University of Anbar

College of Engineering

Civil Engineering Department

LECTURE NOTE

COURSE CODE- CE 2208

CALCULUS III

By

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Course Description:

This course is the third part of our standard four-semester calculus sequence. It includes vector calculus; functions of several variables; differentials and applications; and double and triple integrals.

Course Objectives/Goals:

The goals of this course are to enable students to:

1. Learn the basics of the calculus of functions of two and three variables.
2. Study vectors in three-dimensional space, derivatives, and integrals.
3. Apply these ideas to a wide range of problems like motion in space, optimization, arc length, etc.

Course Learning Outcomes:

By the end of successful completion of this course, the student will be able to:

1. Visualize geometry in three-dimensional space;
2. Perform the calculus of scalar functions of several variables and the calculus of vector functions;
3. Do calculus operations on multivariable functions, including partial derivatives, directional derivatives, and multiple integrals;
4. Apply concepts of multivariable calculus to real world problems.

Text Book(s):

- Anton, Howard, Irl C. Bivens, and Stephen Davis. Calculus Single Variable. John Wiley & Sons, 2012.

Recommended readings:

- Any materials on Calculus III like lecture notes or books that are available online.

Weekly Distribution of Course Topics/Contents

Week	Topic
1.	Rectangular Coordinate systems in 3-space. Vectors
2.	Dot product, projections. Cross product
3.	Parametric equations of a line. Planes in 3-space
4.	Introduction to vector-valued functions. Calculus of vector-valued functions
5.	Change of parameters, Arc Length. Unit Tangent, Normal and Binormal vectors
6.	Curvature
7.	Quadric Surfaces. Functions of two or more variables
8.	**Mid-term Exam**
9.	Limits and continuity. Partial derivatives
10.	Differentiability, Local Linearity. The Chain rule
11.	Directional derivatives and gradients. Tangent planes and normal vectors
12.	Maxima and minima of functions of two variables. Lagrange multipliers
13.	Double integrals. Double integrals over non rectangular regions
14.	Double integrals in polar coordinates. Triple integrals
15.	Cylindrical and spherical coordinates, Triple integrals in cylindrical and Spherical coordinates

Students' Assessment:

Students are assessed as follows:

Assessment Tool(s)	Date	Weight (%)
Semester activities. These include quizzes, homework, and classroom interactions	Week-15	10%
Mid semester exam	Week-7	20%
Progress exam	Week-4 and week-11	10%
Final Exam	Week-16	60%
Total		**100%**

CHAPTER ONE

RECTANGULAR COORDINATE SYSTEMS IN 3-SPACE AND VECTORS

1.1 RECTANGULAR COORDINATE SYSTEMS IN 3-SPACE

- It will be called three-dimensional space ***3-space***, two-dimensional space (a plane) ***2-space***, and one-dimensional space (a line) ***1-space***.
- To locate a point in a plane, this point has 2 dimensional coordinates (a, b). a is called x-coordinate and b is called y-coordinate.
- To locate a point in a space, three coordinates are required. This point has 3 dimensional coordinates (a, b, c). Points in 3-space can be placed in one-to-one correspondence with triples of real numbers by using three mutually perpendicular coordinate lines, called the ***x-axis***, the ***y-axis***, and the ***z-axis***, positioned so that their origins coincide (Figure 1-1).

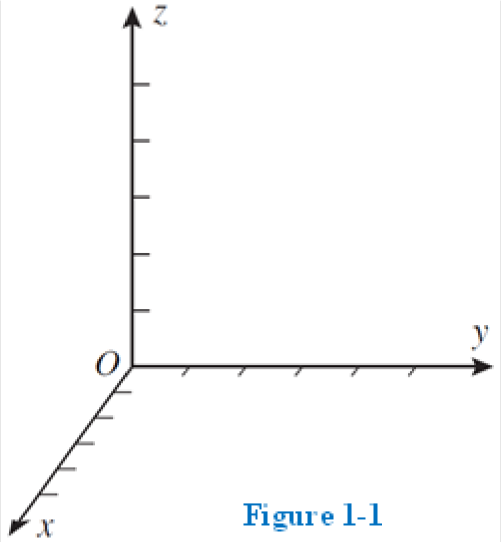


Figure 1-1

- The three coordinate axes form a three dimensional ***rectangular coordinate system*** (or ***Cartesian coordinate system***).
- The point of intersection of the coordinate axes is called the ***origin*** of the coordinate system.
- The coordinate axes, taken in pairs, determine three ***coordinate planes***: the ***xy-plane***, the ***xz-plane***, and the ***yz-plane*** (Figure 1-2).

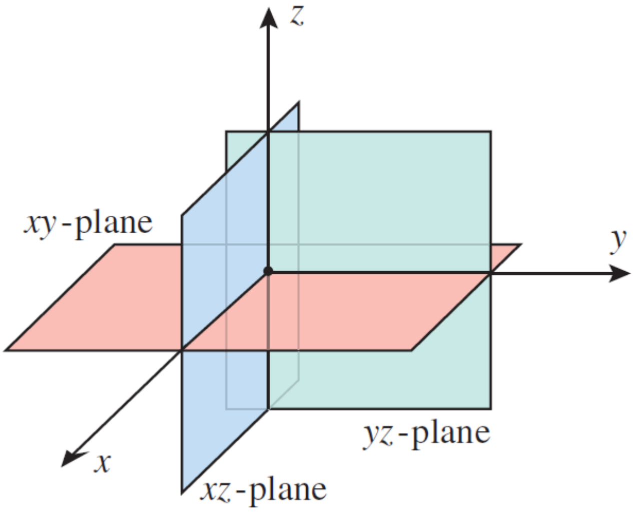


Figure 1-2

- To each point P in 3-space, we can assign a triple of real numbers by passing three planes through P parallel to the coordinate planes and letting a, b, and c be the coordinates of the intersections of those planes with the x-axis, y-axis, and z-axis, respectively (Figure 1-3). We call *a*, *b*, and *c* the ***x-coordinate***, ***y-coordinate***, and ***z-coordinate*** of *P*, respectively, and we denote the point *P* by *(a, b, c)*.

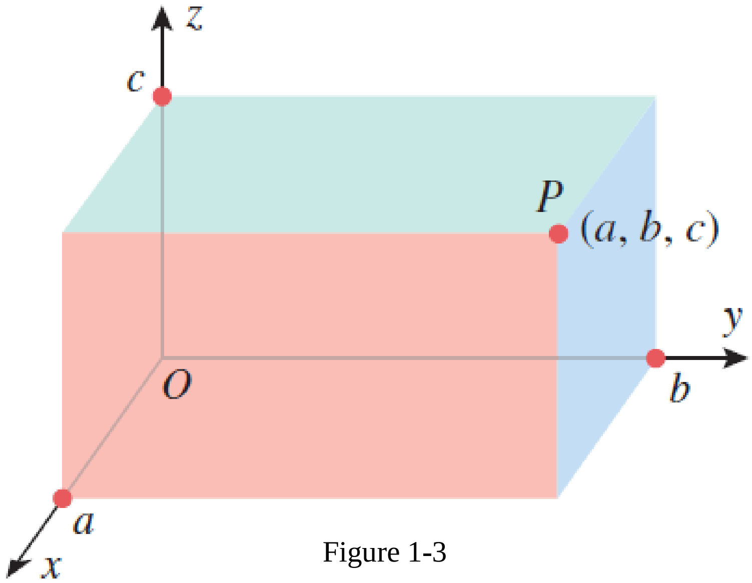


Figure 1-3

-Just as the coordinate axes in a two-dimensional coordinate system divide 2-space into four ***quadrants***, so the coordinate planes of a three-dimensional coordinate system divide 3-space into eight parts, called ***octants***. The set of points with three positive coordinates forms the ***first octant***; the remaining octants have no standard numbering.

-You should be able to visualize the following facts about three-dimensional rectangular coordinate systems:

REGION	DESCRIPTION
xy-plane	Consists of all points of the form $(x, y, 0)$
xz-plane	Consists of all points of the form $(x, 0, z)$
yz-plane	Consists of all points of the form $(0, y, z)$
x-axis	Consists of all points of the form $(x, 0, 0)$
y-axis	Consists of all points of the form $(0, y, 0)$
z-axis	Consists of all points of the form $(0, 0, z)$

1.1.1 Distance in 3-Space

In 2-space, the distance *d* between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance formula in 3-space has the same form, but it has a third term to account for the added dimension. The distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 1.1 Find the distance d between the points (2, 3, −1) and (4, −1, 3).

Solution:

$$d = \sqrt{(4-2)^2 + (-1-3)^2 + (3+1)^2} = \sqrt{36} = 6$$

- The standard equation of the circle in 2-space that has centre (x_0, y_0) and radius r is

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

- The ***standard equation of the sphere*** in 3-space that has centre (x_0, y_0, z_0) and radius r is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Example 1.2:

EQUATION	GRAPH
$(x-3)^2 + (y-2)^2 + (z-1)^2 = 9$	Sphere with center (3, 2, 1) and radius 3
$(x+1)^2 + y^2 + (z+4)^2 = 5$	Sphere with center (−1, 0, −4) and radius $\sqrt{5}$
$x^2 + y^2 + z^2 = 1$	Sphere with center (0, 0, 0) and radius 1

Example 1.3 Find the centre and radius of the sphere

$$x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0$$

$$(x^2 - 2x) + (y^2 - 4y) + (z^2 + 8z) = -17$$

$$(x^2 - 2x + 1) + (y^2 - 4y + 4) + (z^2 + 8z + 16) = -17 + 21$$

$$(x-1)^2 + (y-2)^2) + (z+4)^2 = 4$$

From the equation, the centre of the sphere with (1, 2, −4) and radius 2.

- In general, completing the squares in the previous equation produces an equation of the form

$$\boldsymbol{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = k}$$

If $k > 0$, then the graph of this equation is a sphere with centre (x_0, y_0, z_0) and radius $\sqrt{k}$.

If $k = 0$, then the sphere has radius zero, so the graph is the single point (x_0, y_0, z_0).

If $k < 0$, the equation is not satisfied by any values of x, y, and z (why?), so it has no graph.

1.1.2 CYLINDRICAL SURFACES

- Although it is natural to graph equations in two variables in 2-space and equations in three variables in 3-space, it is also possible to graph equations in two variables in 3-space.
- For example, the graph of the equation $y = x^2$ in an *xy*-coordinate system is a parabola; however, there is nothing to prevent us from inquiring about its graph in an *xyz*-coordinate system. To obtain this graph we need only observe that the equation $y = x^2$ does not impose any restrictions on *z*. Thus, if we find values of *x* and *y* that satisfy this equation, then the coordinates of the point (x, y, z) will also satisfy the equation for *arbitrary* values of *z*.
- Geometrically, the point (x, y, z) lies on the vertical line through the point $(x, y, 0)$ in the *xy*-plane, which means that we can obtain the graph of $y = x^2$ in an *xyz*-coordinate system by first graphing the equation in the *xy*-plane and then translating that graph parallel to the *z*-axis to generate the entire graph (Figure 1-4).

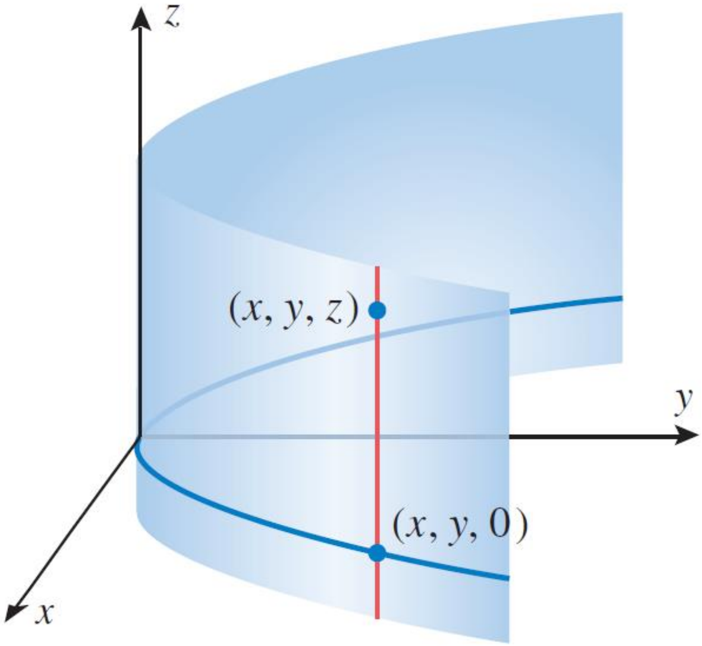
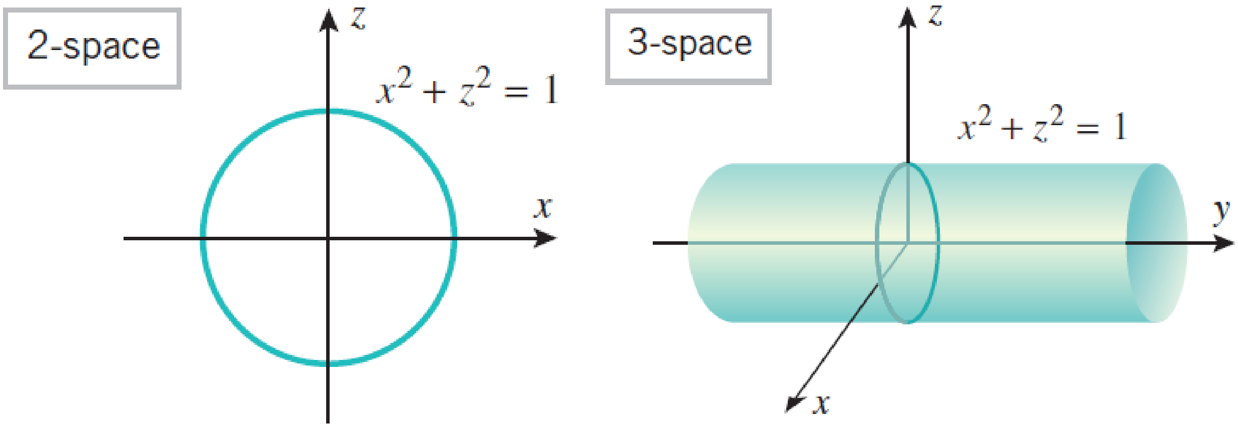


Figure 1-4

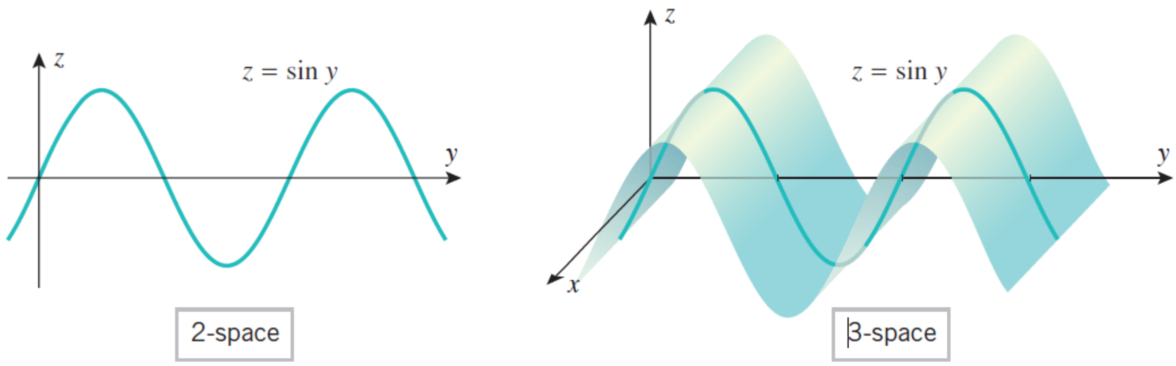
- The process of generating a surface by translating a plane curve parallel to some line is called ***extrusion***, and surfaces that are generated by extrusion are called ***cylindrical surfaces***.
- A familiar example is the surface of a right circular cylinder, which can be generated by translating a circle parallel to the axis of the cylinder.

Theorem: *An equation that contains only two of the variables x, y, and z represents a cylindrical surface in an xyz-coordinate system. The surface can be obtained by graphing the equation in the coordinate plane of the two variables that appear in the equation and then translating that graph parallel to the axis of the missing variable.*

Example 1.4 Sketch the graph of $x^2 + z^2 = 1$ in 3-space.



Example 1.5 Sketch the graph of z = sin(y) in 3-space.



Exercises

1. The distance between the points $(1, -2, 0)$ and $(4, 0, 5)$ is …………..

2. The graph of $(x - 3)2 + (y - 2)2 + (z + 1)2 = 16$ is a ……… of radius centered at ………....

3. The shortest distance from the point $(4, 0, 5)$ to the sphere $(x - 1)^2 + (y + 2)^2 + z^2 = 36$ is ……………….

4. Let S be the graph of $x^2 + z^2 + 6z = 16$ in 3-space.

(a) The intersection of S with the xz-plane is a circle with centre ……… and radius ………. .

(b) The intersection of S with the xy-plane is two lines, x = ……….. and x = …………..

(c) The intersection of S with the yz-plane is two lines, z = ……….. and z = …………..

1.2 VECTOR

- Scalars are physical quantities such as area, length, mass, and temperature and completely described once the magnitude of the quantity is given.
- Other physical quantities, called "vectors," are not completely determined until both a magnitude and a direction are specified. There are many examples like force, velocity and displacement.
- A particle that moves along a line can move in only two directions, so its direction of motion can be described by taking one direction to be positive and the other negative. Thus, the displacement or change in position of the point can be described by a signed real number.
- For example, a displacement of +3 describes a position change of 3 units in the positive direction, and a displacement of −3 describes a position change of 3 units in the negative direction.
- However, for a particle that moves in two dimensions or three dimensions, a plus or minus sign is no longer sufficient to specify the direction of motion—other methods are required.
- One method is to use an arrow, called a ***vector***, that points in the direction of motion and whose length represents the distance from the starting point to the ending point; this is called the ***displacement vector*** for the motion. See Figure 1-5.

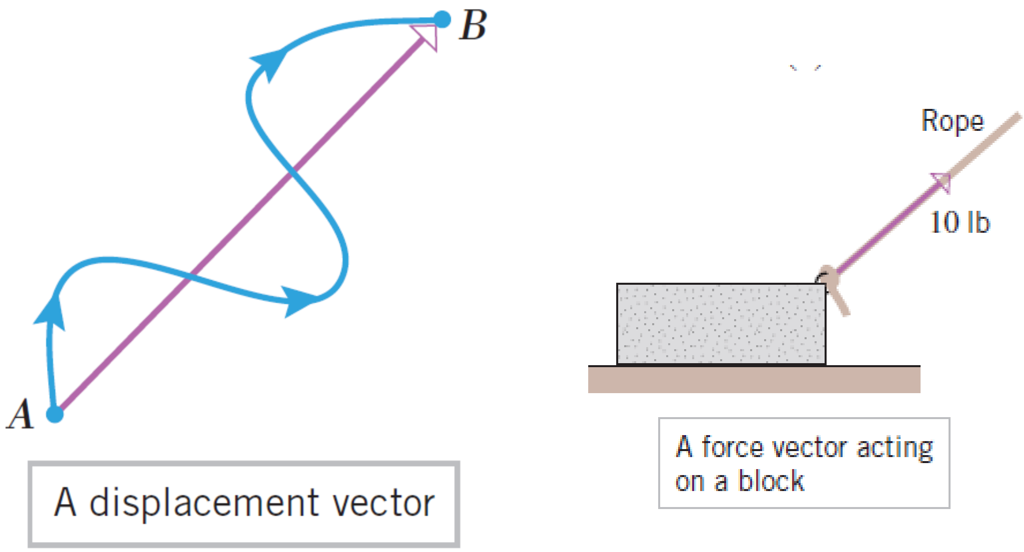


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Figure 1-5

1.2.1 Geometric vectors

- Vectors can be represented geometrically by arrows in 2-space or 3-space; the direction of the arrow specifies the direction of the vector, and the length of the arrow describes its magnitude.
- The tail of the arrow is called the ***initial point*** of the vector, and the tip of the arrow the ***terminal point***.
- We will denote vectors with lowercase boldface type such as **a**, **k**, **v**, **w**, and **x**. Two vectors, **v** and **w**, are considered to be ***equal*** (also called ***equivalent***) if they have the same length and same direction, in which case we write **v** = **w**.
- If the initial and terminal points of a vector coincide, then the vector has length zero; we call this the zero vector and denote it by 0. The zero vector does not have a specific direction.

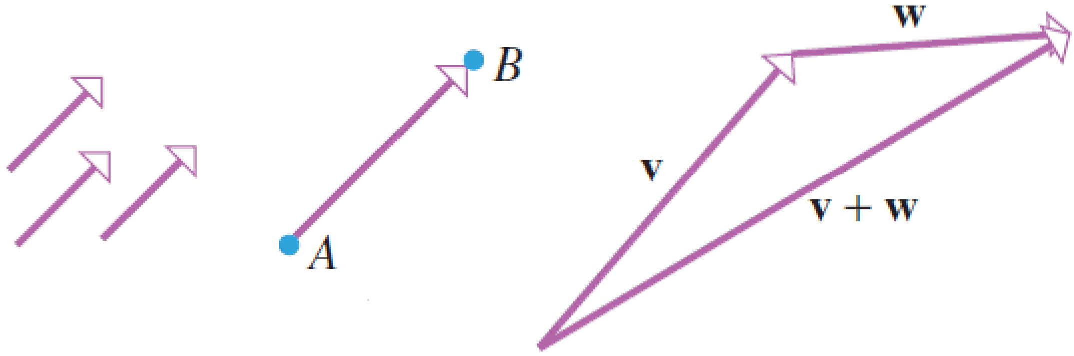


Figure 1-6

Definition If **v** and **w** are vectors, then the ***sum*** **v** + **w** is the vector from the initial point of **v** to the terminal point of **w** when the vectors are positioned so the initial point of **w** is at the terminal point of **v** (Figure 1-6).

- In Figure 1-7, we have constructed two sums, **v** + **w** (from purple arrows) and **w** + **v** (from green arrows). It is evident that

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

- The sum (gray arrow) coincides with the diagonal of the parallelogram determined by **v** and **w** when these vectors are positioned so they have the same initial point. Since the initial and terminal points of 0 coincide, it follows that

$$\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$$

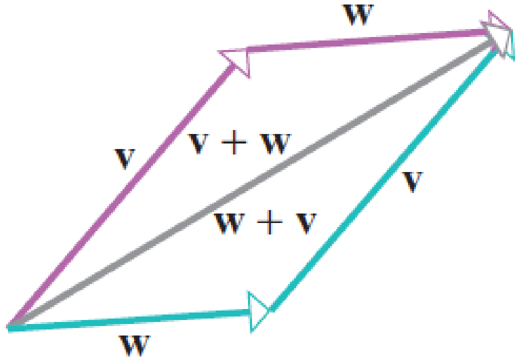
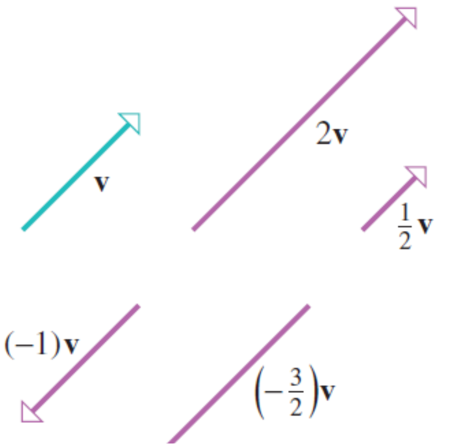


Figure 1-7

Definition If **v** is a nonzero vector and k is a nonzero real number (a scalar), then the ***scalar multiple*** k**v** is defined to be the vector whose length is $|k|$ times the length of **v** and whose direction is the same as that of **v** if $k > 0$ and opposite to that of **v** if $k < 0$. We define $k\mathbf{v} = \mathbf{0}$ if $k = 0$ or $\mathbf{v} = \mathbf{0}$.

Figure 1-8 shows the geometric relationship between a vector **v** and various scalar multiples of it.

- Observe that if k and **v** are nonzero, then the vectors **v** and k**v** lie on the same line if their initial points coincide and lie on parallel or coincident lines if they do not. Thus, we say that **v** and k**v** are ***parallel vectors***.
- Observe also that the vector $(-1)\mathbf{v}$ has the same length as **v** but is oppositely directed. We call $(-1)\mathbf{v}$ the ***negative*** of **v** and denote it by $-\mathbf{v}$ (Figure 1-9). In particular, $-\mathbf{0} = (-1)\mathbf{0} = \mathbf{0}$.



Vector subtraction is defined in terms of addition and scalar multiplication by

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$$

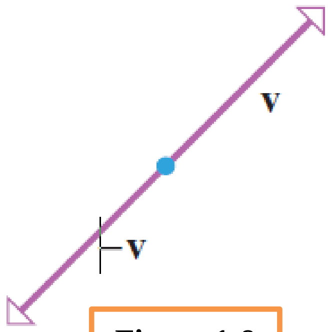


Figure 1-9

- The difference $\mathbf{v} - \mathbf{w}$ can be obtained geometrically by first constructing the vector $-\mathbf{w}$ and then adding **v** and $-\mathbf{w}$, say by the parallelogram method.
- However, if **v** and **w** are positioned so their initial points coincide, then $\mathbf{v} - \mathbf{w}$ can be formed more directly, as shown in Figure 1-10*b*, by drawing the vector from the terminal point of **w** (the second term) to the terminal point of **v** (the first term).

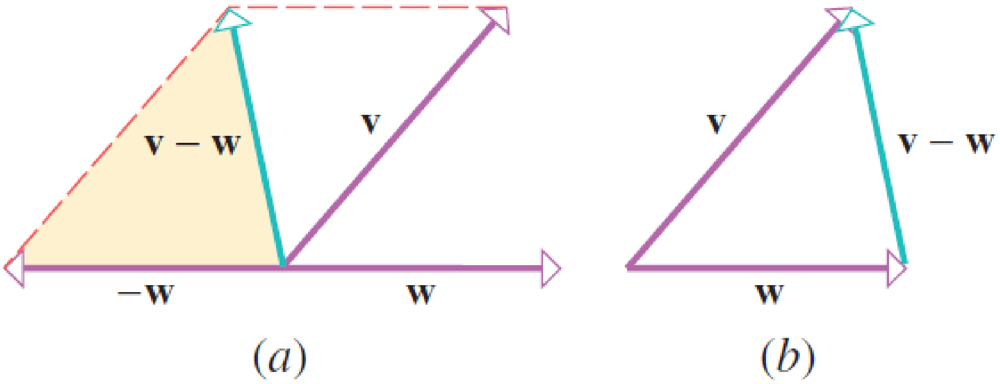


Figure 1-10

In the special case where **v** = **w** the terminal points of the vectors coincide, so their difference is **0**; that is,

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{v} - \mathbf{v} = \mathbf{0}$$

1.2.2 Vectors in coordinate systems

- As shown in figure 1-11, if a vector **v** is positioned with its initial point at the origin of a rectangular coordinate system, then its terminal point will have coordinates of the form (v_1, v_2) or (v_1, v_2, v_3), depending on whether the vector is in 2-space or 3-space.
- We call these coordinates the ***components*** of **v**, and we write **v** in *component form* using the ***bracket notation***

$$\mathbf{v} = \langle v_1, v_2 \rangle \quad \text{or} \quad \mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

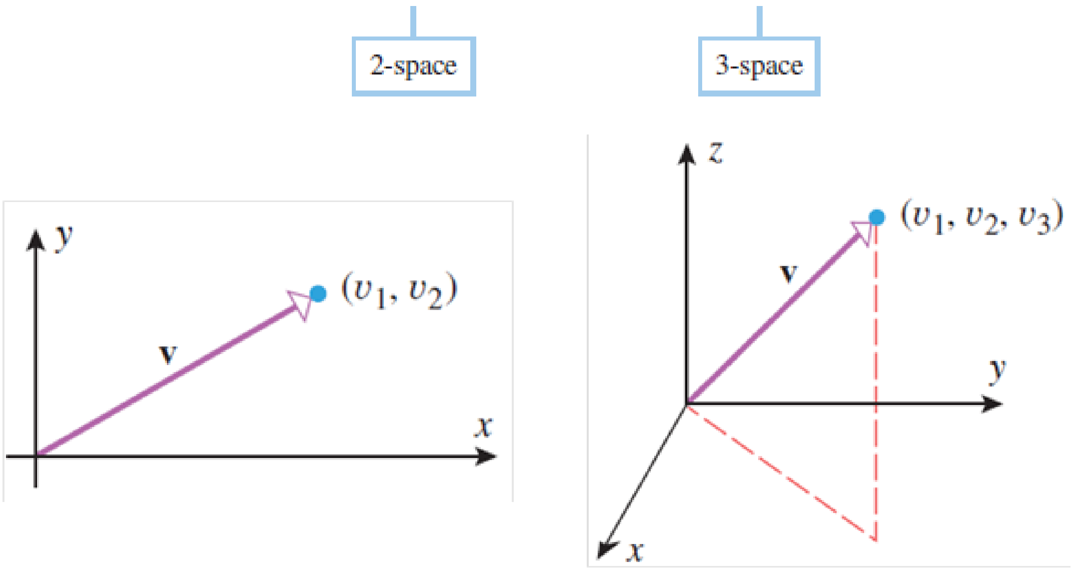


Figure 1-11

- In particular, the zero vectors in 2-space and 3-space are

$$\mathbf{0} = (0, 0) \text{ and } \mathbf{0} = (0, 0, 0)$$

- Considering the vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ in 2-space. If $\mathbf{v} = \mathbf{w}$, then the vectors have the same length and same direction, and this means that their terminal points coincide when their initial points are placed at the origin. It follows that $v_1 = w_1$ and $v_2 = w_2$, so we have shown that equivalent vectors have the same components.
- Conversely, if $v_1 = w_1$ and $v_2 = w_2$, then the terminal points of the vectors coincide when their initial points are placed at the origin. It follows that the vectors have the same length and same direction, so we have shown that vectors with the same components are equivalent.
- A similar argument holds for vectors in 3-space, so we have the following result.

Theorem Two vectors are equivalent if and only if their corresponding components are equal.

1.2.3 Arithmetic Operations on Vectors

Theorem *If* $\mathbf{v} = (v_1, v_2)$ *and* $\mathbf{w} = (w_1, w_2)$ *are vectors in 2-space and k is any scalar, then*

$$v + w = (v_1 + w_1, v_2 + w_2)$$

$$v - w = (v_1 - w_1, v_2 - w_2)$$

$$kv = (kv_1, kv_2)$$

Similarly, if $\mathbf{v} = (v_1, v_2, v_3)$ *and* $\mathbf{w} = (w_1, w_2, w_3)$ *are vectors in 3-space and k is any scalar, then*

$$v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$$

$$v - w = (v_1 - w_1, v_2 - w_2, v_3 - w_3)$$

$$kv = (kv_1, kv_2, kv_3)$$

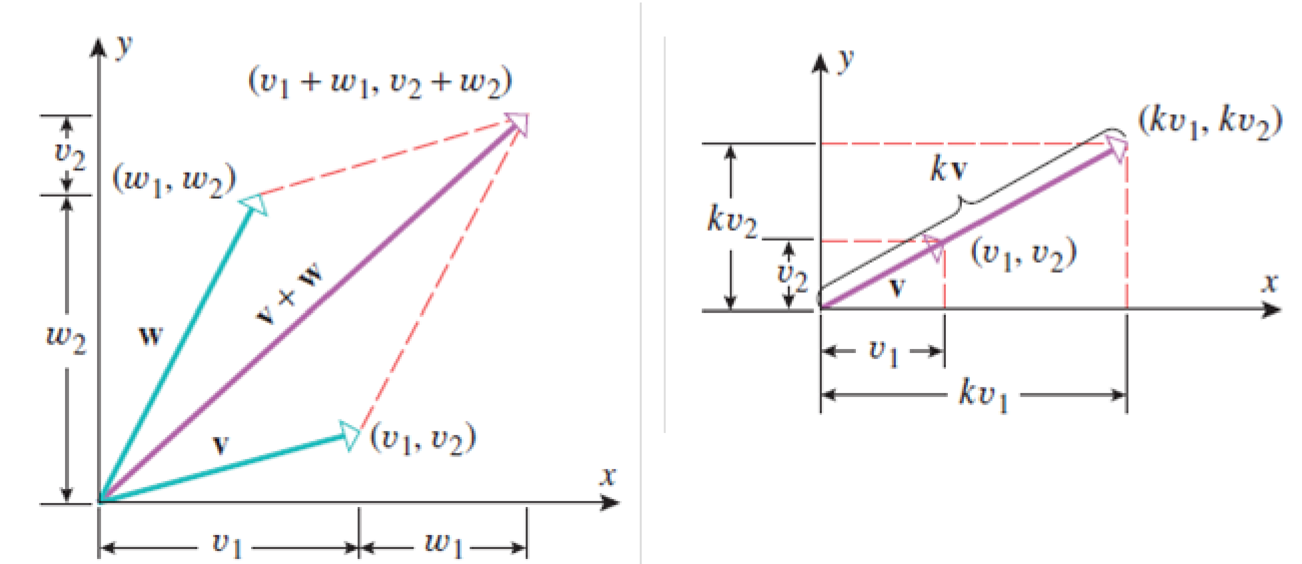


Figure 1-12

Example 1.4 If $\mathbf{v} = (-2, 0, 1)$ and $\mathbf{w} = (3, 5, -4)$, then

$\mathbf{v} + \mathbf{w} = (-2, 0, 1) + (3, 5, -4) = (1, 5, -3)$

$3\mathbf{v} = (-6, 0, 3)$

$-\mathbf{w} = (-3, -5, 4)$

$\mathbf{w} - 2\mathbf{v} = (3, 5, -4) - (-4, 0, 2) = (7, 5, -6)$

1.2.4 Vectors with Initial Point Not at the Origin

To be specific, suppose that $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in 2-space and we are interested in finding the components of the vector $\overrightarrow{P_1P_2}$. As shown in Figure 1-13, we can write this vector as

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1,\ y_2 - y_1)$$

Theorem *If* $\overrightarrow{P_1P_2}$ *is a vector in 2-space with initial point* $P_1(x_1, y_1)$ *and terminal point* $P_2(x_2, y_2)$*, then*

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$$

Similarly, if $\overrightarrow{P_1P_2}$ *is a vector in 3-space with initial point* $P_1(x_1, y_1, z_1)$ *and terminal point* $P_2(x_2, y_2, z_2)$*, then*

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

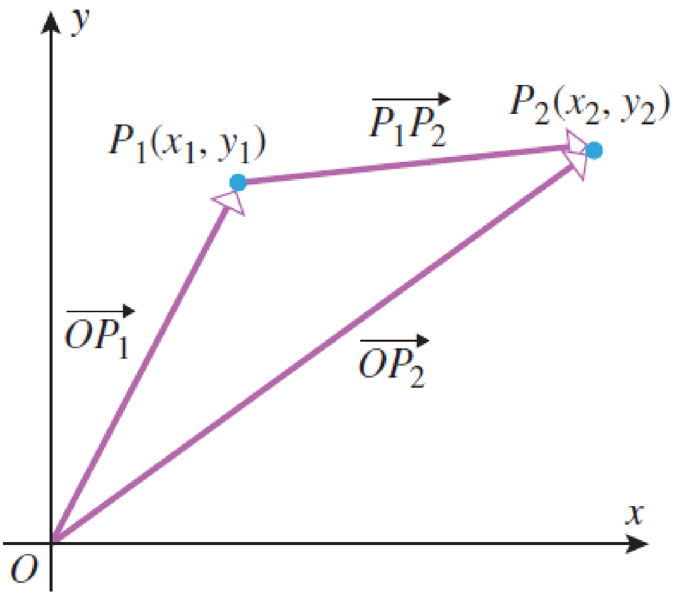


Figure 1-13

Example 1.5 In 2-space the vector from $P_1(1, 3)$ to $P_2(4, -2)$ is

$$\overrightarrow{P_1P_2} = (4 - 1, -2 - 3) = (3, -5)$$

and in 3-space the vector from A(0,−2, 5) to B(3, 4,−1) is

$$\overrightarrow{AB} = (3 - 0, 4 + 2, -1 - 5) = (3, 6, -6)$$

1.2.5 Rules of Vector Arithmetic

Theorem *For any vectors* **u**, **v**, *and* **w** *and any scalars* ***k*** *and* ***l***, *the following relationships hold:*

$(a)\ \mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ $\qquad (e)\ k(l\mathbf{u})=(kl)\mathbf{u}$

$(b)\ (\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ $\qquad (f)\ k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

$(c)\ \mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$ $\qquad (g)\ (k+l)\mathbf{u}=k\mathbf{u}+l\mathbf{u}$

$(d)\ \mathbf{u}+(-\mathbf{u})=\mathbf{0}$ $\qquad (h)\ 1\mathbf{u}=\mathbf{u}$

1.2.6 Norm of a Vector

- The distance between the initial and terminal points of a vector **v** is called the ***length***, the ***norm***, or the ***magnitude*** of **v** and is denoted by ||**v**||.
- This distance does not change if the vector is translated, so for purposes of calculating the norm, we can assume that the vector is positioned with its initial point at the origin (Figure 1-14). This makes it evident that the norm of a vector **v** = (v_1, v_2) in 2-space is given by

$$\|\mathrm{v}\| = \sqrt{v_1^2 + v_2^2}$$

- and the norm of a vector v =(v_1, v_2, v_3) in 3-space is given by

$$\|\mathrm{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

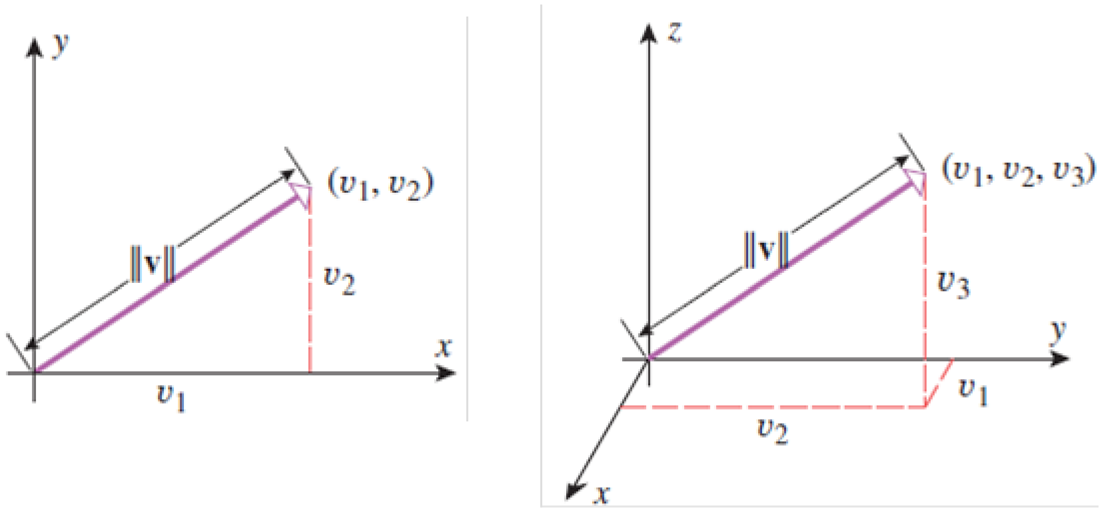


Figure 1-14

Example 1.6 Find the norms of **v** = (−2, 3), 10**v** = (−20, 30), and **w** = (2, 3, 6).

Solution

$$\|v\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$$

$$\|10v\| = \sqrt{(-20)^2 + 30^2} = \sqrt{1300} = 10\sqrt{13}$$

$$\|w\| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$$

Note: $\|kv\| = |k|\|v\|$

For example $\|3v\| = |3|\|v\| = 3\|v\|$

$$\|-2v\| = |-2|\|v\| = 2\|v\|$$

1.2.7 Unit Vectors

- A vector of length 1 is called a ***unit vector***.

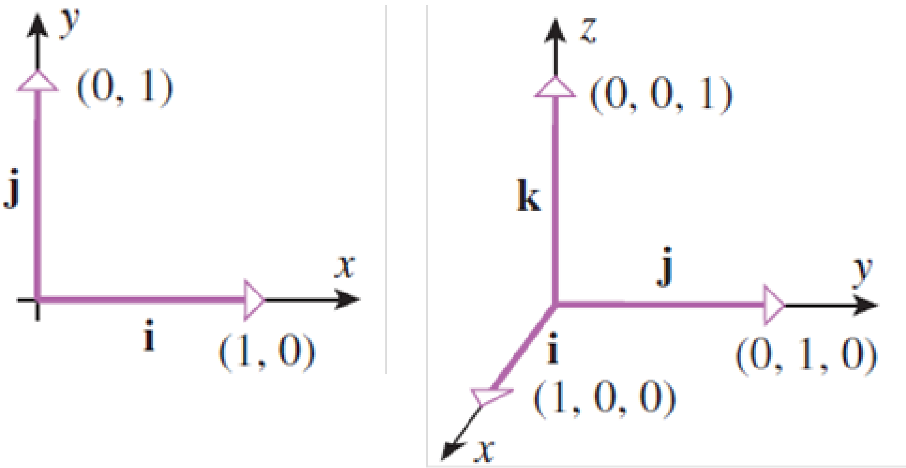


Figure 1-15

In an *xy*-coordinate system the unit vectors along the *x*- and *y*-axes are denoted by **i** and **j**, respectively; and in an *xyz*-coordinate system the unit vectors along the *x*-, *y*-, and *z*-axes are denoted by **i**, **j**, and **k**, respectively.

$\mathbf{i} = \langle 1, 0\rangle, \quad \mathbf{j} = \langle 0, 1\rangle$ In 2-space

$\mathbf{i} = \langle 1, 0, 0\rangle, \quad \mathbf{j} = \langle 0, 1, 0\rangle, \quad \mathbf{k} = \langle 0, 0, 1\rangle$ In 3-space

As shown in figure 1-15, every vector in 2-space is expressible uniquely in terms of **i** and **j**, and every vector in 3-space is expressible uniquely in terms of **i**, **j**, and **k** as follows:

$$\mathbf{v} = (v_1, v_2) = (v_1, 0) + (0, v_2) = v_1(1, 0) + v_2(0, 1) = v_1\mathbf{i} + v_2\mathbf{j}$$

$$\mathbf{v} = (v_1, v_2, v_3) = v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Example 1.7 The following table provides some examples of vector notation in 2-space and 3-space.

2-SPACE	3-SPACE
$\langle 2, 3\rangle = 2\mathbf{i} + 3\mathbf{j}$	$\langle 2, -3, 4\rangle = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$
$\langle -4, 0\rangle = -4\mathbf{i} + 0\mathbf{j} = -4\mathbf{i}$	$\langle 0, 3, 0\rangle = 3\mathbf{j}$
$\langle 0, 0\rangle = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}$	$\langle 0, 0, 0\rangle = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$
$(3\mathbf{i} + 2\mathbf{j}) + (4\mathbf{i} + \mathbf{j}) = 7\mathbf{i} + 3\mathbf{j}$	$(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$
$5(6\mathbf{i} - 2\mathbf{j}) = 30\mathbf{i} - 10\mathbf{j}$	$2(\mathbf{i} + \mathbf{j} - \mathbf{k}) + 4(\mathbf{i} - \mathbf{j}) = 6\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$
$\|2\mathbf{i} - 3\mathbf{j}\| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$	$\|\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$
$\|v_1\mathbf{i} + v_2\mathbf{j}\| = \sqrt{v_1^2 + v_2^2}$	$\|\langle v_1, v_2, v_3\rangle\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

1.2.8 Normalizing a Vector

- A common problem in applications is to find a unit vector **u** that has the same direction as some given nonzero vector **v**. This can be done by multiplying **v** by the reciprocal of its length; that is,

$$\mathrm{u} = \frac{1}{\|\mathrm{v}\|}\,\mathrm{v} = \frac{\mathrm{v}}{\|\mathrm{v}\|}$$

is a unit vector with the same direction as **v**—the direction is the same because $k = 1/\|\mathrm{v}\|$ is a positive scalar, and the length is 1 because

$$\|\mathrm{u}\| = \|k\mathrm{v}\| = |k|\|\mathrm{v}\| = \frac{1}{\|\mathrm{v}\|}\,\|\mathrm{v}\| = 1$$

Example 1.8 find the unit vector that has the same direction as $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

$$\|\mathrm{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

So the unit vector **u** in the same direction as **v** is

$$\mathbf{u} = \frac{1}{3}\mathbf{v} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$$

1.2.9 Vectors Determined by Length and Angle

If **v** is a nonzero vector with its initial point at the origin of an *xy*-coordinate system, and if θ is the angle from the positive *x*-axis to the radial line through **v**, then the *x*-component of **v** can be written as $\|\mathrm{v}\| \cos\theta$ and the *y*-component as $\|\mathrm{v}\| \sin\theta$ (Figure 1-16); and hence **v** can be expressed in trigonometric form as

$$\mathbf{v} = \|\mathrm{v}\| \ (\cos\theta\,, \sin\theta) \quad or \quad \mathbf{v} = \|\mathrm{v}\| \cos\theta\,\mathbf{i} + \|\mathrm{v}\|\sin\theta\,\mathbf{j}$$

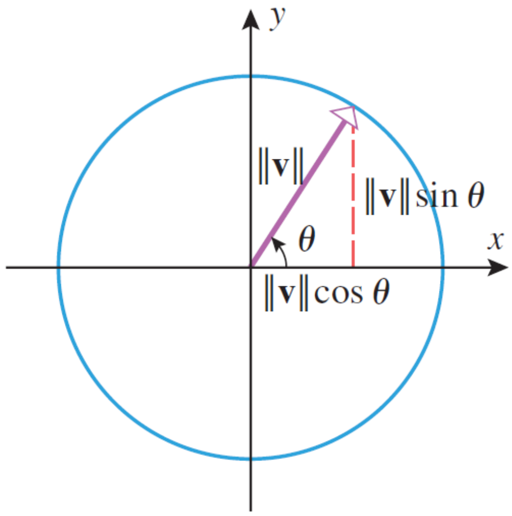


Figure 1-16

In the special case of a unit vector **u** this simplifies to

$$\mathbf{u} = (\cos\theta\,, \sin\theta) \quad or \quad \mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

Example 1.9

(a) Find the vector of length **2** that makes an angle of ***π*/4** with the positive *x*-axis.

