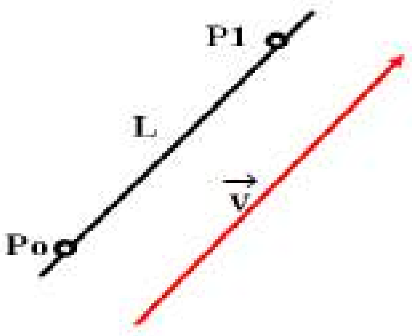
$$\left|\underset{AB}{\to} \times \underset{AC}{\to}\right| = \sqrt{10^2 + 5^2} = 5\sqrt{5}$$

$$= \frac{1}{2}(5\sqrt{5}) = \frac{5}{2}\sqrt{5}\ unit\ area$$

Area of the triangle Δ ABC

1.11 Equation of line in space

Suppose L is a straight line in space and parallel to vector V, L passes through the points Po &P1



$$\vec{V} = ai + bj + ck \qquad P_o(x_o, y_o, z_o), \qquad P_1(x, y, z)$$

$$\overrightarrow{P_oP_1} \text{ is parallel to } \vec{V}$$

$$\overrightarrow{P_oP_1} = t\vec{V} \qquad t \text{ is a scalar}$$

$$\overrightarrow{P_oP_1} = (ta)i + (tb)j + (tc)k$$

$$\overrightarrow{P_oP_1} = (x - x_0)i + (y - y_0)j + (z - z_0)k$$

By equating the two equations

$$ta = x - x_o\ , \qquad t = \frac{x - x_o}{a}$$

$$tb = y - y_o, \qquad t = \frac{y - y_o}{b}$$

$$tc = z - z_o, \qquad t = \frac{z - z_o}{c}$$

And then

$$x = at + x_o$$

$$y = bt + y_o$$

$$z = ct + z_o$$

These equations are called the parametric equations of the line and t is called the parameter.

Example 1.16

Find the parametric equations of a line that passes through the points A(1,2,-1) and B(-1,0,1).

Solution

$$\underset{V}{\rightarrow} = \underset{AB}{\rightarrow} = (-1-1)i + (0-2)j + (1+1)k = -2i - 2j + 2k$$

The parametric equations of the line are

$$x = x_o + at = 1 - 2t$$
$$y = y_o + bt = 2 - 2t$$
$$z = z_o + ct = -1 + 2t$$

Example 1.17

Find the parametric equations for the line that passes through the point (1,2,-3) and parallel to $\underset{V}{\rightarrow} = 4i + 5j - 7k$

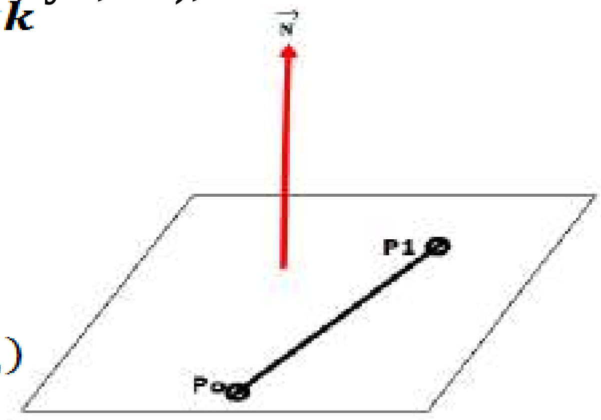
Solution

a=4, b=5, c= -7

$x = 1 + 4t$ $\quad y = 2 + 5t$ $\quad z = -3 - 7t$

1.12 Equation of plane in space

Suppose that a plane passing a through a point Po(xo, yo, zo), and perpendicular to the vector N. $\overrightarrow{N} = ai + bj + ck$



P(x, y, z) is any point in the plane .

Now $\overrightarrow{P_oP} = (x - x_o)i + (y - y_o)j + (z - z_o)k$

$\overrightarrow{P_oP}$ *and* $\overrightarrow{N}$ *are orthogonal*

and then $\overrightarrow{P_oP} \cdot \overrightarrow{N} = 0 = a(x - x_o) + b(y - y_o) + c(z - z_o)$

This is the equation of plane, and can be

$$\boxed{\begin{array}{l} ax + by + cz = d \\ d = ax_o + by_o + cz_o \end{array}}$$

Example 1.18

Find an equation of the plane passing through the point (3,-1,7) and perpendicular to the vector $\overrightarrow{N} == 4i + 2j - 5k$

Solution

$$a(x - x_o) + b(y - y_o) + c(z - z_o) = 0$$

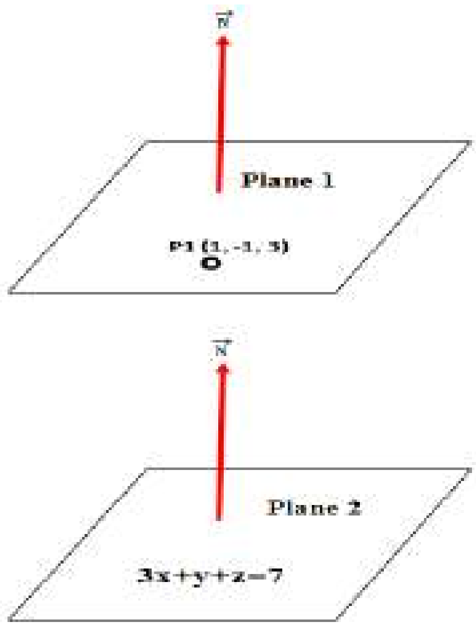
$$4(x - 3) + 2(y + 1) + (5)(z - 7)$$

$$4x - 12 + 2y - 2 - 5z + 35 = 0$$

$$4x + 2y - 5z = -25 \quad \textit{The equation of plane}$$

Example 1.19

Find the equation of the plane that passes through the point $P_o(1, -1, 3)$ and is parallel to the plane 3x + y + z =7.



Solution

$$ax + by + cz = d \qquad 3x + y + z = 7$$

Because both vectors are parallel, Vector N is normal on both planes. $\overrightarrow{N} = 3i + j + k$

$$a(x - x_o) + b(y - y_o) + c(z - z_o) = 0$$

$$3(x - 1) + (y + 1) + (z - 3) = 0$$

$$3x + y + z = 5 \quad \textit{The equation of plane}$$

Example 1.20

Find the equation of the plane that passes through the points A(1,1,-1), B(2, 0, 2), and C (0, -2, 1).

Solution

$$\overrightarrow{AB} = i - 3j + 3k \qquad \overrightarrow{AC} = -i - 3j + 2k$$

Now both vectors AB and AC are on the plane.

From cross vector, we got the normal vector

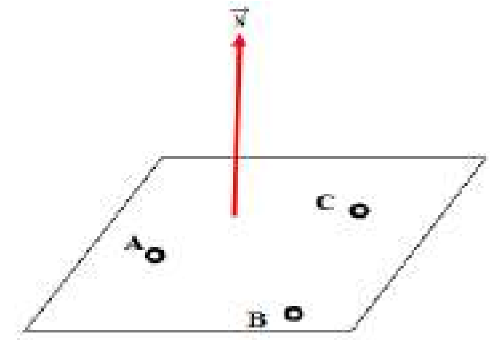
$$\overrightarrow{AB} \times \overrightarrow{AC} = \overrightarrow{N}$$

$$\underset{N}{\rightarrow} = \underset{AB}{\rightarrow} \times \underset{AC}{\rightarrow} = \begin{vmatrix} i & j & k \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix}$$

vector N is normal on the plane

$$= (-2+9)i - (2+3)j + (-3-1)k$$

$$= 7i - 5j - 4k$$



The equation of the plane

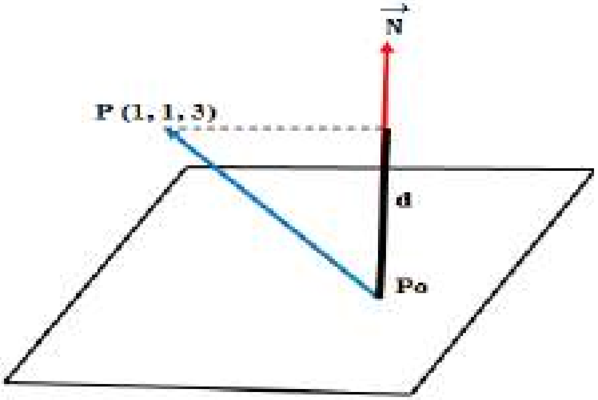
$$a(x - x_o) + b(y - y_o) + c(z - z_o) = 0$$

$$7(x-1) - 5(y-1) - 4(z+1) = 0$$

$$7x - 5y - 4z = 6 \quad \textit{The equation of plane}$$

Example 1.21

Find the distance from the point $P_1(1, 1, 3)$ to the plane $3x+2y+6z=6$



Solution

Let as to take a point on the plane

$x= 0, z = 0$ and then $2y = 6$

The point is $P_o\ (0, 3, 0)$

$$\overrightarrow{P_oP_1} = i - 2j + 3k \ and \ \vec{N} = 3i + 2j + 6k$$

$$Distance = d = Proj_{\vec{N}}^{\overrightarrow{P_oP_1}} = \frac{\overrightarrow{P_oP_1} \cdot \vec{N}}{\left|\vec{N}\right|}$$

$$= \frac{(3)(1) + (-1)(2) + (3)(6)}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{17}{7} \ unit\ length$$

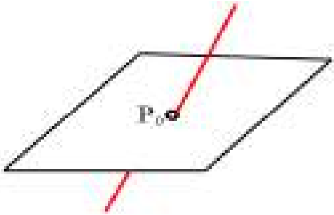
Example 1.22

Find the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z}{4}$ with the plane $x+2y+z=11$.

Solution $t = \frac{x-2}{3} = \frac{y+1}{2} = \frac{z}{4}$

$$x = 2 + 3t \quad y = -1 + 2t \quad z = 4t$$

Then Sub the parametric equations in the equation of the plane



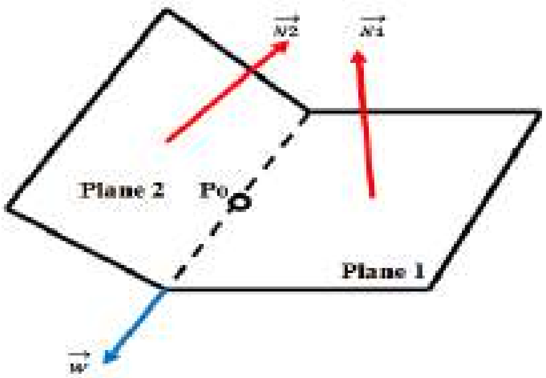
$$(2 + 2t) + 2(-1 + 2t) + (4t) = 11$$

$$11t = 11 \quad then\ t = 1$$

$$x = 2 + 3 = 5, \qquad y = -1 + 2 = 1, \qquad z = 4$$

Example 1.23

Find the parametric equations of the line of the intersection of the two planes x - y +z =3 and x+y+2z=9 .



Solution

$$\overrightarrow{N1} = i - j + k \qquad \overrightarrow{N2} = i + j + 2k$$

$$\overrightarrow{W} = \overrightarrow{N1} \times \overrightarrow{N2} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$\overrightarrow{w} = -3i - j + 2k$$

$$z - y = 3 - - - - (1)$$

$$y + 2z = 9 - - - - (2)$$

To find a point in the intersection line

Let x=0 , and sub it in both planes

$$x = x_o + at = 0 + (-3)t = -3t$$

$$y = y_o + bt = 1 + (-1)t = 1 - t$$

$$z = z_o + ct = 4 + (2)t = 4 + 2t$$

Z= 4 & y=1 and the point (0, 1, 4) lies on the intersection line of both planes

The parametric equations of the line are

=

1.13 Triple Product

1.13.1 Scalar triple product

If

$\underset{A}{\rightarrow} = a_1 i + a_2 j + a_3 k$

$\underset{B}{\rightarrow} = b_1 i + b_2 j + b_3 k$

$\underset{C}{\rightarrow} = c_1 i + c_2 j + c_3 k$

The number $\underset{A}{\rightarrow} \cdot \left(\underset{B}{\rightarrow} \times \underset{C}{\rightarrow}\right)$ **is called the scalar triple product of** $\underset{A}{\rightarrow}, \underset{B}{\rightarrow}$ & $\underset{C}{\rightarrow}$

$$\underset{A}{\rightarrow} \cdot \left(\underset{B}{\rightarrow} \times \underset{C}{\rightarrow}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Note that $\underset{A}{\rightarrow} \cdot \left(\underset{B}{\rightarrow} \times \underset{C}{\rightarrow}\right) = \underset{C}{\rightarrow} \cdot \left(\underset{A}{\rightarrow} \times \underset{B}{\rightarrow}\right) = \underset{B}{\rightarrow} \cdot \left(\underset{C}{\rightarrow} \times \underset{A}{\rightarrow}\right)$

Example 1.24

Find the scalar triple product $\underset{U}{\rightarrow} \cdot \left(\underset{V}{\rightarrow} \times \underset{W}{\rightarrow}\right)$ **of the vectors U= 3i-2j-5k, V= i+4j-4k and W=3j+2k.**

