## Electromagnetic Fields

## 1-Vector Analysis and coordinate systems

Scalar :- refers to a quantity whose value may be represented by a single real number (distance, mass, temperature , density , pressure , volume , voltage ... )

Vector :- quantity has both a magnitude and a direction in space (force, velocity, acceleration , electric field, magnetic field .... )

Vector Algebra :- vectors can be added, divide subtraction vectors and multiplied where for the two vectors $\bar{A}$ and $\bar{B}$ :-

$$
\begin{aligned}
& \bar{A}+\bar{B}=\bar{B}+\bar{A} \text { And } \mathrm{A}+(\bar{B}+\bar{C})=(\mathrm{A}+\bar{B})+\bar{C} \\
& \bar{A}-\bar{B}=\bar{A}+(-\bar{B}) \\
& (\mathrm{r}+\mathrm{s})(\bar{A}+\bar{B}=r(\bar{A}+\bar{B})+s(\bar{A}+\bar{B})=r \bar{A}+r \bar{B}+s \bar{A}+s \bar{B}
\end{aligned}
$$

$$
\frac{\bar{A}}{a}=K \bar{A}\left(\text { Where } K=\frac{1}{a}\right) \text { and } \frac{\bar{A}}{-a}=-K \bar{A}
$$

$$
\bar{A}=\bar{B}(\text { if } \bar{A}-\bar{B}=0)
$$

[^0]The Cartesian coordinate system:-

It is also called Rectangular system using $\mathrm{x}, \mathrm{y}, \mathrm{z}$ axes to describe position, vector, length, directions, angles, projections, or components n this system there coordinate axes at right angles to each other (righthanded) coordinate system

A point is located by giving
Its $\mathrm{x}, \mathrm{y}, \mathrm{z}$ coordinate
P(x,y,z)
P(3, 3, -4)
$\mathrm{Q}(2,-2,1)$

$\mathrm{x} \rightarrow d x=d l$
$\mathrm{y} \rightarrow d y$
$\mathrm{z} \rightarrow d z$

$\mathrm{ds} 1=\mathrm{dx}$ dy $(\mathrm{xy}$-plane $)$
$\mathrm{ds} 2=\mathrm{dx} \mathrm{dz}(\mathrm{xz}-\mathrm{plane})$
$\mathrm{ds} 3=\mathrm{dy} \mathrm{dz}$ (yz-plane)
$d v=d x d y d z$
$\qquad$

ds is differential area
The differential volume is
$d v=d x d y d z$
dl : differential length
ds : differential area ( surface )
dv: differential volume .
dl and ds may be given in vector

## Vector components and unit vectors:-

Any vector can be describe in the Cartesian coordinate system by giving the three components vectors lying a long $\mathrm{x}, \mathrm{y}, \mathrm{z}$-axes. The component vector have a magnitude and direction. We shall use a until vector $\bar{a} x, \bar{a} y$ and $\bar{a} z$ to show the directions of these components :
$|\bar{a} x|=1$ (in x-direction)
$|\bar{a} y|=1$ (in y-direction)
$|\bar{a} z|=1$ (in z-direction)
$\bar{r}=\bar{a} x+2 \bar{a} y+3 \bar{a} z$

Ex1:- Given the two points $\mathrm{P}(1,2,3)$ and $\mathrm{Q}(2,-2,1)$ find the vectors $\bar{r} P, \bar{r} Q$ from the origin. Then estimate the vector $\bar{R} P Q$.

Solution :-

$\bar{r} P=(1-0) \bar{a} x+(2-0) \bar{a} y+(3-0) \bar{a} z$
Then $\bar{r} P=\bar{a} x+2 \bar{a} y+3 \bar{a} z$
$|\bar{r} P|=\sqrt{(1)^{2}+(2)^{2}+(3)^{2}}$
$|\bar{r} P|=\sqrt{14}$
$\bar{r} Q=2 \bar{a} x-2 \bar{a} y+\bar{a} z$
$|\bar{r} Q|=\sqrt{(2)^{2}+(-2)^{2}+(1)^{2}}$
$|\bar{r} Q|=\sqrt{4+4+1}=3$
$\bar{R} P Q=\bar{r} Q-\bar{r} P=(2-1) \bar{a} x+(-2-2) \bar{a} y+(1-3) \bar{a} z$
Then $\bar{R} P Q=\bar{a} x-4 \bar{a} y-2 \bar{a} z=-\bar{R} Q P$
$|\bar{R} Q P|=\sqrt{1+16+4}=\sqrt{21}$
$\bar{R} P Q=\operatorname{Ax} \bar{a} x+A y \bar{a} y+A z \bar{a} z$
$A x=1, A y=-4, A z=-2 \quad$ (component of the vector )
Unit vector of $\bar{R} P Q$ is :-
$\bar{a}=\frac{\bar{R} P Q}{|\bar{R} P Q|}=$ direction of the vector $\bar{R} P Q$

Ex2:- Given the three points $\mathrm{A}(2,-3,1), \mathrm{B}(-4,-2,6)$ and $\mathrm{C}(1,5,-3)$. Find : athe vector from A to C b-the unit vector directed from B to A . $\mathrm{c}-$ the distance from $B$ to $C$. d- the vector from $A$ to the midpoint of the straight line joining $B$ to $C$.