(b) Find the angle that the vector $\mathbf{v} = -\sqrt{3}\,\mathbf{i} + \mathbf{j}$ makes with the positive *x*-axis.

$$\mathbf{v} = 2\ \cos\frac{\pi}{4}\mathbf{i}\,, +2\,\sin\frac{\pi}{4}\mathbf{j} = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$$

We will normalize **v**, then use (previous equation) to find sin θ and cos θ, and then use these values to find θ. Normalizing **v** yields

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{-\sqrt{3}\mathbf{i} + \mathbf{j}}{\sqrt{(-\sqrt{3})^2 + 1^2}} = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

Thus, $\cos\theta = -\sqrt{3}/2$ and sinθ = 1/2, from which we conclude that **θ = 5π/6.**

1.2.10 Vectors Determined by Length and a Vector in the Same Direction

It is a common problem in many applications that a direction in 2-space or 3-space is determined by some known unit vector **u**, and it is of interest to find the components of a vector **v** that has the same direction as **u** and some specified length $\|\mathrm{v}\|$. This can be done by expressing **v** as

$$\mathbf{v} = \|\mathbf{v}\|\mathbf{u}$$

v is equal to its length times a unit vector in the same direction.

and then reading off the components of $\|\mathrm{v}\|\mathrm{u}$.

Example 1.10 Figure 1-17 shows a vector v of length $\sqrt{5}$ that extends along the line through A and B. Find the components of **v**.

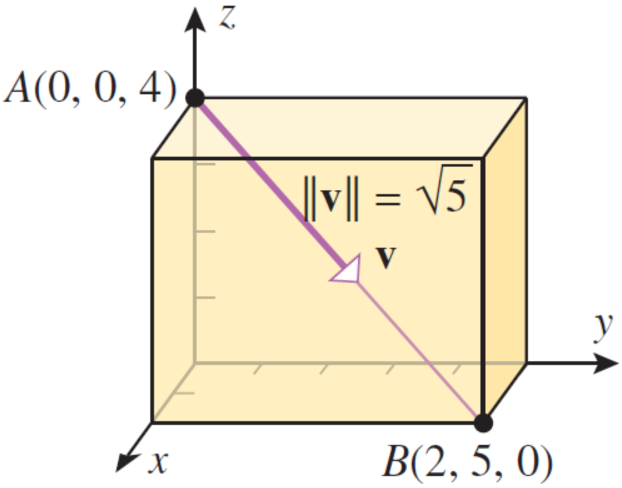


Figure 1-17

Solution:

$$\overrightarrow{AB} = \langle 2, 5, 0\rangle - \langle 0, 0, 4\rangle = \langle 2, 5, -4\rangle$$

$$\|\overrightarrow{AB}\| = \sqrt{2^2 + 5^2 + (-4)^2} = \sqrt{45} = 3\sqrt{5}$$

$$\frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \left\langle \frac{2}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}, -\frac{4}{3\sqrt{5}} \right\rangle$$

$$\mathbf{v} = \|\mathbf{v}\|\left(\frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|}\right) = \sqrt{5}\left\langle \frac{2}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}, -\frac{4}{3\sqrt{5}} \right\rangle = \left\langle \frac{2}{3}, \frac{5}{3}, -\frac{4}{3} \right\rangle$$

1.2.11 Resultant of Two Concurrent Forces

- If two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied at the same point on an object, then the two forces have the same effect on the object as the single force $\mathbf{F}_1 + \mathbf{F}_2$ applied at the point (Figure 1-18).
- Physicists and engineers call $\mathbf{F}_1 + \mathbf{F}_2$ the ***resultant*** of $\mathbf{F}_1$ and $\mathbf{F}_2$, and they say that the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are ***concurrent*** to indicate that they are applied at the same point.

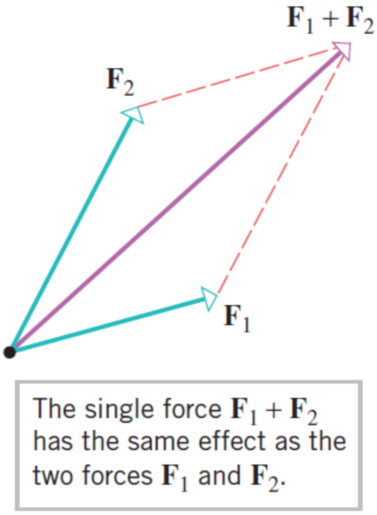


Figure 1-18

Example 1.11 Suppose that two forces are applied to an eye bracket, as shown in Figure 1-19. Find the magnitude of the resultant and the angle θ that it makes with the positive x-axis.

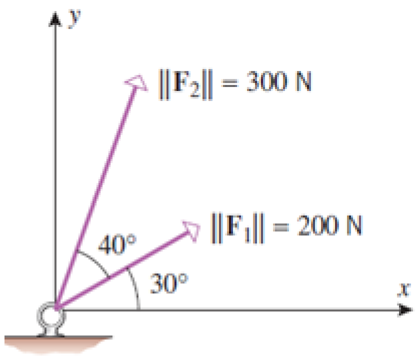


Figure 1-19

Solution. Note that $\mathbf{F}_1$ makes an angle of 30° with the positive x-axis and $\mathbf{F}_2$ makes an angle of 30° + 40° = 70° with the positive x-axis. Since we are given that $\|\mathbf{F}_1\| = 200$ N and $\|\mathbf{F}_2\| = 300$ N,

$$\mathbf{F}_1 = 200\langle \cos 30°, \sin 30° \rangle = \langle 100\sqrt{3}, 100 \rangle$$

$$\mathbf{F}_2 = 300\langle \cos 70°, \sin 70° \rangle = \langle 300\cos 70°, 300\sin 70° \rangle$$

Therefore, the resultant $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ has component form

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \langle 100\sqrt{3} + 300\cos 70^\circ, 100 + 300\sin 70^\circ \rangle$$
$$= 100\langle \sqrt{3} + 3\cos 70^\circ, 1 + 3\sin 70^\circ \rangle \approx \langle 275.8, 381.9 \rangle$$

The magnitude of the resultant is then

$$\|\mathbf{F}\| = 100\sqrt{\left(\sqrt{3} + 3\cos 70^\circ\right)^2 + \left(1 + 3\sin 70^\circ\right)^2} \approx 471 \text{ N}$$

Let θ denote the angle **F** makes with the positive x-axis when the initial point of **F** is at the origin.

$$\|\mathbf{F}\|\cos\theta = 100\sqrt{3} + 300\cos 70^\circ \quad \text{or} \quad \cos\theta = \frac{100\sqrt{3} + 300\cos 70^\circ}{\|\mathbf{F}\|}$$

Since the terminal point of F is in the first quadrant, we have

$$\theta = \cos^{-1}\left(\frac{100\sqrt{3} + 300\cos 70^\circ}{\|\mathbf{F}\|}\right) \approx 54.2^\circ$$

See Figure 1-20

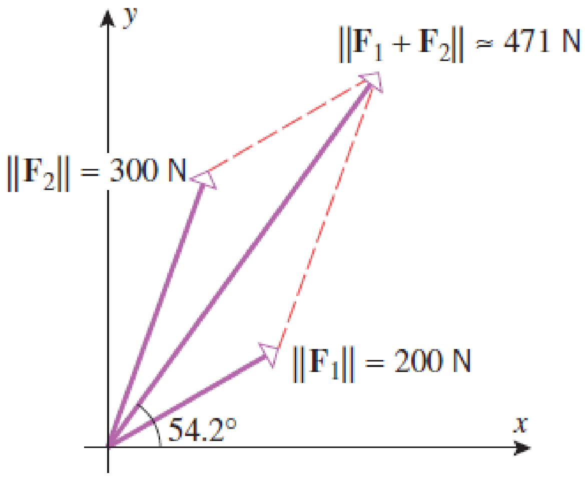


Figure 1-20

1.3 DOT PRODUCT; PROJECTIONS

1.3.1 Definition of the Dot Product

Definition If $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are vectors in 2-space, then the ***dot product*** of **u** and **v** is written as $\mathbf{u}\cdot\mathbf{v}$ and is defined as $\langle \mathbf{u}\cdot\mathbf{v}\rangle = u_1v_1 + u_2v_2$

Similarly, if $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$are vectors in 3-space, then their dot product is defined as $\langle \mathbf{u}\cdot\mathbf{v}\rangle = u_1v_1 + u_2v_2 + u_3v_3$

- In words, the dot product of two vectors is formed by multiplying their corresponding components and adding the resulting products. Note that the dot product of two vectors is a scalar.

Example 1.12

$$\langle 3, 5\rangle \cdot \langle -1, 2\rangle = 3(-1) + 5(2) = 7$$
$$\langle 2, 3\rangle \cdot \langle -3, 2\rangle = 2(-3) + 3(2) = 0$$
$$\langle 1, -3, 4\rangle \cdot \langle 1, 5, 2\rangle = 1(1) + (-3)(5) + 4(2) = -6$$

Here are the same computations expressed another way:

$$(3\mathbf{i} + 5\mathbf{j}) \cdot (-\mathbf{i} + 2\mathbf{j}) = 3(-1) + 5(2) = 7$$
$$(2\mathbf{i} + 3\mathbf{j}) \cdot (-3\mathbf{i} + 2\mathbf{j}) = 2(-3) + 3(2) = 0$$
$$(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) = 1(1) + (-3)(5) + 4(2) = -6$$

1.3.2 Algebraic Properties of the Dot Product

Theorem *If* **u**, **v**, *and* **w** *are vectors in 2- or 3-space and k is a scalar, then:*

(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

(b) $\mathbf{u}\cdot(\mathbf{v} + \mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$

(c) $k(\mathbf{u}\cdot\mathbf{v}) = (k\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(k\mathbf{v})$

(d) $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$

(e) $\mathbf{0}\cdot\mathbf{v} = 0$

1.3.3 Angle between Vectors

Suppose that **u** and **v** are nonzero vectors in 2-space or 3-space that are positioned so their initial points coincide. We define the ***angle between* u *and* v** to be the angle θ determined by the vectors that satisfies the condition $0 \le \theta \le \pi$ (Figure 1-21). In 2-space, θ is the smallest counter clockwise angle through which one of the vectors can be rotated until it aligns with the other.

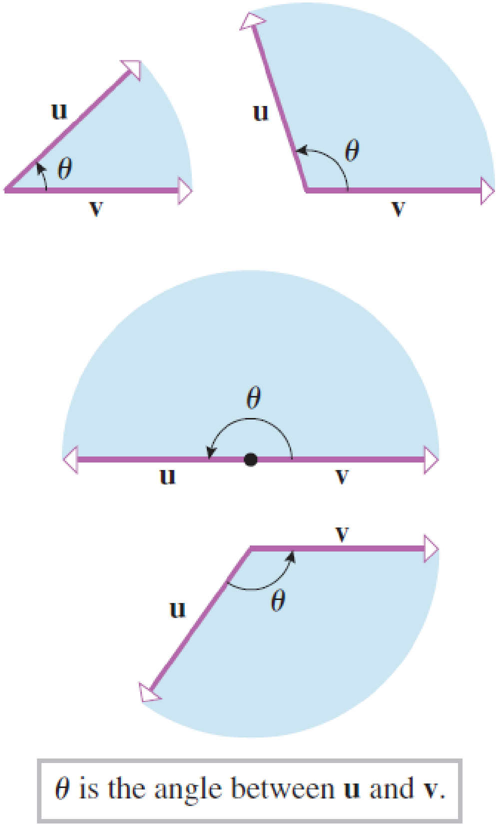


Figure 1-21

Theorem *If* **u** *and* **v** *are nonzero vectors in 2-space or 3-space, and if θ is the angle between them, then*

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Example 1.13 Find the angle between the vector $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and

(a) $\mathbf{v} = -3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ (b) $\mathbf{w} = 2\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$ (c) $\mathbf{z} = -3\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$

Solution (a).

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-11}{(3)(7)} = -\frac{11}{21}$$

$$\theta = \cos^{-1}\left(-\tfrac{11}{21}\right) \approx 2.12 \text{ radians} \approx 121.6^\circ$$

Solution (b).

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{u}\|\,\|\mathbf{w}\|} = \frac{0}{\|\mathbf{u}\|\,\|\mathbf{w}\|} = 0$$

Thus, $\theta = \pi/2$, which means that the vectors are perpendicular.

Solution (c).

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{z}}{\|\mathbf{u}\|\,\|\mathbf{z}\|} = \frac{-27}{(3)(9)} = -1$$

Thus, $\theta = \pi$, which means that the vectors are oppositely directed. (In retrospect, we could have seen this without computing θ, since $\mathbf{z} = -3\mathbf{u}$.)

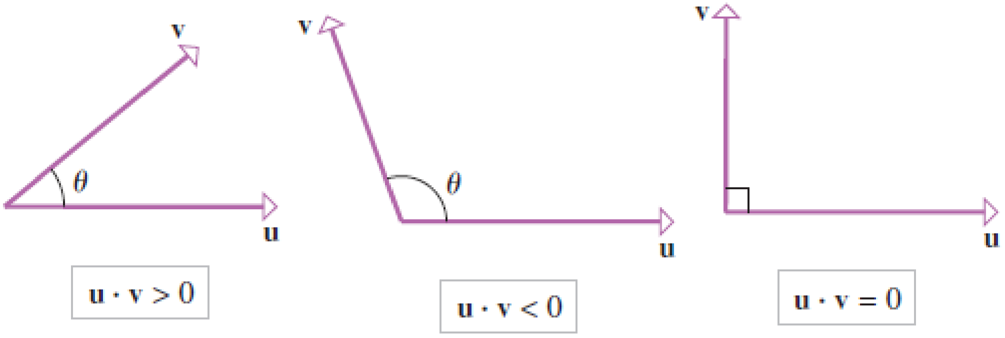


Figure 1-22

1.3.4 Interpreting the Sign of the Dot Product

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \qquad \text{(a)}$$

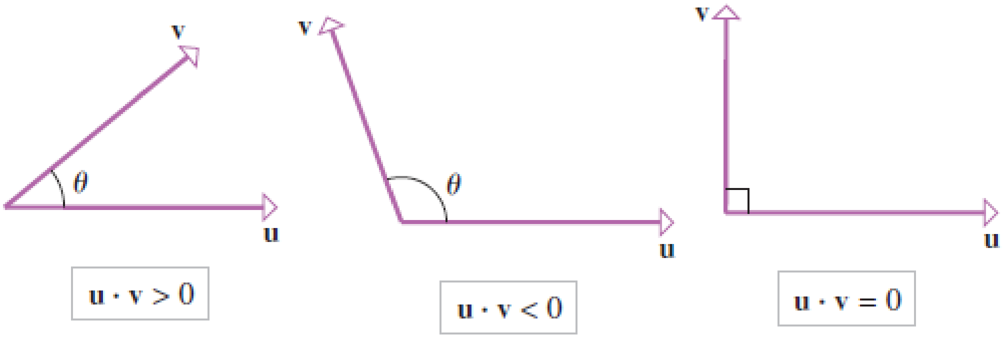


Figure 1-22

Notes:

- The terms “perpendicular,” “orthogonal,” and “normal” are all commonly used to describe geometric objects that meet at right angles.
- Although the zero vector does not make a well-defined angle with other vectors, we will consider 0 to be orthogonal to all vectors. This convention allows us to say that **u** and **v** are orthogonal vectors if and only if $\mathbf{u} \,.\, \mathbf{v} = 0$, and makes Formula (a) valid if **u** or **v** (or both) is zero.

1.3.5 Direction Angles

In an *xy*-coordinate system, the direction of a nonzero vector **v** is completely determined by the angles α and β between **v** and the unit vectors **i** and **j** (Figure 1-23), and in an *xyz*-coordinate system the direction is completely determined by the angles α, β, and γ between **v** and the unit vectors **i**, **j**, and **k** (Figure 1-23).

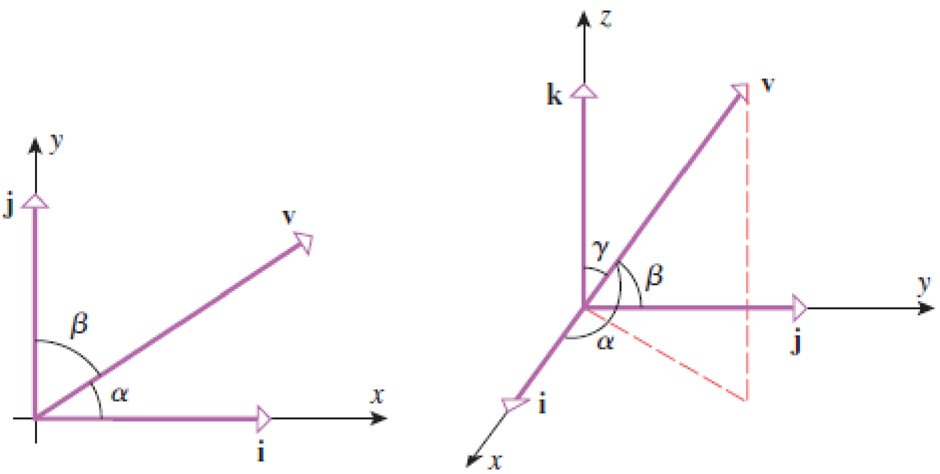


Figure 1-23

- In both 2-space and 3-space the angles between a nonzero vector **v** and the vectors **i**, **j**, and **k** are called the ***direction angles*** of **v**, and the cosines of those angles are called the ***direction cosines*** of **v**.

Theorem *The direction cosines of a nonzero vector* $\mathbf{v} = v_1\mathbf{i} + v_2\,\mathbf{j} + v_3\mathbf{k}$ *are*

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$

Example 1-14 Find the direction cosines of the vector $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, and approximate the direction angles to the nearest degree.

Solution. First we will normalize the vector **v** and then read off the components. We have

$$\|\mathbf{v}\| = \sqrt{4+16+16} = 6, \text{ so that } \mathbf{v}/\|\mathbf{v}\| = \tfrac{1}{3}\mathbf{i} - \tfrac{2}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}. \text{ Thus,}$$

$$\cos\alpha = \tfrac{1}{3}, \quad \cos\beta = -\tfrac{2}{3}, \quad \cos\gamma = \tfrac{2}{3}$$

With the help of a calculating utility we obtain

$$\alpha = \cos^{-1}\left(\tfrac{1}{3}\right) \approx 71°, \quad \beta = \cos^{-1}\left(-\tfrac{2}{3}\right) \approx 132°, \quad \gamma = \cos^{-1}\left(\tfrac{2}{3}\right) \approx 48°$$

Example 1-15 Find the angle between a diagonal of a cube and one of its edges.

Solution. Assume that the cube has side a, and introduce a coordinate system as shown in Figure 1-24. In this coordinate system the vector

$$\mathbf{d} = a\mathbf{i} + a\,\mathbf{j} + a\mathbf{k}$$

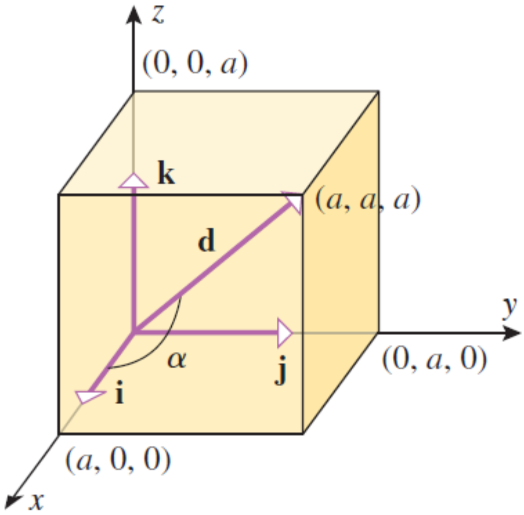


Figure 1-24

is a diagonal of the cube and the unit vectors **i**, **j**, and **k** run along the edges. By symmetry, the diagonal makes the same angle with each edge, so it is sufficient to find the angle between **d** and **i** (the direction angle α). Thus,

$$\cos\alpha = \frac{\mathbf{d}\cdot\mathbf{i}}{\|\mathbf{d}\|\,\|\mathbf{i}\|} = \frac{a}{\|\mathbf{d}\|} = \frac{a}{\sqrt{3a^2}} = \frac{1}{\sqrt{3}}$$

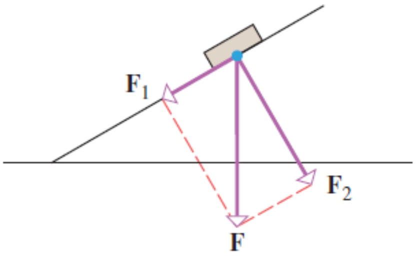
$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.955 \text{ radian} \approx 54.7^\circ$$

1.3.1 Decomposing Vectors into Orthogonal Components

In many applications it is desirable to "decompose" a vector into a sum of two orthogonal vectors with convenient specified directions. For example, Figure 1-25 shows a block on an inclined plane. The downward force **F** that gravity exerts on the block can be decomposed into the sum

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

where the force $\mathbf{F}_1$ is parallel to the ramp and the force $\mathbf{F}_2$ is perpendicular to the ramp. The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are useful because $\mathbf{F}_1$ is the force that pulls the block *along* the ramp, and $\mathbf{F}_2$ is the force that the block exerts *against* the ramp.



The force of gravity pulls the block against the ramp and down the ramp.

Figure 1-25

Thus, our next objective is to develop a computational procedure for decomposing a vector into a sum of orthogonal vectors. For this purpose, suppose that $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthogonal *unit* vectors in 2-space, and suppose that we want to express a given vector **v** as a sum

$$\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$$

so that $\mathbf{w}_1$ is a scalar multiple of $\mathbf{e}_1$ and $\mathbf{w}_2$ is a scalar multiple of $\mathbf{e}_2$ (Figure 1-26*a*).

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{e}_1)\, \mathbf{e}_1 + (\mathbf{v} \cdot \mathbf{e}_2)\, \mathbf{e}_2$$

In this formula we call $(\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1$ and $(\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2$ the ***vector components*** of **v** along $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively;

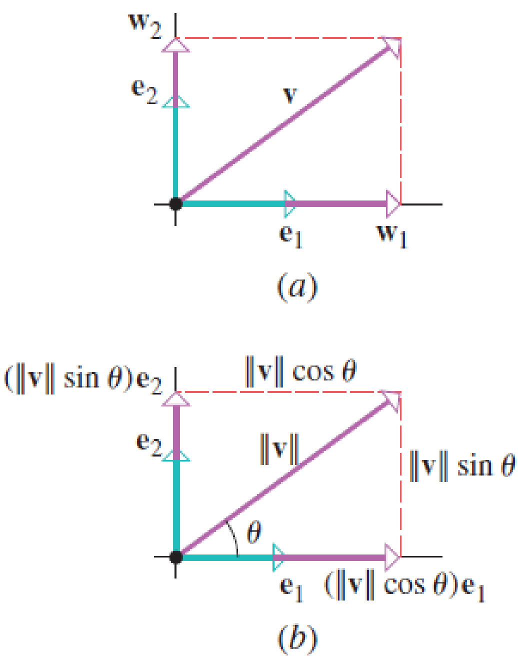


Figure 1-26

and we call $\mathbf{v} \cdot \mathbf{e}_1$ and $\mathbf{v} \cdot \mathbf{e}_2$ the ***scalar components*** of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. If θ denotes the angle between $\mathbf{v}$ and $\mathbf{e}_1$, and the angle between $\mathbf{v}$ and $\mathbf{e}_2$ is $\pi/2$ or less, then the scalar components of $\mathbf{v}$ can be written in trigonometric form as

$$\mathbf{v} \cdot \mathbf{e}_1 = \|\mathbf{v}\| \cos\theta \qquad \text{and} \qquad \mathbf{v} \cdot \mathbf{e}_2 = \|\mathbf{v}\| \sin\theta$$

(Figure 1-26*b*). Moreover, the vector components of $\mathbf{v}$ can be expressed as

$$(\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 = (\|\mathbf{v}\| \cos\theta)\mathbf{e}_1 \qquad \text{and} \qquad (\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 = (\|\mathbf{v}\| \sin\theta)\mathbf{e}_2$$

The decomposition can be expressed as

$$\mathbf{v} = (\|\mathbf{v}\| \cos\theta)\mathbf{e}_1 + (\|\mathbf{v}\| \sin\theta)\mathbf{e}_2$$

provided the angle between $\mathbf{v}$ and $\mathbf{e}_2$ is at most $\pi/2$.

Example 1.16 Let

$$\mathbf{v} = \langle 2, 3 \rangle, \quad \mathbf{e}_1 = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle, \quad \text{and} \quad \mathbf{e}_2 = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Find the scalar components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ and the vector components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$.

Solution. The scalar components of $\mathbf{v}$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ are

$$\mathbf{v} \cdot \mathbf{e}_1 = 2\left(\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{5}{\sqrt{2}}$$

$$\mathbf{v} \cdot \mathbf{e}_2 = 2\left(-\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

so the vector components are

$$(\mathbf{v} \cdot \mathbf{e}_1)\mathbf{e}_1 = \frac{5}{\sqrt{2}}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle \frac{5}{2}, \frac{5}{2} \right\rangle$$

$$(\mathbf{v} \cdot \mathbf{e}_2)\mathbf{e}_2 = \frac{1}{\sqrt{2}}\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle -\frac{1}{2}, \frac{1}{2} \right\rangle$$

Example 1.17 A rope is attached to a 100 lb block on a ramp that is inclined at an angle of 30◦ with the ground (Figure 1-27*a*). How much force does the block exert against the ramp, and how much force must be applied to the rope in a direction parallel to the ramp to prevent the block from sliding down the ramp? (Assume that the ramp is smooth, that is, exerts no frictional forces.)

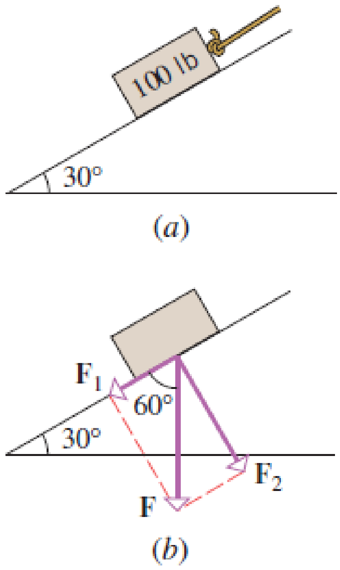


Figure 1-27

Solution. Let **F** denote the downward force of gravity on the block (so ||**F**|| = 100 lb), and let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the vector components of **F** parallel and perpendicular to the ramp (as shown in Figure 1-27*b*). The lengths of $\mathbf{F}_1$ and $\mathbf{F}_2$ are

$$\|\mathbf{F}_1\| = \|\mathbf{F}\|\cos 60^\circ = 100\left(\frac{1}{2}\right) = 50 \text{ lb}$$

$$\|\mathbf{F}_2\| = \|\mathbf{F}\|\sin 60^\circ = 100\left(\frac{\sqrt{3}}{2}\right) \approx 86.6 \text{ lb}$$

Thus, the block exerts a force of approximately 86.6 lb against the ramp, and it requires a force of 50 lb to prevent the block from sliding down the ramp.

1.3.2 Orthogonal Projections

The vector components of **v** along $\mathbf{e}_1$ and $\mathbf{e}_2$ in previous equation are also called the *orthogonal projections* of **v** on $\mathbf{e}_1$ and $\mathbf{e}_2$ and are commonly denoted by

$$\text{proj}_{\mathbf{e}_1}\mathbf{v} = (\mathbf{v}\cdot\mathbf{e}_1)\mathbf{e}_1 \quad \text{and} \quad \text{proj}_{\mathbf{e}_2}\mathbf{v} = (\mathbf{v}\cdot\mathbf{e}_2)\mathbf{e}_2$$

In general, if **e** is a unit vector, then we define the ***orthogonal projection of* v *on* e** to be

$$\text{proj}_{\mathbf{e}}\mathbf{v} = (\mathbf{v}\cdot\mathbf{e})\mathbf{e}$$

$$\text{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}$$

Geometrically, if **b** and **v** have a common initial point, then $\text{proj}_{\mathbf{b}}\mathbf{v}$ is the vector that is determined when a perpendicular is dropped from the terminal point of **v** to the line through **b** (illustrated in Figure 1-28 in two cases).

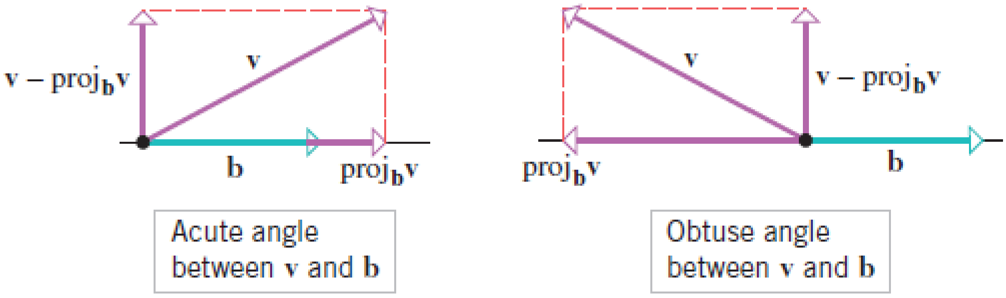


Figure 1-28

Example 1-18 Find the orthogonal projection of **v** = **i** + **j** + **k** on **b** = 2**i** + 2**j**, and then find the vector component of **v** orthogonal to **b**.

Solution. We have

$$\mathbf{v}\cdot\mathbf{b} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}+2\mathbf{j}) = 2+2+0 = 4$$
$$\|\mathbf{b}\|^2 = 2^2 + 2^2 = 8$$

Thus, the orthogonal projection of **v** on **b** is

$$\text{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} = \frac{4}{8}(2\mathbf{i}+2\mathbf{j}) = \mathbf{i}+\mathbf{j}$$

and the vector component of **v** orthogonal to **b** is

$$\mathbf{v} - \text{proj}_{\mathbf{b}}\mathbf{v} = (\mathbf{i}+\mathbf{j}+\mathbf{k}) - (\mathbf{i}+\mathbf{j}) = \mathbf{k}$$

These results are consistent with Figure 1-29.

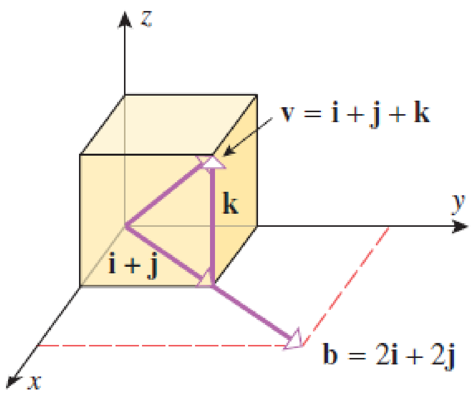


Figure 1-29

1.4 CROSS PRODUCT

Some of the concepts that we will develop in this section require basic ideas about ***determinants***, which are functions that assign numerical values to square arrays of numbers. For example, if a_1, a_2, b_1, and b_2 are real numbers, then we define a **2 × 2 *determinant*** by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

The purpose of the arrows is to help you remember the formula—the determinant is the product of the entries on the rightward arrow minus the product of the entries on the leftward arrow. For example,

$$\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = (3)(5) - (4)(-2) = 15 + 8 = 23$$

A **3 × 3 *determinant*** is defined in terms of 2 × 2 determinants by

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

The right side of this formula is easily remembered by noting that a_1, a_2, and a_3 are the entries in the first "row" of the left side, and the 2 × 2 determinants on the right side arise by deleting the first row and an appropriate column from the left side. The pattern is as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example 1-19:

$$\begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 3 \begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} + (-5) \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}$$

$$= 3(20) + 2(2) - 5(3) = 49$$

There are also definitions of 4 × 4 determinants, 5 × 5 determinants, and higher, but we will not need them in this text. Properties of determinants are studied in a branch of mathematics called ***linear algebra***, but we will only need the two properties stated in the following theorem.

Theorem

(a) If two rows in the array of a determinant are the same, then the value of the determinant is 0.

(b) Interchanging two rows in the array of a determinant multiplies its value by −1.

Proof (*a*)

$$\begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} = a_1a_2 - a_2a_1 = 0$$

Proof (*b*)

$$\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = b_1a_2 - b_2a_1 = -(a_1b_2 - a_2b_1)$$

Definition

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the ***cross product*** $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

or, equivalently,

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\,\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Observe that the right side of Formula has the same form as the right side of Formula, the difference being notation and the order of the factors in the three terms. Thus, we can rewrite as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

However, this is just a mnemonic device and not a true determinant since the entries in a determinant are numbers, not vectors.

Example 1-20 Let **u** = (1, 2, −2) and **v** = (3, 0, 1). Find (a) **u** × **v** (b) **v** × **u**

Solution (a)

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & -2\\ 3 & 0 & 1\end{vmatrix}=\begin{vmatrix}2 & -2\\ 0 & 1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & -2\\ 3 & 1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & 2\\ 3 & 0\end{vmatrix}\mathbf{k}=2\mathbf{i}-7\mathbf{j}-6\mathbf{k}$$

(b)

$$\mathbf{u}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{u})=-2\mathbf{i}+7\mathbf{j}+6\mathbf{k}$$

1.4.1 Algebraic Properties of the Cross Product

Theorem

If **u**, **v**, and **w** are any vectors in 3-space and k is any scalar, then:

(*a*) **u** × **v** = −(**v** × **u**)

(*b*) **u** × (**v** + **w**) = (**u** × **v**) + (**u** × **w**)

(*c*) (**u** + **v**) × **w** = (**u** × **w**) + (**v** × **w**)

(*d*) k(**u** × **v**) = (k**u**) × **v** = **u** × (k**v**)

(*e*) **u** × **0** = **0** × **u** = **0**

(*f*) **u** × **u** = **0**

The following cross products occur so frequently that it is helpful to be familiar with them:

i × **j** =**k**	**j** × **k** = **i**	**k** × **i** = **j**
j × **i** = −**k**	**k** × **j** = −**i**	**i** × **k** = −**j**

Example 1-21

$$\mathbf{i}\times\mathbf{j}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix}=\begin{vmatrix}0 & 0\\ 1 & 0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}\mathbf{k}=\mathbf{k}$$

1.4.2 Geometric Properties of the Cross Product

Theorem

If **u** and **v** are vectors in 3-space, then:

(a) **u . (u** × **v**) = 0 (**u** × **v** is orthogonal to **u**)

(b) **v . (u** × **v**) = 0 (**u** × **v** is orthogonal to **v**)

We will prove part (a). The proof of part (b) is similar.

Proof (a)

Let **u** = (u_1, u_2, u_3) and **v** = (v_1, v_2, v_3). Then

$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$

so that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0$

Example 1-22 Find a vector that is orthogonal to both of the vectors **u** = (2, −1, 3) and **v** = (−7, 2, −1).

Solution:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -7 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ -7 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ -7 & 2 \end{vmatrix}\mathbf{k} = -5\mathbf{i} - 19\mathbf{j} - 3\mathbf{k}$$

It can be proved that if **u** and **v** are nonzero and nonparallel vectors, then the direction of **u** × **v** relative to **u** and **v** is determined by a right-hand rule; that is, if the fingers of the right hand are cupped so they curl from **u** toward **v** in the direction of rotation that takes **u** into **v** in less than 180◦ , then the thumb will point (roughly) in the direction of **u** × **v** (Figure 1-30).

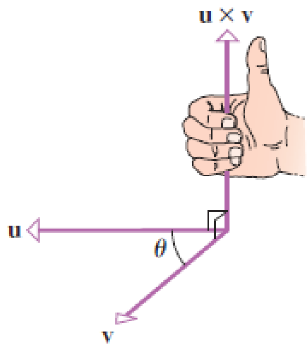


Figure 1-30

Theorem

Let **u** and **v** be nonzero vectors in 3-space, and let θ be the angle between these vectors when they are positioned so their initial points coincide.

(*a*) $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$

(*b*) *The area A of the parallelogram that has* **u** *and* **v** *as adjacent sides is*

$A = \|\mathbf{u}\times\mathbf{v}\|$

(*c*) **u** × **v** = **0** *if and only if* **u** *and* **v** *are parallel vectors, that is, if and only if they are scalar multiples of one another*.

Proof (a)

$$\begin{aligned}
\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta &= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1-\cos^2\theta} \\
&= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1-\frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}} \\
&= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2} \\
&= \sqrt{(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2} \\
&= \sqrt{(u_2v_3-u_3v_2)^2+(u_1v_3-u_3v_1)^2+(u_1v_2-u_2v_1)^2} \\
&= \|\mathbf{u}\times\mathbf{v}\|
\end{aligned}$$

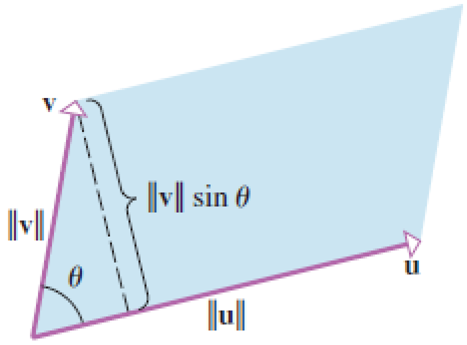


Figure 1-31

Example 1-23 Find the area of the triangle that is determined by the points $P_1(2, 2, 0)$, $P_2(-1, 0, 2)$, and $P_3(0, 4, 3)$.

Solution. The area A of the triangle is half the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ **Figure 1-32** But $\overrightarrow{P_1P_2} = \langle -3, -2, 2\rangle$ and $\overrightarrow{P_1P_3} = \langle -2, 2, 3\rangle$, so

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \langle -10, 5, -10\rangle$$

(verify), and consequently

$$A = \tfrac{1}{2}\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\| = \tfrac{15}{2}$$

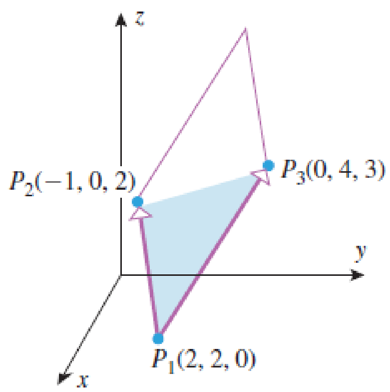


Figure 1-32

1.4.3 Scalar Triple Products

If $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$ are vectors in 3-space, then the number

$$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w})$$

is called the ***scalar triple product*** of **u**, **v**, and **w**. It is not necessary to compute the dot product and cross product to evaluate a scalar triple product—the value can be obtained directly from the formula

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

The validity of which can be seen by writing

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \cdot \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \right)$$

$$= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example 1-24 Calculate the scalar triple product **u . (v × w)** of the vectors **u** = 3**i** − 2**j** − 5**k**, **v** = **i** + 4**j** − 4**k**, **w** = 3**j** + 2**k**

Solution

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 49$$

1.4.4 Geometric Properties of the Scalar Triple Product

If **u**, **v**, and **w** are nonzero vectors in 3-space that are positioned so their initial points coincide, then these vectors form the adjacent sides of a parallelepiped (see figure). The following theorem establishes a relationship between the volume of this parallelepiped and the scalar triple product of the sides.

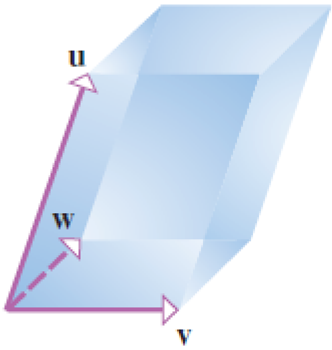
Theorem

Let **u**, **v**, *and* **w** *be nonzero vectors in 3-space.*

(a) *The volume V of the parallelepiped that has* **u**, **v**, *and* **w** *as adjacent edges is*

$$V = |\mathbf{u}\,.\,(\mathbf{v}\times\mathbf{w})|$$

(b) **u . (v × w)** = 0 *if and only if* **u**, **v**, *and* **w** *lie in the same plane.*



1.4.5 Algebraic Properties of the Scalar Triple Product

- The expression **u × v × w** must be avoided because it is ambiguous without parentheses. However, the expression **u . v × w** is not ambiguous—it has to mean **u .** (**v × w**) and not (**u . v**) × **w** because we cannot form the cross product of a scalar and a vector.
- Similarly, the expression **u × v . w** must mean (**u × v**) **. w** and not **u ×** (**v . w**). Thus, when you see an expression of the form **u . v × w** or **u × v . w**, the cross product is formed first and the dot product second.

- Since interchanging two rows of a determinant multiplies its value by −1, making two row interchanges in a determinant has no effect on its value. This being the case, it follows that

$$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \,.\, (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \,.\, (\mathbf{w} \times \mathbf{u})$$

- Since the 3 × 3 determinants that are used to compute these scalar triple products can be obtained from one another by two row interchanges.
- Another useful formula can be obtained by rewriting the first equality as

$$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \,.\, \mathbf{w}$$

  and then omitting the superfluous parentheses to obtain

$$\mathbf{u} \,.\, \mathbf{v} \times \mathbf{w} = \mathbf{u} \times \mathbf{v} \,.\, \mathbf{w}$$

1.5 PARAMETRIC EQUATIONS OF LINES

1.6 PLANES IN 3-SPACE

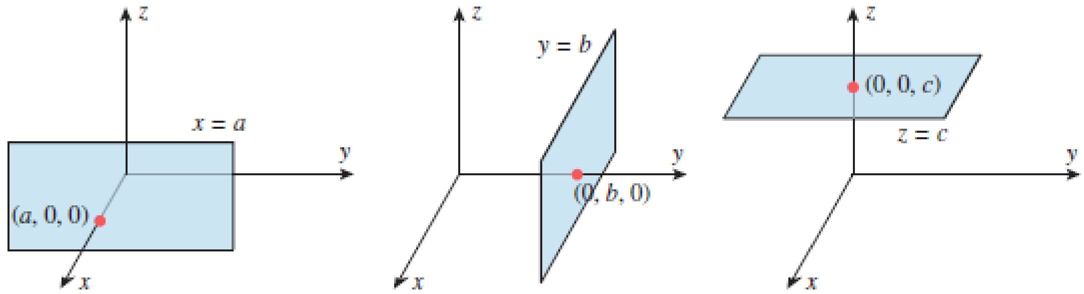
1.6.1 Planes Parallel to the Coordinate Planes

Based on below figure,

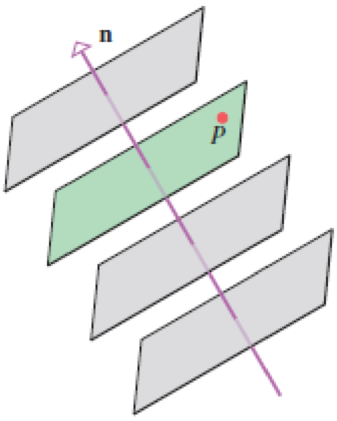
The graph of $x = a$ is the plane through $(a, 0, 0)$ that is parallel to the yz-plane,

The graph of $y = b$ is the plane through $(0, b, 0)$ that is parallel to the xz-plane,

The graph of $z = c$ is the plane through $(0, 0, c)$ that is parallel to the xy-plane.