Solution

$$\underset{U}{\rightarrow} \cdot \left(\underset{V}{\rightarrow} \times \underset{W}{\rightarrow}\right) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 3(8 + 12) + 2(2 - 0) - 5(3 - 0)$$

$$= 49$$

1.13.2 vector triple product

If

$$\underset{A}{\rightarrow} = a_1 i + a_2 j + a_3 k$$
$$\underset{B}{\rightarrow} = b_1 i + b_2 j + b_3 k$$
$$\underset{C}{\rightarrow} = c_1 i + c_2 j + c_3 k$$

Then $\left(\underset{A}{\rightarrow} \times \underset{B}{\rightarrow}\right) \times \underset{C}{\rightarrow} = \left(\underset{A}{\rightarrow} . \underset{C}{\rightarrow}\right) \underset{B}{\rightarrow} - \left(\underset{B}{\rightarrow} . \underset{C}{\rightarrow}\right) \underset{A}{\rightarrow}$ **this called vector triple product**

Example 1.25

If $\underset{A}{\rightarrow} = i - j + 2k$, $\underset{B}{\rightarrow} = 2i + j + k$ **and** $\underset{C}{\rightarrow} = i + 2j - k$ $find$ $\left(\underset{A}{\rightarrow} \times \underset{B}{\rightarrow}\right) \times \underset{C}{\rightarrow}$

Solution

$$\left(\underset{A}{\rightarrow} \times \underset{B}{\rightarrow}\right) \times \underset{C}{\rightarrow} = \left(\underset{A}{\rightarrow} . \underset{C}{\rightarrow}\right) \underset{B}{\rightarrow} - \left(\underset{B}{\rightarrow} . \underset{C}{\rightarrow}\right) \underset{A}{\rightarrow}$$

$$\left(\underset{A}{\rightarrow} . \underset{C}{\rightarrow}\right) = -3 \;\; and \left(\underset{B}{\rightarrow} . \underset{C}{\rightarrow}\right) = 3$$

$$\left(\underset{A}{\rightarrow} \times \underset{B}{\rightarrow}\right) \times \underset{C}{\rightarrow} = (-3)\,(2i + j + k) - (3)\,(i - j + 2k) = -9i - 9k$$

OR

$$\underset{A}{\rightarrow} \times \underset{B}{\rightarrow} = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = -3i + 3j + 3k$$

$$\left(\underset{A}{\rightarrow} \times \underset{B}{\rightarrow}\right) \times \underset{C}{\rightarrow} = \begin{vmatrix} i & j & k \\ -3 & 3 & 3 \\ 1 & 2 & -1 \end{vmatrix} = -9i - 9k$$

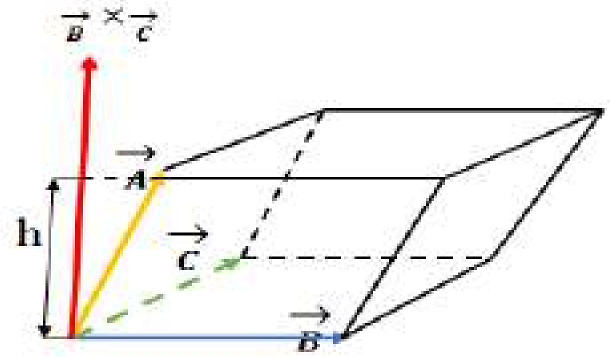
1.13.3 volume of parallelepiped

If

$$\underset{A}{\rightarrow} = a_1 i + a_2 j + a_3 k$$

$$\underset{B}{\rightarrow} = b_1 i + b_2 j + b_3 k$$

$$\underset{C}{\rightarrow} = c_1 i + c_2 j + c_3 k$$



The volume of the parallelepiped is

Volume = $\left|\underset{A}{\rightarrow} \cdot \left(\underset{B}{\rightarrow} \times \underset{C}{\rightarrow}\right)\right|$ = (area of parallelogram) . (height)

$$Height = h = proj_{\underset{B}{\rightarrow} \times \underset{C}{\rightarrow}}^{\underset{A}{\rightarrow}} = \frac{\underset{A}{\rightarrow} \cdot \left(\underset{B}{\rightarrow} \times \underset{C}{\rightarrow}\right)}{\left|\underset{B}{\rightarrow} \times \underset{C}{\rightarrow}\right|}$$

$$Volume = \left|\underset{B}{\rightarrow} \times \underset{C}{\rightarrow}\right| \cdot \frac{\underset{A}{\rightarrow} \cdot \left(\underset{B}{\rightarrow} \times \underset{C}{\rightarrow}\right)}{\left|\underset{B}{\rightarrow} \times \underset{C}{\rightarrow}\right|} = \underset{A}{\rightarrow} \cdot \left(\underset{B}{\rightarrow} \times \underset{C}{\rightarrow}\right)$$

Example 1.26

Find the volume of the box (parallelepiped) that determined by

$$\underset{A}{\rightarrow} = i + 2j - k, \ \underset{B}{\rightarrow} = -2i + 3k, and \ \underset{C}{\rightarrow} = 7j - 4k$$

Solution

$$volume \ is \ equal \ the \ absolute \ of \ \underset{A}{\rightarrow} \cdot \left(\underset{B}{\rightarrow} \times \underset{C}{\rightarrow}\right)$$

$$\underset{A}{\rightarrow} \cdot \left(\underset{B}{\rightarrow} \times \underset{C}{\rightarrow}\right) = (i + 2j - k) \cdot \begin{vmatrix} i & j & k \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}$$

$$= (i + 2j - k) \cdot (-21i - 8j - 14k) = -21 - 16 + 14 = -23$$

$$volume = |-23| = 23 \ unit \ volume$$

Assignment 1 (Vectors)

1- Given A = 2i − 3j − 3k, B = i + j + 2k, and C = 3i − 2j − k,
find the angles between the following pairs of vectors:
(a) A + B, B − 2C. (b) 2A − C, A + B − C. (c) B + 3C, A − 2C.

2- Find the vector AB from the following of pairs of points
(a) A(1, 2, 5) & B (2, -3, 9) (b) A(-3, 0, 7) & B (4, -8, 0)

3- Find the initial point of the vector $\underset{A}{\rightarrow} = 5i + 4j - 6k$ **if the terminal point is**
(a) (5, 4, 1) (b) (4, 1, 3)

4- Find unit vector that has the same direction of the vector from A (5,1,3) to b(3,7,6)

5- By using dot product, find the angle between the following pairs of vectors
(a) $\underset{A}{\rightarrow} = i + 2j - 3k$ $\underset{B}{\rightarrow} = -i + j + 5k$ **(b)** $\underset{A}{\rightarrow} = 4i - 2j$, $\underset{B}{\rightarrow} = 7i + 4j + 2k$

6- Find the cross product of the following pairs of vectors
(a) $\underset{A}{\rightarrow} = 2i - j + 3k,$ $\underset{B}{\rightarrow} = i - 4j + 5k$ **(b)** $\underset{A}{\rightarrow} = i - 2j + 4k,$ $\underset{B}{\rightarrow} = -i + 2k$

7- Given that A = i + 2j + 2k and B = 2i − 3j + k, find (a) the projection of A onto the line of B, and (b) the projection of B onto the line of A.

8- By using vectors rules, Find the area of the triangle that has vertices A(2, 5, 3) B(4, 2, 4) and C(2,1,4).

9- Find the parametric equations of the line that passes through the point Po(3, 4, 5) and parallel to the vector A=2i+5j-6k.

10- Find an equation of the plane that passes through the point Po(2, 2, 2) and parallel to the plane 2x+5y+7z=5.

11- Find the distance between two parallel planes 4x-2y+7z=-12 and 4x-2y+7z=0.

12- Show that the lines L1 and L2 are parallel and also find the distance between them.

L1: x=2-t, y=2t, z=1+t L2: 1+2t, y=3-4t, z=5-2t

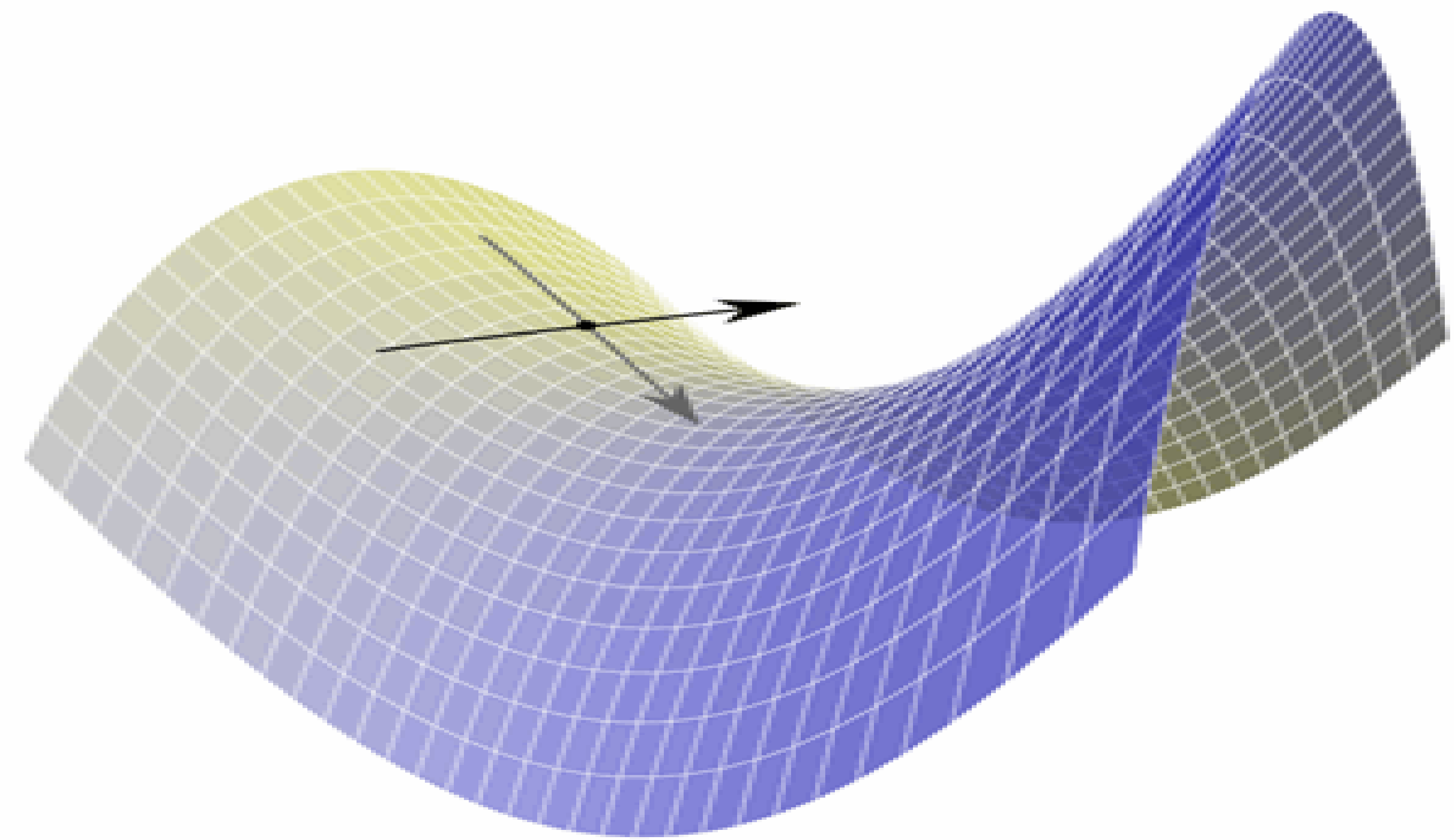
13- Find an equation of plane that passes through the point (-1, 4, 2) and contains the line of intersection of the planes 4x-y+z=2 and 2x+y-2z =3.

14- Find the volume of the parallelepiped that determined by

$$\underset{A}{\rightarrow} = i - 2j + 4k, \quad \underset{B}{\rightarrow} = -i + 2k \text{ and } \underset{C}{\rightarrow} = 2i + 3j - 4k$$

Chapter Two

Partial derivatives



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2.1 Limits and continues of function with two variables

Recall that for a function of one variable, the mathematical statement

$$\lim_{x \to c} f(x) = L$$

means that for x close enough to c, the difference between f(x) and L is "small". Very similar definitions exist for functions of two or more variables;

$$\lim_{(x,y) \to (x_o,y_o)} f(x,y) = L$$

$$|f(x,y) - L| < e$$

$$(x_o, y_o)$$

A function f of two variables is continuous at a point (x_o, y_o) if

$$\lim_{(x,y) \to (x_o,y_o)} f(x,y) = (x_o, y_o)$$

1- $f(x_o, y_o)$ is defined

2- $\lim_{(x,y) \to (x_o,y_o)} f(x,y)$ is exit

3-

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For fully derivative

$$\frac{\delta f(x,y)}{\delta x} = \lim_{\Delta y \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\delta f(x,y)}{\delta y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x,y)}{\Delta y}$$

2.2First and higher order partial derivatives.