1.6.2 Planes Determined by a Point and a Normal Vector



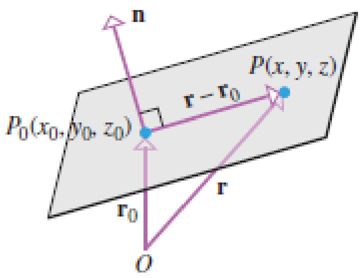
The colored plane is determined uniquely by the point P and the vector $\mathbf{n}$ perpendicular to the plane.

- A plane in 3-space can be determined uniquely by specifying a point in the plane and a vector perpendicular to the plane (see figure). A vector perpendicular to a plane is called a ***normal*** to the plane.
- Suppose that we want to find an equation of the plane passing through $P_0(x_0, y_0, z_0)$ and perpendicular to the vector $\mathbf{n} = (a, b, c)$. Define the vectors $\mathbf{r}_0$ and $\mathbf{r}$ as

$$\mathbf{r}_0 = (x_0, y_0, z_0) \text{ and } \mathbf{r} = (x, y, z)$$

- It should be evident from Figure that the plane consists precisely of those points $P(x, y, z)$ for which the vector $\mathbf{r} - \mathbf{r}_0$ is orthogonal to $\mathbf{n}$; or, expressed as an equation,

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$



If preferred, we can express this vector equation in terms of components as $(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$

from which we obtain

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is called the ***point-normal form*** of the equation of a plane.

Example: Find an equation of the plane passing through the point (3, −1, 7) and perpendicular to the vector **n** = (4, 2, −5).

Solution: a point-normal form of the equation is

$$4(x-3)+2(y+1)-5(z-7)=0$$

$$(4, 2, -5)\cdot(x-3, y+1, z-7)=0$$

we obtain an equation of the form

$$ax+by+cz+d=0$$

$$4x+2y-5z+25=0$$

The following theorem shows that every equation represents a plane in 3-space.

Theorem

If a, b, c, and d are constants, and a, b, and c are not all zero, then the graph of the equation

$$\boldsymbol{ax+by+cz+d=0}$$

is a plane that has the vector **n** = (*a*, *b*, *c*) *as a normal.*

Example: Determine whether the planes $3x-4y+5z=0$ and $-6x+8y-10z-4=0$ are parallel.

Solution: It is clear geometrically that two planes are parallel if and only if their normals are parallel vectors. A normal to the first plane is

$$\mathbf{n}_1=(3,-4,5)$$

and a normal to the second plane is

$$\mathbf{n}_2=(-6,8,-10)$$

Since $\mathbf{n}_2$ is a scalar multiple of $\mathbf{n}_1$, the normals are parallel, and hence so are the planes.

Example: Find an equation of the plane through the points $P_1(1, 2, -1)$, $P_2(2, 3, 1)$, and $P_3(3, -1, 2)$.

Solution: Since the points P_1, P_2, and P_3 lie in the plane, the vectors $\overrightarrow{P_1P_2}=(1,1,2)$ and $\overrightarrow{P_1P_3}=(2,-3,3)$ are parallel to the plane. Therefore,

$$\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 2\\ 2 & -3 & 3\end{vmatrix}$$

is normal to the plane, since it is orthogonal to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. By using this normal and the point $P_1(1, 2, -1)$ in the plane, we obtain the point-normal form

$$9(x-1)+(y-2)-5(z+1)=0$$

which can be rewritten as

$$9x+y-5z-16=0$$

Example: Determine whether the line $x = 3 + 8t$, $y = 4 + 5t$, $z = -3 - t$ is parallel to the plane $x - 3y + 5z = 12$.

Solution: The vector $\mathbf{v} = (8, 5, -1)$ is parallel to the line and the vector $\mathbf{n} = (1, -3, 5)$ is normal to the plane. For the line and plane to be parallel, the vectors **v** and **n** must be orthogonal. But this is not so, since the dot product $\mathbf{v} \cdot \mathbf{n} = (8)(1) + (5)(-3) + (-1)(5) = -12$ is nonzero. Thus, the line and plane are not parallel. ($\mathbf{v} \cdot \mathbf{n} = 0$ then right angle)

Example: Find the intersection of the line and plane in the previous example.

Solution: If we let (x_0, y_0, z_0) be the point of intersection, then the coordinates of this point satisfy both the equation of the plane and the parametric equations of the line. Thus,

$$x_0 - 3y_0 + 5z_0 = 12 \qquad (1)$$

and for some value of t, say $t = t_0$,

$$x_0 = 3 + 8t_0,\ y_0 = 4 + 5t_0,\ z_0 = -3 - t_0 \qquad (2)$$

Substituting (2) in (1) yields

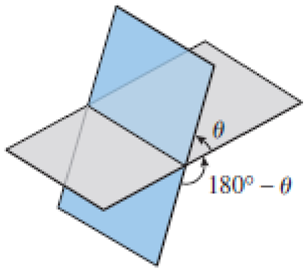
$$(3 + 8t_0) - 3(4 + 5t_0) + 5(-3 - t_0) = 12$$

Solving for t_0 yields $t_0 = -3$ and on substituting this value in (2), we obtain

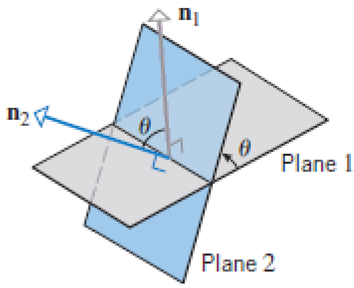
$$(x_0, y_0, z_0) = (-21, -11, 0)$$

1.6.3 Intersecting Planes

- Two distinct intersecting planes determine two positive angles of intersection—an (acute) angle θ that satisfies the condition $0 \le \theta \le \pi/2$ and the supplement of that angle (Figure *a*).
- If $\mathbf{n}_1$ and $\mathbf{n}_2$ are normals to the planes, then depending on the directions of $\mathbf{n}_1$ and $\mathbf{n}_2$, the angle θ is either the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ or the angle between $\mathbf{n}_1$ and $-\mathbf{n}_2$ (Figure *b*).
- In both cases, Theorem yields the following formula for the *acute angle θ between the planes*:



(*a*)



(*b*)

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}$$

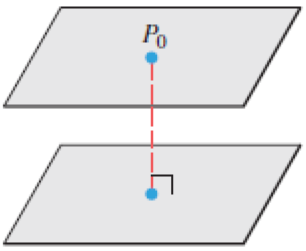
Example: Find the acute angle of intersection between the two planes $2x - 4y + 4z = 6$ and $6x + 2y - 3z = 4$

Solution: The given equations yield the normals $\mathbf{n}_1 = (2,-4, 4)$ and $\mathbf{n}_2 = (6, 2,-3)$.

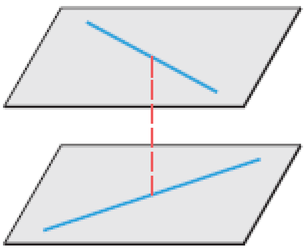
$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{|-8|}{\sqrt{36}\sqrt{49}} = \frac{4}{21}$$

$$\theta = \cos^{-1}\left(\frac{4}{21}\right) \approx 79^\circ$$

1.6.4 Distance Problems Involving Planes



(a)


(b)

Considering three basic distance problems in 3-space:

a. Find the distance between a point and a plane.
b. Find the distance between two parallel planes.
c. Find the distance between two skew lines.

Theorem

The distance D between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$\boldsymbol{D = \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}}}$$

Example: Find the distance D between the point $(1, -4, -3)$ and the plane $2x - 3y + 6z = -1$

Solution: the plane be rewritten in the form $ax + by + cz + d = 0$.

Thus, we rewrite the equation of the given plane as

$$2x - 3y + 6z + 1 = 0$$

from which we obtain $a = 2$, $b = -3$, $c = 6$, and $d = 1$.

$$D = \frac{|(2)(1) + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{|-3|}{7} = \frac{3}{7}$$

Example: The planes $x + 2y - 2z = 3$ and $2x + 4y - 4z = 7$ are parallel since their normals, (1, 2, −2) and (2, 4, −4), are parallel vectors. Find the distance between these planes.

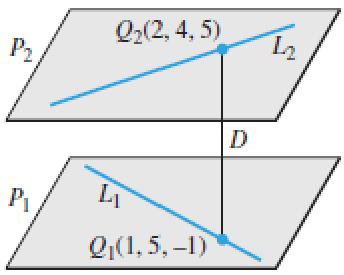
Solution: To find the distance D between the planes, we can select an *arbitrary* point in one of the planes and compute its distance to the other plane. By setting $y = z = 0$ in the equation $x + 2y - 2z = 3$, we obtain the point $P_0(3, 0, 0)$ in this plane.

The distance from P_0 to the plane $2x + 4y - 4z = 7$ is

$$D = \frac{|(2)(3) + 4(0) + (-4)(0) - 7|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{6}$$

Example: It was shown in previous example that the lines L_1: $x = 1 + 4t$, $y = 5 - 4t$, $z = -1 + 5t$ L_2: $x = 2 + 8t$, $y = 4 - 3t$, $z = 5 + t$ are skew. Find the distance between them.

Solution: Let P_1 and P_2 denote parallel planes containing L_1 and L_2, respectively (see figure).



- To find the distance D between L_1 and L_2, we will calculate the distance from a point in $P1$ to the plane P_2.
- Since L_1 lies in plane P_1, we can find a point in P_1 by finding a point on the line L_1; we can do this by substituting any convenient value of t in the parametric equations of L_1. The simplest choice is $t = 0$, which yields the point $Q_1(1, 5, -1)$.
- The next step is to find an equation for the plane P_2. For this purpose, observe that the vector $\mathbf{u}_1 = (4, -4, 5)$ is parallel to line L_1, and therefore also parallel to planes P_1 and P_2.
- Similarly, $\mathbf{u}_2 = (8, -3, 1)$ is parallel to L_2 and hence parallel to P_1 and P_2. Therefore, the cross product

$$\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = 11\mathbf{i} + 36\mathbf{j} + 20\mathbf{k}$$

is normal to both P_1 and P_2. Using this normal and the point $Q_2(2, 4, 5)$ found by setting $t = 0$ in the equations of L_2, we obtain an equation for P_2:

$$11(x - 2) + 36(y - 4) + 20(z - 5) = 0$$

or

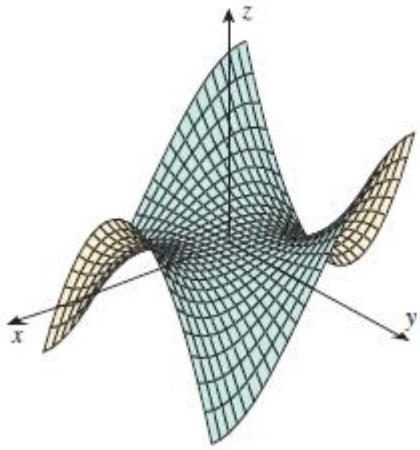
$$11x + 36y + 20z - 266 = 0$$

The distance between $Q_1(1, 5, -1)$ and this plane is

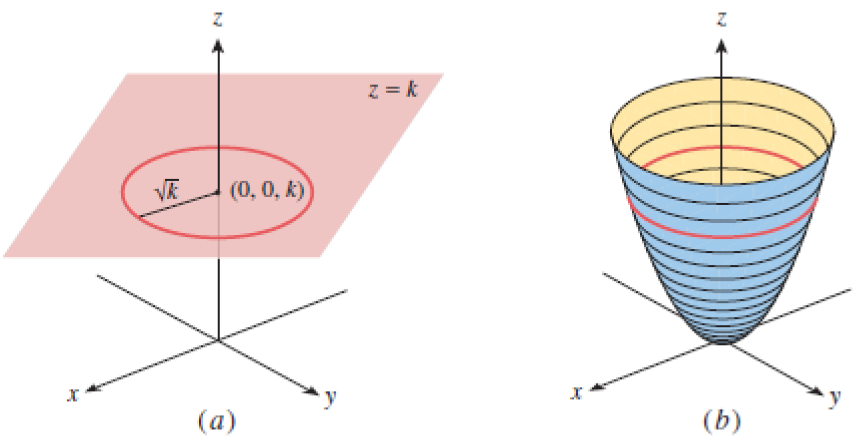
$$D = \frac{|(11)(1) + (36)(5) + (20)(-1) - 266|}{\sqrt{11^2 + 36^2 + 20^2}} = \frac{95}{\sqrt{1817}}$$

1.7 QUADRIC SURFACES

- Although the general shape of a curve in 2-space can be obtained by plotting points, this method is not usually helpful for surfaces in 3-space because too many points are required.
- It is more common to build up the shape of a surface with a network of ***mesh lines***, which are curves obtained by cutting the surface with well-chosen planes.
- For example, the figure shows the graph of $z = x^3 - 3xy^2$ rendered with a combination of mesh lines and colorization to produce the surface detail. This surface is called a "monkey saddle"]].
- The mesh line that results when a surface is cut by a plane is called the ***trace*** of the surface in the plane (see figure).



A monkey saddle



We noted that a second-degree equation

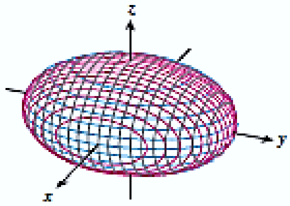
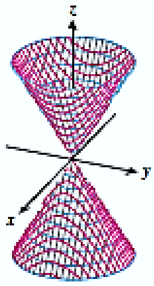
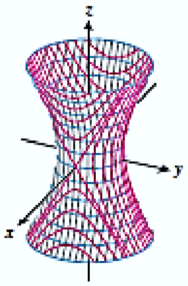
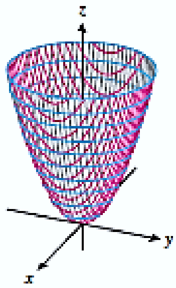
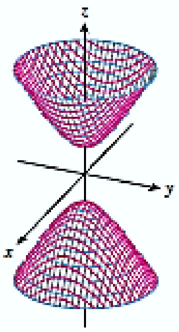
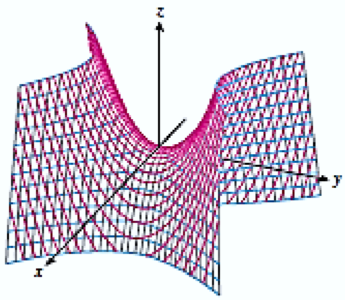
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents a conic section (possibly degenerate). The analog of this equation in an *xyz*-coordinate system is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

which is called a ***second-degree equation in x, y, and z***. The graphs of such equations are called ***quadric surfaces*** or sometimes ***quadrics***.

Six common types of quadric surfaces are shown in the following table—*ellipsoids, hyperboloids of one sheet, hyperboloids of two sheets, elliptic cones, elliptic paraboloids,* and *hyperbolic paraboloids*. (The constants *a*, *b*, and *c* that appear in the equations in the table are assumed to be positive.)

SURFACE	EQUATION	SURFACE	EQUATION
ELLIPSOID 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ The traces in the coordinate planes are ellipses, as are the traces in those planes that are parallel to the coordinate planes and intersect the surface in more than one point.	ELLIPTIC CONE 	$z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ The trace in the *xy*-plane is a point (the origin), and the traces in planes parallel to the *xy*-plane are ellipses. The traces in the *yz*- and *xz*-planes are pairs of lines intersecting at the origin. The traces in planes parallel to these are hyperbolas.
HYPERBOLOID OF ONE SHEET 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ The trace in the *xy*-plane is an ellipse, as are the traces in planes parallel to the *xy*-plane. The traces in the *yz*-plane and *xz*-plane are hyperbolas, as are the traces in those planes that are parallel to these and do not pass through the *x*- or *y*-intercepts. At these intercepts the traces are pairs of intersecting lines.	ELLIPTIC PARABOLOID 	$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ The trace in the *xy*-plane is a point (the origin), and the traces in planes parallel to and above the *xy*-plane are ellipses. The traces in the *yz*- and *xz*-planes are parabolas, as are the traces in planes parallel to these.
HYPERBOLOID OF TWO SHEETS 	$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ There is no trace in the *xy*-plane. In planes parallel to the *xy*-plane that intersect the surface in more than one point the traces are ellipses. In the *yz*- and *xz*-planes, the traces are hyperbolas, as are the traces in those planes that are parallel to these.	HYPERBOLIC PARABOLOID 	$z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$ The trace in the *xy*-plane is a pair of lines intersecting at the origin. The traces in planes parallel to the *xy*-plane are hyperbolas. The hyperbolas above the *xy*-plane open in the *y*-direction, and those below in the *x*-direction. The traces in the *yz*- and *xz*-planes are parabolas, as are the traces in planes parallel to these.

1.7.1 Techniques for Graphing Quadric Surfaces

A rough sketch of an ellipsoid

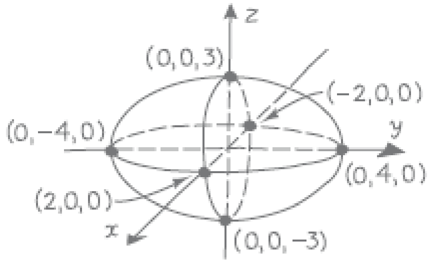
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad\qquad (a>0, b>0, c>0)$$

can be obtained by first plotting the intersections with the coordinate axes, and then sketching the elliptical traces in the coordinate planes.

Example: Sketch the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$$

Solution: The x-intercepts can be obtained by setting $y = 0$ and $z = 0$ in. This yields $x = \pm 2$. Similarly, the y-intercepts are $y = \pm 4$, and the z-intercepts are $z = \pm 3$. Sketching the elliptical traces in the coordinate planes yields the graph in the figure.



Example: Sketch the graph of the hyperboloid of one sheet

$$x^2 + y^2 - \frac{z^2}{4} = 1$$

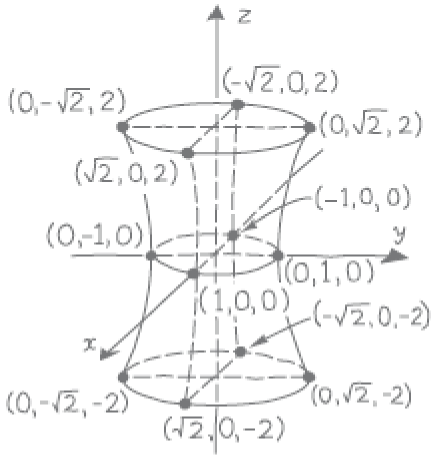
Solution: The trace in the xy-plane, obtained by setting $z = 0$, is

$$x^2 + y^2 = 1 \ (z = 0)$$

which is a circle of radius 1 centered on the z-axis. The traces in the planes $z = 2$ and $z = -2$, obtained by setting $z = \pm 2$, are given by

$$x^2 + y^2 = 2 \ (z = \pm 2)$$

which are circles of radius $\sqrt{2}$ centered on the z-axis. Joining these circles by the hyperbolic traces in the vertical coordinate planes yields the graph in the following figure.

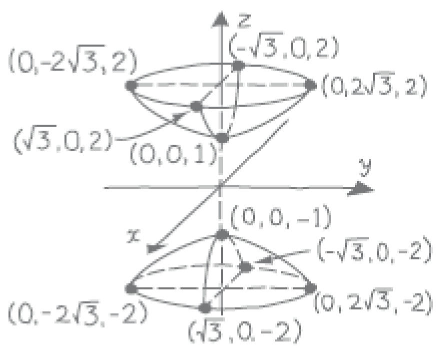


Example: Sketch the graph of the hyperboloid of two sheets

$$z^2 - x^2 - \frac{y^2}{4} = 1$$

Solution: The z-intercepts, obtained by setting $x = 0$ and $y = 0$, are $z = \pm 1$. The traces in the planes $z = 2$ and $z = -2$, obtained by setting $z = \pm 2$ in (10), are given by

$$\frac{x^2}{3} + \frac{y^2}{12} = 1 \qquad (z = \pm 2)$$



Sketching these ellipses and the hyperbolic traces in the vertical coordinate planes yields the following figure.

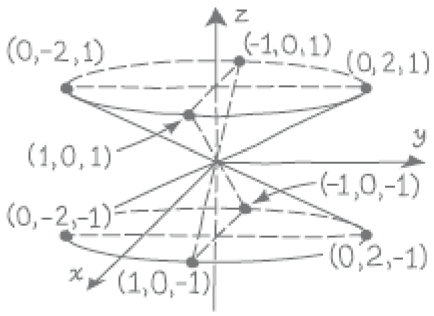
Example: Sketch the graph of the elliptic cone

$$z^2 = x^2 + \frac{y^2}{4}$$

Solution: The traces in the planes z = ±1 are given by

$$x^2 + \frac{y^2}{4} = 1 \qquad (z = \pm 1)$$

Sketching these ellipses and the linear traces in the vertical coordinate planes yields the graph in the figure.



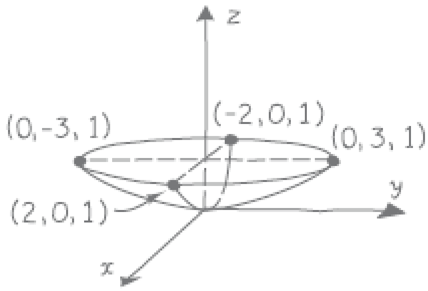
Example: Sketch the graph of the elliptic paraboloid

$$z = \frac{x^2}{4} + \frac{y^2}{9}$$

Solution: The trace in the plane $z = 1$ is

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \qquad (z = 1)$$

Sketching this ellipse and the parabolic traces in the vertical coordinate planes yields the graph in the figure.



Example: Sketch the graph of the hyperbolic paraboloid

$$\boldsymbol{z = \frac{y^2}{4} - \frac{x^2}{9}} \qquad \textbf{(a)}$$

Solution. Setting $x = 0$ in (a) yields

$$\boldsymbol{z = \frac{y^2}{4}}$$

which is a parabola in the *yz*-plane with vertex at the origin and opening in the positive *z*-direction (since $z \geq 0$), and setting $y = 0$ yields

$$\boldsymbol{z = -\frac{x^2}{9}}$$

which is a parabola in the *xz*-plane with vertex at the origin and opening in the negative *z*-direction.

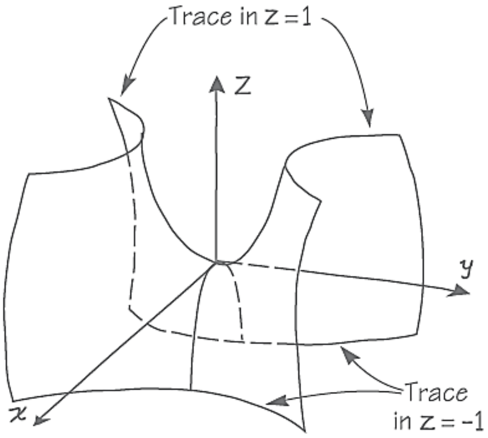
The trace in the plane $z = 1$ is

$$\frac{y^2}{4} - \frac{x^2}{9} = 1 \qquad (z = 1)$$

which is a hyperbola that opens along a line parallel to the *y*-axis, and the trace in the plane $z = -1$ is

$$\frac{x^2}{9} - \frac{y^2}{4} = 1 \qquad (z = -1)$$

which is a hyperbola that opens along a line parallel to the *x*-axis. Combining all of the above information leads to the sketch in Figure.



1.7.2 Translations of Quadric Surfaces

Example: Describe the surface $z = (x - 1)^2 + (y + 2)^2 + 3$.

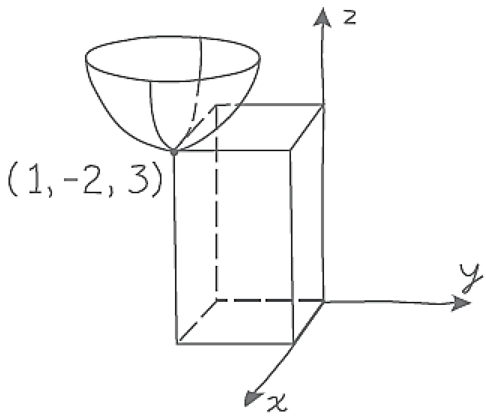
Solution. The equation can be rewritten as

$$z - 3 = (x - 1)^2 + (y + 2)^2$$

This surface is the paraboloid that results by translating the paraboloid

$$z = x^2 + y^2$$

in Figure so that the new "vertex" is at the point (1,−2, 3). A rough sketch of this paraboloid is shown in Figure.



Example: Describe the surface

$$4x^2 + 4y^2 + z^2 + 8y - 4z = -4$$

Solution. Completing the squares yields

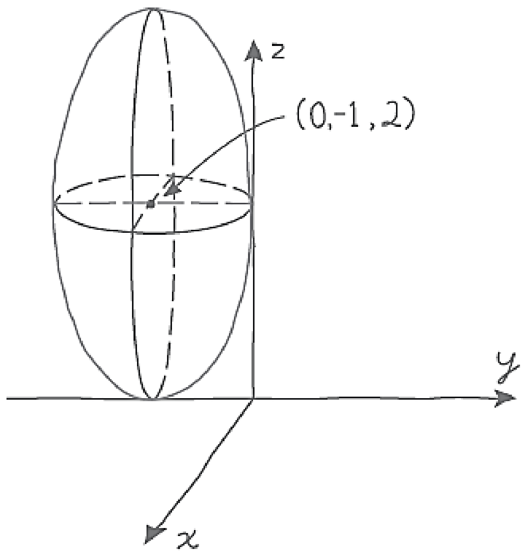
$$4x^2 + 4(y+1)^2 + (z-2)^2 = -4 + 4 + 4$$

$$x^2 + (y+1)^2 + \frac{(z-2)^2}{4} = 1$$

Thus, the surface is the ellipsoid that results when the ellipsoid

$$x^2 + y^2 + \frac{z^2}{4} = 1$$

is translated so that the new "center" is at the point (0,−1, 2). A rough sketch of this ellipsoid is shown in Figure.



1.7.3 A Technique for Identifying Quadric Surfaces

IDENTIFYING A QUADRIC SURFACE FROM THE FORM OF ITS EQUATION

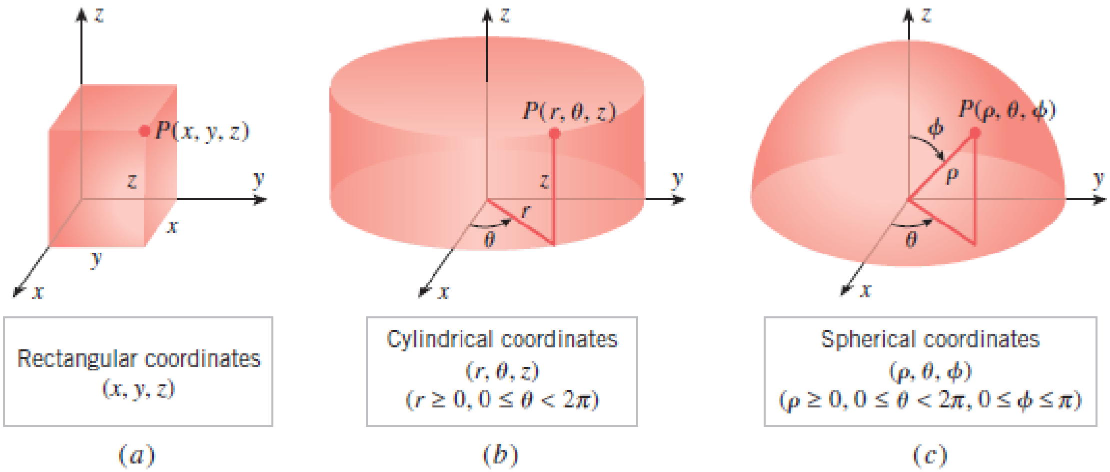
EQUATION	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$z^2-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{y^2}{b^2}+\frac{x^2}{a^2}=0$
CHARACTERISTIC	No minus signs	One minus sign	Two minus signs	No linear terms	One linear term; two quadratic terms with the same sign	One linear term; two quadratic terms with opposite signs
CLASSIFICATION	Ellipsoid	Hyperboloid of one sheet	Hyperboloid of two sheets	Elliptic cone	Elliptic paraboloid	Hyperbolic paraboloid

1.8 CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS

Three coordinates are required to establish the location of a point in 3-space. We have already done this using rectangular coordinates. However, figures (a, b, c) show two other possibilities:

- part (*a*) of the figure shows the ***rectangular coordinates*** (x, y, z) of a point P,
- part (*b*) shows the ***cylindrical coordinates*** (r, θ, z) of P,
- part (*c*) shows the ***spherical coordinates*** (ρ, θ, ϕ) of P.

In a rectangular coordinate system the coordinates can be any real numbers, but in cylindrical and spherical coordinate systems there are restrictions on the allowable values of the coordinates.



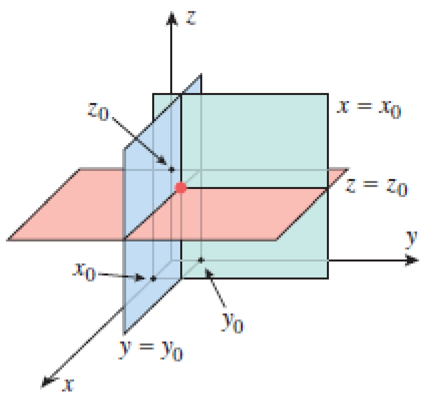
1.8.1 Constant Surfaces

In rectangular coordinates the surfaces represented by equations of the form

$$x = x_0,\ y = y_0, \text{ and } z = z_0$$

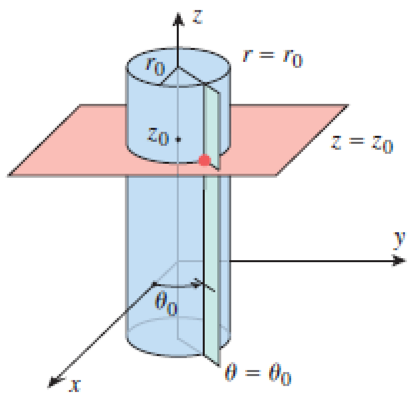
where x_0, y_0, and z_0 are constants, are planes parallel to the *yz*-plane, *xz*-plane, and *xy*-plane, respectively (see figure). In cylindrical coordinates the surfaces represented by equations of the form

$$r = r_0,\ \theta = \theta_0, \text{ and } z = z_0$$



where r_0, θ_0, and z_0 are constants, are shown in the following figure:

- The surface $r = r_0$ is a right circular cylinder of radius r_0 centered on the z-axis.
- The surface $\theta = \theta_0$ is a half-plane attached along the z-axis and making an angle θ_0 with the positive x-axis.
- The surface $z = z_0$ is a horizontal plane.

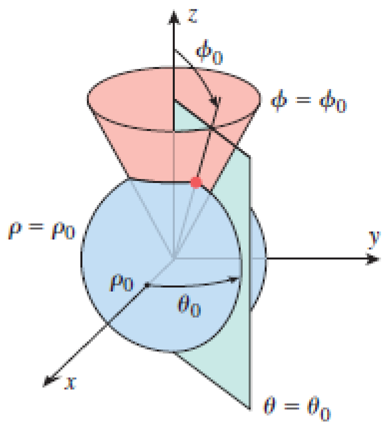


In spherical coordinates the surfaces represented by equations of the form

$$\rho = \rho_0,\ \theta = \theta_0, \text{ and } \varphi = \varphi_0$$

Where ρ_0, θ_0, and φ_0 are constants, are shown in the following figure:

- The surface $\rho = \rho_0$ consists of all points whose distance ρ from the origin is ρ_0. Assuming ρ_0 to be nonnegative, this is a sphere of radius ρ_0 centered at the origin.
- As in cylindrical coordinates, the surface $\theta = \theta_0$ is a half-plane attached along the z-axis, making an angle of θ_0 with the positive x-axis.
- The surface $\varphi = \varphi_0$ consists of all points from which a line segment to the origin makes an angle of φ_0 with the positive z-axis. If $0 < \varphi_0 < \pi/2$, this will be the nappe of a cone opening up, while if $\pi/2 < \varphi_0 < \pi$, this will be the nappe of a cone opening down. (If $\varphi_0 = \pi/2$, then the cone is flat, and the surface is the xy-plane.)



1.8.2 Converting Coordinates

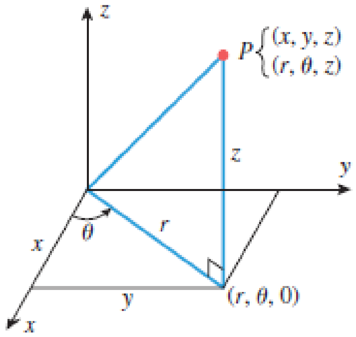
Just as we needed to convert between rectangular and polar coordinates in 2-space, so we will need to be able to convert between rectangular, cylindrical, and spherical coordinates in 3-space. The following table provides formulas for making these conversions.

CONVERSION FORMULAS FOR COORDINATE SYSTEMS

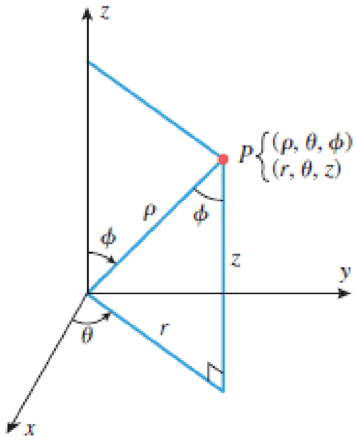
CONVERSION		FORMULAS	RESTRICTIONS
Cylindrical to rectangular	$(r, \theta, z) \to (x, y, z)$	$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z$	
Rectangular to cylindrical	$(x, y, z) \to (r, \theta, z)$	$r = \sqrt{x^2+y^2},\quad \tan\theta = y/x,\quad z = z$	
Spherical to cylindrical	$(\rho, \theta, \phi) \to (r, \theta, z)$	$r = \rho\sin\phi,\quad \theta = \theta,\quad z = \rho\cos\phi$	$r \ge 0,\ \rho \ge 0$ $0 \le \theta < 2\pi$ $0 \le \phi \le \pi$
Cylindrical to spherical	$(r, \theta, z) \to (\rho, \theta, \phi)$	$\rho = \sqrt{r^2+z^2},\quad \theta = \theta,\quad \tan\phi = r/z$	
Spherical to rectangular	$(\rho, \theta, \phi) \to (x, y, z)$	$x = \rho\sin\phi\cos\theta,\quad y = \rho\sin\phi\sin\theta,\quad z = \rho\cos\phi$	
Rectangular to spherical	$(x, y, z) \to (\rho, \theta, \phi)$	$\rho = \sqrt{x^2+y^2+z^2},\quad \tan\theta = y/x,\quad \cos\phi = z/\sqrt{x^2+y^2+z^2}$	

The diagrams in the following figure will help you to understand how the formulas in the table are derived.

For example, part (*a*) of the figure shows that in converting between rectangular coordinates *(x, y, z)* and cylindrical coordinates *(r, θ, z)*, we can interpret *(r, θ)* as polar coordinates of *(x, y)*. Thus, the polar-to-rectangular and rectangular-to-polar conversion formulas (1) and (2) provide the conversion formulas between rectangular and cylindrical coordinates in the table.



(*a*)



(*b*)

Comparison of coordinate systems

Part (*b*) of Figure suggests that the spherical coordinates *(ρ, θ, φ)* of a point *P* can be converted to cylindrical coordinates *(r, θ, z)* by the conversion formulas

$$r = \rho\sin\varphi,\ \theta = \theta,\ z = \rho\cos\varphi \qquad (1)$$

Moreover, since the cylindrical coordinates *(r, θ, z)* of *P* can be converted to rectangular coordinates *(x, y, z)* by the conversion formulas

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z \qquad (2)$$

We can obtain direct conversion formulas from spherical coordinates to rectangular coordinates by substituting (1) in (2). This yields

$$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi \qquad (3)$$

Example:

(a) Find the rectangular coordinates of the point with cylindrical coordinates

$$(r, \theta, z) = (4, \pi/3, -3)$$

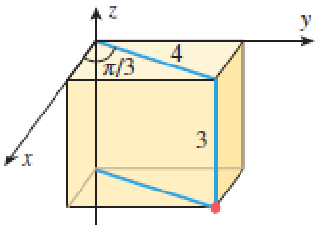
(b) Find the rectangular coordinates of the point with spherical coordinates

$$(\rho, \theta, \varphi) = (4, \pi/3, \pi/4)$$

Solution (a): Applying the cylindrical-to-rectangular conversion formulas in the table yields

$$x = r\cos\theta = 4\cos\frac{\pi}{3} = 2, \quad y = r\sin\theta = 4\sin\frac{\pi}{3} = 2\sqrt{3}, \quad z = -3$$

Thus, the rectangular coordinates of the point are $(x, y, z) = (2, 2\sqrt{3}, -3)$ (see figure).



cylindrical: (4, π/3, −3)
rectangular: (2, 2√3, −3)

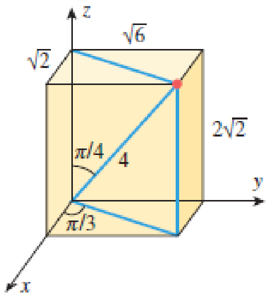
Solution (b): Applying the spherical-to-rectangular conversion formulas in the table yields

$$x = \rho\sin\phi\cos\theta = 4\sin\frac{\pi}{4}\cos\frac{\pi}{3} = \sqrt{2}$$

$$y = \rho\sin\phi\sin\theta = 4\sin\frac{\pi}{4}\sin\frac{\pi}{3} = \sqrt{6}$$

$$z = \rho\cos\phi = 4\cos\frac{\pi}{4} = 2\sqrt{2}$$

The rectangular coordinates of the point are $(x, y, z) = (\sqrt{2}, \sqrt{6}, 2\sqrt{2})$



spherical: (4, π/3, π/4)
rectangular: (√2, √6, 2√2)

Example: Find the spherical coordinates of the point that has rectangular coordinates

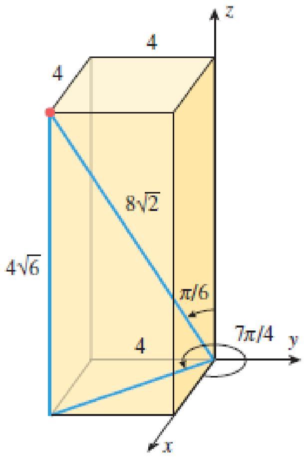
$$(x, y, z) = (4, -4, 4\sqrt{6})$$

Solution: From the rectangular-to-spherical conversion formulas in the table we obtain

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{16 + 16 + 96} = \sqrt{128} = 8\sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

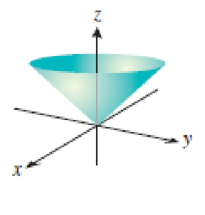
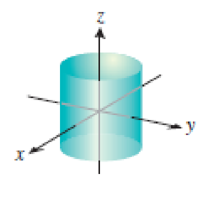
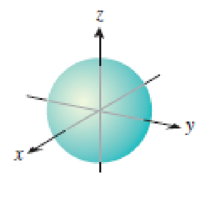
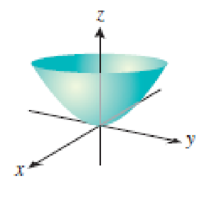
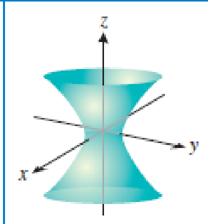
$$\cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{4\sqrt{6}}{8\sqrt{2}} = \frac{\sqrt{3}}{2}$$



rectangular: $(4, -4, 4\sqrt{6})$
spherical: $(8\sqrt{2}, 7\pi/4, \pi/6)$

From the restriction $0 \le \theta < 2\pi$ and the computed value of $\tan\theta$, the possibilities for θ are $\theta = 3\pi/4$ and $\theta = 7\pi/4$. However, the given point has a negative y-coordinate, so we must have $\theta = 7\pi/4$. Moreover, from the restriction $0 \le \varphi \le \pi$ and the computed value of $\cos\varphi$, the only possibility for φ is $\varphi = \pi/6$. Thus, the spherical coordinates of the point are $(\rho, \theta, \varphi) = (8\sqrt{2}, 7\pi/4, \pi/6)$.

1.8.3 Equations of Surfaces in Cylindrical and Spherical Coordinates

	CONE	CYLINDER	SPHERE	PARABOLOID	HYPERBOLOID
					
RECTANGULAR	$z = \sqrt{x^2 + y^2}$	$x^2 + y^2 = 1$	$x^2 + y^2 + z^2 = 1$	$z = x^2 + y^2$	$x^2 + y^2 - z^2 = 1$
CYLINDRICAL	$z = r$	$r = 1$	$z^2 = 1 - r^2$	$z = r^2$	$z^2 = r^2 - 1$
SPHERICAL	$\phi = \pi/4$	$\rho = \csc\phi$	$\rho = 1$	$\rho = \cos\phi \csc^2\phi$	$\rho^2 = -\sec 2\phi$

CHAPTER TWO

VECTOR-VALUED FUNCTIONS

2.1 INTRODUCTION TO VECTOR-VALUED FUNCTIONS

2.1.1 Parametric Curves in 3-Space

- If f and g are well-behaved functions, then the pair of parametric equations

$$x = f(t), \quad y = g(t) \tag{2-1}$$

generates a curve in 2-space that is traced in a specific direction as the parameter t increases.

- We defined this direction to be the ***orientation*** of the curve or the ***direction of increasing parameter***, and we called the curve together with its orientation the *graph* of the parametric equations or the ***parametric curve*** represented by the equations.
- Analogously, if f, g, and h are three well-behaved functions, then the parametric equations

$$x = f(t), \quad y = g(t), \quad z = h(t) \tag{2-2}$$

- Generate a curve in 3-space that is traced in a specific direction as t increases. As in 2-space, this direction is called the ***orientation*** or ***direction of increasing parameter***, and the curve together with its orientation is called the ***graph*** of the parametric equations or the ***parametric curve*** represented by the equations. If no restrictions are stated explicitly or are implied by the equations, then it will be understood that t varies over the interval $(-\infty, +\infty)$.

Example 2-1: The parametric equations $x = 1 - t, \quad y = 3t, \quad z = 2t$

represent a line in 3-space that passes through the point (1, 0, 0) and is parallel to the vector (−1, 3, 2). Since x decreases as t increases, the line has the orientation shown in Figure 2-1.

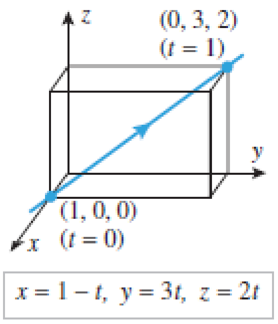


Figure 2-1

Example 2-2 Describe the parametric curve represented by the equations

$$x = a \cos t,\ y = a \sin t,\ z = ct$$

where a and c are positive constants.

Solution: As the parameter t increases, the value of $z = ct$ also increases, so the point (x, y, z) moves upward. However, as t increases, the point (x, y, z) also moves in a path directly over the circle

$$x = a \cos t,\ y = a \sin t$$

in the xy-plane. The combination of these upward and circular motions produces a corkscrew-shaped curve that wraps around a right circular cylinder of radius a centered on the z-axis (Figure 2-2). This curve is called a ***circular helix***.

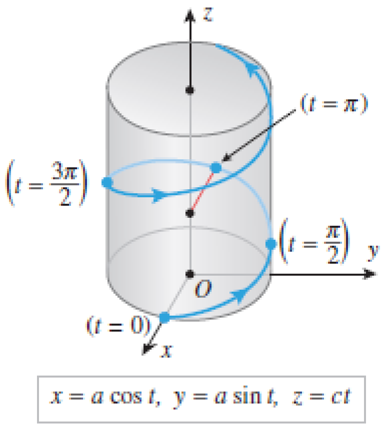


Figure 2-2

2.1.2 Parametric Curves Generated with Technology

Except in the simplest cases, parametric curves can be difficult to visualize and draw without the help of a graphing utility. For example, the ***tricuspoid*** is the graph of the parametric equations

$$x = 2 \cos(t) + \cos(2t), \quad y = 2 \sin(t) - \sin(2t)$$

Although it would be tedious to plot the tricuspoid by hand, a computer rendering is easy to obtain and reveals the significance of the name of the curve (Figure 2-3).

However, note that the depiction of the tricuspoid in Figure 2-3 is incomplete, since the orientation of the curve is not indicated. This is often the case for curves that are generated with a graphing utility.