2.2.1 First order partial derivatives

In mathematics, a partial derivative of a function of several variables is its derivative with respect to one of those variable with the other held constant.

The partial derivative of a function f(x , y, ….) with respect to the variable is variously denoted by

$$f'_x, f_x, \partial_x f, D_x f, D_1 f, \frac{\partial}{\partial x} f, or\ \frac{\partial f}{\partial x}$$

$$f(x,y) = 2x^2 + 5y^3 - 2xy + ysinx + xcosy$$

Example 2.1

Find the first partial derivative of the

$$f_x = 4x - 2y + ycosx + cosy$$

Solution

$$f_y = 15y^2 - 2x + sinx - xsiny$$

$$f(x,y) = x^4 sin(xy^3)$$

Example 2.1

$$f_x = x^4 y^3 cos(xy^3) + 4x^3 sin(xy^3)$$

Find the first partial derivative of the

$$f_y = x^4 cos(xy^3) 3xy^2 = 3x^5 y^2 cos(xy^3)$$

Solution

2.2.2Higher order partial derivatives f(x , y)

2.2.2.1 second-order partial derivatives

It cab be denoted by

$$f_{xx}, \frac{\partial 2f}{\partial x^2}, or\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy}, \frac{\partial 2f}{\partial y^2}, or\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy}, \frac{\partial 2f}{\partial y \partial x}, or\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}, \frac{\partial 2f}{\partial x \partial y}, or\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Example 2.2

Find the second order partial derivatives of $f(x, y) = 5xy^3 - 2xy$

Solution

$$f_x = 5y^3 - 2y\ and\ f_{xx} = 0$$
$$f_y = 15xy^2 - 2x\ and\ f_{yy} = 30xy$$
$$f_{xy} = 15y^2 - 2$$
$$f_{yx} = 15y^2 - 2$$

Example 2.3

if f(x ,t) = sin(x-ct), show that $\frac{\partial 2f}{\partial t^2} = c^2 \frac{\partial 2f}{\partial x^2}$

Solution

$$\frac{\partial f}{\partial t} = (-c)\ cos(x - ct)\ then\ \frac{\partial 2f}{\partial t^2} = -c^2 sin(x - ct)$$
$$\frac{\partial f}{\partial x} = \ cos(x - ct)\ then\ \frac{\partial 2f}{\partial x^2} = -sin(x - ct)$$
$$\frac{\partial 2f}{\partial t^2} = c^2 \frac{\partial 2f}{\partial x^2}$$

2.2.3 Third-order partial derivatives f(x, y)

$$f_{xxx} = \frac{\partial}{\partial x}\left(\frac{\partial 2f}{\partial x^2}\right) \quad f_{yyy} = \frac{\partial}{\partial y}\left(\frac{\partial 2f}{\partial y^2}\right) \quad f_{xyy} = \frac{\partial}{\partial y}\left(\frac{\partial 2f}{\partial y \partial x}\right) \quad f_{yxx} = \frac{\partial}{\partial x}\left(\frac{\partial 2f}{\partial x \partial y}\right)$$

Example 2.4

$Find\ f_{xxx}, f_{yyy}, f_{xyy}\ and\ f_{yxx}\ of\ f(x,y) = \sin xy^2$

Solution

$$f_x = y^2 \cos xy^2, then\ f_{xx} = -y^4 \sin xy^2\ then\ f_{xxx} = -y^6 \cos xy^2$$

$$f_y = 2xy \cos xy^2, then\ f_{yy} = -4x^2y^2 \sin xy^2 + 2x \cos xy^2\ then\ f_{yyy} = \ldots$$

$$f_x = y^2 \cos xy^2, then\ f_{xy} = -2xy^3 \sin xy^2 + 2y \cos xy^2\ then\ f_{xyy} = \ldots$$

$$f_y = 2xy \cos xy^2, then\ f_{yx} = -2xy^3 \sin xy^2 + 2y \cos xy^2\ then\ f_{yxx} = \ldots$$

2.2.4 Fourth-order partial derivatives f(x, y)

$$f_{xxxx} = \frac{\partial}{\partial x}\left(\frac{\partial 3f}{\partial x^3}\right)$$

$$f_{yyyy} = \frac{\partial}{\partial y}\left(\frac{\partial 3f}{\partial y^3}\right)$$

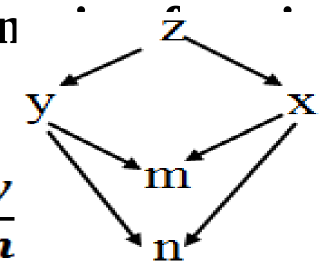
$$f_{xxyy} = \frac{\partial}{\partial y}\left(\frac{\partial 3f}{\partial y \partial x^2}\right)$$

$$f_{yyxx} = \frac{\partial}{\partial x}\left(\frac{\partial 3f}{\partial x \partial y^2}\right)$$

2.3 Chain rule of composite functions and total differential.

2.3.1 Chain rule (Function of function)

If z is a function to x and y, and x is a function to m and n, then n to m and n indirectly.



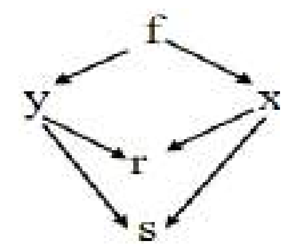
Its possible to find the derivative of z respect to m and n

$$\frac{\partial z}{\partial m} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial m} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial m} \qquad \frac{\partial z}{\partial n} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial n} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial n}$$

and

Example 2.5

$$f = x^2 + y^2, \qquad x = r\cos s \;, y = e^s - \sin r \quad find\ f_r and\ f_s$$



$$f_r = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} = (2x)(\cos s) + (2y)(-\cos r)$$

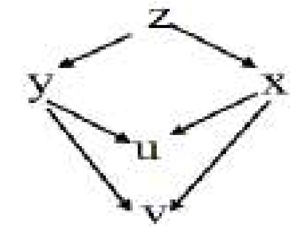
$$= 2x \cos s - 2y \cos r$$

$$f_s = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} = (2x)(-r\sin s) + (2y)(e^s)$$

$$= -2rx \sin s - 2y\, e^s$$

Example 2.6

If $Z = e^{x^2 y}$ $, x = u + v,\qquad y = \frac{2u}{v}\,,\qquad find\; z_u and\; z_v$



Solution

$$z_u = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \left(2xye^{x^2y}\right)(1) + \left(x^2e^{x^2y}\right)\left(\frac{2}{v}\right) = e^{x^2y}\left(2xy + \frac{2x^2}{v}\right)$$

$$= e^{(u+v)^2\left(\frac{2u}{v}\right)}\left(2(u+v)\left(\frac{2u}{v}\right) + \frac{2(u+v)^2}{v}\right) = e^{\frac{2u^3}{v}+4vu^2+2uv}\left(\frac{4u^2}{v} + 8u + 2v\right)$$

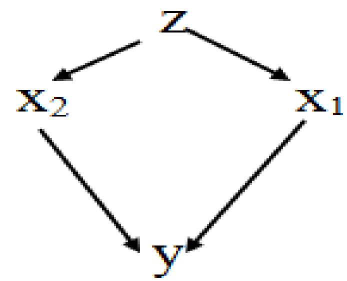
$$z_v = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \left(2xye^{x^2y}\right)(1) + \left(x^2e^{x^2y}\right)\left(-\frac{2u}{v^2}\right) = e^{x^2y}\left(2xy - \frac{2ux^2}{v^2}\right)$$

$$= e^{(u+v)^2\left(\frac{2u}{v}\right)}\left(2(u+v)\left(\frac{2u}{v}\right) - \frac{2u(u+v)^2}{v^2}\right) = e^{\frac{2u^3}{v}+4vu^2+2uv}\left(2u - \frac{2u^3}{v^2}\right)$$

2.3.2 Total differential

If Z is a function of xs $Z = f(x_1, x_2, \dots, x_n)$

$x_1, x_2, \dots, x_n$ *are function of y then*



$$dz = \frac{\partial z}{\partial x_1}.dx_1 + \frac{\partial z}{\partial x_2}.dx_2 + \cdots + \frac{\partial z}{\partial x_n}.dx_n$$

dz is called total differential of z

$$\frac{dz}{dy} = \frac{\partial z}{\partial x_1}.\frac{dx_1}{dy} + \frac{\partial z}{\partial x_2}.\frac{dx_2}{dy} + \cdots + \frac{\partial z}{\partial x_n}.\frac{dx_n}{dy}$$

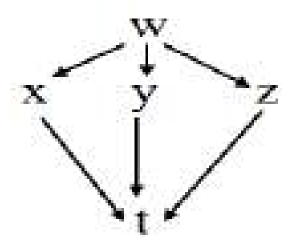
Also

Example 2.7

Given

$$w = x^2 + y^2 + z^2, \qquad where\ x = e^t \sin t, y = e^t cost, \qquad z = e^t$$

$$find\ \frac{dw}{dt}?$$



Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}.\frac{dx}{dt} + \frac{\partial w}{\partial y}.\frac{dy}{dt} + \frac{\partial w}{\partial z}.\frac{dz}{dt}$$

$$\frac{dw}{dt} = (2x)(e^t sint + e^t cost) + (2y)(e^t cost - e^t sint) + (2z)(e^t)$$

$$\frac{dw}{dt} = (2e^t sint)(e^t sint + e^t cost) + (2e^t cost)(e^t cost - e^t sint) + (2e^t)(e^t)$$

$$\frac{dw}{dt} = 2e^{2t}(sin^2 t + sintcost + cos^2 t - sintcost + 1) = 4e^{2t}$$

2.4 Directional derivatives

The directional derivatives of a function ($w = f(x,y)$) is defined as

$$\frac{df}{ds} = \nabla f \cdot \vec{u} = D_u f = |\nabla f| \left|\vec{u}\right| \cos\theta = |\nabla f| \cos\theta$$

Where, $\frac{df}{ds}$ **is the directional derivatives of W in the direction of** $\vec{u}$

$\vec{u}$ *is unit vectoe*

$\nabla f = \nabla w = w_x i + w_y j + w_z k$ **= the gradient of W**

Example 2.8

Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ **at Po(1,1,0) in the direction of vector** $\vec{A} = 2i - 3j + 6k$

Solution

$$\vec{u} = \frac{\vec{A}}{\left|\vec{A}\right|} = \frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}k$$

$f_x = 3x^2 - y^2,\ \ then\ f_x|_{Po} = 2$

$f_y = -2xy,\ \ then\ f_y|_{Po} = -2$

$f_z = -1\ then\ f_z|_{P_o} = 1$

$$\nabla f|_{Po} = f_x|_{P_o} i + f_y\big|_{P_o} j + f_z|_{P_o} k = 2i - 2j - k$$

The directional derivative is $\dfrac{df}{ds} = \nabla f|_{Po} . \underset{u}{\rightarrow}$

$$\frac{df}{ds} = (2i - 2j - k).\left(\frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}k\right) = \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}$$

$$f(x, y, z) = xe^y + yz$$

Example 2.9

Find how much will change if the point P(x,y,z) is moved from Po(2,0,0) straight toward P1(4,1,-2) a distance of ds=0.1 units.

$$\underset{P_oP_1}{\rightarrow} = 2i + j - 2k \qquad \underset{u}{\rightarrow} = \frac{\overrightarrow{P_oP_1}}{\left|\overrightarrow{P_oP_1}\right|} = \frac{2}{3}i + \frac{1}{3}j - \frac{2}{3}k$$

Solution

$$\nabla f|_{Po} = f_x|_{P_o} i + f_y\big|_{P_o} j + f_z|_{P_o} k$$

$$f_x = e^y \quad then\ f_x|_{Po} = 1$$

$$f_y = xe^y + z, \quad then\ f_y\big|_{Po} = 2$$

$$f_z = y \quad then\ f_z|_{Po} = 0$$

$$\nabla f|_{Po} = f_x|_{P_o} i + f_y\big|_{P_o} j + f_z|_{P_o} k = i + 2j$$

$$\frac{df}{ds} = \nabla f|_{Po} . \underset{u}{\rightarrow} = (1 + j)\left(\frac{2}{3}i + \frac{1}{3}j - \frac{2}{3}k\right) = \frac{4}{3}$$

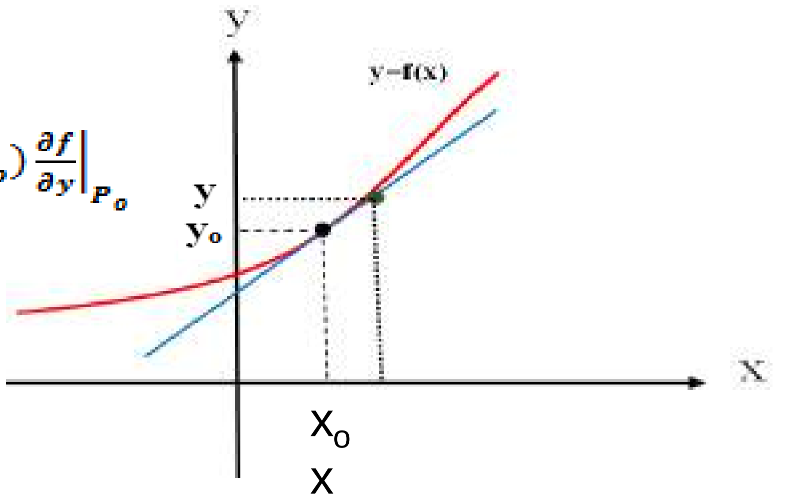
$$ds = (0.1).\left(\frac{4}{3}\right) = 0.13$$

2.5 Linear Approximation of Function

The linear approximation Of function

f(x,y) near the point Po(x_o, y_o) is

$$L(x,y) = f(x,y) \cong f(x_o,y_o) + (x-x_o)\frac{\partial f}{\partial x}\Big|_{P_o} + (y-y_o)\frac{\partial f}{\partial y}\Big|_{P_o}$$



Example 2.10

Show that

$$f(x,y) = \frac{1}{1+x-y}$$

Can be approximation near the Po(0,0) by

1-x+y

Solution

$$f(x,y) \cong f(x_o,y_o) + (x-x_o)\frac{\partial f}{\partial x}\Big|_{P_o} + (y-y_o)\frac{\partial f}{\partial y}\Big|_{P_o}$$

$$\frac{\partial f}{\partial x} = \frac{-1}{(1+x-y)^2}\Big|_{P_o} = -1$$

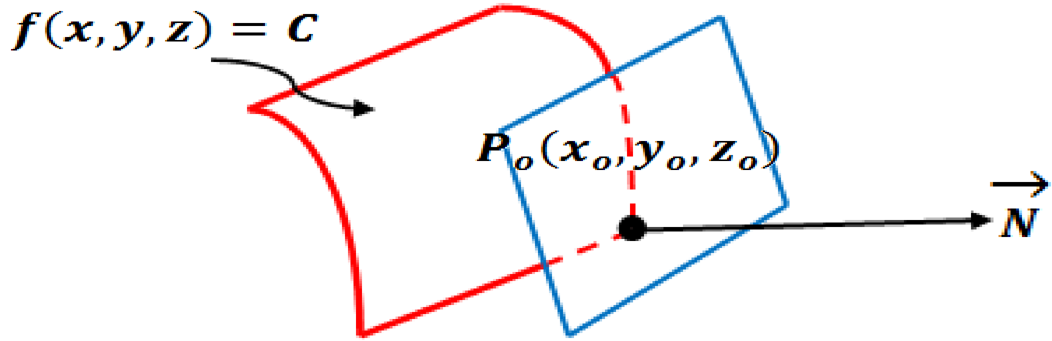
$$\frac{\partial f}{\partial y} = \frac{1}{(1+x-y)^2}\Big|_{P_o} = 1$$

$$f(0,0) = 1$$

$$f(x,y) = 1 + (x-0)(-1) + (y-0)(1) = 1 - x + y$$

2.6 Tangent plane and normal lines

If the equation of a surface is defined by f(x, y ,z)=c and passes through the point Po(x_o , y_o , z_o) as shown.



$$\nabla f = f_x i + f_y j + f_z k = \vec{u}$$

The normal line to the surface at point Po is

$$x = x_o + f_x|_{P_o} . t$$

$$y = y_o + f_y|_{P_o} . t$$

$$z = z_o + f_z|_{P_o} . t$$

$$\frac{x - x_o}{f_x} = \frac{y - y_o}{f_y} = \frac{z - z_o}{f_z}$$

Or

$$f_{x_{P_o}}(x - x_o) + f_{y_{P_o}}(y - y_o) + f_{z_{P_o}}(z - z_o) = 0$$

The tangent plane of surface at point Po is

Example 2.11

Find the equation of the tangent plane and normal line of the surface that has the function $f(x,y,z) = x^2 + y^2 + z - 9 = 0$ **at point Po(1,2,3)**

Solution

$$f_{x_{P_o}}(x - x_o) + f_{y_{P_o}}(y - y_o) + f_{z_{P_o}}(z - z_o) = 0$$

$$\nabla f = f_x i + f_y j + f_z k = 2xi + 2yj + k$$

$$\nabla f|_{P_o} = 2i + 4j + k$$

$$2(x - x_o) + 4(y - y_o) + (z - z_o) = 0$$

$$2(x - 1) + 4(y - 2) + (z - 3) = 0$$

The tangent plane is

$$2x - 2 + 4y - 8 + z - 3 = 0$$

$$2x + 4y + z = 13$$

$$x = 1 + 2t$$

$$y = 2 + 4t$$

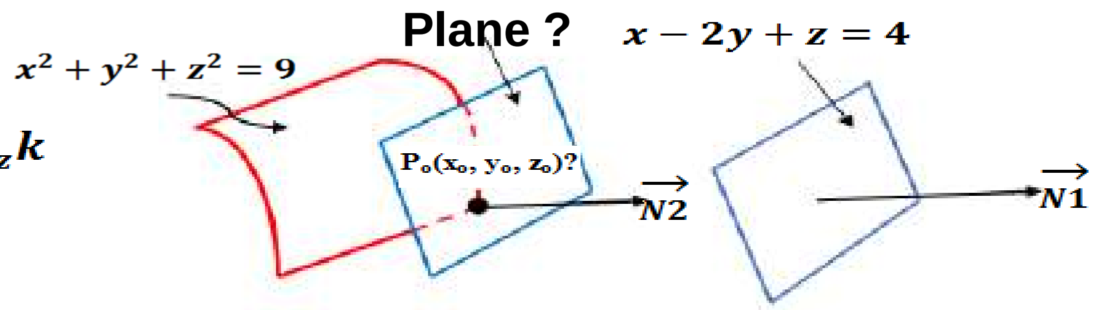
$$z = 3 + t$$

The normal line is

Example 2.12

Find the point on the surface $x^2+y^2+z^2=9$ **at which the tangent palne that is parallel to the plane** $x-2y+z=4$

Solution



$$\overrightarrow{N1}=i-2j+k$$

$$\overrightarrow{N2}=gradf=\nabla f=f_x i+f_y j+f_z k$$

$$\overrightarrow{N2}=2x_o i+2y_o j+2z_o k$$

$$\overrightarrow{N1}//\overrightarrow{N2} \quad then \quad \overrightarrow{N1}\times\overrightarrow{N2}=0$$

$$\overrightarrow{N1}\times\overrightarrow{N2}=\begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 2x_o & 2y_o & 2z_o \end{vmatrix}=0$$

$$(-4z_o-2y_o)i+(2z_o-2x_o)j+(2y_o+4x_o)k=0$$

$$-4z_o-2y_o=0 \ then \ y_o=-2z_o \ \dots \ (1)$$

$$2z_o-2x_o=0 \ then \ x_o=z_o \ \dots \ (2)$$

Sub. Po(xo, yo, zo) in equ. $x^2+y^2+z^2=9$ **To get** $x_o^2+y_o^2+z_o^2=9 \ \dots (3)$

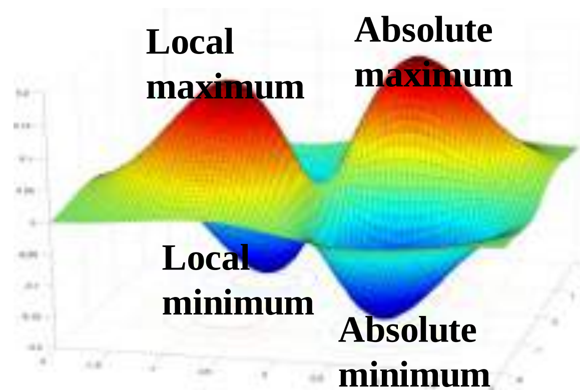
$$z_o^2+4z_o^2+z_o^2=9$$

$$6z_o^2=9 \ \ then \ z_o=\mp\sqrt{\frac{3}{2}} \quad x_o=\mp\sqrt{\frac{3}{2}} \ \ and \ y_o=\mp 2\sqrt{\frac{3}{2}}$$

2.7. Maximum and minimum values

One of the main uses of ordinary derivatives is finding maximum and minimum values. In this section we are going to see how the partial derivatives are used to find the local maximum and minimum values of the function for two or more variables.

$$f_x = 0 \; and \; f_y = 0 \;\; at \; a \; point(a, b)$$



This point called critical point

Whether absolute point or local point

$$D = f_{xx}\big|_{P(a,b)} \cdot f_{yy}\big|_{P(a,b)} - \left(f_{xy}\big|_{P(a,b)}\right)^2$$

Its possible to test the function to know the critical point from this equation

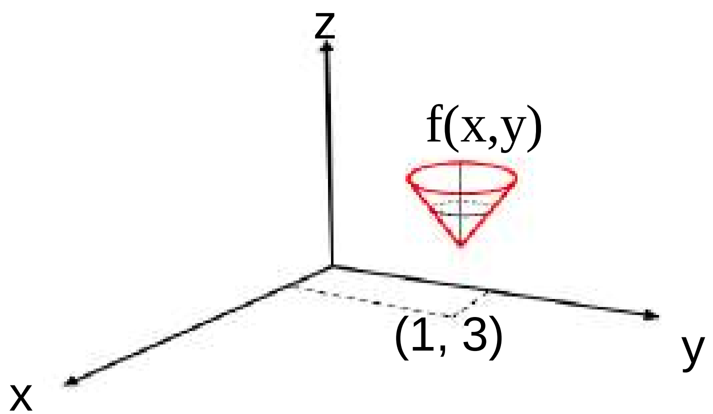
(a) D> 0 and fxx at (a.b) > 0 then f(a,b) is local minimum

(b) D > 0 and fxx at (a.b) < 0 then f(a,b) is local maximum

Example 2.13

Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$ **find the critical point**

Solution



$$f_x = 2x - 2$$

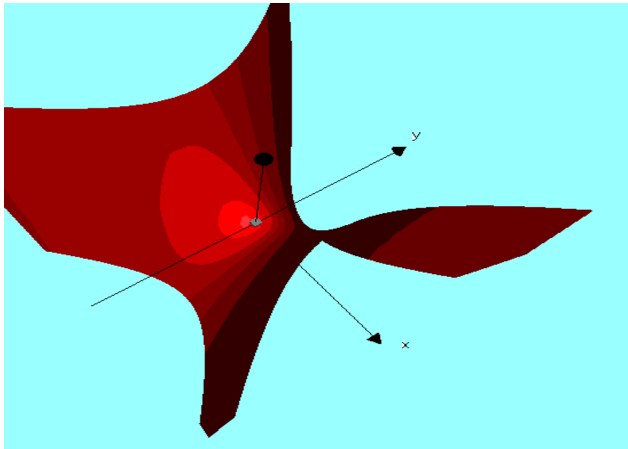
$$f_y = 2y - 6$$

$$if\ f_x = 0\ then\ x = 1$$

$$if\ f_y = 0\ then\ y = 3$$

$$z|_{(1,3)} = 1^2 + 3^2 - 2 - 18 + 14 = 4$$

The critical point is (1,3,4)



$$f(x,y) = y^2 - x^2$$

$$f_x = -2x\ and\ f_y = 2y$$

Example 2.14

Find the critical point

Solution

For points on the y-axis (x=0)

$$f(0,0) = 0$$

$$f(x,y) = -x^2 < 0$$

$$f(x,y) = y^2 > 0$$

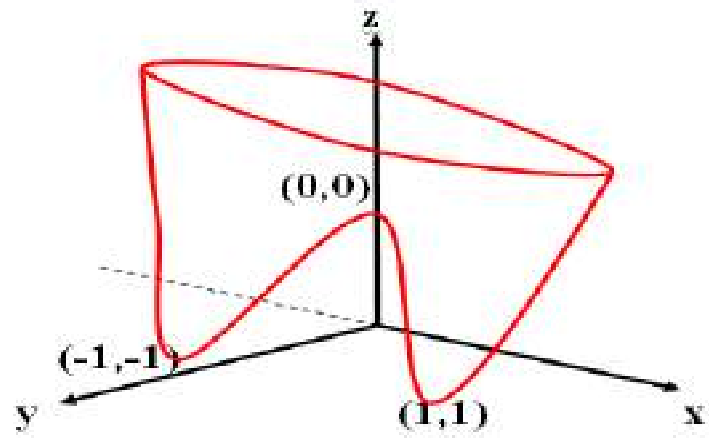
The critical point is (0,0)

For points on the x-axis (y=0)

Example 2.15

Find the local maximum and minimum values a

$$f(x,y) = x^4 + y^4 - 4xy + 1$$



Solution

$$f_x = 4x^3 - 4y = 0\ , \qquad f_y = 4y^3 - 4x = 0$$

To find the critical points

$$x^3 - y = 0 \quad and \quad y^3 - x = 0$$

$$y = x^3 \quad and \quad x = y^3$$

$$x^9 - x = 0 = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

To find the X values

x= 0,1,-1

y=0,1,-1

the critical points

(0,0), (1,1), (-1,-1)

$$f_{xx} = 12x^2 \qquad f_{yy} = 12y^2 \qquad f_{xy} = -4$$

$$D_{(x,y)} = f_{xx}f_{yy} - \left(f_{xy}\right)^2 = 144x^2y^2 - 16$$

$$D_{(0,0)} = 144x^2y^2 - 16 = -16 < 0\ its\ a\ saddle\ point$$

$$D_{(1,1)} = 144 - 16 = 128 > 0 \quad f_{xx(1,1)} = 12 > 0\ it\ is\ a\ local\ \ minimum$$

$$D_{(-1,-1)} = 144 - 16 = 128 > 0 \quad f_{xx(-1,-1)} = 12 > 0\ it\ is\ a\ local\ \ minimum$$

2.8Absolute maximum and minimum values

To find the absolute maximum and minimum values of continuous function f (x,y) on a closed bounded set D.