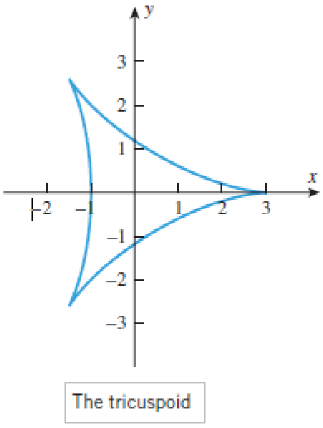


Figure 2-3

Parametric curves in 3-space can be difficult to visualize correctly even with the help of a graphing utility. For example, Figure 2-4*a* shows a parametric curve called a *torus knot* that was produced with a CAS.

Some graphing utilities provide the capability of enclosing the curve within a thin tube, as in Figure 2-4*b*. Such graphs are called ***tube plots***.

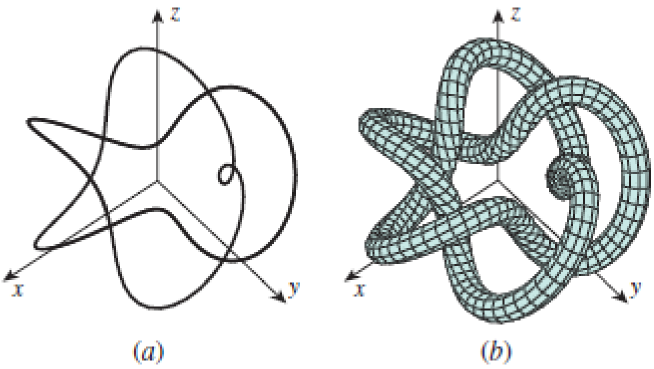


Figure 2-4

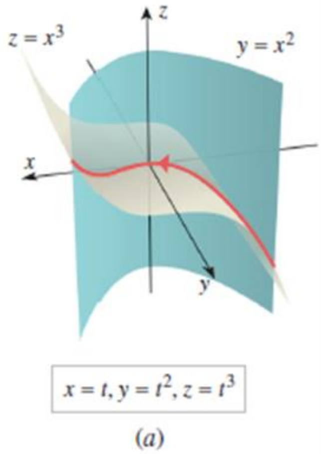
2.1.3 Parametric Equations for Intersections of Surfaces

Curves in 3-space often arise as intersections of surfaces. For example, Figure 2-5*a* shows a portion of the intersection of the cylinders $z = x^3$ and $y = x^2$.

One method for finding parametric equations for the curve of intersection is to choose one of the variables as the parameter and use the two equations to express the remaining two variables in terms of that parameter. In particular, if we choose $x = t$ as the parameter and substitute this into the equations $z = x^3$ and $y = x^2$, we obtain the parametric equations

$$x = t, \qquad y = t^2, \qquad z = t^3 \qquad (2\text{-}3)$$

This curve is called a ***twisted cubic***. The portion of the twisted cubic shown in Figure 2-5*a* corresponds to $t \geq 0$; a computer-generated graph of the twisted cubic for positive and negative values of t is shown in Figure 2-5b. Some other examples and techniques for finding intersections of surfaces are discussed in the exercises.



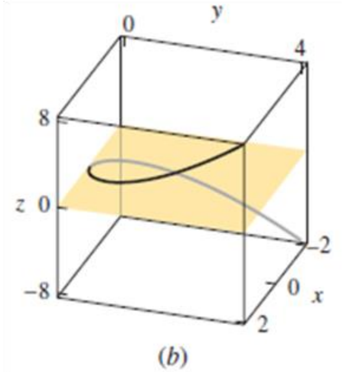


Figure 2-5

2.1.4 Vector-Valued Functions

The twisted cubic defined by the equations in (2-3) is the set of points of the form (t, t^2, t^3) for real values of t. If we view each of these points as a terminal point for a vector **r** whose initial point is at the origin,

$$\mathbf{r} = (x, y, z) = (t, t^2, t^3) = t\,\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

then we obtain **r** as a function of the parameter t , that is, $\mathbf{r} = \mathbf{r}(t)$. Since this function produces a *vector*, we say that $\mathbf{r} = \mathbf{r}(t)$ defines **r** as a ***vector-valued function of a real variable***, or more simply, a ***vector-valued function***. The vectors that we will consider in this text are either in 2-space or 3-space, so we will say that a vector-valued function is in 2-space or in 3-space according to the kind of vectors that it produces.

If $\mathbf{r}(t)$ is a vector-valued function in 3-space, then for each allowable value of t the vector $\mathbf{r} = \mathbf{r}(t)$ can be represented in terms of components as

$$\mathbf{r} = \mathbf{r}(t) = (x(t), y(t), z(t)) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad (2\text{-}4)$$

The functions $x(t)$, $y(t)$, and $z(t)$ are called the ***component functions*** or the ***components*** of $\mathbf{r}(t)$.

Example 2-3: The component functions of $\mathbf{r}(t) = (t, t^2, t^3) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

are $\quad x(t) = t,\ y(t) = t^2,\ z(t) = t^3$

The ***domain*** of a vector-valued function $\mathbf{r}(t)$ is the set of allowable values for t.

If $\mathbf{r}(t)$ is defined in terms of component functions and the domain is not specified explicitly, then it will be understood that the domain is the intersection of the natural domains of the component functions; this is called the ***natural domain*** of $\mathbf{r}(t)$.

Example 2-4: Find the natural domain of

$$\mathbf{r}(t) = \langle \ln|t-1|, e^t, \sqrt{t} \rangle = (\ln|t-1|)\mathbf{i} + e^t\mathbf{j} + \sqrt{t}\mathbf{k}$$

Solution: The natural domains of the component functions

$$x(t) = \ln|t-1|, \quad y(t) = e^t, \quad z(t) = \sqrt{t}$$

are

$$(-\infty, 1) \cup (1, +\infty), \quad (-\infty, +\infty), \quad [0, +\infty)$$

respectively. The intersection of these sets is

$$[0, 1) \cup (1, +\infty)$$

(verify), so the natural domain of $\mathbf{r}(t)$ consists of all values of t such that

$$0 \le t < 1 \quad \text{or} \quad t > 1$$

2.1.5 Graphs of Vector-Valued Functions

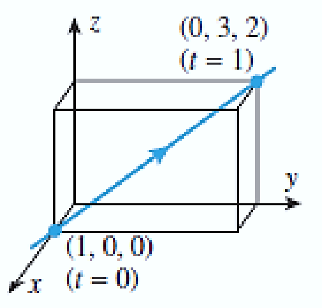
If $\mathbf{r}(t)$ is a vector-valued function in 2-space or 3-space, then we define the ***graph*** of $\mathbf{r}(t)$ to be the parametric curve described by the component functions for $\mathbf{r}(t)$. For example, if

$$\mathbf{r}(t) = (1-t, 3t, 2t) = (1-t)\mathbf{i} + 3t\,\mathbf{j} + 2t\mathbf{k} \qquad \text{(2-5)}$$

then the graph of $\mathbf{r} = \mathbf{r}(t)$ is the graph of the parametric equations

$$x = 1 - t,\ y = 3t,\ z = 2t$$

Thus, the graph of (2-5) is the line in Figure 2-1.



$x = 1 - t,\ y = 3t,\ z = 2t$

Example 2-5: Sketch the graph and a radius vector of

(a) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le 2\pi$

(b) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2\mathbf{k},\ 0 \le t \le 2\pi$

Solution (a): The corresponding parametric equations are

$$x = \cos t,\ y = \sin t\ (0 \le t \le 2\pi)$$

So the graph is a circle of radius 1, centered at the origin, and oriented counter clockwise. The graph and a radius vector are shown in Figure 2-6.

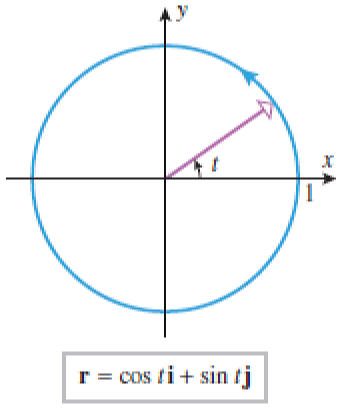


Figure 2-6

Solution (b): The corresponding parametric equations are

$$x = \cos t,\ y = \sin t,\ z = 2\ (0 \le t \le 2\pi)$$

From the third equation, the tip of the radius vector traces a curve in the plane $z = 2$, and from the first two equations, the curve is a circle of radius 1 centered at the point $(0, 0, 2)$ and traced counter clockwise looking down the z-axis. The graph and a radius vector are shown in Figure 2-7.

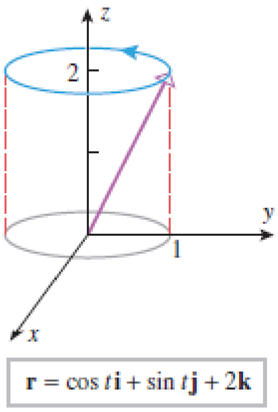


Figure 2-7

2.1.6 Vector Form of a Line Segment

If $\mathbf{r}_0$ is a vector in 2-space or 3-space with its initial point at the origin, then the line that passes through the terminal point of $\mathbf{r}_0$ and is parallel to- the vector $\mathbf{v}$ can be expressed in vector form as

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

In particular, if $\mathbf{r}_0$ and $\mathbf{r}_1$ are vectors in 2-space or 3-space with their initial points at the origin, then the line that passes through the terminal points of these vectors can be expressed in vector form as

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) \qquad (2\text{-}6) \qquad \text{or } \mathbf{r} = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad (2\text{-}7)$$

as indicated in Figure 2-8.

It is common to call either (2-6) or (2-7) the ***two-point vector form of a line*** and to say, for simplicity, that the line passes through the *points* $\mathbf{r}_0$ and $\mathbf{r}_1$ (as opposed to saying that it passes through the *terminal points* of $\mathbf{r}_0$ and $\mathbf{r}_1$).

It is understood in (6) and (7) that t varies from $-\infty$ to $+\infty$. However, if we restrict t to vary over the interval $0 \le t \le 1$, then $\mathbf{r}$ will vary from $\mathbf{r}_0$ to $\mathbf{r}_1$. Thus, the equation

$$\mathbf{r} = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \ (0 \le t \le 1) \ (2\text{-}8)$$

represents the line segment in 2-space or 3-space that is traced from $\mathbf{r}_0$ to $\mathbf{r}_1$.

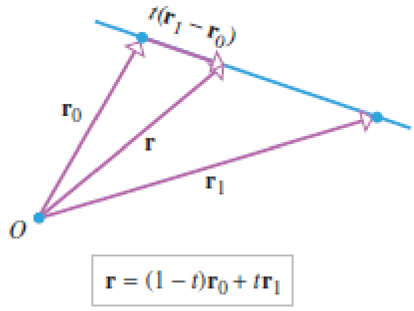


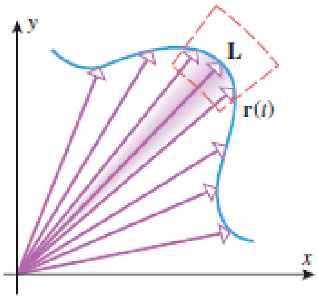
Figure 2-8

2.2 CALCULUS OF VECTOR-VALUED FUNCTIONS

2.2.1 Limits and Continuity

- A vector-valued function **r**(t) in 2-space or 3-space to approach a limiting vector **L** as t approaches a number a. That is, we want to define

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$$

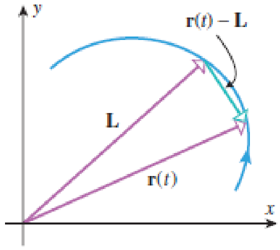


r(t) approaches **L** in length and direction if $\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$.

- position **r**(t) and **L** with their initial points at the origin and interpret this limit to mean that the terminal point of **r**(t) approaches the terminal point of **L** as t approaches a or, equivalently, that the vector **r**(t) approaches the vector **L** in both length and direction at t approaches a (see figure). Algebraically, this is equivalent to stating that

$$\lim_{t \to a} \|\mathbf{r}(t) - \mathbf{L}\| = \mathbf{0}$$

(the following figure). Thus, we make the following definition.



$\|\mathbf{r}(t) - \mathbf{L}\|$ is the distance between terminal points for vectors **r**(t) and **L** when positioned with the same initial points.

Definition Let **r**(t) be a vector-valued function that is defined for all t in some open interval containing the number a, except that **r**(t) need not be defined at a.

We will write

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}$$

if and only if

$$\lim_{t \to a} \|\mathbf{r}(t) - \mathbf{L}\| = \mathbf{0}$$

THEOREM

(*a*) *If* $\mathbf{r}(t) = \langle x(t), y(t)\rangle = x(t)\mathbf{i} + y(t)\mathbf{j}$, *then*

$$\lim_{t\to a}\mathbf{r}(t) = \left\langle \lim_{t\to a} x(t), \lim_{t\to a} y(t)\right\rangle = \lim_{t\to a} x(t)\mathbf{i} + \lim_{t\to a} y(t)\mathbf{j}$$

provided the limits of the component functions exist. Conversely, the limits of the component functions exist provided $\mathbf{r}(t)$ *approaches a limiting vector as* t *approaches* a.

(*b*) *If* $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, *then*

$$\lim_{t\to a}\mathbf{r}(t) = \left\langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t)\right\rangle = \lim_{t\to a} x(t)\mathbf{i} + \lim_{t\to a} y(t)\mathbf{j} + \lim_{t\to a} z(t)\mathbf{k}$$

provided the limits of the component functions exist. Conversely, the limits of the component functions exist provided $\mathbf{r}(t)$ *approaches a limiting vector as* t *approaches* a.

Example: Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\,\mathbf{j} - 2\cos(\pi t)\,\mathbf{k}$. Then

$$\lim_{t\to 0}\mathbf{r}(t) = \left(\lim_{t\to 0} t^2\right)\mathbf{i} + \left(\lim_{t\to 0} e^t\right)\mathbf{j} - \left(\lim_{t\to 0} 2\cos\pi t\right)\mathbf{k} = \mathbf{j} - 2\mathbf{k}$$

Alternatively, using the angle bracket notation for vectors,

$$\lim_{t\to 0}\mathbf{r}(t) = \lim_{t\to 0}\langle t^2, e^t, -2\cos\pi t\rangle = \left\langle \lim_{t\to 0} t^2, \lim_{t\to 0} e^t, \lim_{t\to 0}(-2\cos\pi t)\right\rangle = \langle 0, 1, -2\rangle$$

Motivated by the definition of continuity for real-valued functions, we define a vector valued function $\mathbf{r}(t)$ to be ***continuous*** at $t = a$ if

$$\lim_{t\to a}\mathbf{r}(t) = \mathbf{r}(a)$$

That is, $\mathbf{r}(a)$ is defined, the limit of $\mathbf{r}(t)$ as $t\to a$ exists, and the two are equal. As in the case for real-valued functions, we say that $\mathbf{r}(t)$ is ***continuous on an interval*** I if it is continuous at each point of I [with the understanding that at an endpoint in I the two-sided limit in (above equation) is replaced by the appropriate one-sided limit].

A vector-valued function is continuous at $t = a$ if and only if its component functions are continuous at $t = a$.

2.2.2 Derivatives

The derivative of a vector-valued function is defined by a limit similar to that for the derivative of a real-valued function.

Definition If **r**(*t*) is a vector-valued function, we define the ***derivative of* r *with respect to t*** to be the vector-valued function **r** given by

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

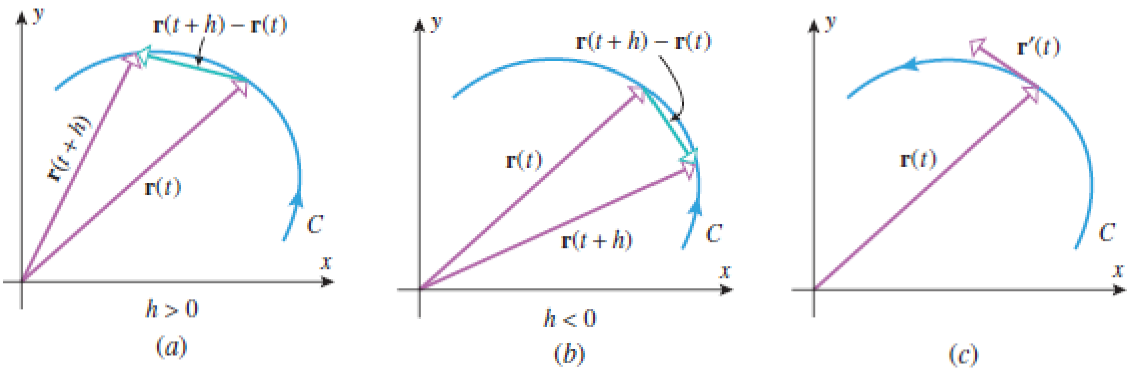
The domain of **r** consists of all values of *t* in the domain of **r**(*t*) for which the limit exists.

-The function **r**(*t*) is ***differentiable*** at *t* if the limit exists.

-The derivative of **r**(*t*) can be expressed as

$$\frac{d}{dt}[\mathbf{r}(t)], \quad \frac{d\mathbf{r}}{dt}, \quad \mathbf{r}'(t), \quad \text{or} \quad \mathbf{r}'$$

It is important to keep in mind that **r**(*t*) is a vector, not a number, and hence has a magnitude and a direction for each value of *t* [except if **r**(*t*) = **0**, in which case **r**(*t*) has magnitude zero but no specific direction].



These illustrations show the graph *C* of **r**(*t*) (with its orientation) and the vectors **r**(*t*), **r**(*t* + *h*), and **r**(*t* + *h*) − **r**(*t*) for positive *h* and for negative *h*.

In both cases, the vector **r**(*t* + *h*) − **r**(*t*) runs along the secant line joining the terminal points of **r**(*t* + *h*) and **r**(*t*), but with opposite directions in the two cases. In the case where *h* is positive the vector **r**(*t* + *h*) − **r**(*t*) points in the direction of increasing parameter, and in the case where *h* is negative it points in the opposite direction. However, in the case where *h* is negative the direction gets reversed when we multiply by 1/*h*, so in both cases the vector

$$\frac{1}{h}[\mathbf{r}(t+h) - \mathbf{r}(t)] = \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

points in the direction of increasing parameter and runs along the secant line. As *h*→0, the secant line approaches the tangent line at the terminal point of **r***(t)*, so we can conclude that the limit

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

(if it exists and is nonzero) is a vector that is tangent to the curve *C* at the tip of **r***(t)* and points in the direction of increasing parameter (Figure *c*). We can summarize all of this as follows.

Geometric interpretation of the derivative

Suppose that *C* is the graph of a vector-valued function **r***(t)* in 2-space or 3-space and that **r**´ *(t)* exists and is nonzero for a given value of *t* . If the vector **r**´ *(t)* is positioned with its initial point at the terminal point of the radius vector **r***(t)*, then **r**´ *(t)* is tangent to *C* and points in the direction of increasing parameter.

Theorem

If **r***(t) is a vector-valued function, then* **r** *is differentiable at t if and only if each of its component functions is differentiable at t, in which case the component functions of* **r**´ *(t) are the derivatives of the corresponding component functions of* **r***(t).*

Proof

For simplicity, we give the proof in 2-space; the proof in 3-space is identical, except for the additional component. Assume that **r**(*t*) = *x*(*t*)**i** + *y*(*t*)**j**. Then

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \\
&= \lim_{h\to 0} \frac{[x(t+h)\mathbf{i} + y(t+h)\mathbf{j}] - [x(t)\mathbf{i} + y(t)\mathbf{j}]}{h} \\
&= \left(\lim_{h\to 0} \frac{x(t+h) - x(t)}{h}\right)\mathbf{i} + \left(\lim_{h\to 0} \frac{y(t+h) - y(t)}{h}\right)\mathbf{j} \\
&= x'(t)\mathbf{i} + y'(t)\mathbf{j}
\end{aligned}$$

Example: Let $\mathbf{r}(t) = t^2\,\mathbf{i} + e^t\,\mathbf{j} - 2\cos(\pi t)\mathbf{k}$. Then

Solution:

$$\mathbf{r}'(t) = \frac{d}{dt}(t^2)\mathbf{i} + \frac{d}{dt}(e^t)\mathbf{j} - \frac{d}{dt}(2\cos \pi t)\mathbf{k}$$
$$= 2t\mathbf{i} + e^t\mathbf{j} + (2\pi \sin \pi t)\mathbf{k}$$

2.2.3 Derivative Rules

Theorem

(*Rules of Differentiation*)

Let $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, *and* $\mathbf{r}_2(t)$ *be differentiable vector-valued functions that are all in 2-space or all in 3-space, and let f(t) be a differentiable real-valued function, k a scalar, and* **c** *a constant vector (that is, a vector whose value does not depend on t). Then the following rules of differentiation hold:*

(*a*) $\dfrac{d}{dt}[\mathbf{c}] = \mathbf{0}$

(*b*) $\dfrac{d}{dt}[k\mathbf{r}(t)] = k\dfrac{d}{dt}[\mathbf{r}(t)]$

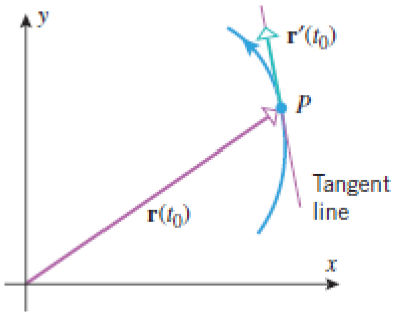
(*c*) $\dfrac{d}{dt}[\mathbf{r}_1(t) + \mathbf{r}_2(t)] = \dfrac{d}{dt}[\mathbf{r}_1(t)] + \dfrac{d}{dt}[\mathbf{r}_2(t)]$

(*d*) $\dfrac{d}{dt}[\mathbf{r}_1(t) - \mathbf{r}_2(t)] = \dfrac{d}{dt}[\mathbf{r}_1(t)] - \dfrac{d}{dt}[\mathbf{r}_2(t)]$

(*e*) $\dfrac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\dfrac{d}{dt}[\mathbf{r}(t)] + \dfrac{d}{dt}[f(t)]\mathbf{r}(t)$

2.2.1 Tangent Lines to Graphs of Vector-Valued Functions

Definition Let P be a point on the graph of a vector-valued function $\mathbf{r}(t)$, and let $\mathbf{r}(t_0)$ be the radius vector from the origin to P (see below figure). If $\mathbf{r}'(t_0)$ exists and $\mathbf{r}'(t_0) \neq \mathbf{0}$, then we call $\mathbf{r}'(t_0)$ a ***tangent vector*** to the graph of $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$, and we call the line through P that is parallel to the tangent vector the ***tangent line*** to the graph of $\mathbf{r}(t)$ at $\mathbf{r}(t_0)$.



Let $\mathbf{r}_0 = \mathbf{r}(t_0)$ and $\mathbf{v}_0 = \mathbf{r}'(t_0)$. The tangent line to the graph of $\mathbf{r}(t)$ at $\mathbf{r}_0$ is given by the vector equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}_0$$

Example: Find parametric equations of the tangent line to the circular helix

$$x = \cos t,\ y = \sin t,\ z = t$$

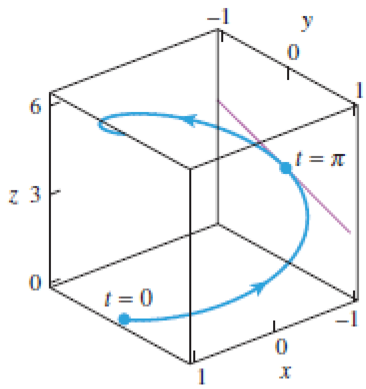
where $t = t_0$, and use that result to find parametric equations for the tangent line at the point where $t = \pi$.

Solution: The vector equation of the helix is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

$$\mathbf{r}_0 = \mathbf{r}(t_0) = \cos t_0\mathbf{i} + \sin t_0\,\mathbf{j} + t_0\mathbf{k}$$

$$\mathbf{v}_0 = \mathbf{r}'(t_0) = (-\sin t_0)\mathbf{i} + \cos t_0\,\mathbf{j} + \mathbf{k}$$



The vector equation of the tangent line at $t = t_0$ is

$$\mathbf{r} = \cos t_0\mathbf{i} + \sin t_0\,\mathbf{j} + t_0\mathbf{k} + t\,[(-\sin t_0)\mathbf{i} + \cos t_0\,\mathbf{j} + \mathbf{k}]$$

$$= (\cos t_0 - t\sin t_0)\mathbf{i} + (\sin t_0 + t\cos t_0)\mathbf{j} + (t_0 + t)\mathbf{k}$$

Thus, the parametric equations of the tangent line at $t = t_0$ are

$$x = \cos t_0 - t\sin t_0,\ y = \sin t_0 + t\cos t_0,\ z = t_0 + t$$

In particular, the tangent line at $t = \pi$ has parametric equations

$$x = -1,\ y = -t,\ z = \pi + t$$

The graph of the helix and this tangent line are shown in figure.

Example: Let

$$\mathbf{r}_1(t) = (\tan^{-1} t)\mathbf{i} + (\sin t)\mathbf{j} + t^2\mathbf{k}$$

and

$$\mathbf{r}_2(t) = (t^2 - t)\mathbf{i} + (2t - 2)\mathbf{j} + (\ln t)\mathbf{k}$$

The graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ intersect at the origin. Find the degree measure of the acute angle between the tangent lines to the graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ at the origin.

Solution: The graph of $\mathbf{r}_1(t)$ passes through the origin at $t = 0$, where its tangent vector is

$$\acute{r}_1(0) = \langle \frac{1}{1+t^2}, \cos t, 2t \rangle \Big|_{t=0} = \langle 1, 1, 0 \rangle$$

The graph of $\mathbf{r}_2(t)$ passes through the origin at $t = 1$ (verify), where its tangent vector is

$$\acute{r}_2(1) = \langle 2t - 1, 2, \frac{1}{t} \rangle \Big|_{t=1} = \langle 1, 2, 1 \rangle$$

the angle θ between these two tangent vectors satisfies

$$\cos\theta = \frac{\langle 1,1,0\rangle \cdot \langle 1,2,1\rangle}{\|\langle 1,1,0\rangle\| \|\langle 1,2,1\rangle\|} = \frac{1+2+0}{\sqrt{2}\sqrt{6}} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$$

It follows that $\theta = \pi/6$ radians, or 30°.

2.2.2 Derivatives of Dot and Cross Products

The following rules, which are derived in the exercises, provide a method for differentiating dot products in 2-space and 3-space and cross products in 3-space.

$$\frac{d}{dt}[\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)] = \mathbf{r}_1(t) \cdot \frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt} \cdot \mathbf{r}_2(t) \qquad \text{(a)}$$

$$\frac{d}{dt}[\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \mathbf{r}_1(t) \times \frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt} \times \mathbf{r}_2(t) \qquad \text{(b)}$$

Theorem

If $\mathbf{r}(t)$ *is a differentiable vector-valued function in 2-space or 3-space and* $\|\mathbf{r}(t)\|$ *is constant for all t, then*

$$\mathbf{r}(t) \,.\, \mathbf{r}'(t) = 0$$

that is, $\mathbf{r}(t)$ *and* $\mathbf{r}'(t)$ *are orthogonal vectors for all t.*

Proof:

It follows from (a) with $\mathbf{r}_1(t) = \mathbf{r}_2(t) = \mathbf{r}(t)$ that

$$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt} \cdot \mathbf{r}(t)$$

or, equivalently,

$$\frac{d}{dt}[\|\mathbf{r}(t)\|^2] = 2\mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt}$$

But $\|\mathbf{r}(t)\|^2$ is constant, so its derivative is zero. Thus

$$2\mathbf{r}(t) \cdot \frac{d\mathbf{r}}{dt} = 0$$

2.2.3 Definite Integrals of Vector-Valued Functions

If $\mathbf{r}(t)$ is a vector-valued function that is continuous on the interval $a \le t \le b$, then we define the ***definite integral*** of $\mathbf{r}(t)$ over this interval as a limit of Riemann sums. Specifically, we define

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} \mathbf{r}(t_k^*)\Delta t_k$$

The definite integral of $\mathbf{r}(t)$ over the interval $a \le t \le b$ can be expressed as a vector whose components are the definite integrals of the component functions of $\mathbf{r}(t)$. For example, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then

$$\begin{aligned}
\int_a^b \mathbf{r}(t)\,dt &= \lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} \mathbf{r}(t_k^*)\Delta t_k \\
&= \lim_{\max \Delta t_k \to 0} \left[\left(\sum_{k=1}^{n} x(t_k^*)\Delta t_k \right)\mathbf{i} + \left(\sum_{k=1}^{n} y(t_k^*)\Delta t_k \right)\mathbf{j} \right] \\
&= \left(\lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} x(t_k^*)\Delta t_k \right)\mathbf{i} + \left(\lim_{\max \Delta t_k \to 0} \sum_{k=1}^{n} y(t_k^*)\Delta t_k \right)\mathbf{j} \\
&= \left(\int_a^b x(t)\,dt \right)\mathbf{i} + \left(\int_a^b y(t)\,dt \right)\mathbf{j}
\end{aligned}$$

In general, we have

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b x(t)\,dt \right)\mathbf{i} + \left(\int_a^b y(t)\,dt \right)\mathbf{j} \qquad \text{2-space}$$

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b x(t)\,dt \right)\mathbf{i} + \left(\int_a^b y(t)\,dt \right)\mathbf{j} + \left(\int_a^b z(t)\,dt \right)\mathbf{k} \qquad \text{3-space}$$

Example: Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\,\mathbf{j} - (2\cos \pi t)\mathbf{k}$. Then

$$\begin{aligned}
\int_0^1 \mathbf{r}(t)\,dt &= \left(\int_0^1 t^2\,dt \right)\mathbf{i} + \left(\int_0^1 e^t\,dt \right)\mathbf{j} - \left(\int_0^1 2\cos \pi t\,dt \right)\mathbf{k} \\
&= \frac{t^3}{3}\Big]_0^1 \mathbf{i} + e^t\Big]_0^1 \mathbf{j} - \frac{2}{\pi}\sin \pi t\Big]_0^1 \mathbf{k} = \frac{1}{3}\mathbf{i} + (e-1)\mathbf{j}
\end{aligned}$$

2.2.4 Rules of Integration

Theorem:

(*Rules of Integration*) *Let* $\mathbf{r}(t)$*,* $\mathbf{r}_1(t)$*, and* $\mathbf{r}_2(t)$ *be vector-valued functions in 2-space or 3-space that are continuous on the interval* $a \le t \le b$*, and let k be a scalar. Then the following rules of integration hold:*

$$(a)\quad \int_a^b k\mathbf{r}(t)\,dt = k\int_a^b \mathbf{r}(t)\,dt$$

$$(b)\quad \int_a^b [\mathbf{r}_1(t) + \mathbf{r}_2(t)]\,dt = \int_a^b \mathbf{r}_1(t)\,dt + \int_a^b \mathbf{r}_2(t)\,dt$$

$$(c)\quad \int_a^b [\mathbf{r}_1(t) - \mathbf{r}_2(t)]\,dt = \int_a^b \mathbf{r}_1(t)\,dt - \int_a^b \mathbf{r}_2(t)\,dt$$

2.2.5 Antiderivatives of Vector-Valued Functions

An ***antiderivative*** for a vector-valued function **r***(t)* is a vector-valued function **R***(t)* such that

$$\mathbf{R}'(t) = \mathbf{r}(t)$$

we express Equation using integral notation as

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}$$

where **C** represents an arbitrary constant *vector*.

Since differentiation of vector-valued functions can be performed componentwise, it follows that anti-differentiation can be done this way as well.

Example:

$$\begin{aligned}\int (2t\mathbf{i} + 3t^2\mathbf{j})\,dt &= \left(\int 2t\,dt\right)\mathbf{i} + \left(\int 3t^2\,dt\right)\mathbf{j}\\ &= (t^2 + C_1)\mathbf{i} + (t^3 + C_2)\mathbf{j}\\ &= (t^2\mathbf{i} + t^3\mathbf{j}) + (C_1\mathbf{i} + C_2\mathbf{j}) = (t^2\mathbf{i} + t^3\mathbf{j}) + \mathbf{C}\end{aligned}$$

where $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j}$ is an arbitrary vector constant of integration.

Most of the familiar integration properties have vector counterparts. For example, vector differentiation and integration are inverse operations in the sense that

$$\frac{d}{dt}\left[\int \mathbf{r}(t)\,dt\right] = \mathbf{r}(t) \quad \text{and} \quad \int \mathbf{r}'(t)\,dt = \mathbf{r}(t) + \mathbf{C}$$

Moreover, if **R***(t)* is an antiderivative of **r***(t)* on an interval containing $t = a$ and $t = b$, then we have the following vector form of the Fundamental Theorem of Calculus:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

Example: Evaluate the definite integral $\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt.$

Solution: Integrating the components yields

$$\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt = t^2\Big]_0^2 \mathbf{i} + t^3\Big]_0^2 \mathbf{j} = 4\mathbf{i} + 8\mathbf{j}$$

Alternative Solution: The function $\mathbf{R}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ is an antiderivative of the integrand since $\mathbf{R}'(t) = 2t\,\mathbf{i} + 3t^2\mathbf{j}$. Thus,

$$\int_0^2 (2t\mathbf{i} + 3t^2\mathbf{j})\,dt = \mathbf{R}(t)\Big]_0^2 = t^2\mathbf{i} + t^3\mathbf{j}\Big]_0^2 = (4\mathbf{i} + 8\mathbf{j}) - (0\mathbf{i} + 0\mathbf{j}) = 4\mathbf{i} + 8\mathbf{j}$$

Example: Find $\mathbf{r}(t)$ given that $\mathbf{r}'(t) = (3, 2t)$ and $\mathbf{r}(1) = (2, 5)$.

Solution: Integrating $\mathbf{r}'(t)$ to obtain $\mathbf{r}(t)$ yields

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int \langle 3, 2t\rangle\,dt = \langle 3t, t^2\rangle + \mathbf{C}$$

where $\mathbf{C}$ is a vector constant of integration. To find the value of $\mathbf{C}$ we substitute $t = 1$ and use the given value of $\mathbf{r}(1)$ to obtain

$$\mathbf{r}(1) = (3, 1) + \mathbf{C} = (2, 5)$$

so that $\mathbf{C} = (-1, 4)$. Thus,

$$\mathbf{r}(t) = (3t, t^2) + (-1, 4) = (3t - 1, t^2 + 4)$$

2.3 CHANGE OF PARAMETER; ARC LENGTH

2.3.1 Arc Length from the Vector Viewpoint

The arc length L of a parametric curve

$$x = x(t), \quad y = y(t) \quad (a \le t \le b)$$

is given by the formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Analogously, the arc length L of a parametric curve

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad (a \le t \le b)$$

in 3-space is given by the formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

vector forms that we can obtain by letting

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad \text{or} \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

It follows that

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \quad \text{or} \quad \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

and hence

$$\left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \text{or} \quad \left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Theorem:

If C is the graph in 2-space or 3-space of a smooth vector-valued function **r**(t)*, then its arc length L from t = a to t = b is*

$$L = \int_a^b \left\|\frac{d\mathbf{r}}{dt}\right\| dt$$

Example: Find the arc length of that portion of the circular helix $x = \cos t$, $y = \sin t$, $z = t$ from $t = 0$ to $t = \pi$.

Solution: Set $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k} = (\cos t, \sin t, t)$. Then

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle \quad \text{and} \quad \|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$$

From Theorem the arc length of the helix is

$$L = \int_0^{\pi} \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_0^{\pi} \sqrt{2}\, dt = \sqrt{2}\pi$$

2.3.2 Arc Length as a Parameter

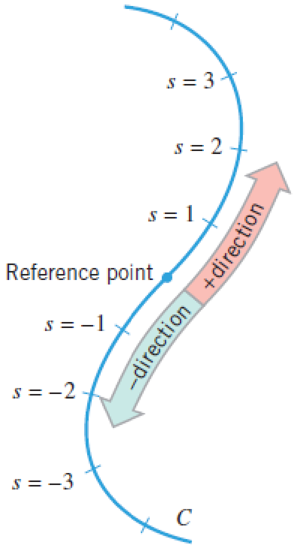
For many purposes the best parameter to use for representing a curve in 2-space or 3-space parametrically is the length of arc measured along the curve from some fixed reference point. This can be done as follows:

Using Arc Length as a Parameter

Step 1. Select an arbitrary point on the curve C to serve as a ***reference point***.

Step 2. Starting from the reference point, choose one direction along the curve to be the ***positive direction*** and the other to be the ***negative direction***.

Step 3. If P is a point on the curve, let s be the “signed” arc length along C from the reference point to P, where s is positive if P is in the positive direction from the reference point and s is negative if P is in the negative direction. The below figure illustrates this idea.



By this procedure, a unique point P on the curve is determined when a value for s is given. For example, $s = 2$ determines the point that is 2 units along the curve in the positive direction from the reference point, and $s = -3/2$ determines the point that is 3/2 units along the curve in the negative direction from the reference point.

Let us now treat s as a variable. As the value of s changes, the corresponding point P moves along C and the coordinates of P become functions of s. Thus, in 2-space the coordinates of P are $(x(s), y(s))$, and in 3-space they are $(x(s), y(s), z(s))$. Therefore, in 2-space or 3-space the curve C is given by the parametric equations

$$x = x(s), y = y(s) \quad \text{or} \quad x = x(s), y = y(s), z = z(s)$$

A parametric representation of a curve with arc length as the parameter is called an ***arc length parametrization*** of the curve. Note that a given curve will generally have infinitely many different arc length parametrizations, since the reference point and orientation can be chosen arbitrarily.

Example: Find the arc length parametrization of the circle $x^2 + y^2 = a^2$ with counter-clockwise orientation and $(a, 0)$ as the reference point.

Solution: The circle with counter-clockwise orientation can be represented by the parametric equations

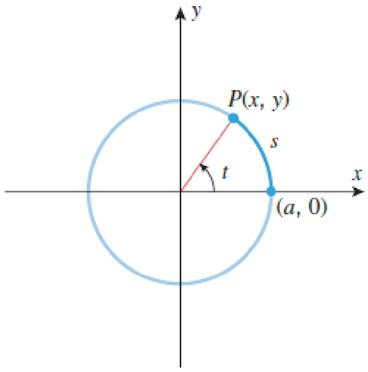
$$x = a\cos t, \quad y = a\sin t \quad (0 \le t \le 2\pi)$$

in which t can be interpreted as the angle in radian measure from the positive x-axis to the radius from the origin to the point $P(x, y)$ (see Figure). If we take the positive direction for measuring the arc length to be counter-clockwise, and we take $(a, 0)$ to be the reference point, then s and t are related by

$$s = at \text{ or } t = s/a$$

Making this change of variable and noting that s increases from 0 to $2\pi a$ as t increases from 0 to 2π yields the following arc length parametrization of the circle:

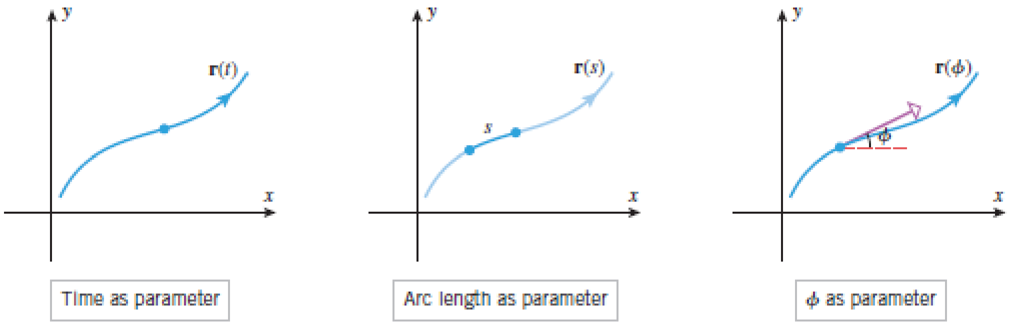
$$x = a\cos(s/a),\ y = a\sin(s/a)\ (0 \le s \le 2\pi a)$$



2.3.1 Change of Parameter

In many situations the solution of a problem can be simplified by choosing the parameter in a vector-valued function or a parametric curve in the right way. The two most common parameters for curves in 2-space or 3-space are time and arc length.

For example, in analyzing the motion of a particle in 2-space, it is often desirable to parametrize its trajectory in terms of the angle φ between the tangent vector and the positive x-axis (see below figures). Thus, our next objective is to develop methods for changing the parameter in a vector-valued function or parametric curve. This will allow us to move freely between different possible parametrizations.



A ***change of parameter*** in a vector-valued function $\mathbf{r}(t)$ is a substitution $t = g(\tau)$ that produces a new vector-valued function $\mathbf{r}(g(\tau))$ having the same graph as $\mathbf{r}(t)$, but possibly traced differently as the parameter τ increases.

Example: Find a change of parameter $t = g(\tau)$ for the circle

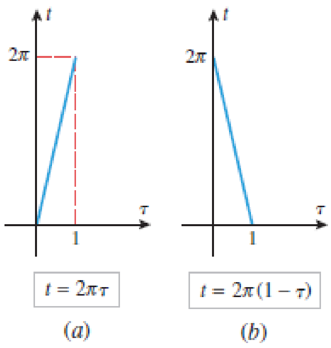
$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} \qquad (0 \le t \le 2\pi)$$

such that

(a) The circle is traced counter-clockwise as τ increases over the interval [0, 1];

(b) The circle is traced clockwise as τ increases over the interval [0, 1].

Solution (a): The given circle is traced counter-clockwise as t increases. Thus, if we choose g to be an increasing function, then it will follow from the relationship $t = g(\tau)$ that t increases when τ increases, thereby ensuring that the circle will be traced counter-clockwise as τ increases. We also want to choose g so that t increases from 0 to 2π as τ increases from 0 to 1. A simple choice of g that satisfies all of the required criteria is the linear function graphed in Figure a. The equation of this line is



$$t = g(\tau) = 2\pi\tau$$

which is the desired change of parameter. The resulting representation of the circle in terms of the parameter τ is

$$\mathbf{r}(g(\tau)) = \cos 2\pi\tau\,\mathbf{i} + \sin 2\pi\tau\,\mathbf{j} \qquad (0 \le \tau \le 1)$$

Solution (b): To ensure that the circle is traced clockwise, we will choose g to be a decreasing function such that t decreases from 2π to 0 as τ increases from 0 to 1. A simple choice of g that achieves this is the linear function

$$t = g(\tau) = 2\pi(1 - \tau)$$

graphed in Figure *b*. The resulting representation of the circle in terms of the parameter τ is

$$\mathbf{r}(g(\tau)) = \cos(2\pi(1 - \tau))\mathbf{i} + \sin(2\pi(1 - \tau))\mathbf{j}\ (0 \le \tau \le 1)$$

which simplifies to (verify)

$$\mathbf{r}(g(\tau)) = \cos 2\pi\tau\mathbf{i} - \sin 2\pi\tau\,\mathbf{j}\ (0 \le \tau \le 1)$$

Theorem (***Chain Rule***) *Let* $\mathbf{r}(t)$ *be a vector-valued function in 2-space or 3- space that is differentiable with respect to t. If* $t = g(\tau)$ *is a change of parameter in which g is differentiable with respect to* τ*, then* $\mathbf{r}(g(\tau))$ *is differentiable with respect to* τ *and*

$$\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau}$$

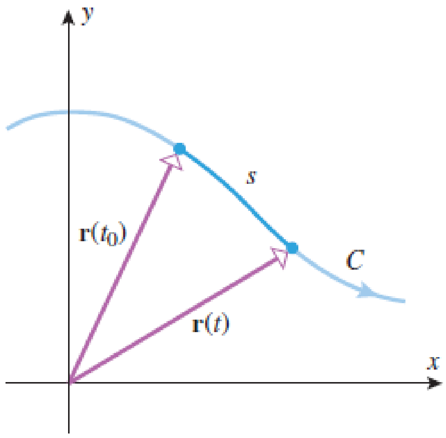
-A change of parameter $t = g(\tau)$ in which $\mathbf{r}(g(\tau))$ is smooth if $\mathbf{r}(t)$ is smooth is called a ***smooth change of parameter***.

-The $t = g(\tau)$ will be a smooth change of parameter if $dt/d\tau$ is continuous and $dt/d\tau \neq 0$ for all values of τ, since these conditions imply that $d\mathbf{r}/d\tau$ is continuous and nonzero if $d\mathbf{r}/dt$ is continuous and nonzero.

-Smooth changes of parameter fall into two categories—those for which $dt/d\tau > 0$ for all τ (called ***positive changes of parameter***) and those for which $dt/d\tau < 0$ for all τ (called ***negative changes of parameter***). A positive change of parameter preserves the orientation of a parametric curve, and a negative change of parameter reverses it.

2.3.2 Finding Arc Length Parametrizations

Theorem *Let C be the graph of a smooth vector-valued function* $\mathbf{r}(t)$ *in 2-space or 3-space, and let* $\mathbf{r}(t_0)$ *be any point on C. Then the following formula defines a positive change*



of parameter from t to s, where s is an arc length parameter having $\mathbf{r}(t_0)$ *as its reference point:*

$$s = \int_{t_0}^{t} \left\| \frac{dr}{du} \right\| du$$

Example: Find the arc length parametrization of the circular helix

$$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$$

that has reference point $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ and the same orientation as the given helix.

Solution: Replacing t by u in $\mathbf{r}$ for integration purposes and taking $t_0 = 0$, we obtain

$$\mathbf{r} = \cos u\mathbf{i} + \sin u\mathbf{j} + u\mathbf{k}$$

$$\frac{d\mathbf{r}}{du} = (-\sin u)\mathbf{i} + \cos u\mathbf{j} + \mathbf{k}$$

$$\left\| \frac{d\mathbf{r}}{du} \right\| = \sqrt{(-\sin u)^2 + \cos^2 u + 1} = \sqrt{2}$$

$$s = \int_0^t \left\| \frac{d\mathbf{r}}{du} \right\| du = \int_0^t \sqrt{2}\,du = \sqrt{2}u \Big]_0^t = \sqrt{2}t$$

Thus, $t = s/\sqrt{2}$, so (13) can be reparametrized in terms of s as

$$\mathbf{r} = \cos\left(\frac{s}{\sqrt{2}}\right)\mathbf{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}$$

Example: A bug walks along the trunk of a tree following a path modeled by the circular helix in previous example. The bug starts at the reference point $(1, 0, 0)$ and walks up the helix for a distance of 10 units. What are the bug's final coordinates?

Solution: the arc length parametrization of the helix relative to the reference point $(1, 0, 0)$ is

$$\mathbf{r} = \cos\left(\frac{s}{\sqrt{2}}\right)\mathbf{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}$$

or, expressed parametrically,

$$x = \cos\left(\frac{s}{\sqrt{2}}\right), \quad y = \sin\left(\frac{s}{\sqrt{2}}\right), \quad z = \frac{s}{\sqrt{2}}$$

Thus, at $s = 10$ the coordinates are

$$\left(\cos\left(\frac{10}{\sqrt{2}}\right), \sin\left(\frac{10}{\sqrt{2}}\right), \frac{10}{\sqrt{2}}\right) \approx (0.705, 0.709, 7.07)$$

Example: Find the arc length parametrization of the line

$$x = 2t + 1, \qquad y = 3t - 2$$

that has the same orientation as the given line and uses $(1, -2)$ as the reference point.

Solution: The line passes through the point $(1, -2)$ and is parallel to $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$. To find the arc length parametrization of the line, we need only rewrite the given equations using $\mathbf{v}/\|\mathbf{v}\|$ rather than $\mathbf{v}$ to determine the direction and replace t by s. Since

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2\mathbf{i} + 3\mathbf{j}}{\sqrt{13}} = \frac{2}{\sqrt{13}}\mathbf{i} + \frac{3}{\sqrt{13}}\mathbf{j}$$

it follows that the parametric equations for the line in terms of s are

$$x = \frac{2}{\sqrt{13}}s + 1, \quad y = \frac{3}{\sqrt{13}}s - 2$$

2.3.3 Properties of Arc Length Parametrizations

Theorem

(a) If C is the graph of a smooth vector-valued function $\mathbf{r}(t)$ *in 2-space or 3-space, where t is a general parameter, and if s is the arc length parameter for C defined by previous formula, then for every value of t the tangent vector has length*

$$\left\|\frac{d\mathbf{r}}{dt}\right\| = \frac{ds}{dt}$$

(b) If C is the graph of a smooth vector-valued function $\mathbf{r}(s)$ *in 2-space or 3-space, where s is an arc length parameter, then for every value of s the tangent vector to C has length*

$$\left\|\frac{d\mathbf{r}}{ds}\right\| = 1$$

(c) If C is the graph of a smooth vector-valued function $\mathbf{r}(t)$ *in 2-space or 3-space, and if* $\|d\mathbf{r}/dt\| = 1$ *for every value of t, then for any value of* t_0 *in the domain of* $\mathbf{r}$*, the parameter* $s = t - t_0$ *is an arc length parameter that has its reference point at the point on C where* $t = t_0$*.*

$$\frac{ds}{dt} = \left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \text{2-space}$$

$$\frac{ds}{dt} = \left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \quad \text{3-space}$$

$$\left\|\frac{d\mathbf{r}}{ds}\right\| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} = 1 \quad \text{2-space}$$

$$\left\|\frac{d\mathbf{r}}{ds}\right\| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = 1 \quad \text{3-space}$$

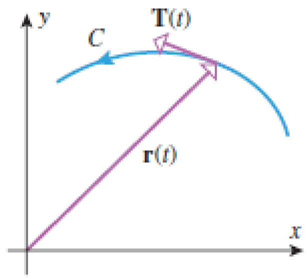
2.4 UNIT TANGENT, NORMAL, AND BINORMAL VECTORS

2.4.1 Unit Tangent Vectors

If C is the graph of a *smooth* vector-valued function **r**(t) in 2-space or 3-space, then the vector $\mathbf{r}'(t)$ is nonzero, tangent to C, and points in the direction of increasing parameter. Thus, by normalizing $\mathbf{r}'(t)$ we obtain a unit vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \qquad (1)$$

that is tangent to C and points in the direction of increasing parameter. We call **T**(t) the ***unit tangent vector*** to C at t.



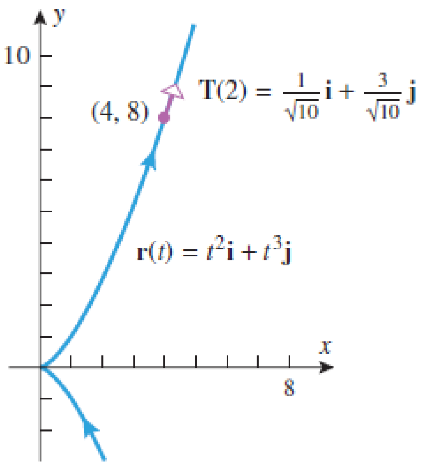
Example: Find the unit tangent vector to the graph of $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j}$ at the point where $t = 2$.

Solution: Since

$$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$$

we obtain

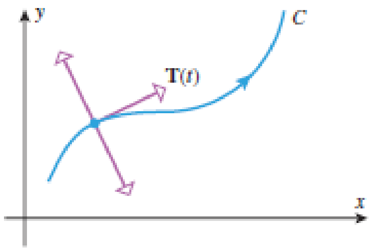
$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{4\mathbf{i} + 12\mathbf{j}}{\sqrt{160}} = \frac{4\mathbf{i} + 12\mathbf{j}}{4\sqrt{10}} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$$



2.4.2 Unit Normal Vectors

If a vector-valued function **r**(*t*) has constant norm, then **r**(*t*) and $\acute{\mathbf{r}}(t)$are orthogonal vectors. In particular, **T**(*t*) has constant norm 1, so **T**(*t*) and $\acute{\mathbf{T}}(t)$ are orthogonal vectors. This implies that $\acute{\mathbf{T}}(t)$ is perpendicular to the tangent line to *C* at *t*, so we say that $\acute{\mathbf{T}}(t)$ is ***normal*** to *C* at *t*. It follows that if $\acute{\mathbf{T}}(t) \neq 0$, and if we normalize $\acute{\mathbf{T}}(t)$, then we obtain a unit vector

$$\mathbf{N}(t) = \frac{\acute{\mathbf{T}}(t)}{\|\acute{\mathbf{T}}(t)\|} \qquad (2)$$



That is normal to *C* and points in the same direction as $\acute{\mathbf{T}}(t)$. We call **N**(*t*) the ***principal unit normal vector*** to *C* at *t* , or more simply, the ***unit normal vector***. Observe that the unit normal vector is defined only at points where $\acute{\mathbf{T}}(t) \neq \mathbf{0}$. Unless stated otherwise, we will assume that this condition is satisfied. In particular, this *excludes* straight lines.

Example: Find **T**(*t*) and **N**(*t*) for the circular helix

$$x = a \cos t,\ y = a \sin t,\ z = ct$$

where $a > 0$.

Solution: The radius vector for the helix is

$$\mathbf{r}(t) = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j} + ct\mathbf{k}$$

(Figure). Thus,

$$\mathbf{r}'(t) = (-a \sin t)\mathbf{i} + a \cos t\, \mathbf{j} + c\mathbf{k}$$

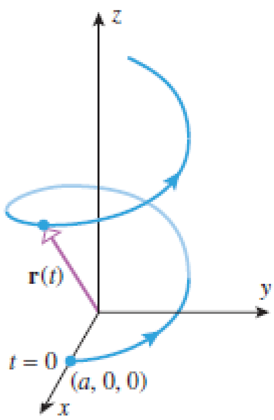
$$\|\mathbf{r}'(t)\| = \sqrt{(-a \sin t)^2 + (a \cos t)^2 + c^2} = \sqrt{a^2 + c^2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\frac{a \sin t}{\sqrt{a^2+c^2}}\mathbf{i} + \frac{a \cos t}{\sqrt{a^2+c^2}}\mathbf{j} + \frac{c}{\sqrt{a^2+c^2}}\mathbf{k}$$

$$\mathbf{T}'(t) = -\frac{a \cos t}{\sqrt{a^2+c^2}}\mathbf{i} - \frac{a \sin t}{\sqrt{a^2+c^2}}\mathbf{j}$$

$$\|\mathbf{T}'(t)\| = \sqrt{\left(-\frac{a \cos t}{\sqrt{a^2+c^2}}\right)^2 + \left(-\frac{a \sin t}{\sqrt{a^2+c^2}}\right)^2} = \sqrt{\frac{a^2}{a^2+c^2}} = \frac{a}{\sqrt{a^2+c^2}}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} = -(\cos t\mathbf{i} + \sin t\mathbf{j})$$



$\mathbf{r}(t) = a \cos t\mathbf{i} + a \sin t\mathbf{j} + ct\mathbf{k}$

2.4.3 Inward Unit Normal Vectors in 2-Space

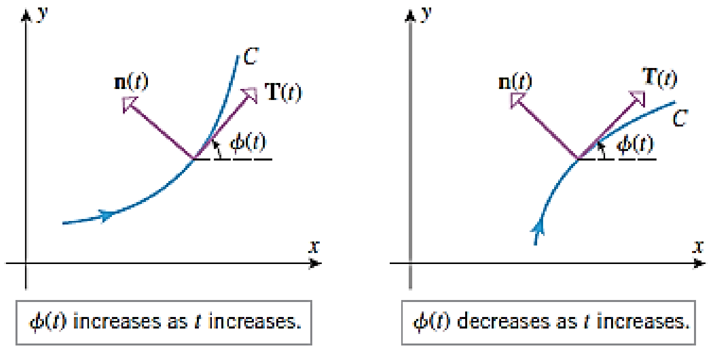
Our next objective is to show that for a nonlinear parametric curve *C* in 2-space the unit normal vector always points toward the concave side of *C*.

For this purpose, let $\varphi(t)$ be the angle from the positive *x*-axis to $\mathbf{T}(t)$, and let $\mathbf{n}(t)$ be the unit vector that results when $\mathbf{T}(t)$ is rotated counter-clockwise through an angle of $\pi/2$ (see below figure). Since $\mathbf{T}(t)$ and $\mathbf{n}(t)$ are unit vectors, that these vectors can be expressed as

$$\mathbf{T}(t) = \cos\varphi(t)\mathbf{i} + \sin\varphi(t)\mathbf{j}$$

$$\mathbf{n}(t) = \cos[\varphi(t) + \pi/2]\mathbf{i} + \sin[\varphi(t) + \pi/2]\,\mathbf{j} = -\sin\varphi(t)\mathbf{i} + \cos\varphi(t)\mathbf{j}$$

Observe that on intervals where $\varphi(t)$ is increasing the vector $\mathbf{n}(t)$ points *toward* the concave side of *C*, and on intervals where $\varphi(t)$ is decreasing it points *away* from the concave side (see below figure).



Now let us differentiate $\mathbf{T}(t)$ by using previous formula and applying the chain rule. This Yields

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{d\phi}\frac{d\phi}{dt} = [(-\sin\phi)\mathbf{i} + (\cos\phi)\mathbf{j}]\frac{d\phi}{dt}$$

and thus

$$\frac{d\mathbf{T}}{dt} = \mathbf{n}(t)\frac{d\phi}{dt}$$

But $d\varphi/dt > 0$ on intervals where $\varphi(t)$ is increasing and $d\varphi/dt < 0$ on intervals where $\varphi(t)$ is decreasing. Thus, $d\mathbf{T}/dt$ has the same direction as $\mathbf{n}(t)$ on intervals where $\varphi(t)$ is increasing and the opposite direction on intervals where $\varphi(t)$ is decreasing. Therefore, $\mathbf{T}'(t) = d\mathbf{T}/dt$ points "inward" toward the concave side of the curve in all cases, and hence so does $\mathbf{N}(t)$. For this reason, $\mathbf{N}(t)$ is also called the ***inward unit normal*** when applied to curves in 2-space.

2.4.4 Computing T and N for Curves Parametrized by Arc Length

In the case where **r**(s) is parametrized by arc length, the procedures for computing the unit tangent vector **T**(s) and the unit normal vector **N**(s) are simpler than in the general case. For example, we showed in Theorem that if s is an arc length parameter, then $\|\acute{\mathbf{r}}\,(s)\| = 1$. Thus, Formula (1) for the unit tangent vector simplifies to

$$\mathbf{T}(s) = \acute{\mathbf{r}}\,(s)$$

and consequently Formula (2) for the unit normal vector simplifies to

$$\mathbf{N}(s) = \frac{\bar{\bar{\mathbf{r}}}(s)}{\|\bar{\bar{\mathbf{r}}}(s)\|}$$

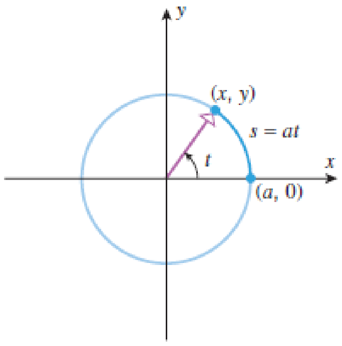
Example: The circle of radius a with counter-clockwise orientation and centered at the origin can be represented by the vector-valued function

$$\mathbf{r} = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j} \; (0 \le t \le 2\pi)$$

Parametrize this circle by arc length and find **T**(s) and **N**(s).

Solution: In (8) we can interpret t as the angle in radian measure from the positive x-axis to the radius vector (below figure). This angle subtends an arc of length $s = at$ on the circle, so we can reparametrize the circle in terms of s by substituting s/a for t. This yields

$$\mathbf{r}(s) = a \cos(s/a)\mathbf{i} + a \sin(s/a)\mathbf{j} \qquad (0 \le s \le 2\pi a)$$



To find **T**(s) and **N**(s) from Formulas (6) and (7), we must compute $\acute{\mathbf{r}}\,(s)$, $\mathbf{r}''(s)$, and $\|\mathbf{r}''(s)\|$. Doing so, we obtain

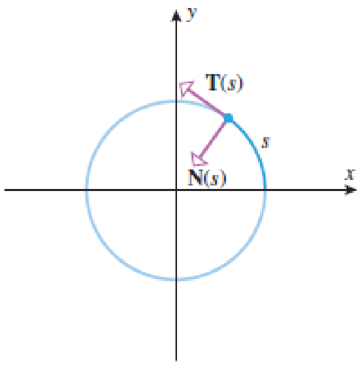
$$\mathbf{r}'(s) = -\sin(s/a)\mathbf{i} + \cos(s/a)\mathbf{j}$$

$$\mathbf{r}''(s) = -(1/a)\cos(s/a)\mathbf{i} - (1/a)\sin(s/a)\mathbf{j}$$

$$\|\mathbf{r}''(s)\| = \sqrt{(-1/a)^2\cos^2(s/a) + (-1/a)^2\sin^2(s/a)} = 1/a$$

$$\mathbf{T}(s) = \mathbf{r}'(s) = -\sin(s/a)\mathbf{i} + \cos(s/a)\mathbf{j}$$

$$\mathbf{N}(s) = \mathbf{r}''(s)/\|\mathbf{r}''(s)\| = -\cos(s/a)\mathbf{i} - \sin(s/a)\mathbf{j}$$



2.4.5 Binormal Vectors In 3-Space

If C is the graph of a vector-valued function $\mathbf{r}(t)$ in 3-space, then we define the ***binormal vector*** to C at t to be

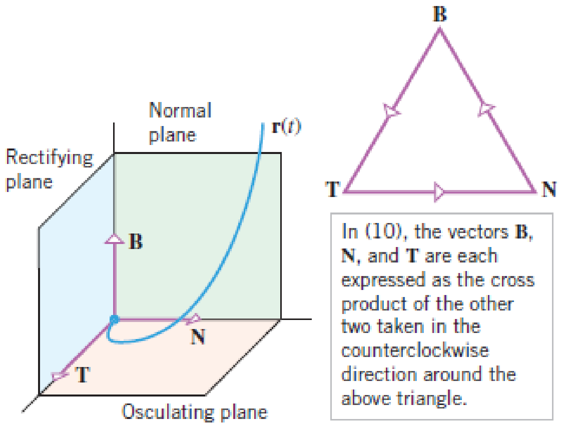
$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \qquad (9)$$

It follows from properties of the cross product that $\mathbf{B}(t)$ is orthogonal to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ and is oriented relative to $\mathbf{T}(t)$ and $\mathbf{N}(t)$ by the right-hand rule. Moreover, $\mathbf{T}(t) \times \mathbf{N}(t)$ is a unit vector since

$$\|\mathbf{T}(t) \times \mathbf{N}(t)\| = \|\mathbf{T}(t)\|\,\|\mathbf{N}(t)\| \sin(\pi/2) = 1$$

Thus, $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ is a set of three mutually orthogonal unit vectors.

Just as the vectors **i**, **j**, and **k** determine a right-handed coordinate system in 3-space, so do the vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$. At each point on a smooth parametric curve C in 3-space, these vectors determine three mutually perpendicular planes that pass through the point— the **TB**-plane (called the ***rectifying plane***), the **TN**-plane (called the ***osculating plane***), and the **NB**-plane (called the ***normal plane***) (Figure). Moreover, one can show that a coordinate system determined by $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ is right-handed in the sense that each of these vectors is related to the other two by the right-hand rule (see figure):



In (10), the vectors **B**, **N**, and **T** are each expressed as the cross product of the other two taken in the counterclockwise direction around the above triangle.

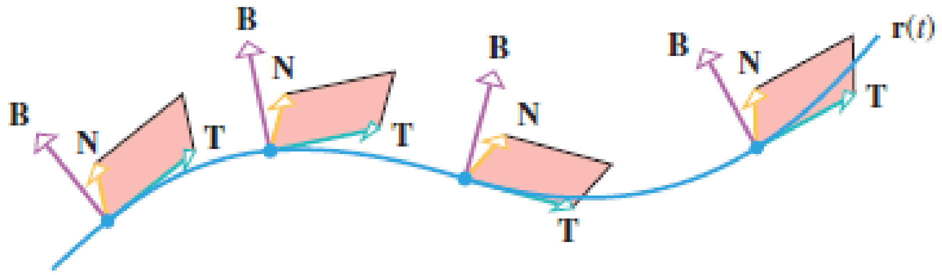
$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t),\ \mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t),\ \mathbf{T}(t) = \mathbf{N}(t) \times \mathbf{B}(t)$$

The coordinate system determined by **T**(*t*), **N**(*t*), and **B**(*t*) is called the **TNB-*frame*** or sometimes the ***Frenet frame*** in honor of the French mathematician Jean Frédéric Frenet (1816– 1900) who pioneered its application to the study of space curves. Typically, the *xyz*-coordinate system determined by the unit vectors **i**, **j**, and **k** remains fixed, whereas the **TNB**-frame changes as its origin moves along the curve *C* (Figure). Formula expresses **B**(*t*) in terms of **T**(*t*) and **N**(*t*). Alternatively, the binormal **B**(*t*) can be expressed directly in terms of **r**(*t*) as

$$\mathbf{B}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}$$

and in the case where the parameter is arc length it can be expressed in terms of **r**(*s*) as

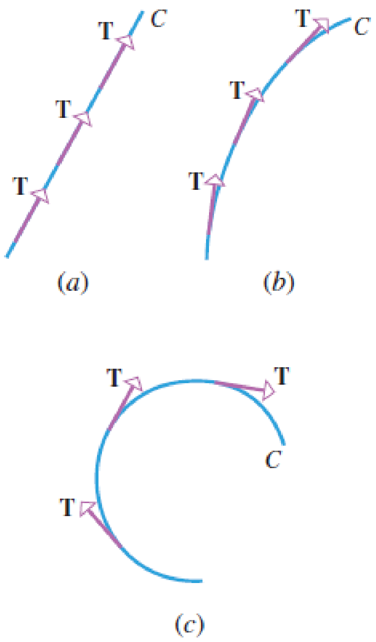
$$\mathbf{B}(s) = \frac{\mathbf{r}'(s) \times \mathbf{r}''(s)}{\|\mathbf{r}''(s)\|}$$



2.5 CURVATURE

2.5.1 Definition of Curvature

Suppose that C is the graph of a smooth vector-valued function in 2-space or 3-space that is parametrized in terms of arc length. Figure suggests that for a curve in 2-space the "sharpness" of the bend in C is closely related to $d\mathbf{T}/ds$, which is the rate of change of the unit tangent vector $\mathbf{T}$ with respect to s. (Keep in mind that $\mathbf{T}$ has constant length, so only its direction changes.) If C is a straight line (no bend), then the direction of $\mathbf{T}$ remains constant (Figure *a*); if C bends slightly, then $\mathbf{T}$ undergoes a gradual change of direction (Figure *b*); and if C bends sharply, then $\mathbf{T}$ undergoes a rapid change of direction (Figure *c*).



The situation in 3-space is more complicated because bends in a curve are not limited to a single plane—they can occur in all directions. To describe the bending characteristics of a curve in 3-space completely, one must take into account $d\mathbf{T}/ds$, $d\mathbf{N}/ds$, and $d\mathbf{B}/ds$. A complete study of this topic would take us too far afield, so we will limit our discussion to $d\mathbf{T}/ds$, which is the most important of these derivatives in applications.

Definition If C is a smooth curve in 2-space or 3-space that is parametrized by arc length, then the ***curvature*** of C, denoted by $\kappa = \kappa(s)$ (κ = Greek "kappa"), is defined by

$$k(s) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \mathbf{r}''(s) \right\| \qquad (1)$$

Observe that $\kappa(s)$ is a real-valued function of s, since it is the *length* of $d\mathbf{T}/ds$ that measures the curvature. In general, the curvature will vary from point to point along a curve; however, the following example shows that the curvature is constant for circles in 2-space, as you might expect.

Example: the circle of radius a, centered at the origin, can be parametrized in terms of arc length as

$$\mathbf{r}(s) = a\cos(s/a)\mathbf{i} + a\sin(s/a)\mathbf{j} \qquad (0 \le s \le 2\pi a)$$

$$\mathbf{r}''(s) = -\frac{1}{a}\cos\left(\frac{s}{a}\right)\mathbf{i} - \frac{1}{a}\sin\left(\frac{s}{a}\right)\mathbf{j}$$

$$\kappa(s) = \|\mathbf{r}''(s)\| = \sqrt{\left[-\frac{1}{a}\cos\left(\frac{s}{a}\right)\right]^2 + \left[-\frac{1}{a}\sin\left(\frac{s}{a}\right)\right]^2} = \frac{1}{a}$$

so the circle has constant curvature 1/a.

2.5.2 Formulas for Curvature

Formula (1) is only applicable if the curve is parametrized in terms of arc length. The following theorem provides two formulas for curvature in terms of a general parameter *t*.

Theorem *If* **r**(t) *is a smooth vector-valued function in 2-space or 3-space, then for each value of t at which* $\mathbf{T}'(t)$ *and* $\mathbf{r}''(t)$ *exist, the curvature κ can be expressed*

a) $k(t) = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$ (2)

b) $k(t) = \dfrac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$ (3)

Proof a:

$$\kappa(t) = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left\|\frac{d\mathbf{T}/dt}{ds/dt}\right\| = \left\|\frac{d\mathbf{T}/dt}{\|d\mathbf{r}/dt\|}\right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

Proof b:

$$\mathbf{r}'(t) = \|\mathbf{r}'(t)\|\mathbf{T}(t)$$
$$\mathbf{r}''(t) = \|\mathbf{r}'(t)\|'\mathbf{T}(t) + \|\mathbf{r}'(t)\|\mathbf{T}'(t)$$

$$\mathbf{T}'(t) = \|\mathbf{T}'(t)\|\mathbf{N}(t) \quad \text{and} \quad \|\mathbf{T}'(t)\| = \kappa(t)\|\mathbf{r}'(t)\|$$

$$\mathbf{T}'(t) = \kappa(t)\|\mathbf{r}'(t)\|\mathbf{N}(t)$$

$$\mathbf{r}''(t) = \|\mathbf{r}'(t)\|'\mathbf{T}(t) + \kappa(t)\|\mathbf{r}'(t)\|^2\mathbf{N}(t)$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \|\mathbf{r}'(t)\|\|\mathbf{r}'(t)\|'(\mathbf{T}(t) \times \mathbf{T}(t)) + \kappa(t)\|\mathbf{r}'(t)\|^3(\mathbf{T}(t) \times \mathbf{N}(t))$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \kappa(t)\|\mathbf{r}'(t)\|^3(\mathbf{T}(t) \times \mathbf{N}(t)) = \kappa(t)\|\mathbf{r}'(t)\|^3\mathbf{B}(t)$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \kappa(t)\|\mathbf{r}'(t)\|^3$$

Example: Find κ(t) for the circular helix

x = a cos t, y = a sin t, z = ct where a > 0.

Solution: The radius vector for the helix is

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k}$$

Thus,

$$\mathbf{r}'(t) = (-a\sin t)\mathbf{i} + a\cos t\,\mathbf{j} + c\mathbf{k}$$

$$\mathbf{r}''(t) = (-a\cos t)\mathbf{i} + (-a\sin t)\mathbf{j}$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = (ac\sin t)\mathbf{i} - (ac\cos t)\mathbf{j} + a^2\mathbf{k}$$

Therefore,

$$\|\mathbf{r}'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} = \sqrt{a^2+c^2}$$

and

$$\|\mathbf{r}'(t)\times\mathbf{r}''(t)\| = \sqrt{(ac\sin t)^2 + (-ac\cos t)^2 + a^4}$$
$$= \sqrt{a^2c^2 + a^4} = a\sqrt{a^2+c^2}$$

so

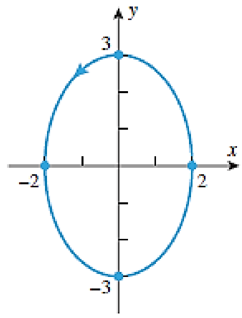
$$\kappa(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{a\sqrt{a^2+c^2}}{\left(\sqrt{a^2+c^2}\right)^3} = \frac{a}{a^2+c^2}$$

Note that κ does not depend on t , which tells us that the helix has constant curvature.

Example: The graph of the vector equation

$$\mathbf{r} = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} \qquad (0 \le t \le 2\pi)$$

is the ellipse as shown in Figure. Find the curvature of the ellipse at the endpoints of the major and minor axes, and use a graphing utility to generate the graph of $\kappa(t)$.



$\mathbf{r} = 2\cos t\mathbf{i} + 3\sin t\mathbf{j}$

Solution: To apply Formula (3), we must treat the ellipse as a curve in the *xy*-plane of an *xyz*-coordinate system by adding a zero **k** component and writing its equation as

$$\mathbf{r} = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 0\mathbf{k}$$

It is not essential to write the zero **k** component explicitly as long as you assume it to be there when you calculate a cross product. Thus,

$$\mathbf{r}'(t) = (-2\sin t)\mathbf{i} + 3\cos t\,\mathbf{j}$$

$$\mathbf{r}''(t) = (-2\cos t)\mathbf{i} + (-3\sin t)\mathbf{j}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin t & 3\cos t & 0 \\ -2\cos t & -3\sin t & 0 \end{vmatrix} = [(6\sin^2 t) + (6\cos^2 t)]\mathbf{k} = 6\mathbf{k}$$

Therefore,

$$\|\mathbf{r}'(t)\| = \sqrt{(-2\sin t)^2 + (3\cos t)^2} = \sqrt{4\sin^2 t + 9\cos^2 t}$$
$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = 6$$

so

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{6}{[4\sin^2 t + 9\cos^2 t]^{3/2}}$$

The endpoints of the minor axis are $(2, 0)$ and $(-2, 0)$, which correspond to $t = 0$ and $t = \pi$, respectively. Substituting these values in (7) yields the same curvature at both points, namely

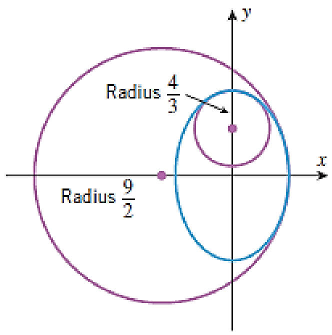
$$\kappa = \kappa(0) = \kappa(\pi) = \frac{6}{9^{3/2}} = \frac{6}{27} = \frac{2}{9}$$

The endpoints of the major axis are $(0, 3)$ and $(0, -3)$, which correspond to $t = \pi/2$ and $t = 3\pi/2$, respectively; from (7) the curvature at these points is

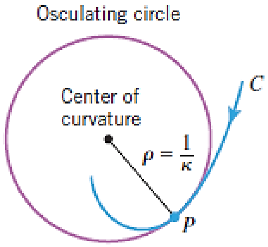
$$\kappa = \kappa\left(\frac{\pi}{2}\right) = \kappa\left(\frac{3\pi}{2}\right) = \frac{6}{4^{3/2}} = \frac{3}{4}$$

RADIUS OF CURVATURE

In the last example we found the curvature at the ends of the minor axis to be 2/9 and the curvature at the ends of the major axis to be 3/4. To obtain a better understanding of the meaning of these numbers, recall from Example 1 that a circle of radius *a* has a constant curvature of $1/a$; thus, the curvature of the ellipse at the ends of the minor axis is the same as that of a circle of radius 9/2, and the curvature at the ends of the major axis is the same as that of a circle of radius 4/3 (Figure).



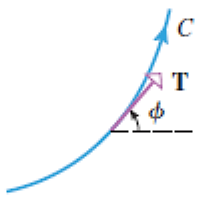
In general, if a curve C in 2-space has nonzero curvature κ at a point P, then the circle of radius $\rho = 1/\kappa$ sharing a common tangent with C at P, and centered on the concave side of the curve at P, is called the ***osculating circle*** or ***circle of curvature*** at P (Figure).



The osculating circle and the curve C not only touch at P but they have equal curvatures at that point. In this sense, the osculating circle is the circle that best approximates the curve C near P. The radius ρ of the osculating circle at P is called the ***radius of curvature*** at P, and the center of the circle is called the ***center of curvature*** at P (previous Figure).

2.5.3 An Interpretation of Curvature in 2-Space

A useful geometric interpretation of curvature in 2-space can be obtained by considering the angle φ measured counter-clockwise from the direction of the positive x-axis to the unit tangent vector $\mathbf{T}$ (see below figure). By previous formula, we can express $\mathbf{T}$ in terms of φ as



$$\mathbf{T}(\varphi) = \cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j}$$

Thus,

$$\frac{d\mathbf{T}}{d\phi} = (-\sin\phi)\mathbf{i} + \cos\phi\,\mathbf{j}$$

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{d\phi}\frac{d\phi}{ds}$$

from which we obtain

$$\kappa(s) = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left|\frac{d\phi}{ds}\right|\left\|\frac{d\mathbf{T}}{d\phi}\right\| = \left|\frac{d\phi}{ds}\right|\sqrt{(-\sin\phi)^2 + \cos^2\phi} = \left|\frac{d\phi}{ds}\right|$$

In summary, we have shown that

$$\kappa(s) = \left|\frac{d\phi}{ds}\right|$$

which tells us that curvature in 2-space can be interpreted as the magnitude of the rate of change of φ with respect to s—the greater the curvature, the more rapidly φ changes with s (Figure a). In the case of a straight line, the angle φ is constant (Figure b) and consequently $\kappa(s) = |d\varphi/ds| = 0$, which is consistent with the fact that a straight line has zero curvature at every point.

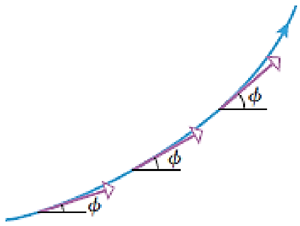


Figure a

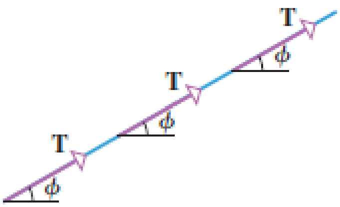


Figure b

2.6 MOTION ALONG A CURVE

2.6.1 Velocity, Acceleration, and Speed

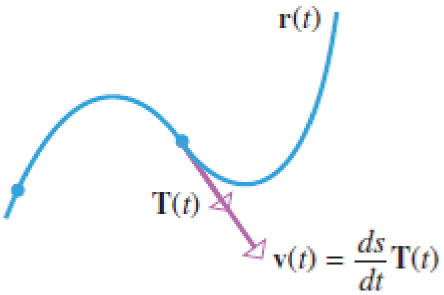
Definition

If $\mathbf{r}(t)$ is the position function of a particle moving along a curve in 2-space or 3-space, then the ***instantaneous velocity***, ***instantaneous acceleration***, and ***instantaneous speed*** of the particle at time t are defined by

$$\text{velocity} = \mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

$$\text{acceleration} = \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

$$\text{speed} = \|\mathbf{v}(t)\| = \frac{ds}{dt}$$



The length of the velocity vector is the speed of the particle, and the direction of the velocity vector is the direction of motion.

FORMULAS FOR POSITION, VELOCITY, ACCELERATION, AND SPEED

	2-SPACE	3-SPACE
POSITION	$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$	$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$
VELOCITY	$\mathbf{v}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$	$\mathbf{v}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$
ACCELERATION	$\mathbf{a}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$	$\mathbf{a}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$
SPEED	$\|\mathbf{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$	$\|\mathbf{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

Example: A particle moves along a circular path in such a way that its x- and y-coordinates at time t are

$$x = 2\cos t,\ y = 2\sin t$$

(a) Find the instantaneous velocity and speed of the particle at time t .

(b) Sketch the path of the particle, and show the position and velocity vectors at time $t = \pi/4$ with the velocity vector drawn so that its initial point is at the tip of the position vector.

(c) Show that at each instant the acceleration vector is perpendicular to the velocity vector.

Solution (a). At time t , the position vector is

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$$

so the instantaneous velocity and speed are

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$$

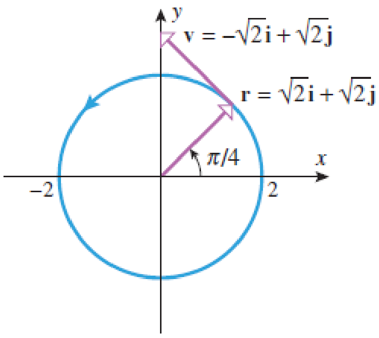
$$\|\mathbf{v}(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2$$

Solution (b). The graph of the parametric equations is a circle of radius 2 centered at the origin. At time $t = \pi/4$ the position and velocity vectors of the particle are

$$\mathbf{r}(\pi/4) = 2\cos(\pi/4)\mathbf{i} + 2\sin(\pi/4)\mathbf{j} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

$$\mathbf{v}(\pi/4) = -2\sin(\pi/4)\mathbf{i} + 2\cos(\pi/4)\mathbf{j} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

These vectors and the circle are shown in Figure



Solution (c). At time t , the acceleration vector is

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -2\cos t\mathbf{i} - 2\sin t\mathbf{j}$$

One way of showing that **v**(t) and **a**(t) are perpendicular is to show that their dot product is zero (try it). However, it is easier to observe that **a**(t) is the negative of **r**(t), which implies that **v**(t) and **a**(t) are perpendicular, since at each point on a circle the radius and tangent line are perpendicular.

Example: A particle moves through 3-space in such a way that its velocity is

$$\mathbf{v}(t) = \mathbf{i} + t\,\mathbf{j} + t^2\,\mathbf{k}$$

Find the coordinates of the particle at time $t = 1$ given that the particle is at the point $(-1, 2, 4)$ at time $t = 0$.

Solution. Integrating the velocity function to obtain the position function yields

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (\mathbf{i} + t\mathbf{j} + t^2\mathbf{k})\,dt = t\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{t^3}{3}\mathbf{k} + \mathbf{C}$$

where **C** is a vector constant of integration. Since the coordinates of the particle at time $t = 0$ are $(-1, 2, 4)$, the position vector at time $t = 0$ is

$$\mathbf{r}(0) = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

It follows on substituting $t = 0$ in (5) and equating the result with (6) that

$$\mathbf{C} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

Substituting this value of **C** in (5) and simplifying yields

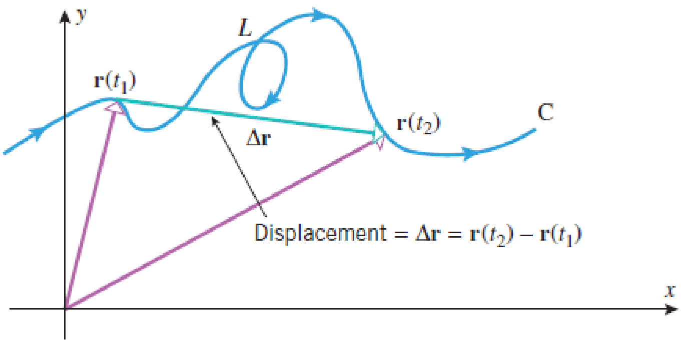
$$\mathbf{r}(t) = (t - 1)\mathbf{i} + \left(\frac{t^2}{2} + 2\right)\mathbf{j} + \left(\frac{t^3}{3} + 4\right)\mathbf{k}$$

Thus, at time $t = 1$ the position vector of the particle is

$$\mathbf{r}(1) = 0\mathbf{i} + \frac{5}{2}\mathbf{j} + \frac{13}{3}\mathbf{k}$$

so its coordinates at that instant are $\left(0, \frac{5}{2}, \frac{13}{3}\right)$.

2.6.2 Displacement and Distance Traveled



$$\Delta\mathbf{r} = \int_{t_1}^{t_2} \mathbf{v}(t)\,dt = \int_{t_1}^{t_2} \frac{d\mathbf{r}}{dt}\,dt = \mathbf{r}(t)\Big]_{t_1}^{t_2} = \mathbf{r}(t_2) - \mathbf{r}(t_1)$$

Displacement

$$s = \int_{t_1}^{t_2} \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_{t_1}^{t_2} \|\mathbf{v}(t)\|\, dt$$

Distance traveled

Example: Suppose that a particle moves along a circular helix in 3-space so that its position vector at time t is

$$\mathbf{r}(t) = (4\cos \pi t)\mathbf{i} + (4\sin \pi t)\,\mathbf{j} + t\mathbf{k}$$

Find the distance traveled and the displacement of the particle during the time interval $1 \le t \le 5$.

Solution. We have

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = (-4\pi \sin \pi t)\mathbf{i} + (4\pi \cos \pi t)\mathbf{j} + \mathbf{k}$$
$$\|\mathbf{v}(t)\| = \sqrt{(-4\pi \sin \pi t)^2 + (4\pi \cos \pi t)^2 + 1} = \sqrt{16\pi^2 + 1}$$

that the distance traveled by the particle from time $t = 1$ to $t = 5$ is

$$s = \int_1^5 \sqrt{16\pi^2 + 1}\, dt = 4\sqrt{16\pi^2 + 1}$$

that the displacement over the time interval is

$$\begin{aligned}\Delta\mathbf{r} &= \mathbf{r}(5) - \mathbf{r}(1)\\ &= (4\cos 5\pi\mathbf{i} + 4\sin 5\pi\mathbf{j} + 5\mathbf{k}) - (4\cos \pi\mathbf{i} + 4\sin \pi\mathbf{j} + \mathbf{k})\\ &= (-4\mathbf{i} + 5\mathbf{k}) - (-4\mathbf{i} + \mathbf{k}) = 4\mathbf{k}\end{aligned}$$

which tells us that the change in the position of the particle over the time interval was 4 units straight up.

Find **T**$(t\ )$, **N**(t), and **B**(t) for the given value of t.

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k};\ t = \pi/4$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j}),\ \mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{i}),$$

$$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{k};$$

Find an arc length parametrization of the curve that has the same orientation as the given curve and for which the reference point corresponds to $t = 0$.

$$\mathbf{r}(t) = (3 + \cos t)\mathbf{i} + (2 + \sin t)\,\mathbf{j};\ 0 \leq t \leq 2\pi$$

$$x = 3 + \cos s,\ y = 2 + \sin s,\ 0 \leq s \leq 2\pi$$

CHAPTER FOUR

DOUBLE INTEGRALS

4.1 DOUBLE INTEGRALS

4.1.1 Volume

Recall that the definite integral of a function of one variable

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k \qquad (1)$$

The volume problem Given a function *f* of two variables that is continuous and nonnegative on a region *R* in the *xy*-plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the region *R* (Figure 1).

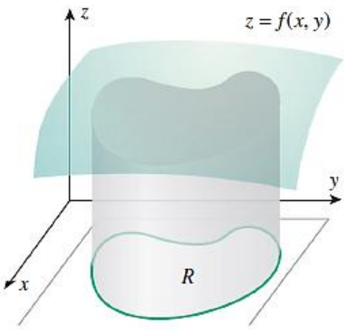


Figure 1

Definition 4.1 (***Volume Under a Surface***) If *f* is a function of two variables that is continuous and nonnegative on a region *R* in the *xy*-plane, then the volume of the solid enclosed between the surface $z = f(x, y)$ and the region *R* is defined by

$$V = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \qquad (2)$$

Here, $n \to +\infty$ indicates the process of increasing the number of sub-rectangles of the rectangle enclosing *R* in such a way that both the lengths and the widths of the sub-rectangles approach zero.

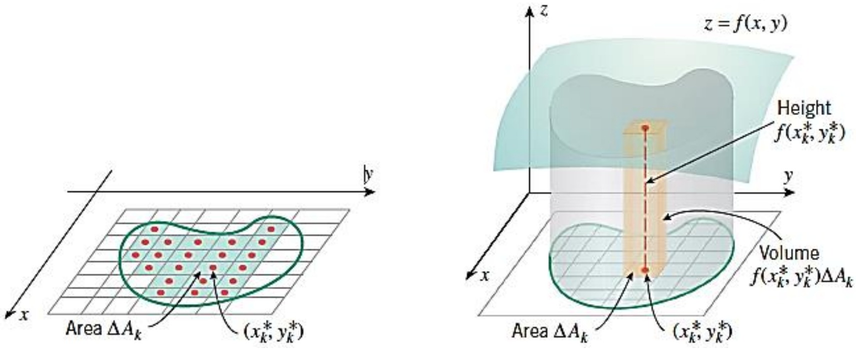


Figure 2

4.1.2 Definition of a Double Integral

As in Definition 4.1, the notation $n \to +\infty$ encapsulate a process in which the enclosing rectangle for R is repeatedly subdivided in such a way that both the lengths and the widths of the sub-rectangles approach zero.

$$\iint_R f(x, y)\, dA = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k$$

which is called the ***double integral*** of *f(x, y)* over *R*.

If f is continuous and nonnegative on the region R, then the volume formula in (2) can be expressed as

$$V = \iint_R f(x, y)\, dA$$

4.1.3 Evaluating Double Integrals

The partial derivatives of a function *f(x, y)* are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Let us consider the reverse of this process, ***partial integration***. The symbols

$$\int_a^b f(x, y)\, dx \quad \text{and} \quad \int_c^d f(x, y)\, dy$$

denote ***partial definite integrals***; the first integral, called the ***partial definite integral with respect to x***, is evaluated by holding y fixed and integrating with respect to x, and the second integral, called the ***partial definite integral with respect to y***, is evaluated by holding x fixed and integrating with respect to y. As the following example shows, the partial definite integral with respect to x is a function of y, and the partial definite integral with respect to y is a function of x.