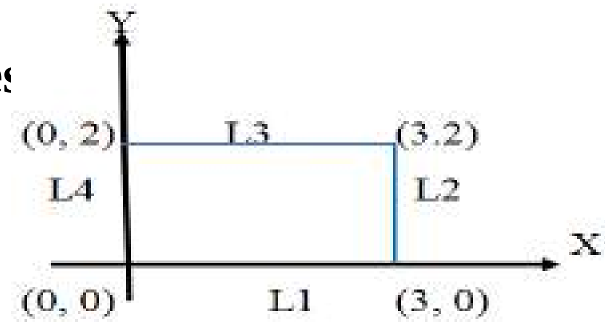
1- Find the value of f at the critical point of f in D

2- Find the extreme values of f

3- The largest of the values from steps 1 and 2 is the absolute maximum and the smallest of these values is the absolute minimum value.

Example 2.16

Find the absolute maximum and minimum values $f(x,y) = x^2 - 2xy + 2y$ **on the rectangular** $D = [(x,y)|0 \le x \le 3, \quad 0 \le y \le 2]$



Solution

To find the critical points

$f_x = 2x - 2y = 0 \; and \; f_y = -2x + 2 = 0$

$x = 1, \quad y = 1$

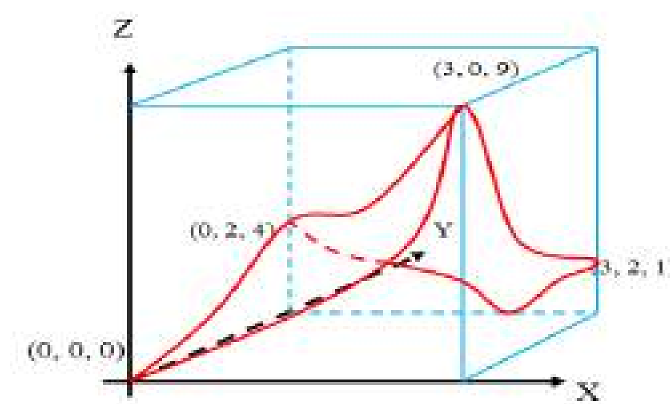
The critical point is (1,1)

To find the points on the boundary

<u>L1</u>

$y = 0, \quad x = 0 \to 3$

$f(x,0) = x^2$



L2

$x = 3, \qquad y = 0 \to 2$

$f(3, y) = 9 - 4y$

Maximum value is f(3,0)=9

Minimum value is f(3,2)=1

$y = 2, \qquad x = 0 \to 3$

L3

$f(x, 2) = x^2 - 4x + 4 = (x - 2)^2$

Maximum value is f(0,2)=4

Minimum value is f(2,2)=0

$x = 0, \qquad y = 0 \to 2$

$f(0, y) = 2y$

Maximum value is f(0,2)=4

L4

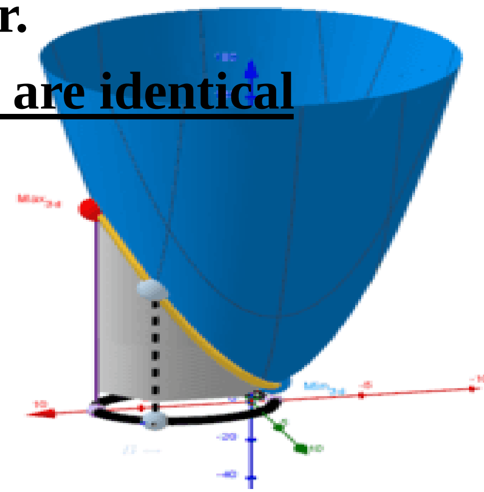
Minimum value is f(0,0)=0

Absolute maximum value of f f(x,y) on D is f(3,0)=9

2.9 Lagrange Multipliers Method

This method is used to find the stationary points (maximum and minimum) of the function w=f(x,y,z) with constraint g(x,y,z)=k as shown in Figure below.

The figure shows a g(x,y) curve together with several curves of f(x,y). To maximize f(x,y) subject to g(x,y)=k to find largest value of C such that the level curve f(x,y)=c intersect g(x,y)=k. its appear from the figure that this happens when these curves just touch each other.



<u>This mean the normal lines at intersection point (xo,yo) are identical</u>

$\nabla f(x_o, y_o) = \lambda \nabla g(x_o, y_o) \quad \lambda\ is\ a\ calar$

<u>The gradient vectors are parallel</u>

$$\nabla f(x_o, y_o, z_o) = \lambda \nabla g(x_o, y_o, z_o)$$

For 3D (three λ variables)

$$\lambda \quad \nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

The number in the equation is called a Lagrange Multiplier

$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z$

To find the maximum and minimum values of f(x,y,z) subject to the

Example 2.17

A rectangular box with out cover is to be made from $12m^2$ of cardboard, find the maximum value of such box.

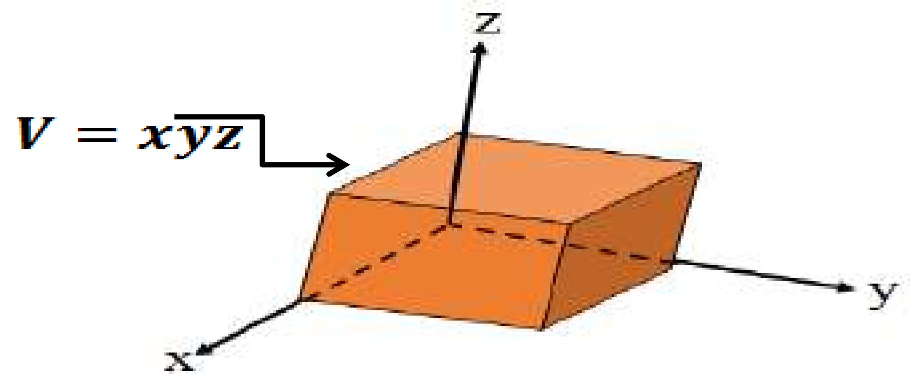
Solution

$V = xyz$

$g(x,y,z) = 2xz + 2yz + xy = 12$

$\nabla f = \lambda \nabla g$

$v_x = \lambda g_x \quad v_y = \lambda g_y \quad v_z = \lambda g_z$



$yz = \lambda(2z + y) \qquad (1)$

$xz = \lambda(2z + x) \qquad (2)$

$xy = \lambda(2x + 2y) \qquad (3)$

$2xz + 2yz + xy = 12 \qquad (4)$

Multiply eq. 1 by x, eq. 2 by y and eq. 3 by z

$xyz = \lambda(2xz + xy) \qquad (5)$

$xyz = \lambda(2yz + xy) \qquad (6)$

$xyz = \lambda(2xz + 2yz) \qquad (7)$

$From\ Eqs.(5) and\ (6) \quad 2xz + yx = 2yz + xy\ then\ \ y = x$

$From\ Eqs.(6) and\ (7) \quad 2yz + yx = 2xz + 2yz\ then\ \ y = x = 2z$

$Sub.in\ eq.(4) \quad 4z^2 + 4z^2 + 4z^2 = 12$

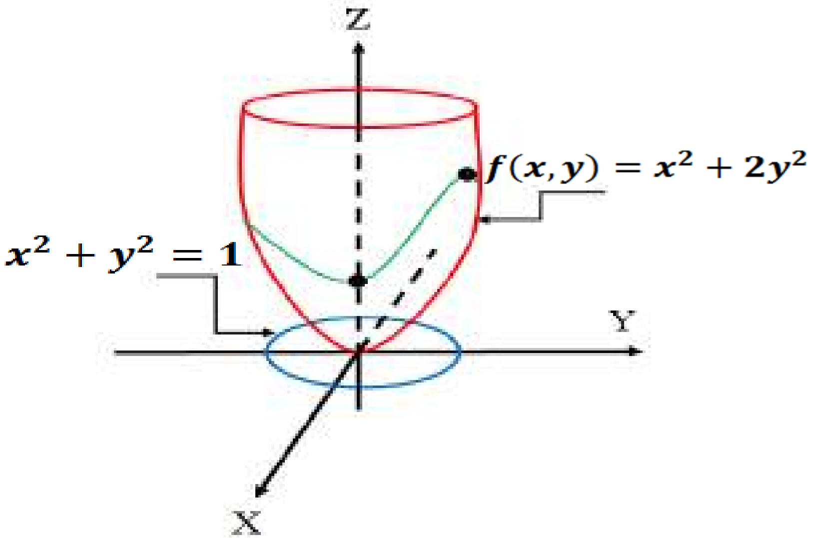
$z^2 = 1\ \ then\ z = 1 \quad x = 2\ y = 2$

$V = 2 * 2 * 1 = 4m^2$

Example 2.18

Find the extreme values of the function $f(x,y) = x^2 + 2y^2$ **on the circle** $x^2 + y^2 = 1$

Solution



$$g(x,y) = x^2 + y^2 = 1$$

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z$$

$$2x = \lambda 2x \qquad (1)$$

$$4y = \lambda 2y \qquad (2)$$

$$x^2 + y^2 = 1 \qquad (3)$$

$$From\ eq.(1)\ \ x = 0\ \ or\ \ \lambda = 1$$

$$if\ x = 0\ \ y = \pm 1\ from\ Eq.3$$

$$if\ \lambda = 1\ \ y = 0\ from\ Eq.2$$

Therefore the possible extreme values at the points (0,1), (0,-1) (1,0) and (-1,0)

$$f(0,1) = 2$$

$$f(0,-1) = 2$$

$$f(1,0) = 1$$

$$f(-1,0) = 1$$

The maximum value of f is f(0,1) = f(0,-1) =2

The minimum value of f is f(1,0) =f(-1,0) =1

Assignment 2(Partial Derivatives)

(1) Find the first partial derivatives of the following functions

(a) $f(x,y) = y^5 - 3xy$ (b) $f(x,y) = e^{-t}\cos \pi x$ (c) $f(x,y,z) = xyZ^2 tan(yz)$

(2) Find the second partial derivatives of the functions

(a) $f(x,y) = x^3y^5 + 2x^4y$ (b) $f(x,y) = \sin^2(mx+ny)$ (c) $f(x,y) = \dfrac{xy}{x-y}$

(3) Show that $u_{xy} = u_{yx}$ for the following

(a) $u = xsin(x+2y)$ (b) $u = x^4y^2 - 2xy^5$ (c) $u = xye^y$

(4) Find the indicated partial derivatives

(a) $f(x,y) = 3xy^4 + x^3y^2;\ f_{xxx}, f_{yyy}$ (b) $w = \dfrac{x}{y+2z}\ ;\ \dfrac{\partial 3w}{\partial z \partial y \partial z}, \dfrac{\partial 3w}{\partial x^2 \partial y}$

(5) Verify that the function $Z = \ln(e^x + e^y)$ is a solution of the deferential equation $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1$

(6) Show that the function $P = bL^{\alpha}k^{\beta}$ satisfies the equation $L\dfrac{\partial P}{\partial L} + k\dfrac{\partial P}{\partial k} = (\alpha+\beta)P$

(7) Use the chain Rule to find $\dfrac{dz}{dt}$ or $\dfrac{dw}{dt}$

(a) $z = cos(x+4y), x = 5t^4, y = e^t$ (b) $w = xe^{\frac{y}{z}}, x = t^2, y = 1-t, z = 1+2z$

(8) The temperature at a point (x,y) on a flat plate is given by $T(x,y) = \frac{60}{(1+x^2+y^2)}$ Where T is measured in C^o and x,y in meters. Find the rate of change of temperature with respect to distance at the point (2.1) in

(a) The x-direction (b) the y-direction

(9) Use the chain Rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

(a) $z = \sin\theta\cos\emptyset, \quad \theta = st^2, \emptyset = s^2t$ (b) $z = e^r\cos\theta, \quad r = st, \theta = \sqrt{s^2+t^2}$

(10) Find the directional derivative of f at the given point in the direction indicated by the angle θ.