Example 4.1

$$\int_0^1 xy^2\,dx = y^2\int_0^1 x\,dx = \left.\frac{y^2x^2}{2}\right]_{x=0}^{1} = \frac{y^2}{2}$$

$$\int_0^1 xy^2\,dy = x\int_0^1 y^2\,dy = \left.\frac{xy^3}{3}\right]_{y=0}^{1} = \frac{x}{3}$$

A partial definite integral with respect to x is a function of y and hence can be integrated with respect to y; similarly, a partial definite integral with respect to y can be integrated with respect to x. This two-stage integration process is called ***iterated*** (or ***repeated***) ***integration***. We introduce the following notation:

$$\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$$

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx$$

These integrals are called ***iterated integrals***.

Example 4.2 Evaluate

(a) $\displaystyle\int_1^3\int_2^4 (40-2xy)\,dy\,dx$ (b) $\displaystyle\int_2^4\int_1^3 (40-2xy)\,dx\,dy$

Solution (a):

$$\begin{aligned}\int_1^3\int_2^4 (40-2xy)\,dy\,dx &= \int_1^3\left[\int_2^4 (40-2xy)\,dy\right]dx\\ &= \int_1^3 (40y-xy^2)\big]_{y=2}^{4}\,dx\\ &= \int_1^3 [(160-16x)-(80-4x)]\,dx\\ &= \int_1^3 (80-12x)\,dx\\ &= (80x-6x^2)\big]_1^3 = 112\end{aligned}$$

Solution (b):

$$\begin{aligned}\int_2^4\int_1^3 (40-2xy)\,dx\,dy &= \int_2^4\left[\int_1^3 (40-2xy)\,dx\right]dy\\ &= \int_2^4 (40x-x^2y)\big]_{x=1}^{3}\,dy\\ &= \int_2^4 [(120-9y)-(40-y)]\,dy\\ &= \int_2^4 (80-8y)\,dy\\ &= (80y-4y^2)\big]_2^4 = 112\end{aligned}$$

Consider the solid S bounded above by the surface $z = 40 - 2xy$ and below by the rectangle R defined by $1 \leq x \leq 3$ and $2 \leq y \leq 4$. The volume of S is given by

$$V = \int_1^3 A(x)\,dx$$

where *A(x)* is the area of a vertical cross section of S taken perpendicular to the x-axis (Figure 3). For a fixed value of x, $1 \leq x \leq 3$, $z = 40 - 2xy$ is a function of y, so the

$$A(x) = \int_2^4 (40 - 2xy)\,dy$$

represents the area under the graph of this function of y. Thus,

$$V = \int_1^3 \left[\int_2^4 (40 - 2xy)\,dy\right] dx = \int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx$$

is the volume of S. Similarly, by the method of slicing with cross sections of S taken perpendicular to the y-axis, the volume of S is given by

$$V = \int_2^4 A(y)\,dy = \int_2^4 \left[\int_1^3 (40 - 2xy)\,dx\right] dy = \int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy$$

(Figure 4). Thus, the iterated integrals in parts (a) and (b) of Example both measure the volume of S, which is the double integral of $z = 40 - 2xy$ over R. That is,

$$\int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx = \iint_R (40 - 2xy)\,dA = \int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy$$

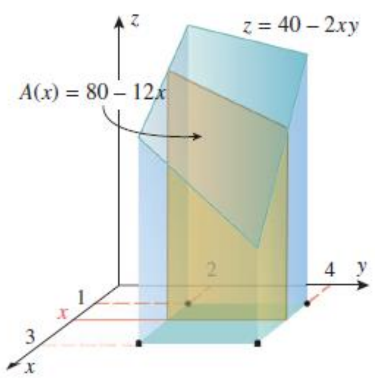


Figure 3

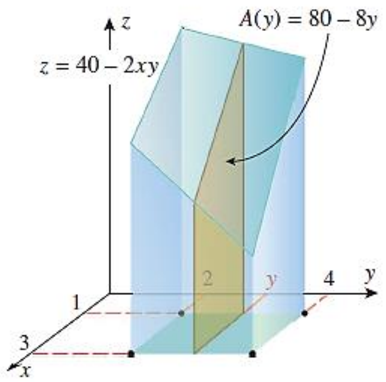


Figure 4

Theorem (***Fubini's Theorem***) *Let R be the rectangle defined by the inequalities*

$$a \leq x \leq b,\ c \leq y \leq d$$

If f(x, y) is continuous on this rectangle, then

$$\iint_R f(x, y)\,dA = \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx$$

Example 4.3 Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$ (Figure 5).

Solution: The volume is the double integral of $z = 4 - x - y$ over R. Using Theorem, this can be obtained from either of the iterated integrals

$$\int_0^2 \int_0^1 (4 - x - y)\, dx\, dy \quad \text{or} \quad \int_0^1 \int_0^2 (4 - x - y)\, dy\, dx$$

Using the first of these, we obtain

$$V = \iint_R (4 - x - y)\, dA = \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy$$

$$= \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^1 dy = \int_0^2 \left(\frac{7}{2} - y\right) dy$$

$$= \left[\frac{7}{2}y - \frac{y^2}{2}\right]_0^2 = 5$$

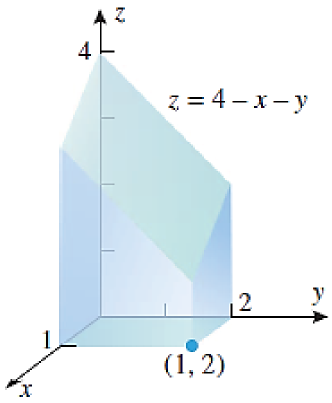


Figure 5

4.1.4 Properties of Double Integrals

$$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA \quad (c \text{ a constant})$$

$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

$$\iint_R [f(x, y) - g(x, y)]\, dA = \iint_R f(x, y)\, dA - \iint_R g(x, y)\, dA$$

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

4.2 DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

4.2.1 Iterated Integrals with Non-constant Limits of Integration

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right] dx$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)\, dx \right] dy$$

Example 4.4 Evaluate

$$\text{(a)} \int_0^1 \int_{-x}^{x^2} y^2 x\, dy\, dx \qquad \text{(b)} \int_0^{\pi/3} \int_0^{\cos y} x \sin y\, dx\, dy$$

Solution (a):

$$\int_{-x}^{x^2} y^2 x\, dy\, dx = \int_0^1 \left[\int_{-x}^{x^2} y^2 x\, dy \right] dx = \int_0^1 \left. \frac{y^3 x}{3} \right]_{y=-x}^{x^2} dx$$
$$= \int_0^1 \left[\frac{x^7}{3} + \frac{x^4}{3} \right] dx = \left. \left(\frac{x^8}{24} + \frac{x^5}{15} \right) \right]_0^1 = \frac{13}{120}$$

Solution (b):

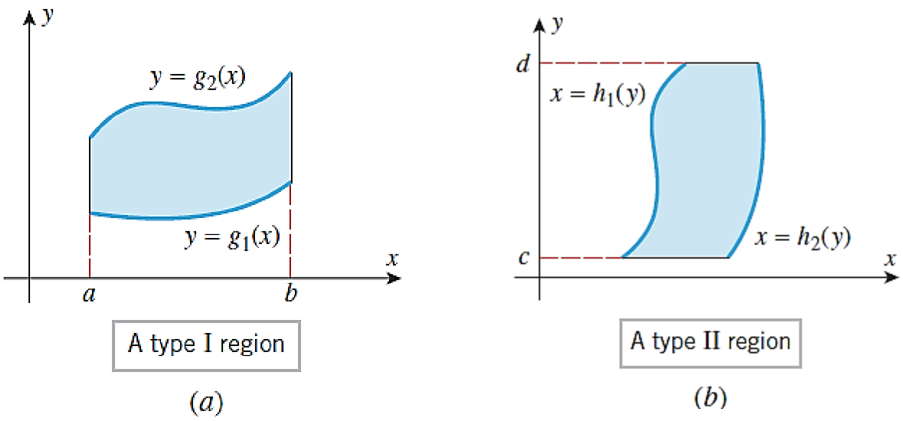
$$\int_0^{\pi/3} \int_0^{\cos y} x \sin y\, dx\, dy = \int_0^{\pi/3} \left[\int_0^{\cos y} x \sin y\, dx \right] dy = \int_0^{\pi/3} \left. \frac{x^2}{2} \sin y \right]_{x=0}^{\cos y} dy$$
$$= \int_0^{\pi/3} \left[\frac{1}{2} \cos^2 y \sin y \right] dy = -\frac{1}{6} \cos^3 y \Big]_0^{\pi/3} = \frac{7}{48}$$

4.2.2 Double Integrals over Nonrectangular Regions

Definition

(a) A ***type I region*** is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$ (Figure *a*).

(b) A ***type II region*** is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \leq h_2(y)$ for $c \leq y \leq d$ (Figure *b*).



Theorem

(*a*) *If R is a type I region on which f(x, y) is continuous, then*

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx \qquad (1)$$

(*b*) *If R is a type II region on which f(x, y) is continuous, then*

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy \qquad (2)$$

Example 4.5 Each of the iterated integrals in Example 4.4 is equal to a double integral over a region R. Identify the region R in each case.

Solution: Using Theorem, the integral in Example 4.4(a) is the double integral of the function $f(x, y) = y^2x$ over the type I region R bounded on the left and right by the vertical lines $x = 0$ and $x = 1$ and bounded below and above by the curves $y = -x$ and $y = x^2$ (Figure a). The integral in Example 4.4(b) is the double integral of the function $f(x, y) = x \sin y$ over the type II region R bounded below and above by the horizontal lines $y = 0$ and $y = \pi/3$ and bounded on the left and right by the curves $x = 0$ and $x = \cos y$ (Figure b).

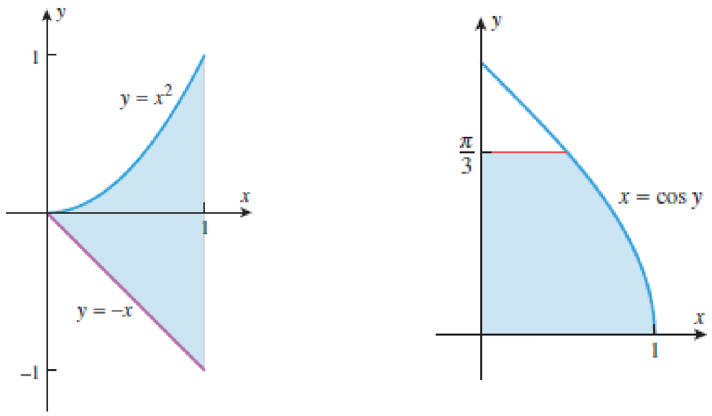
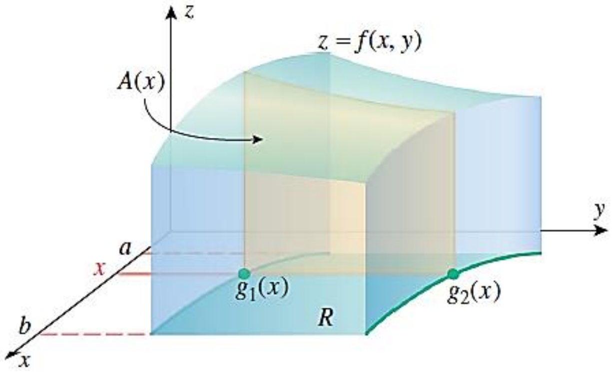


Figure a   Figure b

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

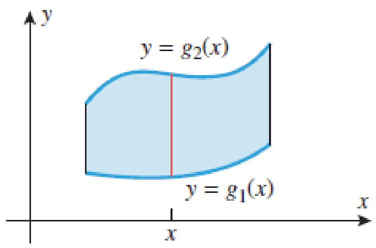


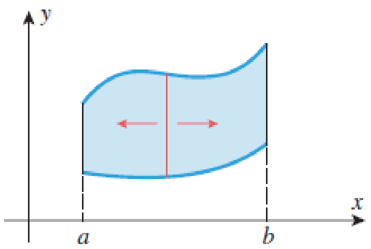
4.2.3 Setting up Limits of Integration for Evaluating Double Integrals

Determining Limits of Integration: Type I Region

Step 1. Since x is held fixed for the first integration, we draw a vertical line through the region R at an arbitrary fixed value x (below figure). This line crosses the boundary of R twice. The lower point of intersection is on the curve $y = g_1(x)$ and the higher point is on the curve $y = g_2(x)$. These two intersections determine the lower and upper y-limits of integration in Formula (1).

Step 2. Imagine moving the line drawn in Step 1 first to the left and then to the right (below figure). The leftmost position where the line intersects the region R is $x = a$, and the rightmost position where the line intersects the region R is $x = b$. This yields the limits for the x-integration in Formula (1).





Example 4.6 Evaluate

$$\iint_R xy\, dA$$

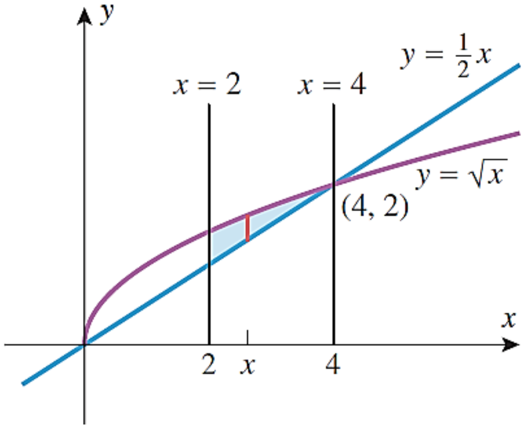
over the region R enclosed between $y = 1/2\, x$, $y = \sqrt{x}$, $x = 2$, and $x = 4$.

Solution:

We view R as a type I region. The region R and a vertical line corresponding to a fixed x are shown in Figure a. This line meets the region R at the lower boundary $y = (½)x$ and the upper boundary $y = \sqrt{x}$. These are the y-limits of integration. Moving this line first left and then right yields the x-limits of integration, $x = 2$ and $x = 4$. Thus,

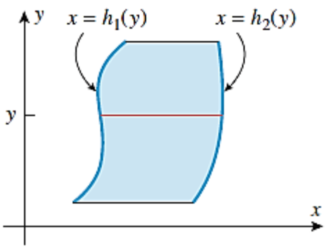
$$\iint_R xy\, dA = \int_2^4 \int_{x/2}^{\sqrt{x}} xy\, dy\, dx = \int_2^4 \left[\frac{xy^2}{2}\right]_{y=x/2}^{\sqrt{x}} dx = \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx$$

$$= \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4 = \left(\frac{64}{6} - \frac{256}{32}\right) - \left(\frac{8}{6} - \frac{16}{32}\right) = \frac{11}{6}$$

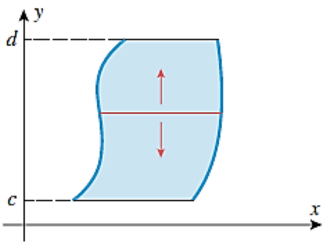


Determining Limits of Integration: Type II Region

Step 1. Since y is held fixed for the first integration, we draw a horizontal line through the region R at a fixed value y (following figure). This line crosses the boundary of R twice. The leftmost point of intersection is on the curve $x = h_1(y)$ and the rightmost point is on the curve $x = h_2(y)$. These intersections determine the x-limits of integration in (2).



Step 2. Imagine moving the line drawn in Step 1 first down and then up (following figure). The lowest position where the line intersects the region R is $y = c$, and the highest position where the line intersects the region R is $y = d$. This yields the y-limits of integration in (2).



Example 4.7 Evaluate

$$\iint_R (2x - y^2)\, dA$$

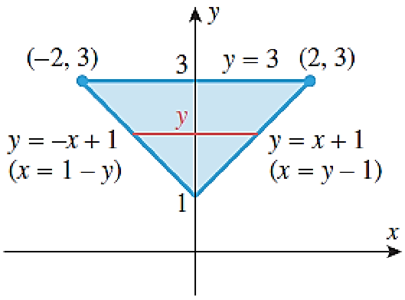
over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.

Solution: We view R as a type II region. The region R and a horizontal line corresponding to a fixed y are shown in below figure. This line meets the region R at its left-hand boundary $x = 1 - y$ and its right-hand boundary $x = y - 1$. These are the x-limits of integration. Moving this line first down and then up yields the y-limits, $y = 1$ and $y = 3$. Thus,

$$\iint_R (2x - y^2)\, dA = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy = \int_1^3 \left[x^2 - y^2 x\right]_{x=1-y}^{y-1} dy$$

$$= \int_1^3 [(1 - 2y + 2y^2 - y^3) - (1 - 2y + y^3)]\, dy$$

$$= \int_1^3 (2y^2 - 2y^3)\, dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3 = -\frac{68}{3}$$



Example 4.8 Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

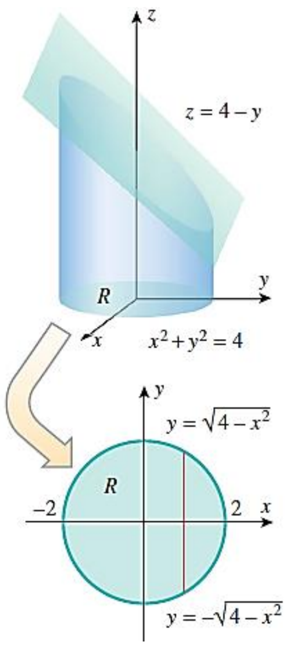
Solution: The solid shown in below figure is bounded above by the plane $z = 4 - y$ and below by the region R within the circle $x^2 + y^2 = 4$. The volume is given by

$$V = \iint_R (4 - y)\, dA$$

Treating R as a type I region we obtain

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\, dy\, dx = \int_{-2}^{2} \left[4y - \frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 8\sqrt{4 - x^2}\, dx = 8(2\pi) = 16\pi$$

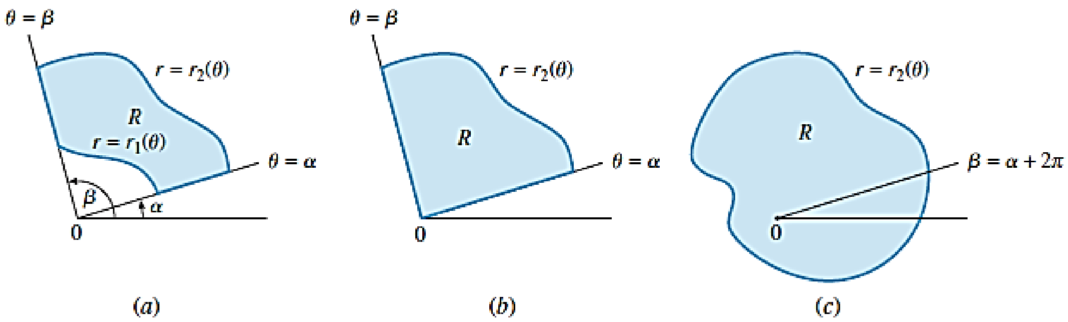


4.3 DOUBLE INTEGRALS IN POLAR COORDINATES

4.3.1 Simple Polar Regions

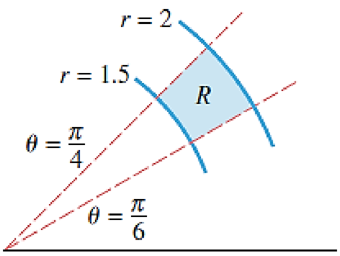
Definition A ***simple polar region*** in a polar coordinate system is a region that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

(i) $\alpha \le \beta$ (ii) $\beta - \alpha \le 2\pi$ (iii) $0 \le r_1(\theta) \le r_2(\theta)$



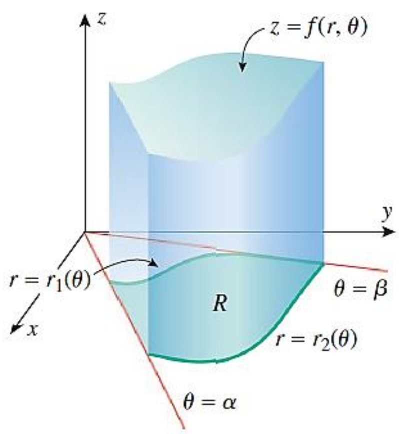
Simple polar regions

A ***polar rectangle*** is a simple polar region for which the bounding polar curves are circular arcs. For example, the following Figure shows the polar rectangle R given by $1.5 \le r \le 2$, $\pi/6 \le \theta \le \pi/4$



4.3.2 Double Integrals in Polar Coordinates

The volume problem in polar coordinates Given a function $f(r, \theta)$ that is continuous and non-negative on a simple polar region R, find the volume of the solid that is enclosed between the region R and the surface whose equation in cylindrical coordinates is $z = f(r, \theta)$ (see the figure).



If $f(r, \theta)$ is continuous on R and has both positive and negative values, then the limit

$$\lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) \Delta A_k$$

represents the net signed volume between the region R and the surface $z = f(r, \theta)$ (as with double integrals in rectangular coordinates). The sums are called ***polar Riemann sums***, and the limit of the polar Riemann sums is denoted by

$$\iint_R f(r, \theta)\, dA = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) \Delta A_k$$

which is called the ***polar double integral*** of $f(r, \theta)$ over R. If $f(r, \theta)$ is continuous and nonnegative on R, then the volume can be expressed as

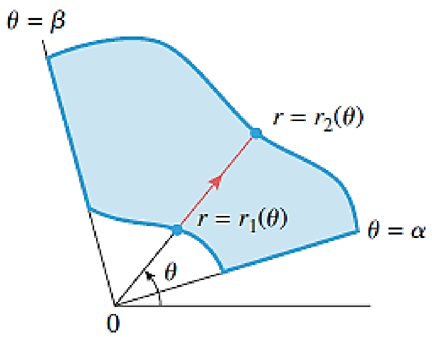
$$V = \iint_R f(r, \theta)\, dA$$

4.3.3 Evaluating Polar Double Integrals

Theorem

If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ shown in the below figure, and if $f(r, \theta)$ is continuous on R, then

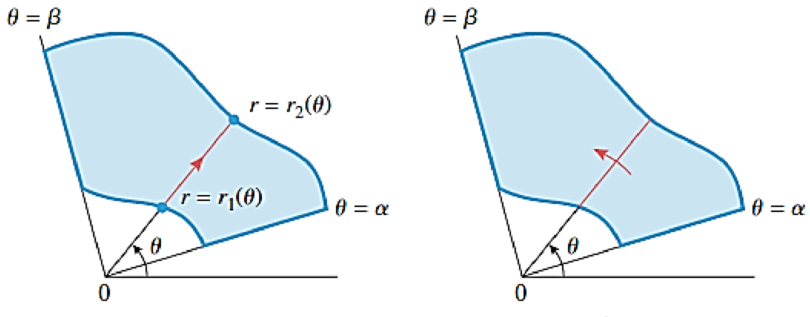
$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) r\, dr\, d\theta \quad (1)$$



Determining Limits of Integration for a Polar Double Integral: Simple Polar Region

Step 1. Since θ is held fixed for the first integration, draw a radial line from the origin through the region R at a fixed angle θ (Figure *a*). This line crosses the boundary of R at most twice. The innermost point of intersection is on the inner boundary curve $r = r_1(\theta)$ and the outermost point is on the outer boundary curve $r = r_2(\theta)$. These intersections determine the r-limits of integration in (1).

Step 2. Imagine rotating the radial line from Step 1 about the origin, thus sweeping out the region R. The least angle at which the radial line intersects the region R is $\theta = \alpha$and the greatest angle is $\theta = \beta$ (Figure b). This determines the θ-limits of integration.



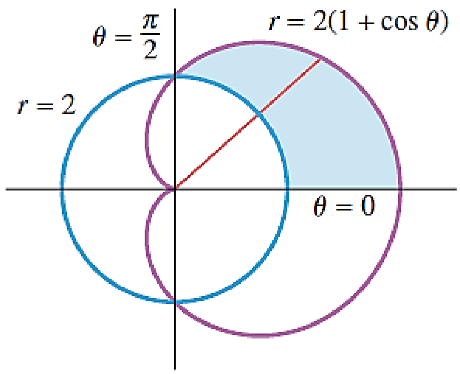
(a) (b)

Example 4.11 Evaluate

$$\iint_R \sin\theta \, dA$$

where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$.

Solution: The region R is sketched in the following figure. Following the two steps outlined above we obtain



$$\iint_R \sin\theta \, dA = \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} (\sin\theta) r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{2} r^2 \sin\theta\right]_{r=2}^{2(1+\cos\theta)} d\theta$$

$$= 2\int_0^{\pi/2} [(1+\cos\theta)^2 \sin\theta - \sin\theta] \, d\theta$$

$$= 2\left[-\frac{1}{3}(1+\cos\theta)^3 + \cos\theta\right]_0^{\pi/2}$$

$$= 2\left[-\frac{1}{3} - \left(-\frac{5}{3}\right)\right] = \frac{8}{3}$$

Example 4.12 The sphere of radius *a* centered at the origin is expressed in rectangular coordinates as $x^2 + y^2 + z^2 = a^2$, and hence its equation in cylindrical coordinates is $r^2 + z^2 = a^2$. Use this equation and a polar double integral to find the volume of the sphere.

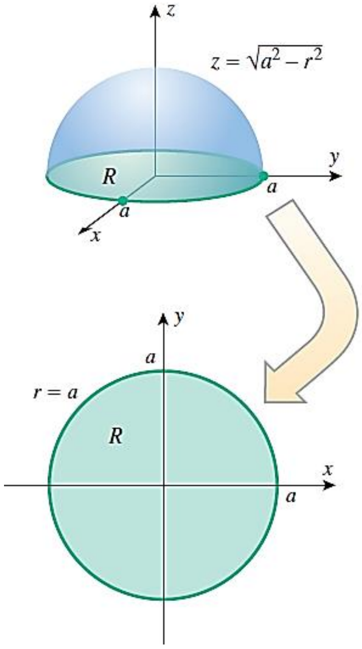
Solution: In cylindrical coordinates the upper hemisphere is given by the equation

$$z = \sqrt{a^2 - r^2}$$

so the volume enclosed by the entire sphere is

$$V = 2\iint_R \sqrt{a^2 - r^2}\, dA$$

where *R* is the circular region shown in following figure. Thus,



$$V = 2\iint_R \sqrt{a^2 - r^2}\, dA = \int_0^{2\pi}\int_0^a \sqrt{a^2 - r^2}(2r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[-\frac{2}{3}(a^2 - r^2)^{3/2}\right]_{r=0}^{a} d\theta = \int_0^{2\pi} \frac{2}{3}a^3\, d\theta$$

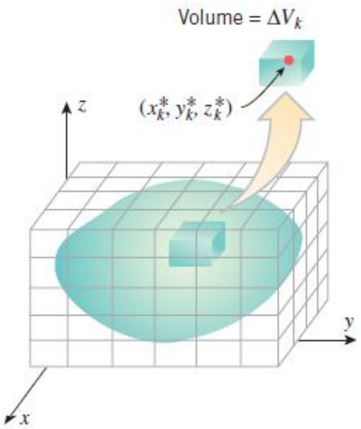
$$= \left[\frac{2}{3}a^3\theta\right]_0^{2\pi} = \frac{4}{3}\pi a^3$$

4.4 TRIPLE INTEGRALS

4.4.1 Definition of a Triple Integral

To define the triple integral of $f(x, y, z)$ over *G*, we first divide the box into *n* "sub-boxes" by planes parallel to the coordinate planes. We then discard those sub-boxes that contain any points outside of *G* and choose an arbitrary point in each of the remaining sub-boxes. As shown in the figure, we denote the volume of the *k*th remaining sub-box by ΔV_k and the point selected in the *k*th sub-box by (x_k^*, y_k^*, z_k^*). Next we form the product

$$f(x_k^*, y_k^*, z_k^*)\Delta V_k$$



for each sub-box, then add the products for all of the sub-boxes to obtain the ***Riemann sum***

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

Finally, we repeat this process with more and more subdivisions in such a way that the length, width, and height of each sub-box approach zero, and n approaches $+\infty$. The limit

$$\iiint_G f(x, y, z)\, dV = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

is called the ***triple integral*** of *f(x, y, z)* over the region *G*.

4.4.2 Properties of Triple Integrals

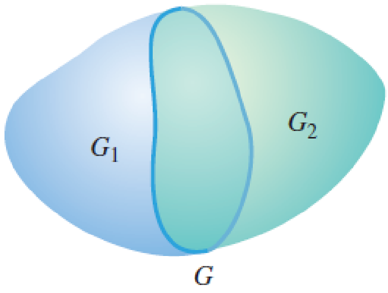
$$\iiint_G cf(x, y, z)\, dV = c\iiint_G f(x, y, z)\, dV \quad (c \text{ a constant})$$

$$\iiint_G [f(x, y, z) + g(x, y, z)]\, dV = \iiint_G f(x, y, z)\, dV + \iiint_G g(x, y, z)\, dV$$

$$\iiint_G [f(x, y, z) - g(x, y, z)]\, dV = \iiint_G f(x, y, z)\, dV - \iiint_G g(x, y, z)\, dV$$

Moreover, if the region G is subdivided into two sub-regions G_1 and G_2 (following figure), then

$$\iiint_G f(x, y, z)\, dV = \iiint_{G_1} f(x, y, z)\, dV + \iiint_{G_2} f(x, y, z)\, dV$$



4.4.3 Evaluating Triple Integrals over Rectangular Boxes

Theorem (***Fubini's Theorem****) *Let G be the rectangular box defined by the inequalities*

$$a \le x \le b,\ c \le y \le d,\ k \le z \le l$$

If f is continuous on the region G, then

$$\iiint_G f(x, y, z)\, dV = \int_a^b \int_c^d \int_k^l f(x, y, z)\, dz\, dy\, dx \qquad (1)$$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

Example 4.15 Evaluate the triple integral

$$\iiint_G 12xy^2z^3\, dV$$

over the rectangular box G defined by the inequalities $-1 \le x \le 2$, $0 \le y \le 3$, $0 \le z \le 2$.

Solution: Of the six possible iterated integrals we might use, we will choose the one in (1). Thus, we will first integrate with respect to z, holding x and y fixed, then with respect to y, holding x fixed, and finally with respect to x.

$$\iiint_G 12xy^2z^3\, dV = \int_{-1}^2 \int_0^3 \int_0^2 12xy^2z^3\, dz\, dy\, dx$$

$$= \int_{-1}^2 \int_0^3 \left[3xy^2z^4\right]_{z=0}^2 dy\, dx = \int_{-1}^2 \int_0^3 48xy^2\, dy\, dx$$

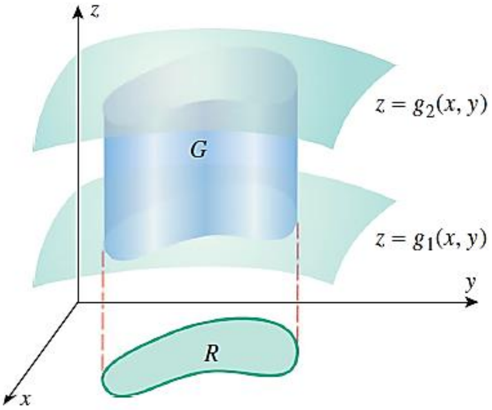
$$= \int_{-1}^2 \left[16xy^3\right]_{y=0}^3 dx = \int_{-1}^2 432x\, dx$$

$$= 216x^2\Big]_{-1}^2 = 648$$

4.4.4 Evaluating Triple Integrals over More General Regions

Theorem *Let G be a simple xy-solid with upper surface $z = g_2(x, y)$ and lower surface $z = g_1(x, y)$, and let R be the projection of G on the xy-plane. If $f(x, y, z)$ is continuous on G, then*

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right] dA \qquad (2)$$



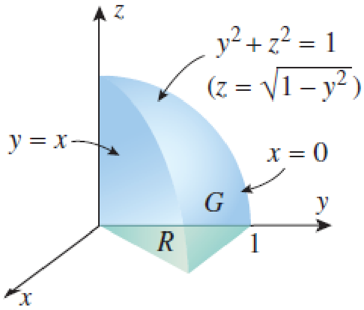
Determining Limits of Integration: Simple xy-Solid

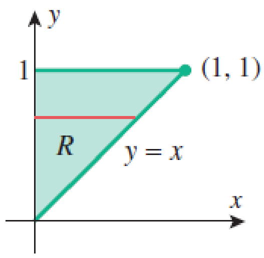
Step 1. Find an equation $z = g_2(x, y)$ for the upper surface and an equation $z = g_1(x, y)$ for the lower surface of G. The functions $g_1(x, y)$ and $g_2(x, y)$ determine the lower and upper z-limits of integration.

Step 2. Make a two-dimensional sketch of the projection R of the solid on the xy-plane. From this sketch determine the limits of integration for the double integral over R in (2).

Example 4.16 Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \leq 1$ by the planes $y = x$ and $x = 0$. Evaluate

$$\iiint_G z\,dV$$





Solution. The solid G and its projection R on the xy-plane are shown in the figure. The upper surface of the solid is formed by the cylinder and the lower surface by the xy-plane. Since the portion of the cylinder $y^2 + z^2 = 1$ that lies above the xy-plane has the equation $z = \sqrt{1 - y^2}$, and the xy-plane has the equation $z = 0$, it follows from (2) that

$$\iiint_G z\,dV = \iint_R \left[\int_0^{\sqrt{1-y^2}} z\,dz\right] dA$$

For the double integral over R, the x- and y-integrations can be performed in either order, since R is both a type I and type II region. We will integrate with respect to x first. With this choice, yields

$$\iiint_G z\,dV = \int_0^1\int_0^y\int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy = \int_0^1\int_0^y \frac{1}{2}z^2\Big]_{z=0}^{\sqrt{1-y^2}}\,dx\,dy$$

$$= \int_0^1\int_0^y \frac{1}{2}(1-y^2)\,dx\,dy = \frac{1}{2}\int_0^1 (1-y^2)x\Big]_{x=0}^{y}\,dy$$

$$= \frac{1}{2}\int_0^1 (y-y^3)\,dy = \frac{1}{2}\left[\frac{1}{2}y^2-\frac{1}{4}y^4\right]_0^1 = \frac{1}{8}$$

4.4.5 Volume Calculated As a Triple Integral

$$\text{volume of } G = \iiint_G dV$$

Example 4.17 Use a triple integral to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.

Solution: The solid G and its projection R on the xy-plane are shown in Figure. The lower surface of the solid is the plane $z = 1$ and the upper surface is the plane $x + z = 5$ or, equivalently, $z = 5 - x$. Thus,

$$\text{volume of } G = \iiint_G dV = \iint_R \left[\int_1^{5-x} dz\right] dA$$

For the double integral over R, we will integrate with respect to y first. Thus,

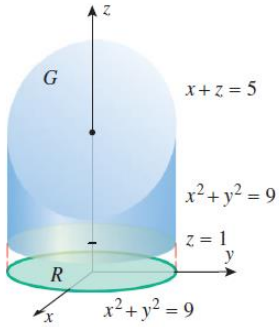
$$\text{volume of } G = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_1^{5-x} dz\,dy\,dx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\Big]_{z=1}^{5-x}\,dy\,dx$$

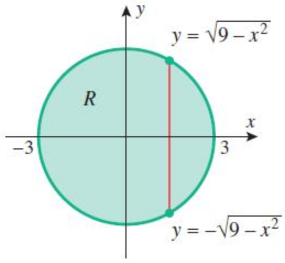
$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-x)\,dy\,dx = \int_{-3}^{3}(8-2x)\sqrt{9-x^2}\,dx$$

$$= 8\int_{-3}^{3}\sqrt{9-x^2}\,dx - \int_{-3}^{3}2x\sqrt{9-x^2}\,dx$$

$$= 8\left(\frac{9}{2}\pi\right) - \int_{-3}^{3}2x\sqrt{9-x^2}\,dx$$

$$= 8\left(\frac{9}{2}\pi\right) - 0 = 36\pi \blacktriangleleft$$





Example 4.18 Find the volume of the solid enclosed between the paraboloids $z = 5x^2 + 5y^2$ and $z = 6 - 7x^2 - y^2$

Solution: The solid G and its projection R on the xy-plane are shown in Figure. The projection R is obtained by solving the given equations simultaneously to determine where the paraboloids intersect. We obtain

$$5x^2 + 5y^2 = 6 - 7x^2 - y^2$$

or

$$2x^2 + y^2 = 1$$

which tells us that the paraboloids intersect in a curve on the elliptic cylinder given by ($2x^2 + y^2 = 1$). The projection of this intersection on the xy-plane is an ellipse with this same equation. Therefore,

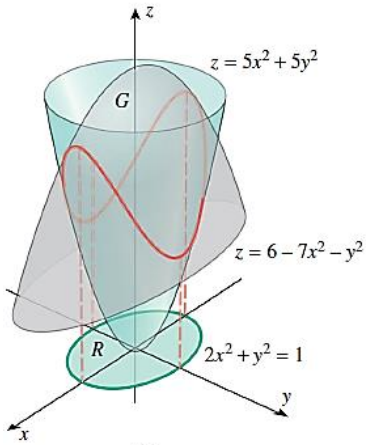
$$\text{volume of } G = \iiint_G dV = \iint_R \left[\int_{5x^2+5y^2}^{6-7x^2-y^2} dz\right] dA$$

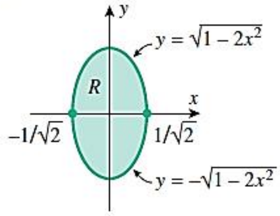
$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} dz\, dy\, dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (6 - 12x^2 - 6y^2)\, dy\, dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[6(1-2x^2)y - 2y^3\right]_{y=-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx$$

$$= 8\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2}\, dx = \frac{8}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta = \frac{3\pi}{\sqrt{2}}$$





CHAPTER THREE

PARTIAL DERIVATIVES

3.1 FUNCTIONS OF TWO OR MORE VARIABLES

3.1.1 Notation and Terminology

There are many familiar formulas in which a given variable depends on two or more other variables. For example, the area A of a triangle depends on the base length b and height h by the formula $A = \frac{1}{2}bh$; the volume V of a rectangular box depends on the length l, the width w, and the height h by the formula $V = lwh$; and the arithmetic average $\bar{x}$ of n real numbers, x_1, $x_2, \ldots, x_n$, depends on those numbers by the formula

$$\bar{x} = \frac{1}{n}(x_1 + \ x_2, + \cdots + x_n)$$

Thus, we say that

A is a function of the two variables b and h;

V is a function of the three variables l, w, and h;

$\bar{x}$ is a function of the n variables $x_1, x_2, \ldots, x_n$.

The terminology and notation for functions of two or more variables is similar to that for functions of one variable. For example, the expression

$$z = f(x, y)$$

means that z is a function of x and y in the sense that a unique value of the dependent variable z is determined by specifying values for the independent variables x and y. Similarly,

$$w = f(x, y, z)$$

expresses w as a function of x, y, and z, and

$$u = f(x_1, x_2, \ldots, x_n)$$

expresses u as a function of $x_1, x_2, \ldots, x_n$.

As with functions of one variable, the independent variables of a function of two or more variables may be restricted to lie in some set D, which we call the ***domain*** of f.

The domain consists of all points for which the formula yields a real value for the dependent variable. We call this the ***natural domain*** of the function.

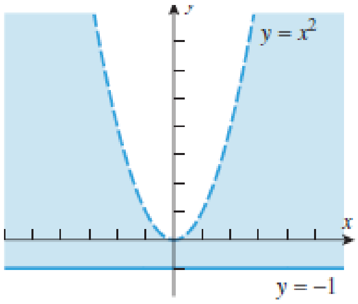
Definition 3.1 A ***function f of two variables***, *x* and *y*, is a rule that assigns a unique real number $f(x, y)$ to each point (x, y) in some set D in the xy-plane.

Definition 3.2 A ***function f of three variables***, *x*, *y*, and *z*, is a rule that assigns a unique real number $f(x, y, z)$ to each point (x, y, z) in some set D in three dimensional space.

Example 3.1 Let $f(x, y) = \sqrt{y+1} + \ln(x^2 - y)$. Find $f(e, 0)$ and sketch the natural

Solution: By substitution,

$$f(e,0) = \sqrt{0+1} + \ln(e^2 - 0) = \sqrt{1} + \ln(e^2) = 1 + 2 = 3$$



To find the natural domain of f, we note that $\sqrt{y} + 1$ is defined only when $y \geq -1$, while $\ln(x^2 - y)$ is defined only when $0 < x^2 - y$ or $y < x^2$. Thus, the natural domain of f consists of all points in the xy-plane for which $-1 \leq y < x^2$. To sketch the natural domain, we first sketch the parabola $y = x^2$ as a "dashed" curve and the line $y = -1$ as a solid curve. The natural domain of f is then the region lying above or on the line $y = -1$ and below the parabola $y = x^2$.

Example 3.2 Let $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$ Find $f(0,1/2,-1/2)$ and the natural domain of f.

Solution: By substitution,

$$f\left(0, \tfrac{1}{2}, -\tfrac{1}{2}\right) = \sqrt{1 - (0)^2 - \left(\tfrac{1}{2}\right)^2 - \left(-\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{2}}$$

Because of the square root sign, we must have $0 \leq 1 - x^2 - y^2 - z^2$ in order to have a real value for f(x, y, z). Rewriting this inequality in the form

$$x^2 + y^2 + z^2 \leq 1$$

We see that the natural domain of f consists of all points on or within the sphere

$$x^2 + y^2 + z^2 = 1$$

3.1.2 Graphs of Functions of Two Variables

Example 3.3 In each part, describe the graph of the function in an *xyz*-coordinate system.

(a) $f(x, y) = 1 - x - \frac{1}{2}y$ (b) $f(x, y) = \sqrt{1 - x^2 - y^2}$

(c) $f(x, y) = -\sqrt{x^2 + y^2}$

Solution (a): By definition, the graph of the given function is the graph of the equation

$$z = 1 - x - \frac{1}{2}y \qquad (1)$$

which is a plane. A triangular portion of the plane can be sketched by plotting the intersections with the coordinate axes and joining them with line segments (Figure *a*).

Solution (b): By definition, the graph of the given function is the graph of the equation

$$z = \sqrt{1 - x^2 - y^2} \qquad (2)$$

After squaring both sides, this can be rewritten as

$$x^2 + y^2 + z^2 = 1$$

which represents a sphere of radius 1, centered at the origin. Since (2) imposes the added condition that $z \geq 0$, the graph is just the upper hemisphere (Figure *b*).

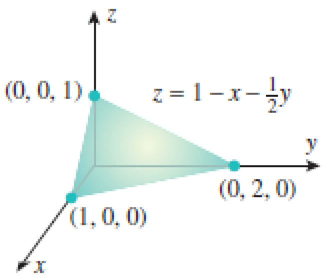
Solution (c): The graph of the given function is the graph of the equation

$$z = -\sqrt{x^2 + y^2} \qquad (3)$$

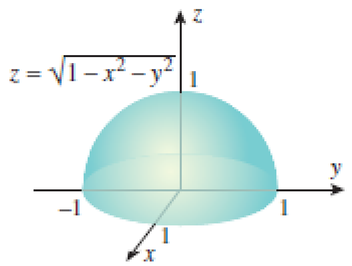
After squaring, we obtain

$$z^2 = x^2 + y^2$$

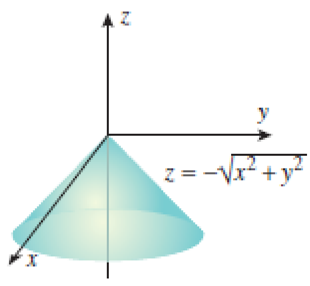
which is the equation of a circular cone. Since (3) imposes the condition that $z \leq 0$, the graph is just the lower nappe of the cone (Figure *c*).



(*a*)

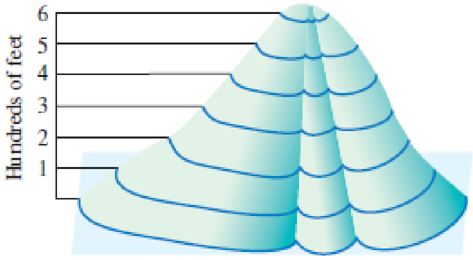


(*b*)

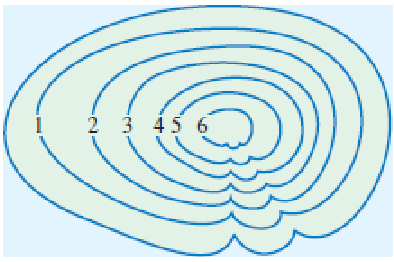


(*c*)

3.1.3 Level Curves

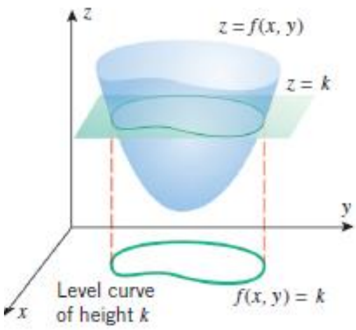


A perspective view of a model hill with two gullies



A contour map of the model hill

Contour maps are also useful for studying functions of two variables. If the surface $z = f(x, y)$ is cut by the horizontal plane $z = k$, then at all points on the intersection we have $f(x, y) = k$. The projection of this intersection onto the xy-plane is called the ***level curve of height k*** or the ***level curve with constant k*** (below figure). A set of level curves for $z = f(x, y)$ is called a ***contour plot*** or ***contour map*** of f.

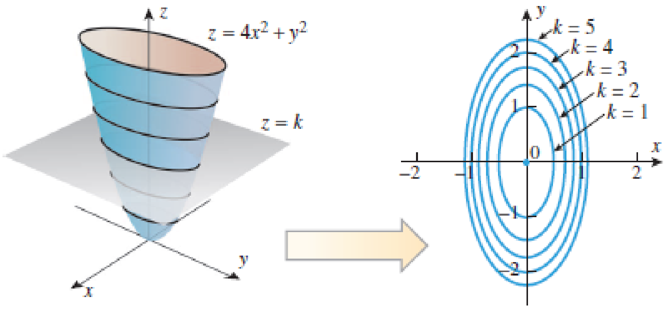


Example 3.4 Sketch the contour plot of $f(x, y) = 4x^2 + y^2$ using level curves of height $k = 0, 1, 2, 3, 4, 5$.

Solution: The graph of the surface $z = 4x^2 + y^2$ is the paraboloid shown in the left part of the below figure, so we can reasonably expect the contour plot to be a family of ellipses centered at the origin. The level curve of height k has the equation $4x^2 + y^2 = k$. If $k = 0$, then the graph is the single point $(0, 0)$. For $k > 0$ we can rewrite the equation as

$$\frac{x^2}{k/4} + \frac{y^2}{k} = 1$$

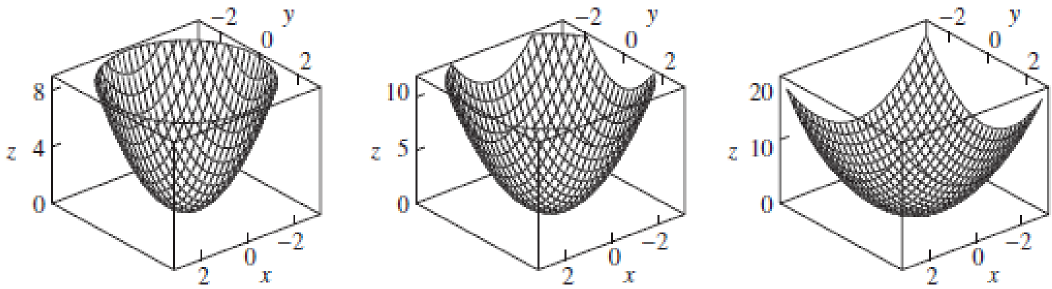
which represents a family of ellipses with x-intercepts $\pm\sqrt{k}/2$ and y-intercepts $\pm\sqrt{k}$. The contour plot for the specified values of k is shown in the right part of the following figure.



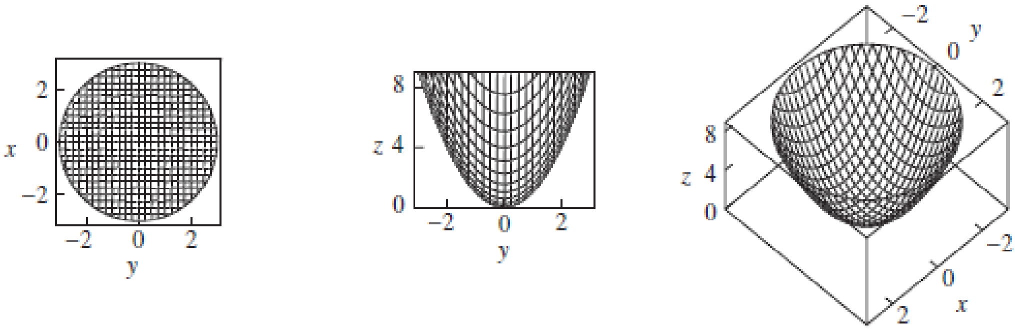
3.1.4 Graphing Functions Using Technology

Graphing utilities can only show a portion of xyz-space in a viewing screen, so the first step in graphing a surface is to determine which portion of xyz-space you want to display. This region is called the ***viewing box*** or ***viewing window***.

For example, the following figure shows the graph of the paraboloid $z = x^2 + y^2$ from three different viewpoints using the first viewing box.



Varying the viewing box.



Varying the viewpoint.

3.2 LIMITS AND CONTINUITY

3.2.1 Limits along Curves

For a function of one variable there are two one-sided limits at a point x_0, namely,

$$\lim_{x \to x_0^+} f(x) \quad \text{and} \quad \lim_{x \to x_0^-} f(x)$$

reflecting the fact that there are only two directions from which x can approach x_0, the right or the left.

For functions of two or three variables the situation is more complicated because there are infinitely many different curves along which one point can approach another.

The limit of f(x, y) as (x, y) approaches a point (x_0, y_0) along a curve C (and similarly for functions of three variables).

If C is a smooth parametric curve in 2-space or 3-space that is represented by the equations

$$x = x(t),\ y = y(t) \text{ or } x = x(t),\ y = y(t),\ z = z(t)$$

and if $x_0 = x(t_0)$, $y_0 = y(t_0)$, and $z_0 = z(t_0)$, then the limits

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (\text{along } C)}} f(x,y) \quad \text{and} \quad \lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ (\text{along } C)}} f(x,y,z)$$

are defined by

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (\text{along } C)}} f(x,y) = \lim_{t\to t_0} f(x(t), y(t)) \qquad (1)$$

$$\lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ (\text{along } C)}} f(x,y,z) = \lim_{t\to t_0} f(x(t), y(t), z(t)) \qquad (2)$$

In these formulas the limit of the function of t must be treated as a one-sided limit if (x_0, y_0) or (x_0, y_0, z_0) is an endpoint of C.

Example 3.5 below figure shows a computer-generated graph of the function

$$f(x,y) = -\frac{xy}{x^2 + y^2}$$

The graph reveals that the surface has a ridge above the line $y = -x$, which is to be expected since $f(x, y)$ has a constant value of 1/2 for $y = -x$, except at $(0, 0)$ where f is undefined (verify). Moreover, the graph suggests that the limit of $f(x, y)$ as $(x, y)\to(0, 0)$ along a line through the origin varies with the direction of the line. Find this limit along

(a) the x-axis (b) the y-axis (c) the line $y = x$ (d) the line $y = -x$ (e) the parabola $y = x^2$

Solution (a): The x-axis has parametric equations $x = t$, $y = 0$, with (0, 0) corresponding to t = 0, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=0)}} f(x,y) = \lim_{t\to 0} f(t,0) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

Solution (b): The y-axis has parametric equations $x = 0$, $y = t$, with (0, 0) corresponding to t = 0, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } x=0)}} f(x,y) = \lim_{t\to 0} f(0,t) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

Solution (c): The line $y = x$ has parametric equations $x = t$, $y = t$, with (0, 0) corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x)}} f(x,y) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0}\left(-\frac{t^2}{2t^2}\right) = \lim_{t\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

Solution (d): The line $y = -x$ has parametric equations $x = t$, $y = -t$, with (0, 0) corresponding to t = 0, so

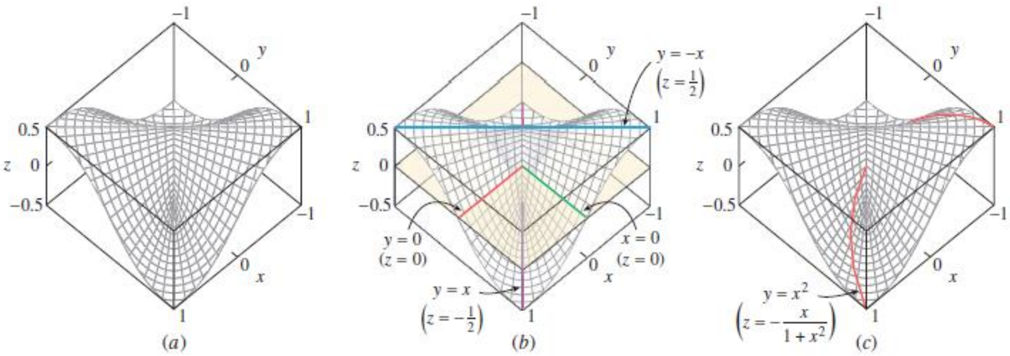
$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=-x)}} f(x,y) = \lim_{t\to 0} f(t,-t) = \lim_{t\to 0}\frac{t^2}{2t^2} = \lim_{t\to 0}\frac{1}{2} = \frac{1}{2}$$

Solution (e): The parabola $y = x^2$ has parametric equations $x = t$, $y = t^2$, with (0, 0) corresponding to t = 0, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x^2)}} f(x,y) = \lim_{t\to 0} f(t,t^2) = \lim_{t\to 0}\left(-\frac{t^3}{t^2+t^4}\right) = \lim_{t\to 0}\left(-\frac{t}{1+t^2}\right) = 0$$

This is consistent with Figure c, which shows the parametric curve

$$x = t, \quad y = t^2, \quad z = -\frac{t}{1+t^2}$$



3.2.2 Open and Closed Sets

Let C be a circle in 2-space that is centered at (x_0, y_0) and has positive radius δ.

-The set of points that are enclosed by the circle, but do not lie on the circle, is called the ***open disk.***

- The set of points that lie on the circle together with those enclosed by the circle is called the ***closed disk.***

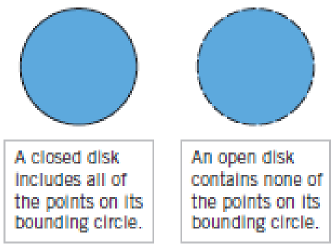
- If S is a sphere in 3-space that is centered at (x_0, y_0, z_0) and has positive radius δ:

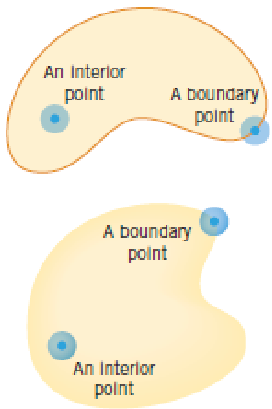
-The set of points that are enclosed by the sphere, but do not lie on the sphere, is called the ***open ball***

-The set of points that lie on the sphere together with those enclosed by the sphere is called the ***closed ball***.

- If D is a set of points in 2-space, then a point (x_0, y_0) is called an ***interior point*** of D if there is *some* open disk centered at (x_0, y_0) that contains only points of D,

- (x_0, y_0) is called a ***boundary point*** of D if *every* open disk centered at (x_0, y_0) contains both points in D and points not in D.





3.2.3 General Limits of Functions of Two Variables

Definition Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0), except possibly at (x_0, y_0). We will write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

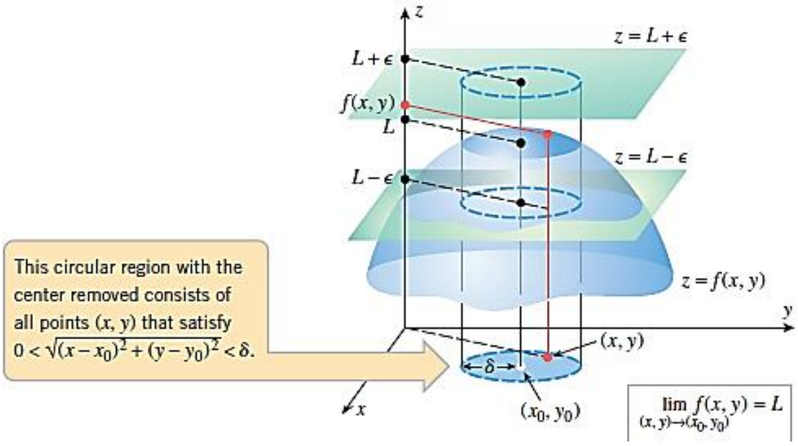
if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y)$ satisfies

$$|f(x, y) - L| < \epsilon$$

whenever the distance between (x, y) and (x_0, y_0) satisfies

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

As in below figure, this figure is intended to convey the idea that the values of $f(x, y)$ can be forced within ϵ units of L on the z-axis by restricting (x, y) to lie within δ units of (x_0, y_0) in the xy-plane. We used a white dot at (x_0, y_0) to suggest that the epsilon condition need not hold at this point.



Example 3.6

$$\lim_{(x,y)\to(1,4)} [5x^3y^2 - 9] = \lim_{(x,y)\to(1,4)} [5x^3y^2] - \lim_{(x,y)\to(1,4)} 9$$
$$= 5\left[\lim_{(x,y)\to(1,4)} x\right]^3 \left[\lim_{(x,y)\to(1,4)} y\right]^2 - 9$$
$$= 5(1)^3(4)^2 - 9 = 71$$

3.2.4 Relationships between General Limits and Limits along Smooth Curves

Theorem

(a) If $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$, then $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$ along any smooth curve.

(b) If the limit of $f(x, y)$ fails to exist as $(x, y) \to (x_0, y_0)$ along some smooth curve, or if $f(x, y)$ has different limits as $(x, y) \to (x_0, y_0)$ along two different smooth curves, then the limit of $f(x, y)$ does not exist as $(x, y) \to (x_0, y_0)$.

Example 3.7 The limit

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2 + y^2}$$

does not exist because in previous example we found two different smooth curves along which this limit had different values. Specifically,

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } x=0)}} -\frac{xy}{x^2+y^2} = 0 \quad \text{and} \quad \lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x)}} -\frac{xy}{x^2+y^2} = -\frac{1}{2}$$

3.2.5 CONTINUITY

Definition

A function $f(x, y)$ is said to be ***continuous at*** (x_0, y_0) if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

In addition, if f is continuous at every point in an open set D, then we say that f is ***continuous on D***, and if f is continuous at every point in the xy-plane, then we say that f is ***continuous everywhere***.

Theorem

(a) If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0).

(b) If $h(x, y)$ is continuous at (x_0, y_0) and $g(u)$ is continuous at $u = h(x_0, y_0)$, then the composition $f(x, y) = g(h(x, y))$ is continuous at (x_0, y_0).

(c) If $f(x, y)$ is continuous at (x_0, y_0), and if $x(t)$ and $y(t)$ are continuous at t_0 with $x(t_0) = x_0$ and $y(t_0) = y_0$, then the composition $f(x(t), y(t))$ is continuous at t_0.

Example 3.8 Use Theorem to show that the functions $f(x, y) = 3x^2y^5$ and $f(x, y) = \sin(3x^2y^5)$ are continuous everywhere.

Solution: The polynomials $g(x) = 3x^2$ and $h(y) = y^5$ are continuous at every real number, and therefore by part (*a*) of Theorem, the function $f(x, y) = 3x^2y^5$ is continuous at every point (x, y) in the xy-plane. Since $3x^2y^5$ is continuous at every point in the xy-plane and $\sin u$ is continuous at every real number u, it follows from part (*b*) of Theorem that the composition $f(x, y) = \sin(3x^2y^5)$ is continuous everywhere.

Recognizing Continuous Functions

- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

Example 3.9 Evaluate

$$\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2 + y^2}$$

Solution: Since $f(x, y) = xy/(x^2 + y^2)$ is continuous at $(-1, 2)$ (why?), it follows from the definition of continuity for functions of two variables that

$$\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2} = \frac{(-1)(2)}{(-1)^2+(2)^2} = -\frac{2}{5}$$

Example 3.10 Since the function

$$f(x, y) = \frac{x^3y^2}{1-xy}$$

is a quotient of continuous functions, it is continuous except where $1 - xy = 0$. Thus, $f(x, y)$ is continuous everywhere except on the hyperbola $xy = 1$.

3.2.6 Limits at Discontinuities

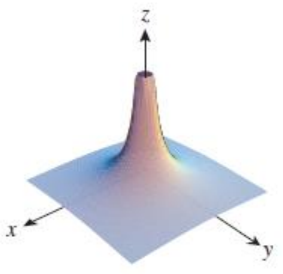
Sometimes it is easy to recognize when a limit does not exist. For example, it is evident that

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = +\infty$$

which implies that the values of the function approach $+\infty$ as $(x, y)\to(0, 0)$ along any smooth curve (below figure). However, it is not evident whether the limit

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

exists because it is an indeterminate form of type $0 \cdot \infty$. Although L'Hôpital's rule cannot be applied directly, the following example illustrates a method for finding this limit by converting to polar coordinates.



$$z = \frac{1}{x^2+y^2}$$

Example 3.11 Find

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

Solution: Let (r, θ) be polar coordinates of the point (x, y) with $r \geq 0$. Then we have

$$x = r\cos\theta,\ y = r\sin\theta,\ r^2 = x^2 + y^2$$

Moreover, since $r \geq 0$ we have $r = \sqrt{x^2 + y^2}$, so that $r \to 0^+$ if and only if $(x, y) \to (0, 0)$. Thus, we can rewrite the given limit as

$$\lim_{(x,y)\to(0,0)} (x^2 + y^2)\ln(x^2 + y^2) = \lim_{r\to 0^+} r^2 \ln r^2$$

$$= \lim_{r\to 0^+} \frac{2\ln r}{1/r^2} \quad \text{This converts the limit to an indeterminate form of type } \infty/\infty.$$

$$= \lim_{r\to 0^+} \frac{2/r}{-2/r^3} \quad \text{L'Hôpital's rule}$$

$$= \lim_{r\to 0^+} (-r^2) = 0$$

3.2.7 Continuity at Boundary Points

Recall that in our study of continuity for functions of one variable, we first defined continuity at a point, then continuity on an open interval, and then, by using one-sided limits, we extended the notion of continuity to include the boundary points of the interval. Similarly, for functions of two variables one can extend the notion of continuity of $f(x, y)$ to the boundary of its domain by modifying previous definition appropriately so that (x, y) is restricted to approach (x_0, y_0) through points lying wholly in the domain of f.

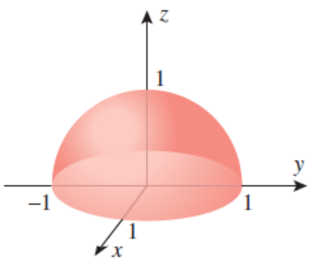
Example 3.12 The graph of the function $f(x,y) = \sqrt{1 - x^2 - y^2}$ is the upper hemisphere shown in below figure, and the natural domain of f is the closed unit disk

$$x^2 + y^2 \leq 1$$

The graph of f has no tears or holes, so it passes our "intuitive test" of continuity. In this case the continuity at a point (x_0, y_0) on the boundary reflects the fact that

$$\lim_{(x,y)\to(x_0,y_0)} \sqrt{1 - x^2 - y^2} = \sqrt{1 - x_0^2 - y_0^2} = 0$$

when (x, y) is restricted to points on the closed unit disk $x^2 + y^2 \leq 1$. It follows that f is continuous on its domain.



$z = \sqrt{1 - x^2 - y^2}$

3.2.8 Extensions to Three Variables

Definition Let f be a function of three variables, and assume that f is defined at all points within a ball centered at (x_0, y_0, z_0), except possibly at (x_0, y_0, z_0). We will write

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y, z)$ satisfies

$$|f(x,y,z) - L| < \epsilon$$

whenever the distance between (x, y, z) and (x_0, y_0, z_0) satisfies

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta$$

As with functions of one and two variables, we define a function $f(x, y, z)$ of three variables to be continuous at a point (x_0, y_0, z_0) if the limit of the function and the value of the function are the same at this point; that is,

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0, y_0, z_0)$$

3.3 PARTIAL DERIVATIVES

3.3.1 Partial Derivatives of Functions of Two Variables

Definition

If $z = f(x, y)$ and (x_0, y_0) is a point in the domain of f, then the ***partial derivative of f with respect to x*** at (x_0, y_0) [also called the ***partial derivative of z with respect to x*** at (x_0, y_0)] is the derivative at x_0 of the function that results when $y = y_0$ is held fixed and x is allowed to vary. This partial derivative is denoted by $f_x(x_0, y_0)$ and is given by

$$f_x(x_0, y_0) = \frac{d}{dx}[f(x, y_0)]\bigg|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \qquad (1)$$

Similarly, the ***partial derivative of f with respect to y*** at (x_0, y_0) [also called the ***partial derivative of z with respect to y*** at (x_0, y_0)] is the derivative at y_0 of the function that results when $x = x_0$ is held fixed and y is allowed to vary. This partial derivative is denoted by $f_y(x_0, y_0)$ and is given by

$$f_y(x_0, y_0) = \frac{d}{dy}[f(x_0, y)]\bigg|_{y=y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \qquad (2)$$

Example 3.13 Find $f_x(1, 3)$ and $f_y(1, 3)$ for the function $f(x, y) = 2x^3y^2 + 2y + 4x$.

Solution: Since

$$f_x(x, 3) = \frac{d}{dx}[f(x, 3)] = \frac{d}{dx}[18x^3 + 4x + 6] = 54x^2 + 4$$

we have $f_x(1, 3) = 54 + 4 = 58$. Also, since

$$f_y(1, y) = \frac{d}{dy}[f(1, y)] = \frac{d}{dy}[2y^2 + 2y + 4] = 4y + 2$$

we have $f_y(1, 3) = 4(3) + 2 = 14$.

3.3.2 The Partial Derivative Functions

Formulas (1) and (2) define the partial derivatives of a function at a specific point (x_0, y_0). However, often it will be desirable to omit the subscripts and think of the partial derivatives as functions of the variables x and y. These functions are

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \qquad f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Example 3.14 Find $f_x(x, y)$ and $f_y(x, y)$ for $f(x, y) = 2x^3y^2 + 2y + 4x$, and use those partial derivatives to compute $f_x(1, 3)$ and $f_y(1, 3)$.

Solution: Keeping y fixed and differentiating with respect to x yields

$$f_x(x, y) = \frac{d}{dx}[2x^3y^2 + 2y + 4x] = 6x^2y^2 + 4$$

and keeping x fixed and differentiating with respect to y yields

$$f_y(x, y) = \frac{d}{dy}[2x^3y^2 + 2y + 4x] = 4x^3y + 2$$

$$f_x(1, 3) = 6(1^2)(3^2) + 4 = 58 \quad \text{and} \quad f_y(1, 3) = 4(1^3)3 + 2 = 14$$

3.3.3 Partial Derivative Notation

If $z = f(x, y)$, then the partial derivatives f_x and f_y are also denoted by the symbols

$$\frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}, \quad \frac{\partial z}{\partial y}$$

Some typical notations for the partial derivatives of $z = f(x, y)$ at a point (x_0, y_0) are

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0, y=y_0}, \quad \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}, \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}, \quad \frac{\partial f}{\partial x}(x_0, y_0), \quad \frac{\partial z}{\partial x}(x_0, y_0)$$

Example 3.15 Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = x^4 \sin(xy^3)$.

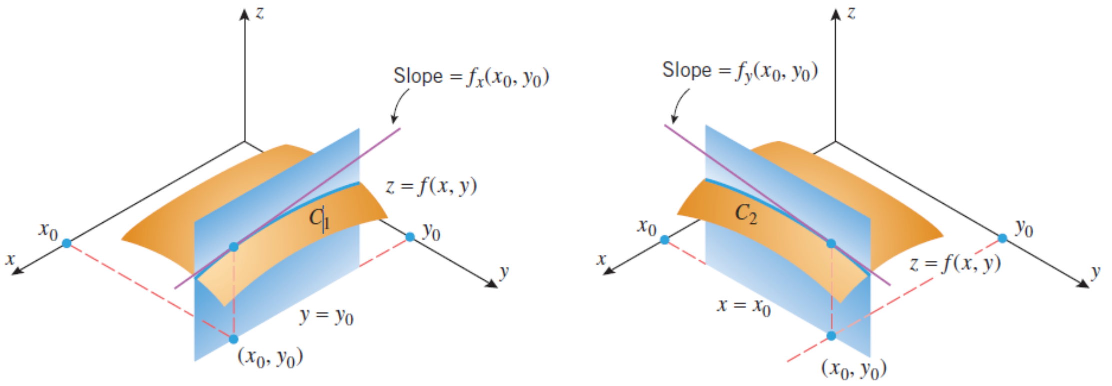
Solution:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial x}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial x}(x^4) \\ &= x^4 \cos(xy^3) \cdot y^3 + \sin(xy^3) \cdot 4x^3 = x^4y^3 \cos(xy^3) + 4x^3 \sin(xy^3) \\ \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial y}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial y}(x^4) \\ &= x^4 \cos(xy^3) \cdot 3xy^2 + \sin(xy^3) \cdot 0 = 3x^5y^2 \cos(xy^3)\end{aligned}$$

3.3.4 Partial Derivatives Viewed As Rates of Change and Slopes

Recall that if $y = f(x)$, then the value of $f(x_0)$ can be interpreted either as the rate of change of y with respect to x at x_0 or as the slope of the tangent line to the graph of f at x_0. Partial derivatives have analogous interpretations. To see that this is so, suppose that C_1 is the intersection of the surface $z = f(x, y)$ with the plane $y = y_0$ and that C_2 is its intersection with the plane $x = x_0$ (below figure). Thus, $fx(x, y_0)$ can be interpreted as the rate of change of z

with respect to x along the curve C_1, and $f_y(x_0, y)$ can be interpreted as the rate of change of z with respect to y along the curve C_2. In particular, $f_x(x_0, y_0)$ is the rate of change of z with respect to x along the curve C_1 at the point (x_0, y_0), and $f_y(x_0, y_0)$ is the rate of change of z with respect to y along the curve C_2 at the point (x_0, y_0).



Example 3.16 Let $f(x, y) = x^2y + 5y^3$.

(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point $(1, -2)$.

(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point $(1, -2)$.

Solution (a): Differentiating f with respect to x with y held fixed yields

$$f_x(x, y) = 2xy$$

Thus, the slope in the x-direction is $f_x(1,-2) = -4$; that is, z is decreasing at the rate of 4 units per unit increase in x.

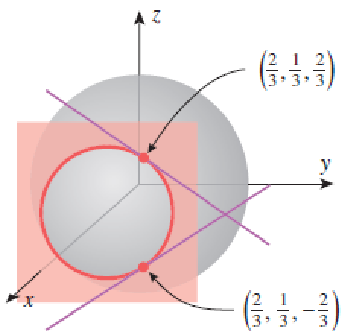
Solution (b): Differentiating f with respect to y with x held fixed yields

$$f_y(x, y) = x^2 + 15y^2$$

Thus, the slope in the y-direction is $f_y(1,-2) = 61$; that is, z is increasing at the rate of 61 units per unit increase in y

3.3.5 Implicit Partial Differentiation

Example 3.17 Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y-direction at the points (2/3, 1/3, 2/3) and (2/3, 1/3, -2/3) (see figure).



Solution: The point (2/3, 1/3, 2/3) lies on the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$, and the point (2/3, 1/3, -2/3) lies on the lower hemisphere $z = -\sqrt{1 - x^2 - y^2}$. We could find the slopes by differentiating each expression for z separately with respect to y and then evaluating the derivatives at $x = 2/3$ and $y = 1/3$. However, it is more efficient to differentiate the given equation

$$x^2 + y^2 + z^2 = 1$$

To perform the implicit differentiation, we view z as a function of x and y and differentiate both sides with respect to y, taking x to be fixed. The computations are as follows:

$$\frac{\partial}{\partial y}[x^2+y^2+z^2]=\frac{\partial}{\partial y}[1]$$
$$0+2y+2z\frac{\partial z}{\partial y}=0$$
$$\frac{\partial z}{\partial y}=-\frac{y}{z}$$

Substituting the y- and z-coordinates of the points (2/3, 1/3, 2/3) and (2/3, 1/3, -2/3) in this expression, we find that the slope at the point (2/3, 1/3, 2/3) is -1/2 and the slope at (2/3, 1/3, -2/3) is ½.

3.3.6 Partial Derivatives and Continuity

In contrast to the case of functions of a single variable, the existence of partial derivatives for a multivariable function does not guarantee the continuity of the function. This fact is shown in the following example.

Example 3.18 Let

$$f(x,y)=\begin{cases}-\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

(a) Show that $f_x(x, y)$ and $f_y(x, y)$ exist at all points (x, y).

(b) Explain why f is not continuous at $(0, 0)$.

Solution (a):

Except that here we have assigned f a value of 0 at $(0, 0)$. Except at this point, the partial derivatives of f are

$$f_x(x,y)=-\frac{(x^2+y^2)y-xy(2x)}{(x^2+y^2)^2}=\frac{x^2y-y^3}{(x^2+y^2)^2}$$
$$f_y(x,y)=-\frac{(x^2+y^2)x-xy(2y)}{(x^2+y^2)^2}=\frac{xy^2-x^3}{(x^2+y^2)^2}$$

It is not evident from previous formula whether f has partial derivatives at $(0, 0)$, and if so, what the values of those derivatives are. To answer that question we will have to use the definitions of the partial derivatives (Definition). Applying previous formulas and we obtain

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$
$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

Solution (b):

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2+y^2}$$

does not exist. Thus, f is not continuous at $(0, 0)$.

3.3.7 Partial Derivatives of Functions with More Than Two Variables

For a function $f(x, y, z)$ of three variables, there are three ***partial derivatives***:

$$f_x(x, y, z),\ f_y(x, y, z),\ f_z(x, y, z)$$

The partial derivative f_x is calculated by holding y and z constant and differentiating with respect to x. For f_y the variables x and z are held constant, and for f_z the variables x and y are held constant. If a dependent variable

$$w = f(x, y, z)$$

is used, then the three partial derivatives of f can be denoted by

$$\frac{\partial w}{\partial x},\quad \frac{\partial w}{\partial y},\quad \text{and}\quad \frac{\partial w}{\partial z}$$

Example 3.18

If $f(x, y, z) = x^3y^2z^4 + 2xy + z$, then

$$f_x(x, y, z) = 3x^2y^2z^4 + 2y$$
$$f_y(x, y, z) = 2x^3yz^4 + 2x$$
$$f_z(x, y, z) = 4x^3y^2z^3 + 1$$
$$f_z(-1, 1, 2) = 4(-1)^3(1)^2(2)^3 + 1 = -31$$

3.3.8 Higher-Order Partial Derivatives

Suppose that f is a function of two variables x and y. Since the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are also functions of x and y, these functions may themselves have partial derivatives. This gives rise to four possible ***second-order*** partial derivatives of f, which are defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Differentiate twice with respect to x. | Differentiate twice with respect to y.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

Differentiate first with respect to x and then with respect to y. | Differentiate first with respect to y and then with respect to x.

The last two cases are called the ***mixed second-order partial derivatives*** or the ***mixed second partials***. Also, the derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are often called the ***first-order partial derivatives*** when it is necessary to distinguish them from higher-order partial derivatives. Similar conventions apply to the second-order partial derivatives of a function of three variables.

Example 3.20 Find the second-order partial derivatives of $f(x, y) = x^2y^3 + x^4y$.

Solution: We have

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy^3 + 4x^3y) = 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2y^2 + x^4) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2 + x^4) = 6xy^2 + 4x^3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy^3 + 4x^3y) = 6xy^2 + 4x^3$$

Third-order, fourth-order, and higher-order partial derivatives can be obtained by successive differentiation. Some possibilities are

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = f_{xxx} \qquad \frac{\partial^4 f}{\partial y^4} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y^3}\right) = f_{yyyy}$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = f_{xyy} \qquad \frac{\partial^4 f}{\partial y^2 \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y \partial x^2}\right) = f_{xxyy}$$

3.3.9 Equality of Mixed Partials

Theorem *Let f be a function of two variables. If f_{xy} and f_{yx} are continuous on some open disk, then $f_{xy} = f_{yx}$ on that disk.*

3.4 DIFFERENTIABILITY, DIFFERENTIALS, AND LOCAL LINEARITY

3.4.1 Differentiability

Recall that a function f of one variable is called differentiable at x_0 if it has a derivative at x_0, that is, if the limit

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \qquad (1)$$

exists. As a consequence of (1) a differentiable function enjoys a number of other important properties:

• The graph of $y = f(x)$ has a non-vertical tangent line at the point $(x_0, f(x_0))$;

• f may be closely approximated by a linear function near x_0;

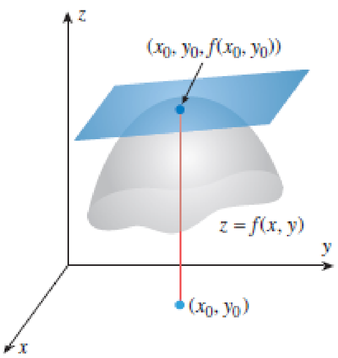
• f is continuous at x_0.

Our primary objective in this section is to extend the notion of differentiability to functions of two or three variables in such a way that the natural analogs of these properties hold. For example, if a function $f(x, y)$ of two variables is differentiable at a point (x_0, y_0), we want it to be the case that

• the surface $z = f(x, y)$ has a non-vertical tangent plane at the point $(x_0, y_0, f(x_0, y_0))$ (see below figure);

• the values of f at points near (x_0, y_0) can be very closely approximated by the values of a linear function;

• f is continuous at (x_0, y_0).



Definition A function f of two variables is said to be ***differentiable*** at (x_0, y_0) provided $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist and

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0 \qquad (4)$$

Example 3.21 Use Definition prove that $f(x, y) = x^2 + y^2$ is differentiable at (0, 0).

Solution: The increment is

$$\Delta f = f(0+\Delta x, 0+\Delta y) - f(0,0) = (\Delta x)^2 + (\Delta y)^2$$

Since $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, we have $f_x(0,0) = f_y(0,0) = 0$, and (4) becomes

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \frac{(\Delta x)^2 + (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim_{(\Delta x, \Delta y)\to(0,0)} \sqrt{(\Delta x)^2 + (\Delta y)^2} = 0$$

Therefore, f is differentiable at $(0, 0)$.

We now derive an important consequence of limit (4). Define a function

$$\epsilon = \epsilon(\Delta x, \Delta y) = \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \quad \text{for } (\Delta x, \Delta y) \neq (0,0)$$

and define $\epsilon(0,0)$ to be 0. Equation (4) then implies that

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \epsilon(\Delta x, \Delta y) = 0$$

Furthermore, it immediately follows from the definition of ϵ that

$$\Delta f = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon\sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{5}$$

In other words, if f is differentiable at (x_0, y_0), then Δf may be expressed as shown in (5), where $\epsilon \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$ and where $\epsilon = 0$ if $(\Delta x, \Delta y) = (0, 0)$.

For functions of three variables we have an analogous definition of differentiability in terms of the increment $\Delta f = f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0)$.

Definition A function f of three variables is said to be ***differentiable*** at (x_0, y_0, z_0) provided $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$, and $f_z(x_0, y_0, z_0)$ exist and

$$\lim_{(\Delta x, \Delta y, \Delta z)\to(0,0,0)} \frac{\Delta f - f_x(x_0, y_0, z_0)\Delta x - f_y(x_0, y_0, z_0)\Delta y - f_z(x_0, y_0, z_0)\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}} = 0$$

3.4.2 Differentiability and Continuity

Theorem *If a function is differentiable at a point, then it is continuous at that point.*

Theorem *If all first-order partial derivatives of f exist and are continuous at a point, then f is differentiable at that point.*

3.4.3 Differentials

As with the one-variable case, the approximations

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

for a function of two variables and the approximation

$$\Delta f \approx f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z \quad (1)$$

for a function of three variables have a convenient formulation in the language of differentials. If $z = f(x, y)$ is differentiable at a point (x_0, y_0), we let

$$dz = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy \quad (2)$$

denote a new function with dependent variable dz and independent variables dx and dy. We refer to this function (also denoted df) as the ***total differential of z*** at (x_0, y_0) or as the ***total differential of f*** at (x_0, y_0). Similarly, for a function $w = f(x, y, z)$ of three variables

we have the ***total differential of w*** at (x_0, y_0, z_0),

$$dw = f_x(x_0, y_0, z_0)\,dx + f_y(x_0, y_0, z_0)\,dy + f_z(x_0, y_0, z_0)\,dz \quad (3)$$

which is also referred to as the ***total differential of f*** at (x_0, y_0, z_0). It is common practice to omit the subscripts and write Equations (2) and (3) as

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy \quad (4)$$

and

$$dw = f_x(x, y, z)\,dx + f_y(x, y, z)\,dy + f_z(x, y, z)\,dz \quad (5)$$

In the two-variable case, the approximation

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

can be written in the form

$$\Delta f \approx df \quad (6)$$

for $dx = \Delta x$ and $dy = \Delta y$. Equivalently, we can write approximation (6) as

$$\Delta z \approx dz \quad (7)$$

In other words, we can estimate the change Δz in z by the value of the differential dz where dx is the change in x and dy is the change in y. Furthermore, it follows from (4) that <u>if Δx and Δy are close to 0, then the magnitude of the error in approximation (7) will be much smaller than the distance $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ between (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$.</u>

Example 3.22 Use (7) to approximate the change in $z = xy^2$ from its value at $(0.5, 1.0)$ to its value at $(0.503, 1.004)$. Compare the magnitude of the error in this approximation with the distance between the points $(0.5, 1.0)$ and $(0.503, 1.004)$.

Solution: For $z = xy^2$ we have $dz = y^2\, dx + 2xy\, dy$. Evaluating this differential at $(x, y) = (0.5, 1.0)$, $dx = \Delta x = 0.503 - 0.5 = 0.003$, and $dy = \Delta y = 1.004 - 1.0 = 0.004$ yields

$$dz = 1.0^2(0.003) + 2(0.5)(1.0)(0.004) = 0.007$$

Since $z = 0.5$ at $(x, y) = (0.5, 1.0)$ and $z = 0.507032048$ at $(x, y) = (0.503, 1.004)$, we have

$$\Delta z = 0.507032048 - 0.5 = 0.007032048$$

and the error in approximating Δz by dz has magnitude

$$|dz - \Delta z| = |0.007 - 0.007032048| = 0.000032048$$

Since the distance between $(0.5, 1.0)$ and $(0.503, 1.004) = (0.5 + \Delta x, 1.0 + \Delta y)$ is

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(0.003)^2 + (0.004)^2} = \sqrt{0.000025} = 0.005$$

we have

$$\frac{|dz - \Delta z|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{0.000032048}{0.005} = 0.0064096 < \frac{1}{150}$$

3.4.4 Local Linear Approximations

If a function f is differentiable at a point, then it can be very closely approximated by a linear function near that point. For example, suppose that $f(x, y)$ is differentiable at the point (x_0, y_0). Then approximation can be written in the form

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

If we let $x = x_0 + \Delta x$ and $y = x_0 + \Delta y$, this approximation becomes

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \quad (1)$$

Since the error in this approximation is equal to the error in approximation (3), we conclude that for (x, y) close to (x_0, y_0), the error in (1) will be much smaller than the distance between these two points. When $f(x, y)$ is differentiable at (x_0, y_0) we get

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

and refer to $L(x, y)$ as the ***local linear approximation to f at*** (x_0, y_0).

Example 3.23 Let $L(x, y)$ denote the local linear approximation to $f(x,y) = \sqrt{x^2 + y^2}$ at the point $(3, 4)$. Compare the error in approximating

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$$

by $L(3.04, 3.98)$ with the distance between the points (3, 4) and (3.04, 3.98).

Solution: We have

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$$

with $f_x(3, 4) = \frac{3}{5}$ and $f_y(3, 4) = \frac{4}{5}$. Therefore, the local linear approximation to f at (3, 4) is given by

$$L(x, y) = 5 + \tfrac{3}{5}(x - 3) + \tfrac{4}{5}(y - 4)$$

Consequently,

$$f(3.04, 3.98) \approx L(3.04, 3.98) = 5 + \tfrac{3}{5}(0.04) + \tfrac{4}{5}(-0.02) = 5.008$$

Since

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2} \approx 5.00819$$

the error in the approximation is about $5.00819 - 5.008 = 0.00019$. This is less than $\frac{1}{200}$ of the distance

$$\sqrt{(3.04 - 3)^2 + (3.98 - 4)^2} \approx 0.045$$

between the points (3, 4) and (3.04, 3.98).

For a function $f(x, y, z)$ that is differentiable at (x_0, y_0, z_0), the local linear approximation is

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$

3.5 THE CHAIN RULE

3.5.1 Chain Rules for Derivatives

Theorem (***Chain Rules for Derivatives***) *If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

If each of the functions $x = x(t)$, $y = y(t)$, and $z = z(t)$ is differentiable at t, and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(t), y(t), z(t))$, then the function $w = f(x(t), y(t), z(t))$ is differentiable at t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y, z).

Example 3.24 Suppose that

$$z = x^2y, \quad x = t^2, \quad y = t^3$$

Use the chain rule to find dz/dt, and check the result by expressing z as a function of t and differentiating directly.

Solution: By the chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2xy)(2t) + (x^2)(3t^2)$$
$$= (2t^5)(2t) + (t^4)(3t^2) = 7t^6$$

Alternatively, we can express z directly as a function of t,

$$z = x^2y = (t^2)^2(t^3) = t^7$$

and then differentiate to obtain $dz/dt = 7t^6$. However, this procedure may not always be convenient.

Example 3.25 Suppose that

$$w = \sqrt{x^2 + y^2 + z^2}, \quad x = \cos\theta, \quad y = \sin\theta, \quad z = \tan\theta$$

Use the chain rule to find $dw/d\theta$ when $\theta = \pi/4$.

Solution:

$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\frac{dz}{d\theta}$$
$$= \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x)(-\sin\theta) + \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2y)(\cos\theta)$$
$$+ \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2z)(\sec^2\theta)$$

When $\theta = \pi/4$, we have

$$x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad z = \tan\frac{\pi}{4} = 1$$

Substituting $x = 1/\sqrt{2}, y = 1/\sqrt{2}, z = 1, \theta = \pi/4$ in the formula for $dw/d\theta$ yields

$$\left.\frac{dw}{d\theta}\right|_{\theta=\pi/4} = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2})\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2})\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(2)(2)$$
$$= \sqrt{2}$$

3.5.2 Chain Rules for Partial Derivatives

Theorem (*Chain Rules for Partial Derivatives*)

If x = x(u, v) and y = y(u, v) have first-order partial derivatives at the point (u, v), and if z = f(x, y) is differentiable at the point (x, y) = (x(u, v), y(u, v)), then z = f(x(u, v), y(u, v)) has first-order partial derivatives at the point (u, v) given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

If each function x = x(u, v), y = y(u, v), and z = z(u, v) has first-order partial derivatives at the point (u, v), and if the function w = f(x, y, z) is differentiable at the point (x, y, z) = (x(u, v), y(u, v), z(u, v)), then w = f(x(u, v), y(u, v), z(u, v)) has first-order partial derivatives at the point (u, v) given by

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \quad \text{and} \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

Example 3.26 Given that $z = e^{xy}$, $x = 2u + v$, $y = u/v$ find $\partial z/\partial u$ and $\partial z/\partial v$ using the chain rule.

Solution:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (ye^{xy})(2) + (xe^{xy})\left(\frac{1}{v}\right) = \left[2y + \frac{x}{v}\right]e^{xy}$$

$$= \left[\frac{2u}{v} + \frac{2u+v}{v}\right]e^{(2u+v)(u/v)} = \left[\frac{4u}{v} + 1\right]e^{(2u+v)(u/v)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (ye^{xy})(1) + (xe^{xy})\left(-\frac{u}{v^2}\right)$$

$$= \left[y - x\left(\frac{u}{v^2}\right)\right]e^{xy} = \left[\frac{u}{v} - (2u+v)\left(\frac{u}{v^2}\right)\right]e^{(2u+v)(u/v)}$$

$$= -\frac{2u^2}{v^2}e^{(2u+v)(u/v)}$$

3.5.3 Implicit Differentiation

Theorem *If the equation f(x, y) = c defines y implicitly as a differentiable function of x, and if* $\partial f/\partial y \neq 0$, *then*

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

Example 3.27 Given that $x^3 + y^2x - 3 = 0$

find *dy/dx using the above equation*, and check the result using implicit differentiation.

Solution:

$$f(x, y) = x^3 + y^2x - 3,$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2 + y^2}{2yx}$$

differentiating implicitly yields

$$3x^2 + y^2 + x\left(2y\frac{dy}{dx}\right) - 0 = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{3x^2 + y^2}{2yx}$$

Theorem *If the equation f(x, y, z) = c defines z implicitly as a differentiable function of x and y, and if* $\partial f/\partial z \neq 0$, *then*

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad \textit{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

Example 3.28 Consider the sphere $x^2 + y^2 + z^2 = 1$. Find $\partial z/\partial x$ and $\partial z/\partial y$ at the point (2/3, 1/3, 2/3)

Solution:

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{2x}{2z} = -\frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{2y}{2z} = -\frac{y}{z}$$

At the point (2/3, 1/3, 2/3), evaluating these derivatives gives $\partial z/\partial x = -1$ and $\partial z/\partial y = -1/2$.