(a) $f(x,y) = x^2y^3 - y^4$ $(2,1)$, $\theta = \frac{\pi}{4}$, (b) $f(x,y) = x\sin(xy),$ $(2,0)$, $\theta = \frac{\pi}{3}$

(11) Find the directional derivative of the function

(a) $f(x,y) = 1 + 2x\sqrt{y},\ P = (3,4), V = (4,-3)$

(b) $f(x,y,z) = xe^y + ye^z + ze^x, P = (2,3,1), V = (4,-2,1)$

(12) Find equation of the tangent plane and the normal line to the given surface at the specified point

(a) $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10, (3,3,5)$ (b) $z + 1 = xe^y\cos z, (1,0,0)$

(13) Find the local maximum and minimum values and saddle point of the following functions

(a) $f(x,y) = x^3y + 12x^2 - 8y$ (b) $f(x,y) = e^y\cos y$

(14) **Find the absolute maximum and minimum values of f on the set D.**

(a) $f(x,y) = 3 + xy - x - 2y$

D is the closed triangular region with vertices (1,0), (5,0), and (1,4)

(b) $f(x,y) = x^2 + y^2 + x^2y + 4$ $\quad D = \{(x,y) \mid |x| \le 1, |y| \le 1\}$

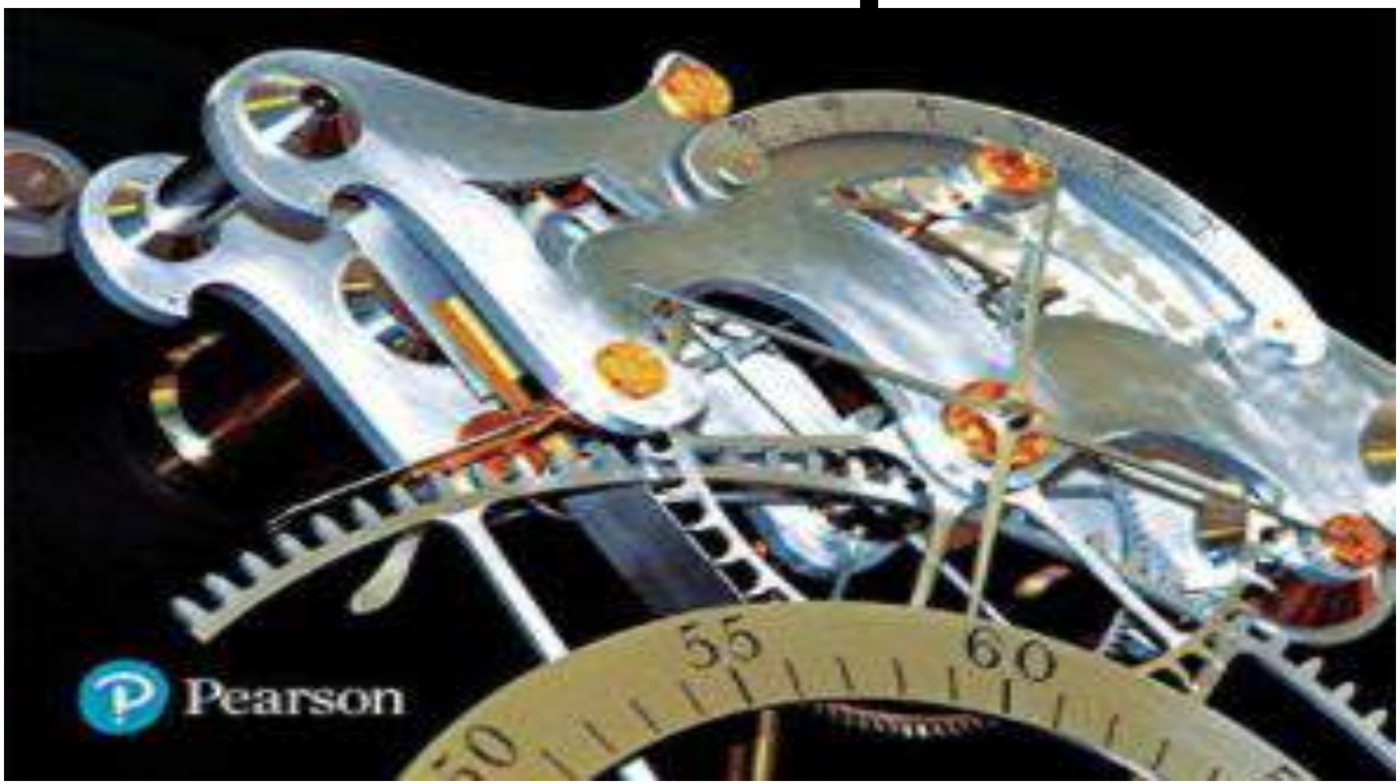
(15) By Lagrange multipliers

(a) Find the three positive numbers whose their sum is 48 and such that their product is a large as possible

(b) Find the maximum volume of box with three faces in the coordinate planes and vertex in the first octant of the plane $x + y + z = 1$

Chapter Three

Differential Equations



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3.1 Introduction

A differential equation is an equation that contains unknown factors and one or more of its derivative. The order of differential equation is the order of the highest derivative that occurs in the equation.

3.2 order and degree of differential equation

Differential equation	Order	Degree
$\frac{dy}{dx} = 3x + e^x$	1	1
$\left(\frac{dy}{dx}\right)^5 - \left(\frac{d2y}{dx^2}\right)^3 + \left(\frac{d3y}{dx^3}\right)^2 = sinx$	3	5

3.3 First order Differential Equations

That equations which can be classified to the following types

1- 1st order differential equation (Separable Type)

2- 1st order differential equation (Homogenous Type)

3- 1st order differential equation (Exact Type)

4- 1st order differential equation (Linear Type)

5- 1st order differential equation (Bernoulli's Type)

3.3.1 1st order differential equation (Separable Type)

A separable equation is a first-order differential equation in which the expression for dx/dy can be factored as a function of x times a function of y. In other words it can written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

The name of separable comes from the fact that expression on the right side can be separable and can put the equation

$$h(y)dy = g(x)dx \quad h(y) = \frac{1}{f(y)}$$

The solution is $\int h(y)dy = \int g(x)dx + c$

Example 3.1

Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$ and find the solution of this equation satisfies the initial condition y(0)=2

Solution $\frac{dy}{dx} = \frac{x^2}{y^2} \rightarrow y^2 dy = x^2 dx$

$$\int y^2 dy + c_1 = \int x^2 dx + c_2 \qquad \frac{1}{3}y^3 = \frac{1}{3}x^2 + c \qquad c = c_2 - c_1$$

y= $\sqrt[3]{x^3 + 3c}$ $\quad let\ k = 3c$

$$y = \sqrt[3]{x^3 + k} \quad x = 0\ y = 2 \quad \rightarrow 2 = \sqrt[3]{0 + k} \rightarrow k = 8$$

$$y = \sqrt[3]{x^3 + 8}$$

Example 3.2

Solve the differential equation $\frac{dy}{dx} = \frac{6x^2}{2y + cosy}$

Solution

$$(2y + cosy)dy = 6x^2 dx$$

$$\int (2y + cosy)dy = \int 6x^2 dx + c$$

$$y^2 + siny = 2x^3 + c$$

3.3.2 1st order differential equation (Homogenous Type)

The general form is $\frac{dy}{dx} = f(\frac{y}{x})$

Put $V = \frac{y}{x}$

1st order differential equation is said to be homogenous if it satisfy the following condition $F(\lambda x, \lambda y) = f(x, y)$

sometime 1st order differential equation can be written as following $M(x,y)dx + N(x,y)dy = 0$

$$\frac{dy}{dx} = -\frac{(x,y)}{N(x,y)} \; and \; \frac{M(x,y)}{N(x,y)} = \frac{M(\lambda x, \lambda y)}{N(\lambda x, \lambda y)}$$

The general solution $\ln x = \int \frac{dV}{F(V) - V} + c$

Example 3.3

Solve $x^2y\,dx = (x^3 - y^3)dy \quad y(1) = 1$

Solution $\dfrac{dy}{dx} = \dfrac{x^2y}{x^3 - y^3} = f(x,y) \quad V = \dfrac{y}{x} \qquad f(x,y) = \dfrac{x^2y}{x^3 - y^3}$

$$F(\lambda x, \lambda y) = \frac{(\lambda x)^2 \lambda y}{(\lambda x)^3 - (\lambda y)^3} = \frac{\lambda^3 x^2 y}{\lambda^3 (x^3 - y^3)} = \frac{x^2 y}{(x^3 - y^3)} = f(x,y)$$

The equation is homogenous

$$\frac{dy}{dx} = F(V) = \frac{\frac{x^2y}{x^3}}{\frac{x^3}{x^3} - \frac{y^3}{x^3}} = \frac{V}{1 - V^3}$$

$$\ln x = \int \frac{dV}{F(V) - V} + c$$

$$\ln x = \int \frac{dV}{\frac{V}{1 - V^3} - V} + c = \int \frac{dV}{\frac{V - V + V^4}{1 - V^3}} + c \rightarrow \ln x = \int \frac{(1 - V^3)dV}{V^4} + c$$

General solution

$$\ln x = \int \left(V^{-4} - \frac{1}{V}\right) dV + c \rightarrow \ln x = -\frac{1}{3V^3} - \ln V + c$$

$$\ln x = -\frac{x^3}{3y^3} - \ln\frac{y}{x} + c \qquad at\ x = 1 \quad y = 1$$

$$\ln(1) = -\frac{(1)^3}{3(1)^3} - \ln(1) + c \;\rightarrow\; 0 = -\frac{1}{3} - 0 + c \rightarrow \quad c = \frac{1}{3}$$

$$\ln x = -\frac{x^3}{3y^3} - \ln\frac{y}{x} + \frac{1}{3}$$

Example 3.4

Solve $(x^2 - y^2)dx + xydy = 0$

Solution

$$\frac{dy}{dx} = -\frac{(x^2 + y^2)}{xy}\ , \quad f(x, y) = -\frac{(x^2 + y^2)}{xy}$$

$$F(\lambda x, \lambda y) = -\frac{[(\lambda x)^2 + (\lambda y)^2]}{\lambda x\, \lambda y} = -\frac{\lambda^2}{\lambda^2}\frac{(x^2 + y^2)}{xy} = -\frac{(x^2 + y^2)}{xy} = f(x, y)$$

The equation is homogeneous

$$\frac{dy}{dx} = F(V) = \frac{\left(\frac{x^2}{x^2} + \frac{y^2}{x^2}\right)}{\frac{xy}{x^2}} = -\frac{(1 + V^2)}{V}$$

$$\ln x = \int \frac{dV}{F(V) - V} + c$$

General Solution

$$\ln x = \int \frac{-V dV}{\frac{(1 + V^2)}{V} - V} + c = \int \frac{dV}{\frac{-(1 + 2V^2)}{V}} + c$$

$$\ln x = \int \frac{-V}{1 + 2V^2} + c \rightarrow \ln x = -\frac{1}{4}\ln|1 + 2V^2| + c$$

$$\ln x = -\frac{1}{4}\ln\left|1 + 2\left(\frac{y}{x}\right)^2\right| + c$$

3.3.3 1st order differential equation (Exact Type)

The general form is

$$M(x,y)dx + N(x,y)dy = 0$$

or

$$\frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy = 0$$

f(x,y) represents the general solution of the above equation

$$M(x,y) = \frac{\partial f(x,y)}{\partial x}dx \quad and \quad N(x,y) = \frac{\partial f(x,y)}{\partial y}dy$$

1st order differential equation is said to be exact if it satisfy the following condition

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

The general solution shall be undergoes the following routes as below

1-

$$f(x,y) = \int M(x,y)dx + \varnothing(y)$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + \varnothing(y)^{/}$$

$\varnothing(y)^{/}$ $\quad \varnothing(y) = \int \varnothing(y)^{/} + c$ ....(*)

$$f(x,y) = \int N(x,y)dy + g(x)$$

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x}\left(\int N(x,y)dy\right) + g(x)^{/} + c$$

To find $g(x) = \int g(x)^{/} + c$ sub. In eq. (*) to get the G.S.

2- ....(**)

3- $$f(x,y) = \int M(x,y)dx + \int N(x,y)dy + c$$

(regardless all terms containing variable x)

4- $$f(x,y) = \int N(x,y)dy + \int M(x,y)dx + c$$

(regardless all terms containing variable y)

Example 3.5

Solve

$$(2xy + e^y)dx + \left(x^2 + xe^y\right)dy = 0$$

Solution

Compare it with

$$M(x,y)dx + N(x,y)dy = 0$$

$$M(x,y) = 2xy + e^y = \frac{\partial f(x,y)}{\partial x} \qquad N(x,y) = x^2 + xe^y = \frac{\partial f(x,y)}{\partial y}$$

$$\frac{\partial M(x,y)}{\partial y} = 2x + e^y \qquad \frac{\partial N(x,y)}{\partial x} = 2x + e^y$$

Since

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

It is exact

$$f(x,y) = \int M(x,y)dx + \emptyset(y)$$

$$f(x,y) = \int (2xy + e^y)dx + \emptyset(y)$$

$$f(x,y) = x^2y + xe^y + \emptyset(y)$$

$$\frac{\partial f(x,y)}{\partial y} = N(x,y) = x^2 + xe^y + \emptyset(y)' = x^2 + xe^y \rightarrow \emptyset(y)' = 0$$

$$\emptyset(y) = \int \emptyset(y)' + c \rightarrow = c$$

$$f(x,y) = x^2 y + xe^y + c$$

Method 2 Practice for you

Method 3

$$f(x,y) = \int M(x,y)dx + \int N(x,y)dy + c$$

(regardless all terms containing variable x)

$$f(x,y) = \int (2xy + e^y)dx + \int 0 + c$$

$$f(x,y) = x^2 y + xe^y + c$$

$$(2xy + x^2)dx + (x^2 + y^2)dy = 0$$

Method 4 practice for you

Other practices

3.3.4 1st order differential equation (Linear Type)

The general form is $\dfrac{dy}{dx} + P(x)y = Q(x)$

The general solution shall be

$$y\,I(x) = \int I(x)\;Q(x)\;dx + c$$

$I(x)$ is an integrating factor $= e^{\int P(x)dx}$

$$\frac{dx}{dy} + P(y)x = Q(y)$$

Or the general form can be written as

$$x\,I(y) = \int I(y)\;Q(y)\;dy + c \qquad I(y) \;= e^{\int P(y)dy}$$

The general solution can be written as

$$y' + y\tan x = \sec x$$

$$\frac{dy}{dx} + y\tan x = \sec x \qquad \frac{dy}{dx} + P(x)y = Q(x)$$

Example 3.6

$$y\,I(x) = \int I(x)\;Q(x)\;dx + c \qquad I(x) \;= e^{\int P(x)dx}$$

Solve

$$I(x) = e^{\int \tan x dx} = e^{\ln \sec x} = \sec x$$

Solution

$$y\sec x = \int \sec x \sec x\;dx + c$$

$$y\sec x = \int \sec x^2\;\;dx + c$$

compare with

$$y\sec x = \tan x\;\;dx + c$$

G.S

3.3.5 1st order differential equation (Bernoulli's Eq.)

The general form is $\frac{dy}{dx} + P(x)y = Q(x)y^n$

It can be reduced to linear form by considering the following transformation

$$z = y^{1-n} \quad where \quad \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\frac{dx}{dy} + P(y)x = Q(y)x^n \qquad z = x^{1-n} \quad where \quad \frac{dz}{dy} = (1-n)x^{-n}\frac{dx}{dy}$$

Or

$$y' + \frac{y}{x} = \ln x y^2$$

Example 3.7 $\frac{dy}{dx} + P(x)y = Q(x)y^n$

Solve

Solution

Compare with its Bernoulli's equation it can be reduce to linear

$$z = y^{1-n} \quad n = 2 \quad z = y^{-1}$$

$$\frac{dz}{dx} = (-1)y^{-2}\frac{dy}{dx} \rightarrow \frac{dy}{dx} = \frac{-1}{y^{-2}}\frac{dz}{dx} = -y^2\frac{dz}{dx} \quad sub\ in\ above\ Eq.$$

$$-y^2\frac{dz}{dx} + \frac{y}{x} = \ln x\, y^2 \rightarrow \frac{dz}{dx} - \frac{1}{x}y^{-1} = -\ln x \rightarrow \frac{dz}{dx} - \frac{1}{x}z = -\ln x$$

$$z\, I(x) = \int I(x)\, Q(x)\, dx + c$$

$$I(x) = e^{-\int\frac{1}{x}} = e^{-\ln x} = e^{\ln\frac{1}{x}} = \frac{1}{x}$$

$$z\left(\frac{1}{x}\right) = -\int \frac{1}{x}\ln x\, dx + c$$

$$\frac{1}{yx} = -\frac{(\ln x)^2}{2} + c$$

G.S.

$$HW \quad y' + \frac{y}{x} + \frac{y^2}{x} = 0$$

3.4 Second order differential equations

Those equation which can be classified to the following types

3.4.1 2nd order differential equation, linear, homogenous with constant coefficients

$$y^{//}+Py^{/}+gy=0$$

The general form is

$$\frac{d2y}{dx^2}+P\frac{dy}{dx}+qy=0\rightarrow\left(\frac{d2}{dx^2}+p\frac{d}{dx}+q\right)y=0$$

Where P and q are constant

Or $(D^2+PD+g)y=0$ $\quad D=\frac{d}{dx}=\text{Differential operator}$

$$y^{//}+5y^{/}+6y=0$$

To get **where**

Likewise

$$m^2+Pm+g=0\ \ldots\ldots(*)$$

the above equation can be solved by introducing a certain equation that is called "characteristic equation" by considering the coefficient of y in differential operator from than equating to zero after replacing each D certain parameter "m"

The equation takes the following routes

1-

if $m_1\neq m_2$ and both m_1 and m_2 are real roots

$$y(x)=C_1e^{m_1x}+C_2e^{m_2x}$$

if $m_1=m_2$ and both m_1 and m_2 are real roots

$$y(x)=C_1e^{mx}+C_2xe^{mx}$$

if m_1 and m_2 are both imagenary and they are of form $m_{1,2}=a+ib$

$$y(x)=e^{ax}(C_1\cos bx+C_2\sin bx)$$

Example 3.8

Solve $y^{//} + 5y^{/} + 6y = 0$

Solution

$$\frac{d2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \qquad (D^2 + 5D + 6)y = 0$$

$$m^2 + 5m + 6 = 0 \quad \rightarrow \ (m+3)(m+2) = 0 \quad \rightarrow \ m = -3\ , m = -2$$

$$G.S.\ \ y(x) = C_1e^{-3x} + C_2e^{-2x}$$

Example 3.9

Solve

$$\frac{d2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

Solution

$$(D^2 - 2D + 1)y = 0$$

$$\rightarrow\ m^2 - 2m + 1 = 0 \quad \rightarrow\ (m-1)(m-1) = 0 \quad m = 1, \qquad m = 1$$

$$G.S.\ \ y(x) = C_1e^x + C_2xe^x$$

Example 3.10

Solve

$$\frac{d2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0 \quad y(0) = y(0)^{/} = 1$$

Solution

$$(D^2 - 2D + 5)y = 0$$

$$m^2 - 2m + 5 = 0 \rightarrow m_{1,2} = \frac{2 \mp \sqrt{2^2 - 4(1)(5)}}{2(1)} = \frac{2 \mp 4i}{2} = 1 \mp 2i \qquad \mathrm{a = 1, b = 2}$$

$$G.S.\ y(x) = e^{ax}(C_1\cos bx + C_2 \sin bx) = e^x(C_1\cos 2x + C_2 \sin 2x)$$

$$at\ x = 0, y = 1 \rightarrow e^0(C_1\cos 2(0) + C_2\sin 2(0)) \rightarrow C_1 = 1$$

$$at\ x = 0, y^{/} = 1 \rightarrow e^x(-2C_1\sin 2x + 2C_2\cos 2x) + e^x(C_1\cos 2x + C_2 \sin 2x)$$

$$1 = e^0(0 + 2C_2) + e^0(C_1 + 0) \rightarrow 1 = 2C_2 + C_1 \rightarrow C_2 = 0$$

$$y(x) = e^x cos2x$$

3.4.2 2nd order differential equation, non, homogenous, linear with constant coefficients

The general form is

$$y^{//} + Py^{/} + qy = f(x)$$

$$\frac{d2y}{dx^2} + P\frac{dy}{dx} + qy = f(x) \qquad (D^2 + PD + q)y = f(x)$$

Or

$$y(x) = yh + yP$$

The general solution of above equation shall be

yh : Transient solution $\qquad (y^{//} + Py^{/} + qy = 0)$

yP : Steady state solution

yh can be ... that discussed previo...

yP can be ... on f(x) where if f(x) is of a standa... following table

f(x)	Suggested solution
C: constant	K: constant
e^{ax}	Ke^{ax}
x^n	$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$
$\sin ax$ $\cos ax$ $\sin ax + \cos ax$	$k_1 \cos ax + k_2 \sin ax$
$\sinh ax$ $\cosh ax$ $\sinh ax + \cosh ax$	$k_1 \cosh ax + k_2 \sinh ax$

Note

Each solution taken from the previous table shall be compared with yh.

If there is certain similarities between them suggested solution shall be multiplied by (X).

If f(x) is non of these mentioned before, then yP shall be evaluated using (variation parameters)

$$yP = y_1u_1 + y_2u_2$$

y1 and y2 shall be evaluated from yh regardless their constant while u1 and u2 shall be evaluated by using "" technique

$$w(x) = Det\begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = y_1y_2{}' - y_2y_1{}'$$

Where w(x) is a wronskian function. Grammar-wronskian

$$u_1 = \int \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2{}' \end{vmatrix}}{w(x)} dx \qquad u_2 = \int \frac{\begin{vmatrix} y_1 & 0 \\ y_1{}' & f(x) \end{vmatrix}}{w(x)} dx$$

$$y^{//} + y = tanx + 4e^{3x} + x^2 + sinx + 5$$

$$y(x) = yh + yP$$

Example 3.11

Solve

Solution

$$\frac{d2y}{dx^2} + y = 0 \rightarrow (D^2 + 1)y = 0$$

$$m^2 + 1 = 0 \rightarrow m^2 = -1 \rightarrow m = \mp\sqrt{-1} = 0 \mp i \;\; compare\, with\, a \mp ib$$

$$yh = e^{ax}(c_1 \cos bx + c_2 \sin bx) \rightarrow \ yh = e^{0}(c_1 \cos x + c_2 \sin x)$$

$$yh = c_1 \cos x + c_2 \sin x$$

$$yP = yP_1 + yP_2 + yP_3 + yP_4 + yP_5$$

To find yP

$y^{//} + y = tanx$ $\qquad f(x) = tanx$

To find yP1

$$yp_1 = y_1 u_1 + y_2 u_2 \qquad y_1 = \cos x \quad y_2 = \sin x$$

$$w(x) = Det \begin{vmatrix} y_1 & y_2 \\ {y_1}^{/} & {y_2}^{/} \end{vmatrix}$$

$$w(x) = Det \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + sin^2 x = 1$$

$$u_1 = \int \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & {y_2}^{/} \end{vmatrix}}{w(x)} dx = \int \frac{\begin{vmatrix} 0 & sinx \\ tanx & cosx \end{vmatrix}}{1} dx = \int -sinxtanxdx = \int \frac{-sin^2 x}{cosx} dx$$

$$= -\int \frac{(1 - \cos^2 x)}{\cos x} dx = -\int (\sec x - \cos x)\, dx = \sin x - \ln(\sec x + \tan x)$$

$$u_1 = \int \frac{\begin{vmatrix} y_1 & 0 \\ {y_1}^{/} & f(x) \end{vmatrix}}{w(x)} dx = \int \frac{\begin{vmatrix} cosx & 0 \\ -sinx & tanx \end{vmatrix}}{1} dx = \int cosxtanxdx = \int sinx\, dx = conx$$

$$yp_1 = y_1 u_1 + y_2 u_2 \rightarrow yp_1 = \cos x\,[\sin x - \ln(\sec x + \tan x)] - \sin x \cos x$$

$$yp_1 = \cos x \sin x - \cos x \ln(\sec x + \tan x) - \sin x \cos x$$

$$yp_1 = \frac{1}{\ln(\sec x + \tan x)} \cos x$$

To find yP2

$$yP_2{}' = 3ke^{3x} \quad yP_2{}'' = 9ke^{3x} \quad sub\ in\ eq.(*)$$

$$9ke^{3x} + ke^{3x} = 4e^{3x} \to 10ke^{3x} = 4e^{3x} \to k = \frac{2}{5}$$

$$yP_2 = \frac{2}{5}e^{3x}$$

$$\to let\ yp_3 = a_0 + a_1x + a_2x^2$$

To find yP3

$$y'' + y = x^2 \dots(*) \qquad \to let\ yp_3 = a_0 + a_1x + a_2x^2$$

$$yp_3{}' = a_1 + 2a_2x$$

$$yp_3{}'' = 2a_2 \quad sub.in\ (*)$$

$$2a_2 + a_0 + a_1x + a_2x^2 = x^2$$

$$a_2 = 1, \qquad a_1 = 0$$

$$2a_2 + a_0 = 0 \to a_0 = -2$$

$$yp_3 = -2 + x^2$$

To find yp4

$$y'' + y = sinx \dots(*)$$

$$yp_4 = k_1\cos x + k_2\sin x \quad not\ OK$$

$$yp_4 = x(k_1\cos x + k_2\sin x) \quad OK$$

$$yp_4{}' = x(-k_1 sinx + k_2\cos x) + (k_1\cos x + k_2\sin x)$$

$$yp_4{}'' = x(-k_1 cosx - k_2\sin x) + (-k_1 sinx + k_2\cos x) + (-k_1\sin x + k_2\cos x)$$

$$yp_4{}'' = -x(k_1 cosx + k_2\sin x) + 2(-k_1 sinx + k_2\cos x)$$

$$-x(k_1 cosx + k_2 \sin x) + \; 2(-k_1 sinx + k_2 \cos x) + x(k_1 \cos x + k_2 \sin x) = sinx$$

$$-2k_1 sinx - 2k_2 cosx = sinx \;\rightarrow\; -2k_1 = 1 \;\rightarrow k_1 = -\frac{1}{2}\,, \qquad k_2 = 0 \; sub.in\ (*)$$

$$yp_4 = x\left(-\frac{1}{2}cosx + 0sinx\right) = -\frac{1}{2}xcosx$$

To find yp5

$$y^{//} + y = 5 \ldots (*) \rightarrow let\; yP_5 = k$$

$$yp_5{}^{/} = yp_5{}^{//} = 0 \quad sub.in\ (*)$$

$$0 + k = 5$$

$$yp_5 = 5$$

$$yp = yp_1 + yp_2 + yp_3 + yp_4 + yp_5 = \ln\frac{1}{\sec x \tan x}cosx + \frac{2}{5}e^{3x} + x^2 - \frac{1}{2}xcosx + 5$$

$$y(x) = yh + yp$$

$$y(x) = c_1 cosx + c_2 sinx + \ln\frac{1}{\sec x \tan x}cosx + \frac{2}{5}e^{3x} + x^2 - \frac{1}{2}xcosx + 5$$

$$(D^2 - 16)y = e^{4x}$$

$$y^{//} + y = \frac{1}{1 + cosx}$$

Practices

1-

3.5 Higher Order Differential Equations

3.5.1 Third order differential equations, Linear with constant coefficient

general form is

$(D^3 + PD^2 + qD + s)y = 0 \; homogenous$

$y(x) = yh : homogenous\ solution$

$m^3 + Pm^2 + qm + s = 0$

Homogenous solution can be achieved by considering

$if\ m_1 \neq m_2 \neq m_3 \;\; Real\ roots$

$y(x) = yh = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$

There are m1, m2, and m3 roots with the following arrangements.