3.6 DIRECTIONAL DERIVATIVES AND GRADIENTS

3.6.1 Directional Derivatives

Definition If $f(x, y)$ is a function of x and y, and if $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ is a unit vector, then the ***directional derivative of f in the direction of* u** at (x_0, y_0) is denoted by $D_{\mathbf{u}}f(x_0, y_0)$ and is defined by

$$D_{\mathbf{u}} f(x_0, y_0) = \frac{d}{ds}\left[f(x_0 + su_1, y_0 + su_2)\right]_{s=0}$$

provided this derivative exists.

Geometrically, $D_{\mathbf{u}}f(x_0, y_0)$ can be interpreted as the ***slope of the surface z = f (x, y) in the direction of* u** at the point $(x_0, y_0, f(x_0, y_0))$ (Figure a). Usually the value of $D_{\mathbf{u}}f(x_0, y_0)$ will depend on both the point (x_0, y_0) and the direction **u**. Thus, at a fixed point the slope of the surface may vary with the direction (Figure b). Analytically, the directional derivative represents the ***instantaneous rate of change of f (x, y) with respect to distance in the direction of* u** at the point (x_0, y_0).

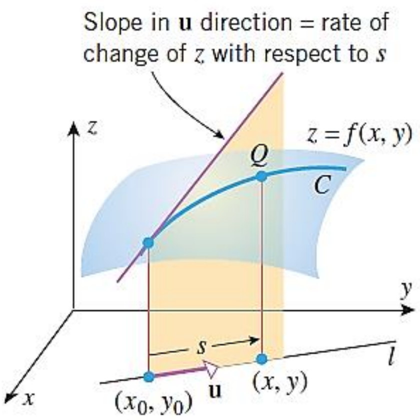


Figure a

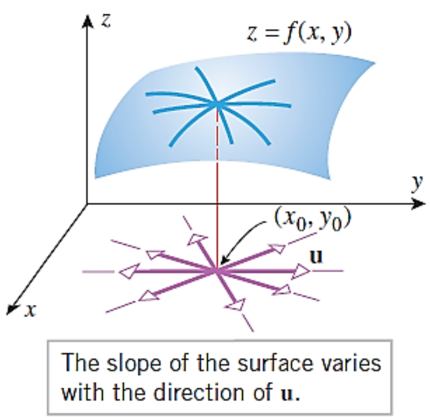


Figure b

Example 3.28 Let $f(x, y) = xy$. Find and interpret $D_{\mathbf{u}}f(1, 2)$ for the unit vector

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

Solution:

$$D_{\mathbf{u}} f(1, 2) = \frac{d}{ds}\left[f\left(1 + \frac{\sqrt{3}s}{2}, 2 + \frac{s}{2}\right)\right]_{s=0}$$

Since

$$f\left(1 + \frac{\sqrt{3}s}{2}, 2 + \frac{s}{2}\right) = \left(1 + \frac{\sqrt{3}s}{2}\right)\left(2 + \frac{s}{2}\right) = \frac{\sqrt{3}}{4}s^2 + \left(\frac{1}{2} + \sqrt{3}\right)s + 2$$

we have

$$D_{\mathbf{u}} f(1, 2) = \frac{d}{ds}\left[\frac{\sqrt{3}}{4}s^2 + \left(\frac{1}{2} + \sqrt{3}\right)s + 2\right]_{s=0}$$

$$= \left[\frac{\sqrt{3}}{2}s + \frac{1}{2} + \sqrt{3}\right]_{s=0} = \frac{1}{2} + \sqrt{3}$$

Since $1/2+\sqrt{3} \approx 2.23$, we conclude that if we move a small distance from the point $(1, 2)$ in the direction of $\mathbf{u}$, the function $f(x, y) = xy$ will increase by about 2.23 times the distance moved.

Definition

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, and if $f(x, y, z)$ is a function of x, y, and z, then the ***directional derivative of f in the direction of*** $\mathbf{u}$ at (x_0, y_0, z_0) is denoted by $D_{\mathbf{u}}f(x_0, y_0, z_0)$ and is defined by

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \frac{d}{ds}\left[f(x_0 + su_1, y_0 + su_2, z_0 + su_3)\right]_{s=0}$$

provided this derivative exists.

Theorem

(a) If $f(x, y)$ is differentiable at (x_0, y_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ exists and is given by

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

(b) If $f(x, y, z)$ is differentiable at (x_0, y_0, z_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0, z_0)$ exists and is given by

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)u_1 + f_y(x_0, y_0, z_0)u_2 + f_z(x_0, y_0, z_0)u_3$$

Example 3.29 Find the directional derivative of $f(x, y) = e^{xy}$ at $(-2, 0)$ in the direction of the unit vector that makes an angle of $\pi/3$ with the positive x-axis.

Solution: The partial derivatives of f are

$$f_x(x, y) = ye^{xy}, \quad f_y(x, y) = xe^{xy}$$

$$f_x(-2, 0) = 0, \quad f_y(-2, 0) = -2$$

The unit vector **u** that makes an angle of $\pi/3$ with the positive x-axis is

$$\mathbf{u} = \cos(\pi/3)\mathbf{i} + \sin(\pi/3)\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$D_{\mathbf{u}} f(-2, 0) = f_x(-2, 0)\cos(\pi/3) + f_y(-2, 0)\sin(\pi/3)$$
$$= 0(1/2) + (-2)(\sqrt{3}/2) = -\sqrt{3}$$

Example 3.30 Find the directional derivative of $f(x, y, z) = x^2y - yz^3 + z$ at the point (1, −2, 0) in the direction of the vector $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Solution:

$$f_x(x, y, z) = 2xy, \quad f_y(x, y, z) = x^2 - z^3, \quad f_z(x, y, z) = -3yz^2 + 1$$
$$f_x(1, -2, 0) = -4, \quad f_y(1, -2, 0) = 1, \quad f_z(1, -2, 0) = 1$$

Since **a** is not a unit vector, we normalize it, getting

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{9}}(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Formula then yields

$$D_{\mathbf{u}} f(1, -2, 0) = (-4)\left(\frac{2}{3}\right) + \frac{1}{3} - \frac{2}{3} = -3$$

3.6.2 The Gradient

Definition

(a) If f is a function of x and y, then the ***gradient of f*** is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

(c) If f is a function of x, y, and z, then the ***gradient of f*** is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

The symbol ∇ (read "del") is a "nabla"

Formulas can now be written as

$D_{\mathbf{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \,.\, \mathbf{u}$

$D_{\mathbf{u}}f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \,.\, \mathbf{u}$

For example, using above formula our solution to **Example 3.30** would take the form

$$D_{\mathbf{u}}f(1, -2, 0) = \nabla f(1, -2, 0) \,.\, \mathbf{u} = (-4\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2/3\,\mathbf{i} + 1/3\mathbf{j} - 2/3\mathbf{k}$$
$$= (-4)(2/3) + 1/3 - 2/3 = -3$$

3.6.3 Properties of the Gradient

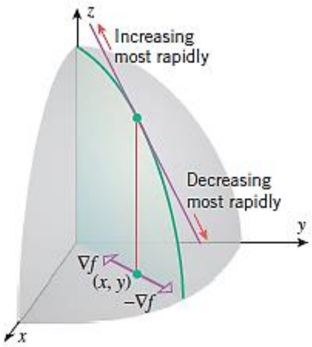
Theorem

Let f be a function of either two variables or three variables, and let P denote the point $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$, respectively. Assume that f is differentiable at P.

(a) If $\nabla f = \mathbf{0}$ at P, then all directional derivatives of f at P are zero.

(b) If $\nabla f \neq \mathbf{0}$ at P, then among all possible directional derivatives of f at P, the derivative in the direction of ∇f at P has the largest value. The value of this largest directional derivative is $\|\nabla f\|$ at P.

(c) If $\nabla f \neq \mathbf{0}$ at P, then among all possible directional derivatives of f at P, the derivative in the direction opposite to that of ∇f at P has the smallest value. The value of this smallest directional derivative is $-\|\nabla f\|$ at P.



Example 3.31 Let $f(x, y) = x^2e^y$. Find the maximum value of a directional derivative at (−2, 0), and find the unit vector in the direction in which the maximum value occurs.

Solution:

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 2xe^y\mathbf{i} + x^2e^y\mathbf{j}$$

the gradient of f at (−2, 0) is

$$\nabla f(-2, 0) = -4\mathbf{i} + 4\mathbf{j}$$

By Theorem [illegible], the maximum value of the directional derivative is

$$\|\nabla f(-2, 0)\| = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

This maximum occurs in the direction of $\nabla f(-2, 0)$. The unit vector in this direction is

$$\mathbf{u} = \frac{\nabla f(-2, 0)}{\|\nabla f(-2, 0)\|} = \frac{1}{4\sqrt{2}}(-4\mathbf{i} + 4\mathbf{j}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

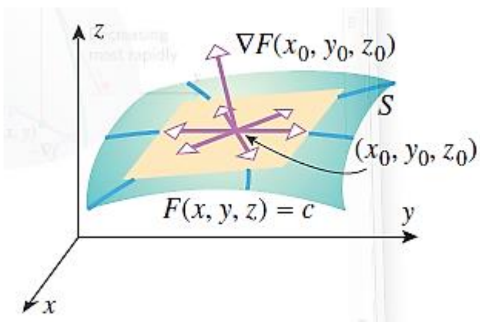
3.7 TANGENT PLANES AND NORMAL VECTORS

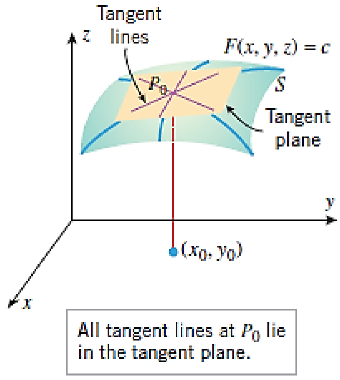
3.7.1 Tangent Planes and Normal Vectors to Level Surfaces $F(x, y, z) = c$

Definition

Assume that $F(x, y, z)$ has continuous first-order partial derivatives and that $P_0(x_0, y_0, z_0)$ is a point on the level surface S: $F(x, y, z) = c$. If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\mathbf{n} = \nabla F(x_0, y_0, z_0)$ is a ***normal vector*** to S at P_0 and the ***tangent plane*** to S at P_0 is the plane with equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$





Example 3.32 Consider the ellipsoid $x^2 + 4y^2 + z^2 = 18$.

(a) Find an equation of the tangent plane to the ellipsoid at the point (1, 2, 1).

(b) Find parametric equations of the line that is normal to the ellipsoid at the point (1, 2, 1).

(c) Find the acute angle that the tangent plane at the point (1, 2, 1) makes with the xy-plane.

Solution:

Solution (a): We apply Definition with $F(x, y, z) = x^2 + 4y^2 + z^2$ and $(x_0, y_0, z_0) = (1, 2, 1)$. Since

$$\nabla F(x, y, z) = (F_x(x, y, z), F_y(x, y, z), F_z(x, y, z)) = (2x, 8y, 2z)$$

we have

$$\mathbf{n} = \nabla F(1, 2, 1) = (2, 16, 2)$$

Hence, the equation of the tangent plane is

$$2(x - 1) + 16(y - 2) + 2(z - 1) = 0 \text{ or } x + 8y + z = 18$$

Solution (b): Since $\mathbf{n} = (2, 16, 2)$ at the point (1, 2, 1), it follows that parametric equations for the normal line to the ellipsoid at the point (1, 2, 1) are

$$x = 1 + 2t,\ y = 2 + 16t,\ z = 1 + 2t$$

Solution (c): To find the acute angle θ between the tangent plane and the xy-plane,

$\mathbf{n}_1 = \mathbf{n} = (2, 16, 2)$ and $\mathbf{n}_2 = (0, 0, 1)$. This yields

$$\cos\theta = \frac{|\langle 2, 16, 2\rangle \cdot \langle 0, 0, 1\rangle|}{\|\langle 2, 16, 2\rangle\| \, \|\langle 0, 0, 1\rangle\|} = \frac{2}{(2\sqrt{66})(1)} = \frac{1}{\sqrt{66}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{66}}\right) \approx 83°$$

3.7.2 Tangent Planes to Surfaces of The Form $z = f(x, y)$

Example 3.33 Find an equation for the tangent plane and parametric equations for the normal line to the surface $z = x^2y$ at the point $(2, 1, 4)$.

Solution: Let $F(x, y, z) = z - x^2y$. Then $F(x, y, z) = 0$ on the surface, so we can find the find the gradient of F at the point $(2, 1, 4)$:

$$\nabla F(x, y, z) = -2xy\mathbf{i} - x^2\mathbf{j} + \mathbf{k}$$

$$\nabla F(2, 1, 4) = -4\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

the tangent plane has equation

$$-4(x - 2) - 4(y - 1) + 1(z - 4) = 0 \text{ or } -4x - 4y + z = -8$$

and the normal line has equations

$$x = 2 - 4t,\ y = 1 - 4t,\ z = 4 + t$$

Theorem *If $f(x, y)$ is differentiable at the point (x_0, y_0), then the tangent plane to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, f(x_0, y_0))$ [or (x_0, y_0)] is the plane*

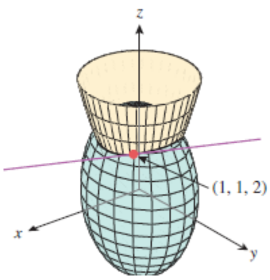
$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

3.7.3 Using Gradients to Find Tangent Lines to Intersections of Surfaces

Example 3.34 Find parametric equations of the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ at the point $(1, 1, 2)$

Solution: We begin by rewriting the equations of the surfaces as

$$x^2 + y^2 - z = 0 \text{ and } 3x^2 + 2y^2 + z^2 - 9 = 0$$



and we take

$$F(x, y, z) = x^2 + y^2 - z \text{ and } G(x, y, z) = 3x^2 + 2y^2 + z^2 - 9$$

We will need the gradients of these functions at the point $(1, 1, 2)$. The computations are

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k},\ \nabla G(x, y, z) = 6x\mathbf{i} + 4y\,\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(1, 1, 2) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k},\ \nabla G(1, 1, 2) = 6\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

Thus, a tangent vector at $(1, 1, 2)$ to the curve of intersection is

$$\nabla F(1,1,2) \times \nabla G(1,1,2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 6 & 4 & 4 \end{vmatrix} = 12\mathbf{i} - 14\mathbf{j} - 4\mathbf{k}$$

Since any scalar multiple of this vector will do just as well, we can multiply by 1/2 to reduce the size of the coefficients and use the vector of $6\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}$ to determine the direction of the tangent line. This vector and the point $(1, 1, 2)$ yield the parametric equations

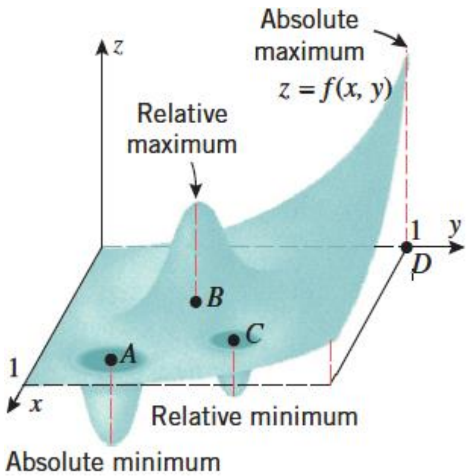
$$x = 1 + 6t,\ y = 1 - 7t,\ z = 2 - 2t$$

3.8 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

3.8.1 Extrema

Definition A function f of two variables is said to have a ***relative maximum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute maximum*** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.

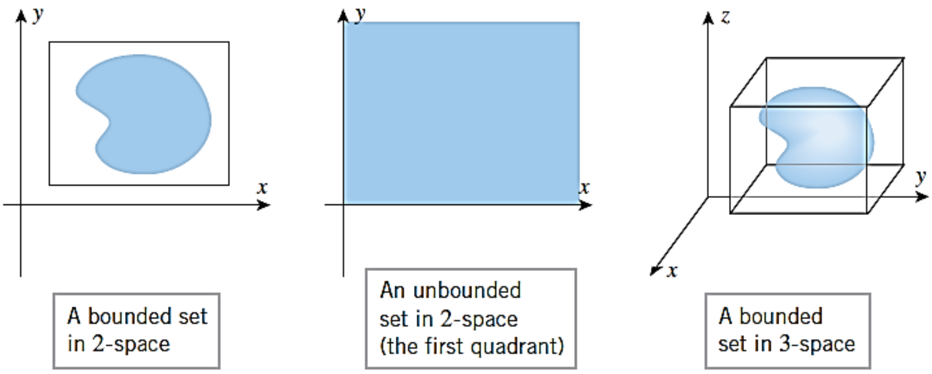
Definition A function f of two variables is said to have a ***relative minimum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute minimum*** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.



If f has a relative maximum or a relative minimum at (x_0, y_0), then we say that f has a ***relative extremum*** at (x_0, y_0), and if f has an absolute maximum or absolute minimum at (x_0, y_0), then we say that f has an ***absolute extremum*** at (x_0, y_0).

3.8.2 Bounded Sets

- (finite intervals and infinite intervals on the real line),
- Distinguish between regions of "finite extent" and regions of "infinite extent" in 2-space and 3-space.
- A set of points in 2-space is called ***bounded*** if the entire set can be contained within some rectangle,
- called ***unbounded*** if there is no rectangle that contains all the points of the set.
- Similarly, a set of points in 3-space is ***bounded*** if the entire set can be contained within some box, and is **unbounded** otherwise (see below Figure).



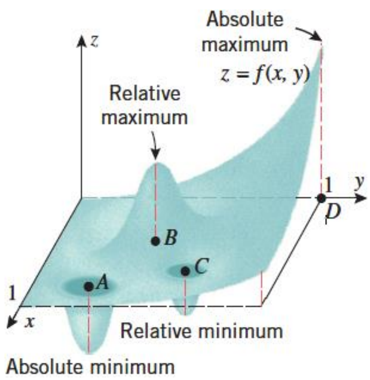
3.8.3 The Extreme-Value Theorem

Theorem (***Extreme-Value Theorem***) *If f(x, y) is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum on R.*

Example 3.35 The square region R whose points satisfy the inequalities

$$0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1$$

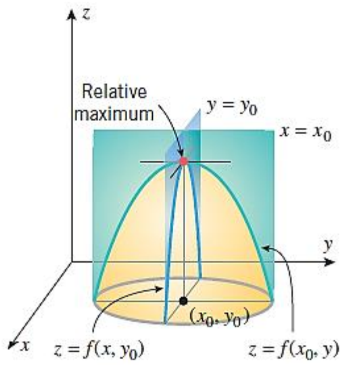
is a closed and bounded set in the xy-plane. The function f whose graph is shown in Figure is continuous on R; thus, it is guaranteed to have an absolute maximum and minimum on R by the last theorem. These occur at points D and A that are shown in the figure.



3.8.4 Finding Relative Extrema

Theorem *If f has a relative extremum at a point (x_0, y_0), and if the first-order partial derivatives of f exist at this point, then*

$$f_x(x_0, y_0) = 0 \text{ and } f_y(x_0, y_0) = 0$$



Definition A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a ***critical point*** of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

Example: consider the function

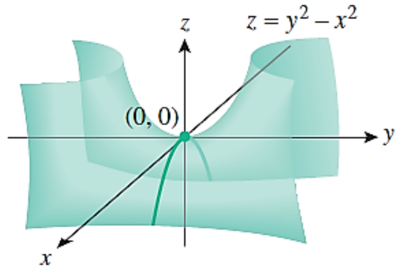
$$f(x, y) = y^2 - x^2$$

This function, whose graph is the hyperbolic paraboloid shown in the figure, has a critical point at (0, 0), since

$$f_x(x, y) = -2x \text{ and } f_y(x, y) = 2y$$

from which it follows that

$$f_x(0, 0) = 0 \text{ and } f_y(0, 0) = 0$$

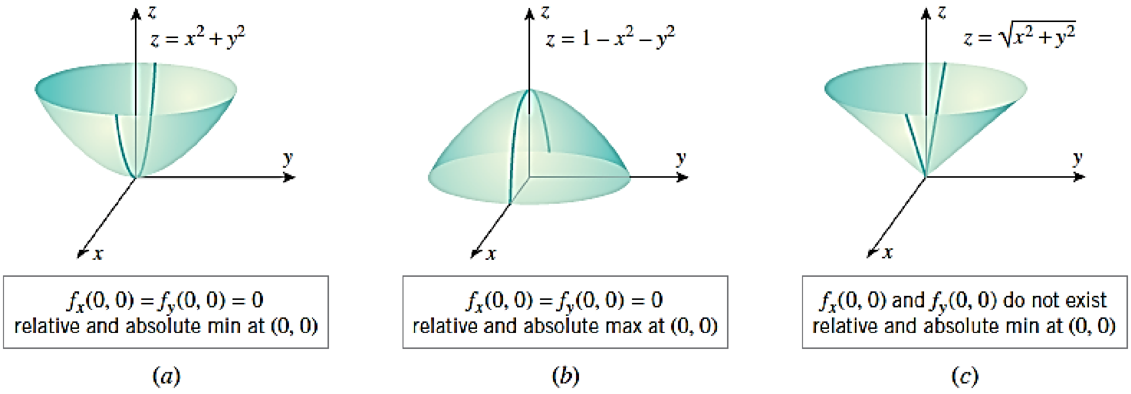


The function $f(x, y) = y^2 - x^2$ has neither a relative maximum nor a relative minimum at the critical point (0, 0).

The function f has neither a relative maximum nor a relative minimum at $(0, 0)$. For obvious reasons, the point $(0, 0)$ is called a ***saddle point*** of f.

In general, we will say that a surface $z = f(x, y)$ has a ***saddle point*** at (x_0, y_0) if there are two distinct vertical planes through this point such that the trace of the surface in one of the planes has a relative maximum at (x_0, y_0) and the trace in the other has a relative minimum at (x_0, y_0).

Example The three functions graphed in the following figure all have critical points at (0, 0). For the paraboloids, the partial derivatives at the origin are zero. You can check this algebraically by evaluating the partial derivatives at (0, 0), but you can see it geometrically by observing that the traces in the *xz*-plane and *yz*-plane have horizontal tangent lines at (0, 0).



3.8.5 The Second Partials Test

Theorem (***The Second Partials Test***) *Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point* (x_0, y_0)*, and let*

$$D = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0)$$

(a) If $D > 0$ *and* $f_{xx}(x_0, y_0) > 0$*, then f has a relative minimum at* (x_0, y_0)*.*

(b) If $D > 0$ *and* $f_{xx}(x_0, y_0) < 0$*, then f has a relative maximum at* (x_0, y_0)*.*

(c) If $D < 0$*, then f has a saddle point at* (x_0, y_0)*.*

(d) If $D = 0$*, then no conclusion can be drawn.*

Example 3.36 Locate all relative extrema and saddle points of

$$f(x, y) = 3x^2 - 2xy + y^2 - 8y$$

Solution: Since $fx(x, y) = 6x - 2y$ and $fy(x, y) = -2x + 2y - 8$, the critical points of f satisfy the equations

$$6x - 2y = 0$$
$$-2x + 2y - 8 = 0$$

Solving these for x and y yields $x = 2$, $y = 6$ (verify), so (2, 6) is the only critical point.

To apply Theorem, we need the second-order partial derivatives

$$f_{xx}(x, y) = 6, \quad f_{yy}(x, y) = 2, \quad f_{xy}(x, y) = -2$$

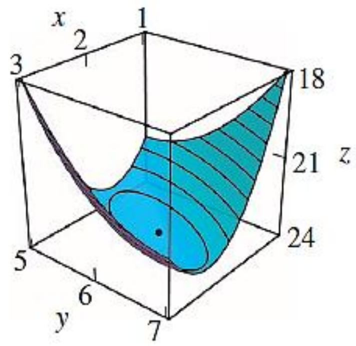
At the point (2, 6) we have

$$D = f_{xx}(2, 6)\, f_{yy}(2, 6) - f^2_{xy}(2, 6) = (6)(2) - (-2)^2 = 8 > 0$$

and

$$f_{xx}(2, 6) = 6 > 0$$

so f has a relative minimum at (2, 6) by part (*a*) of the second partials test. The below figure shows a graph of f in the vicinity of the relative minimum.



$f(x, y) = 3x^2 - 2xy + y^2 - 8y$

Example 3.37 Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

Solution: Since

$$f_x(x, y) = 4y - 4x^3$$
$$f_y(x, y) = 4x - 4y^3 \qquad (1)$$

the critical points of f have coordinates satisfying the equations

$$4y - 4x^3 = 0 \qquad\qquad y = x^3$$

or

$$4x - 4y^3 = 0 \qquad\qquad x = y^3 \qquad (2)$$

Substituting the top equation in the bottom yields $x = (x^3)^3$ or, equivalently, $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0$, $x = 1$, $x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y-values $y = 0$, $y = 1$, $y = -1$. Thus, the critical points of f are (0, 0), (1, 1), and (−1, −1).

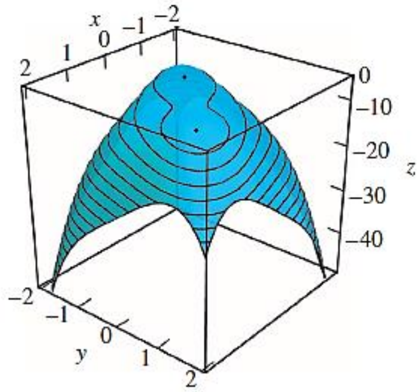
From (1),

$$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

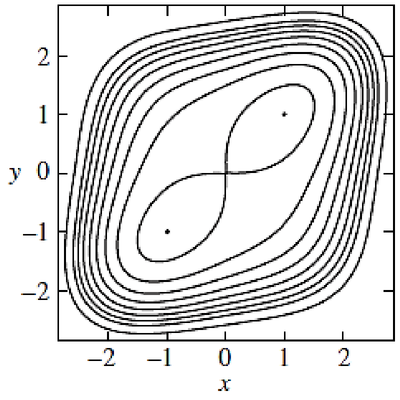
which yields the following table:

CRITICAL POINT (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx}f_{yy} - f_{xy}^2$
(0, 0)	0	0	4	−16
(1, 1)	−12	−12	4	128
(−1, −1)	−12	−12	4	128

At the points (1, 1) and (−1,−1), we have D > 0 and f_{xx} < 0, so relative maxima occur at these critical points. At (0, 0) there is a saddle point since D < 0. The surface and a contour plot are shown in the below figure.



$f(x, y) = 4xy - x^4 - y^4$



Theorem *If a function f of two variables has an absolute extremum (either an absolute maximum or an absolute minimum) at an interior point of its domain, then this extremum occurs at a critical point.*

3.8.6 Finding Absolute Extrema on Closed and Bounded Sets

How to Find the Absolute Extrema of a Continuous Function f of Two Variables on a Closed and Bounded Set R

Step 1. Find the critical points of f that lie in the interior of R.

Step 2. Find all boundary points at which the absolute extrema can occur.

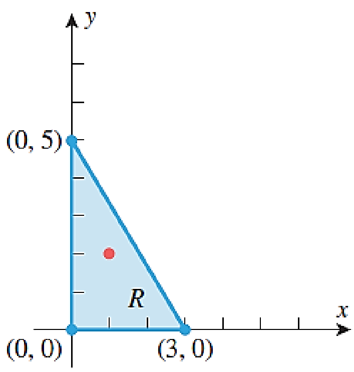
Step 3. Evaluate $f(x, y)$ at the points obtained in the preceding steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

Example 3.38 Find the absolute maximum and minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7 \qquad (1)$$

on the closed triangular region R with vertices (0, 0), (3, 0), and (0, 5).

Solution: The region R is shown in Figure.



Step 1: find critical points

$$\partial f/\partial x = 3y - 6 \text{ and}$$

$$\partial f/\partial y = 3x - 3$$

so all critical points occur where

$$3y - 6 = 0 \text{ and } 3x - 3 = 0$$

Solving these equations yields $x = 1$ and $y = 2$, so (1, 2) is the only critical point. As shown in Figure, this critical point is in the interior of R.

Step 2: Determine the locations of the points on the boundary of R at which the absolute extrema might occur. The boundary of R consists of three line segments, each of which we will treat separately:

The line segment between (0, 0) and (3, 0): On this line segment we have $y = 0$, so (1) simplifies to a function of the single variable x,

$$u(x) = f(x, 0) = -6x + 7,\ 0 \leq x \leq 3$$

This function has no critical points because $u(x) = -6$ is nonzero for all x. Thus the extreme values of $u'(x)$ occur at the endpoints $x = 0$ and $x = 3$, which correspond to the points (0, 0) and (3, 0) of R.

The line segment between (0, 0) and (0, 5): On this line segment we have $x = 0$, so (1) simplifies to a function of the single variable y,

$$v(y) = f(0, y) = -3y + 7,\ 0 \leq y \leq 5$$

This function has no critical points because $v'(y) = -3$ is nonzero for all y. Thus, the extreme values of $v(y)$ occur at the endpoints $y = 0$ and $y = 5$, which correspond to the points (0, 0) and (0, 5) of R.

The line segment between (3, 0) and (0, 5): In the xy-plane, an equation for this line segment is

$$y = -\tfrac{5}{3}x + 5, \quad 0 \leq x \leq 3$$

so (1) simplifies to a function of the single variable x,

$$\begin{aligned} w(x) &= f\left(x, -\tfrac{5}{3}x + 5\right) = 3x\left(-\tfrac{5}{3}x + 5\right) - 6x - 3\left(-\tfrac{5}{3}x + 5\right) + 7 \\ &= -5x^2 + 14x - 8, \qquad 0 \leq x \leq 3 \end{aligned}$$

Since $w'(x) = -10x + 14$, the equation $w(x) = 0$ yields $x = 7/5$ as the only critical point of w. Thus, the extreme values of w occur either at the critical point $x = 7/5$ or at the endpoints $x = 0$ and $x = 3$. The endpoints correspond to the points (0, 5) and (3, 0) of R, and from (4) the critical point corresponds to (7/5, 8/3).

Final step: the below table lists the values of $f(x, y)$ at the interior critical point and at the points on the boundary where an absolute extremum can occur. From the table we conclude that the absolute maximum value of f is $f(0, 0) = 7$ and the absolute minimum value is $f(3, 0) = -11$.

(x, y)	$(0, 0)$	$(3, 0)$	$(0, 5)$	$\left(\frac{7}{5}, \frac{8}{3}\right)$	$(1, 2)$
$f(x, y)$	7	-11	-8	$\frac{9}{5}$	1

Example 3.39 Determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution: Let

x = length of the box (in feet)

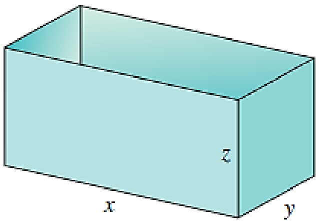
y = width of the box (in feet)

z = height of the box (in feet)

S = surface area of the box (in square feet)

We may reasonably assume that the box with least surface area requires the least amount of material, so our objective is to minimize the surface area

$$S = xy + 2xz + 2yz \qquad (1)$$



Two sides each have area xz.
Two sides each have area yz.
The base has area xy.

(Figure) subject to the volume requirement

$$xyz = 32 \qquad (2)$$

From (2) we obtain $z = 32/xy$, so (1) can be rewritten as

$$S = xy + \frac{64}{y} + \frac{64}{x} \qquad (3)$$

Differentiating (3) we obtain

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}, \quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2}$$

so the coordinates of the critical points of S satisfy

$$y - \frac{64}{x^2} = 0, \quad x - \frac{64}{y^2} = 0$$

Solving the first equation for y yields

$$y = \frac{64}{x^2}$$

and substituting this expression in the second equation yields

$$x - \frac{64}{(64/x^2)^2} = 0$$

which can be rewritten as

$$x\left(1 - \frac{x^3}{64}\right) = 0$$

The solutions of this equation are $x = 0$ and $x = 4$. Since we require $x > 0$, the only solution of significance is $x = 4$. Substituting this value into ($y=64/x^2$) yields $y = 4$. We conclude that the point $(x, y) = (4, 4)$ is the only critical point of S in the first quadrant. Since $S = 48$ if $x = y = 4$, this suggests we try to show that the minimum value of S on the open first quadrant is 48.

It immediately follows from Equation (3) that $48 < S$ at any point in the first quadrant for which at least one of the inequalities

$$xy > 48, \quad 64/y > 48, \quad 64/x > 48$$

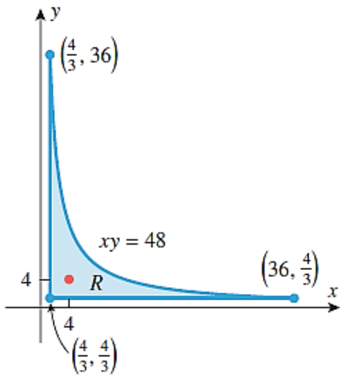
is satisfied. Therefore, to prove that $48 \le S$, we can restrict attention to the set of points in the first quadrant that satisfy the three inequalities

$$xy \le 48, \quad 64/y \le 48, \quad 64/x \le 48$$

These inequalities can be rewritten as

$$xy \le 48, \quad y \ge 4/3, \quad x \ge 4/3$$

and they define a closed and bounded region R within the first quadrant (below figure). The function S is continuous on R, so Theorem guarantees that S has an absolute minimum value somewhere on R. Since the point $(4, 4)$ lies within R, and $48 < S$ on the boundary of R (why?), the minimum value of S on R must occur at an interior point. It then follows from Theorem that the mimimum value of S on R must occur at a critical point of S. Hence, the absolute minimum of S on R (and therefore on the entire open first quadrant) is $S = 48$ at the point $(4, 4)$. Substituting $x = 4$ and $y = 4$ into (6) yields $z = 2$, so the box using the least material has a height of 2 ft and a square base whose edges are 4ft long.



3.9 LAGRANGE MULTIPLIERS

3.9.1 Extremum Problems with Constraints

Three-Variable Extremum Problem with One Constraint

Maximize or minimize the function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

Two-Variable Extremum Problem with One Constraint

Maximize or minimize the function $f(x, y)$ subject to the constraint $g(x, y) = 0$.

Theorem (***Constrained-Extremum Principle for Two Variables and One Constraint***) *Let f and g be functions of two variables with continuous first partial derivatives on some open set containing the constraint curve* $g(x, y) = 0$, *and assume that* $\nabla g \neq \mathbf{0}$ *at any point on this curve. If f has a constrained relative extremum, then this extremum occurs at a point* (x_0, y_0) *on the constraint curve at which the gradient vectors* $\nabla f(x_0, y_0)$ *and* $\nabla g(x_0, y_0)$ *are parallel; that is, there is some number* λ *such that*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Example 3.40 At what point or points on the circle $x^2 + y^2 = 1$ does $f(x, y) = xy$ have an absolute maximum, and what is that maximum?

Solution: The circle $x^2 + y^2 = 1$ is a closed and bounded set and $f(x, y) = xy$ is a continuous function, so it follows from the Extreme-Value Theorem that f has an absolute maximum and an absolute minimum on the circle. To find these extrema, we will use Lagrange multipliers to find the constrained relative extrema, and then we will evaluate f at those relative extrema to find the absolute extrema.

We want to maximize $f(x, y) = xy$ subject to the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0 \qquad (1)$$

First we will look for constrained *relative* extrema. For this purpose we will need the gradients

$$\nabla f = y\,\mathbf{i} + x\,\mathbf{j} \quad \text{and} \quad \nabla g = 2x\,\mathbf{i} + 2y\,\mathbf{j}$$

From the formula for ∇g we see that $\nabla g = \mathbf{0}$ if and only if $x = 0$ and $y = 0$, so $\nabla g \neq \mathbf{0}$ at any point on the circle $x^2 + y^2 = 1$. Thus, at a constrained relative extremum we must have

$$\nabla f = \lambda \nabla g \quad \text{or} \quad y\,\mathbf{i} + x\,\mathbf{j} = \lambda(2x\,\mathbf{i} + 2y\,\mathbf{j})$$

which is equivalent to the pair of equations

$$y = 2x\lambda \text{ and } x = 2y\lambda$$

It follows from these equations that if $x = 0$, then $y = 0$, and if $y = 0$, then $x = 0$. In either case we have $x^2 + y^2 = 0$, so the constraint equation $x^2 + y^2 = 1$ is not satisfied. Thus, we can assume that x and y are nonzero, and we can rewrite the equations as

$$\lambda = y/2x \qquad \text{and} \qquad \lambda = x/2y$$

from which we obtain

$$y/2x = x/2y$$

or

$$y^2 = x^2 \qquad (2)$$

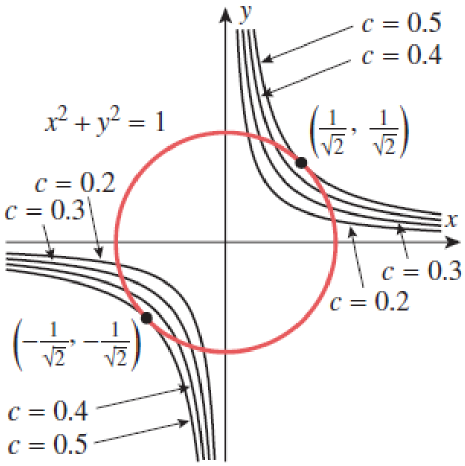
Substituting this in (1) yields

$$2x^2 - 1 = 0$$

from which we obtain $x = \pm 1/\sqrt{2}$. Each of these values, when substituted in Equation (2), produces y-values of $y = \pm 1/\sqrt{2}$. Thus, constrained relative extrema occur at the points $(1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$, and $(-1/\sqrt{2}, -1/\sqrt{2})$. The values of xy at these points are as follows:

(x, y)	$(1/\sqrt{2}, 1/\sqrt{2})$	$(1/\sqrt{2}, -1/\sqrt{2})$	$(-1/\sqrt{2}, 1/\sqrt{2})$	$(-1/\sqrt{2}, -1/\sqrt{2})$
xy	1/2	−1/2	−1/2	1/2

Thus, the function $f(x, y) = xy$ has an absolute maximum of 1/2 occurring at the two points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. Although it was not asked for, we can also see that f has an absolute minimum of −1/2 occurring at the points $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$. The below figure shows some level curves $xy = c$ and the constraint curve



Example 3.41 Use the method of Lagrange multipliers to find the dimensions of a rectangle with perimeter p and maximum area.

Solution: Let

x = length of the rectangle, y = width of the rectangle, A = area of the rectangle

We want to maximize $A = xy$ on the line segment

$$2x + 2y = p,\ 0 \leq x, y \qquad (1)$$

that corresponds to the perimeter constraint. This segment is a closed and bounded set, and since $f(x, y) = xy$ is a continuous function, it follows from the Extreme-Value Theorem that f has an absolute maximum on this segment. This absolute maximum must also be a constrained relative maximum since f is 0 at the endpoints of the segment and positive elsewhere on the segment. If $g(x, y) = 2x + 2y$, then we have

$$\nabla f = y\mathbf{i} + x\,\mathbf{j} \text{ and } \nabla g = 2\mathbf{i} + 2\mathbf{j}$$

Noting that $\nabla g \neq \mathbf{0}$, it follows from (4) that

$$y\,\mathbf{i} + x\,\mathbf{j} = \lambda(2\,\mathbf{i} + 2\,\mathbf{j})$$

at a constrained relative maximum. This is equivalent to the two equations

$$y = 2\lambda \quad \text{and} \quad x = 2\lambda$$

Eliminating λ from these equations we obtain $x = y$, which shows that the rectangle is actually a square. Using this condition and constraint (1), we obtain $x = p/4$, $y = p/4$.

3.9.2 Three Variables and One Constraint

Theorem (***Constrained-Extremum Principle for Three Variables and One Constraint***) *Let f and g be functions of three variables with continuous first partial derivatives on some open set containing the constraint surface $g(x, y, z) = 0$, and assume that $\nabla g \neq \mathbf{0}$ at any point on this surface. If f has a constrained relative extremum, then this extremum occurs at a point (x_0, y_0, z_0) on the constraint surface at which the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ are parallel; that is, there is some number λ such that*

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

Example 3.41 Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and farthest from the point (1, 2, 2).

Solution: To avoid radicals, we will find points on the sphere that minimize and maximize the *square* of the distance to (1, 2, 2). Thus, we want to find the relative extrema of

$$f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 2)^2$$

subject to the constraint

$$x^2 + y^2 + z^2 = 36 \qquad (1)$$

If we let $g(x, y, z) = x^2 + y^2 + z^2$, then $\nabla g = 2x\mathbf{i} + 2y\,\mathbf{j} + 2z\mathbf{k}$. Thus, $\nabla g = \mathbf{0}$ if and only if $x = y = z = 0$. It follows that $\nabla g \neq \mathbf{0}$ at any point of the sphere (1), and hence the constrained relative extrema must occur at points where

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

That is,

$$2(x-1)\mathbf{i} + 2(y-2)\,\mathbf{j} + 2(z-2)\mathbf{k} = \lambda(2x\mathbf{i} + 2y\,\mathbf{j} + 2z\mathbf{k})$$

which leads to the equations

$$2(x-1) = 2x\lambda,\ 2(y-2) = 2y\lambda,\ 2(z-2) = 2z\lambda \qquad (2)$$

We may assume that x, y, and z are nonzero since $x = 0$ does not satisfy the first equation, $y = 0$ does not satisfy the second, and $z = 0$ does not satisfy the third. Thus, we can rewrite (2) as

$$\frac{x-1}{x} = \lambda, \quad \frac{y-2}{y} = \lambda, \quad \frac{z-2}{z} = \lambda$$

The first two equations imply that

$$\frac{x-1}{x} = \frac{y-2}{y}$$

from which it follows that

$$y = 2x \qquad (3)$$

Similarly, the first and third equations imply that

$$z = 2x \qquad (4)$$

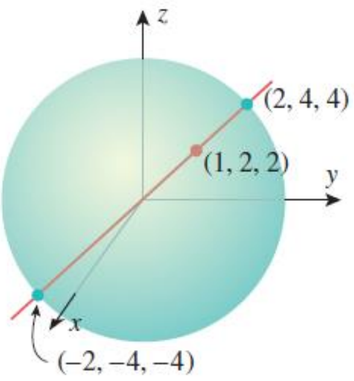
Substituting (3) and (4) in the constraint equation (1), we obtain

$$9x^2 = 36 \text{ or } x = \pm 2$$

Substituting these values in (3) and (4) yields two points:

$$(2, 4, 4) \text{ and } (-2, -4, -4)$$

Since $f(2, 4, 4) = 9$ and $f(-2,-4,-4) = 81$, it follows that $(2, 4, 4)$ is the point on the sphere closest to (1, 2, 2), and $(-2,-4,-4)$ is the point that is farthest (the following figure).



Example 3.42 Use Lagrange multipliers to determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution: the problem is to minimize the surface area

$$S = xy + 2xz + 2yz$$

subject to the volume constraint

$$xyz = 32 \qquad (1)$$

If we let $f(x, y, z) = xy + 2xz + 2yz$ and $g(x, y, z) = xyz$, then

$$\nabla f = (y + 2z)\,\mathbf{i} + (x + 2z)\,\mathbf{j} + (2x + 2y)\,\mathbf{k} \text{ and } \nabla g = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$$

It follows that $\nabla g \neq \mathbf{0}$ at any point on the surface $xyz = 32$, since x, y, and z are all nonzero on this surface. Thus, at a constrained relative extremum we must have $\nabla f = \lambda \nabla g$, that is,

$$(y + 2z)\mathbf{i} + (x + 2z)\mathbf{j} + (2x + 2y)\mathbf{k} = \lambda(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k})$$

This condition yields the three equations

$$y + 2z = \lambda yz,\ x + 2z = \lambda xz,\ 2x + 2y = \lambda xy$$

Because x, y, and z are nonzero, these equations can be rewritten as

$$\frac{1}{z} + \frac{2}{y} = \lambda, \quad \frac{1}{z} + \frac{2}{x} = \lambda, \quad \frac{2}{y} + \frac{2}{x} = \lambda$$

From the first two equations,

$$y = x \qquad (2)$$

and from the first and third equations,

$$z = (½)\, x \qquad (3)$$

Substituting (2) and (3) in the volume constraint (1) yields

$$(1/2)\, x^3 = 32$$

This equation, together with (13) and (14), yields

$$x = 4,\ y = 4,\ z = 2$$