1- $if\ m_1 = m_2 = m_3 \;\; Real\ roots$

$y(x) = yh = C_1 e^{mx} + C_2 x e^{mx} + C_3 e x^{2^{mx}}$

$if\ (m_1 = m_2 = m) \neq m_3\ Real\ roots$

2- $y(x) = yh = C_1 e^{mx} + C_2 x e^{mx} + C_3 e^{m_3 x}$

$if\ \left(m_{1,2} = a + ib\right) \neq m_3$

$y(x) = yh = e^{ax}(C_1 \cos bx + C_2 \sin bx) + C_3 e^{m_3 x}$

3-

While If $(D^3 + PD^2 + qD + s)y = f(x)$ represents third order-non homogenous differential equation. It can be solve as

$$y(x) = yh + yp$$

yp shall be taken out from the suggested solution table , if f(x) is of a standard form. But if it is not, yp shall be

$$yp = y_1 u_1 + y_2 u_2 + y_3 u_3$$

Where y1, y2 and y3 shall be from yh that is mentioned before while u1,u2 and u3 shall be evaluated by using "Grammer-wronskian" method.

$$u_1 = \int \frac{\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2{}' & y_3{}' \\ f(x) & y_2{}'' & y_3{}'' \end{vmatrix}}{w(x)} dx \qquad u_2 = \int \frac{\begin{vmatrix} y_1 & 0 & y_3 \\ y_1{}' & 0 & y_3{}' \\ y_1{}'' & f(x) & y_3{}'' \end{vmatrix}}{w(x)} dx$$

$$u_3 = \int \frac{\begin{vmatrix} y_1 & y_2 & 0 \\ y_1{}' & y_2{}' & 0 \\ y_1{}'' & y_2{}'' & f(x) \end{vmatrix}}{w(x)} dx \qquad w(x) = Det. \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1{}' & y_2{}' & y_3{}' \\ y_1{}'' & y_2{}'' & y_3{}'' \end{vmatrix}$$

yh shall be evaluated by considering

$$(D^3 + PD^2 + qD + s)y = 0$$

3.5.2 Forth order differential equation, Linear with constant coefficient.

The general form is

$$\left(D^4 + PD^3 + qD^2 + sD + R\right)y = 0 \; homogenous$$

$$m^4 + Pm^3 + qm^2 + sm + R = 0$$

It can be solve by

$$if\; m_1 \neq m_2 \neq m_3 \neq m_4 \quad Real\, roots$$

$$y(x) = yh = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + C_4 e^{m_4 x}$$

Where P,q, s and R are constants, m1,m2,m3 and m4 are four roots can be

$$if\; m_1 = m_2 = m_3 = m_4 \quad Real\, roots$$

1- $y(x) = yh = C_1 e^{mx} + C_2 x e^{mx} + C_3 x^2 e^{mx} + C_4 x^3 e^{mx}$

$$if\; (m_1 = m_2 = m_3 = m) \neq m_4 \quad Real\, roots$$

2- $y(x) = yh = C_1 e^{mx} + C_2 x e^{mx} + C_3 x^2 e^{mx} + C_4 e^{m_4 x}$

$$if\; (m_1 = m_2 = m) \neq m_3 \neq m_4 \quad Real\, roots$$

$$y(x) = yh = C_1 e^{mx} + C_2 x e^{mx} + C_3 e^{m_3 x} + C_4 e^{m_4 x}$$

3- $if\; (m_1 = m_2 = m) and\, (m_3 = m_4 = m^-) \quad Real\, roots$

$$y(x) = yh = C_1 e^{mx} + C_2 x e^{mx} + C_3 e^{m^- x} + C_4 x e^{m^- x}$$

$$if\; (m_1 \neq m_2) and\, m_{3,4} = a \mp ib$$

4- $y(x) = yh = C_1 e^{m_1 x} + C_2 e^{m_2 x} + e^{ax}(C_3 \cos bx + C_4 \sin bx)$

$$if\; (m_1 = m_2 = m) and\, m_{3,4} = a \mp ib$$

5- $y(x) = yh = C_1 e^{m_1 x} + C_2 x e^{m_2 x} + e^{ax}(C_3 \cos bx + C_4 \sin bx)$

$$if\; m_{1,2} = a \mp ib \; and \; m_{3,4} = u \mp iv$$

$$y(x) = yh = e^{ax}(C_3 \cos bx + C_4 \sin bx) + e^{ux}(C_3 \cos vx + C_4 \sin vx)$$

6- $if\; m_{1,2} = m_{3,4} = a \mp ib$

$$y(x) = yh = e^{ax}(C_3 \cos bx + C_4 \sin bx) + x e^{ax}(C_3 \cos bx + C_4 \sin bx)$$

While the equation of form $(D^4 + PD^3 + qD^2 + sD + R)y = f(x)$ it is 4th order differential equation it can be solve by

$$y(x) = yh + yp$$

yh : shall be as mentioned before

yp: shall be taken out from suggested solution in the table that mentioned previously if f(x) is at the standard form. But if it is not, shall be as following

$$yp = y_1u_1 + y_2u_2 + y_3u_3 + y_4u_4$$

y1,y2,y3 and y4 shall be from yh regardless their constants. But u1,u2,u3 and u4 shall be from following

$$u_1 = \int \frac{\begin{vmatrix} 0 & y_2 & y_3 & y_4 \\ 0 & y_2{}' & y_3{}' & y_4{}' \\ 0 & y_2{}'' & y_3{}'' & y_4{}'' \\ f(x) & y_2{}''' & y_3{}''' & y_4{}''' \end{vmatrix}}{w(x)} dx \qquad u_2 = \int \frac{\begin{vmatrix} y_1 & 0 & y_3 & y_4 \\ y_1{}' & 0 & y_3{}' & y_4{}' \\ y_1{}'' & 0 & y_3{}'' & y_4{}'' \\ y_1{}''' & f(x) & y_3{}''' & y_4{}''' \end{vmatrix}}{w(x)} dx$$

$$u_3 = \int \frac{\begin{vmatrix} y_1 & y_2 & 0 & y_4 \\ y_1{}' & y_2{}' & 0 & y_4{}' \\ y_1{}'' & y_2{}'' & 0 & y_4{}'' \\ y_1{}''' & y_2{}''' & f(x) & y_4{}''' \end{vmatrix}}{w(x)} dx \qquad u_4 = \int \frac{\begin{vmatrix} y_1 & y_2 & y_3 & 0 \\ y_1{}' & y_2{}' & y_3{}' & 0 \\ y_1{}'' & y_2{}'' & y_3{}'' & 0 \\ y_1{}''' & y_2{}''' & y_3{}''' & f(x) \end{vmatrix}}{w(x)} dx$$

$$w(x) = Det. \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1{}' & y_2{}' & y_3{}' & y_4{}' \\ y_1{}'' & y_2{}'' & y_3{}'' & y_4{}'' \\ y_1{}''' & y_2{}''' & y_3{}''' & y_4{}''' \end{vmatrix}$$

Example 3.12

Solve $y^{///} - 6y^{//} + 11y^{/} - 6y = 4e^{4x}$

Solution

$y(x) = yh + yp$

To find yh consider

$y^{///} - 6y^{//} + 11y^{/} - 6y = 0$

$(D^3 - 6D^2 + 11D - 6)y = 0 \rightarrow m^3 - 6m^2 + 11m - 6 = 0$

$m_1 = 1$ satisfy the equation, using long divition principle to get $m_2 and\ m_2$

$(m-1)(m^2 - 5m + 6) = 0$

$(m-1)(m-3)(m-2) = 0$

$m_1 = 1, m_2 = 3, m_3 = 2$

$yh = C_1 e^{x} + C_2 e^{3x} + C_3 e^{2x}$

$$
\begin{array}{r|l}
 & m^2 - 5m + 6 \\ \hline
m - 1 & m^3 - 6m^2 + 11m - 6 \\
 & m^3 - m^2 \\ \hline
 & -5m^2 + 11m - 6 \\
 & -5m^2 + 5m \\ \hline
 & 6m - 6 \\
 & 6m - 6 \\ \hline
 & 0 \quad 0
\end{array}
$$

To find yp

$yp = ke^{4x}$

$yp^{/} = 4ke^{4x}, yp^{//} = 16ke^{4x}, yp^{///} = 64ke^{4x}$ sub. in the eq.

$64ke^{4x} - 96ke^{4x} + 44ke^{4x} - 6ke^{4x} = 4e^{4x}$

$6ke^{4x} = 4e^{4x} \rightarrow k = \frac{2}{3} \rightarrow yp = \frac{2}{3}e^{4x}$

$y(x) = C_1 e^{x} + C_2 e^{3x} + C_3 e^{2x} + \frac{2}{3}e^{4x}$

Example 3.13

Solve $(D^4 - 1)y = 0$

Solution

Consider

$$m^4 - 1 = 0 \rightarrow (m^2 - 1)(m^2 + 1) = 0 \rightarrow (m - 1)(m + 1)(m^2 + 1) = 0$$

$$m_1 = 1, m_2 = -1, m_{3,4} = 0 \mp i \quad a = 0, b = 1$$

$$y(x) = C_1 e^x + C_2 e^{-x} + e^0 (C_3 \cos x + C_4 \sin x)$$

$$y(x) = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$$

Practices

1- $\big((D-1)(D-2)(D-3)(D-4)\big)y = 4e^{5x}$

2- $y^{////} - 5y^{//} + 4y = x^4 + 8e^{-3x}$

Sheet 3 (Differential Equations)

1st order differential equations

1. $y' = -2xy$
2. $2(xy + x)y' = y$
3. $ye^{x+y}dy = dx$
4. $2xdx - dy = x(xdy - 2ydx) \quad y(-3) = 1$
5. $(x^2 + y^2)dx = 2xydy$
6. $(xy + y^2)dx = (x^2 + xy + y^2)dy$
7. $x^2dy = (xy - y^2)dx$
8. $(2xy + x^2)dx + (x^2 + y^2)dy = 0$
9. $(siny - ysinxy)dx + (xcosy - xsinxy)dy = 0$
10. $(x^2 - y^2)y' + (2xy + 1) = 0$
11. $(5x^2 + 1)y' - (20xy) = 10x \quad y(0) = \frac{1}{2}$
12. $y' + y = e^{-x} \quad y(0) = 3$
13. $(x^2 + 1)dy = (x^3 - 2xy + x)dx \quad y(1) = 1$
14. $y' + 2xy - x = e^{-x^2}$
15. $yy' + xy^2 - x = 0 \quad y(0) = -1$
16. $ydy = (x - y^2)dx$

2^{nd} order differential equations

1. $(D^2+3D+2)y=\dfrac{-e^{-x}}{x}+x^2$
2. $(D^2+D)y=cos^2x+sin^2xx^2$
3. $y^{//}-2y^{/}+2y=e^{-x}cosx$
4. $y^{//}+4y^{/}+3y=x-1$
5. $y^{//}-5y^{/}+6y=coshx$
6. $y^{//}+y^{/}=sinx+2cos2x$
7. $y^{//}+5y^{/}+6y=3e^{-2x}+4x^2$
8. $(D^2-2D+1)y=xlnx$

Higher order differential equations

1. $(D+2)(D^2+2D+2)y=x-sinx$
2. $(D^3+D)y=4cos2x$
3. $(D^4-16)y=e^x$
4. $(D^3+D^2+3D-5)y=e^x$
5. $(D+1)^4y=e^x+12$
6. $(D^2+1)(D^2+5)y=e^x$