Solution :-

a-
$\bar{A} C=(1-2) \bar{a} x+(5-(-3) \bar{a} y+(-3-1) \bar{a} z$
$\bar{A} C=-\bar{a} x+8 \bar{a} y-4 \bar{a} z$
b-
$\overline{B A}=(2-(-4) \bar{a} x+(-3(-2) \bar{a} y+(1-6) \bar{a} z$
$\overline{B A}=6 \bar{a} x-\bar{a} y-5 \bar{a} z$
$|\overline{B A}|=\sqrt{6^{2}+1^{2}+5^{2}}$
$|\overline{B A}|=\sqrt{62}$
$\bar{a} A B=\frac{\overline{B A}}{|\overline{B A}|}=\frac{6}{\sqrt{62}} \bar{a} x-\frac{1}{\sqrt{62}} \bar{a} y-\frac{5}{\sqrt{62}} \bar{a} z$
Then $\bar{a} B A=0.762 \bar{a} x-0.127 \bar{a} y-0.635 \bar{a} z$
c-
$\bar{B} C=(1+4) \bar{a} x+(5+2) \bar{a} y+(-3-6) \bar{a} z$
$\bar{B} C=5 \bar{a} x+7 \bar{a} y-9 \bar{a} z$
Then $|\bar{B} C|=\sqrt{25+49+81}=\sqrt{155}=12-45$
d-
$\mathrm{x} 1=\frac{1-4}{2}=-1.5, \mathrm{y} 1=\frac{5-2}{2}=1.5, z 1=\frac{-3+6}{3}=1.5$
$\mathrm{AD}=(-1.5-2) \bar{a} x+(1.5+3) \bar{a} y+(1.5-1) \bar{a} z=-3.5 \bar{a} x+$ $4.5 \bar{a} y+0.5 \bar{a} z$

## The vector field:-

It is defined as vector function of a position vector. The magnitude and direction of the function will change as we more throughout the region depends on the coordinate values of the point (vector function of $x, y, z$ )

Ex 1:-A vector field is given as:-
$\bar{w}=4 x^{2} y \bar{a} x-(7 x+2 z) \bar{a} y+\left(4 x y+2 z^{2}\right) \bar{a} z$.
a-what is the magnitude of the field at the point $\mathrm{P}(2,-3,4)$.
b-give a unit vector that shows the direction of the field at P .
c- At what point or points on the z -axis is the magnitude of $\bar{w} e q u a l$ to unity.

## Solution:-

a-
$\bar{w}=(4 \times 4 \times-3) \bar{a} x-(7 \times 2+2 \times 4) \bar{a} y+\left(8 \times-3+2 \times 4^{2}\right) \bar{a} z$
Then $\bar{w}=-48 \bar{a} x-22 \bar{a} y+8 \bar{a} z$
$|\bar{w}|=\sqrt{\left(-48^{2}\right)+\left(-22^{2}\right)+\left(8^{2}\right)}$
$|\bar{w}|=53.4$
b-
$\bar{a} w=\frac{\bar{w}}{|\bar{w}|}=\frac{-48 a x-22 \bar{a} y+8 \bar{a} z}{53.4}$
$\bar{a} w=-0.899 \bar{a} x-0.412 \bar{a} y+0.15 \bar{a} z$
c-
$|\bar{w}|=\sqrt{\left(4 x^{2} y\right)^{2}+(7 x+2 z)^{2}+\left(4 x y+2 z^{2}\right)^{2}}=1$
On the z -axis then $\mathrm{x}=0, \mathrm{y}=0$ then
$|\bar{w}|=\sqrt{4 z^{2}+4 z^{4}}=1$
$4 z^{4}+4 z^{2}-1=0$
Then $\mathrm{z} 1=0.455$ and $\mathrm{z} 2=-0.455$

The dot product:-
For the two vectors
$\bar{A}$ and $\bar{B}$ the dot product, or scalar product , is defined as: -

$$
\begin{equation*}
\bar{A} \cdot \bar{B}=|\bar{A}||\bar{B}| \cos \theta \tag{1}
\end{equation*}
$$

Also $\bar{A} \cdot \bar{B}=\bar{B} \cdot \bar{A}$


Then $\bar{a} x \cdot \bar{a} x=1, \bar{a} y \cdot \bar{a} y=1, \bar{a} z \cdot \bar{a} z=1$

$\bar{a} x . \bar{a} y=0$


For $\bar{A}=A x \bar{a} x+A y \bar{a} y+A z \bar{a} z$ and

$$
\bar{B}=B x \bar{a} x+B y \bar{a} y+B z \bar{a} z \text { then }
$$

$\bar{A} \cdot \bar{B}=(A x \bar{a} x+A y \bar{a} y+A z \bar{a} z) \cdot(B x \bar{a} x+B y \bar{a} y+B z \bar{a} z)$
And by using then dot product of the unit vector then we have a terms and
$\bar{a} x \cdot \bar{a} y=\bar{a} y \cdot \bar{a} x=\bar{a} x \cdot \bar{a} z=\bar{a} z \cdot \bar{a} x=\bar{a} y \cdot \bar{a} z=\bar{a} z \cdot \bar{a} y=0$
And $\bar{a} x \cdot \bar{a} x=1, \bar{a} y \cdot \bar{a} y=1, \bar{a} z \cdot \bar{a} z=1$
Then $\bar{A} \cdot \bar{B}=A x B x+A y B y+A z B z$
$1=2$ gives
$|\bar{A}||\bar{B}| \cos \theta=A x B x+A y B y+A z B z$
Then $\quad \cos \theta=\frac{A x B x+A y B y+A z B z}{|\bar{A}||\bar{B}|}$
Also $\bar{A} \cdot \bar{A}=|\bar{A}|^{2}=A^{2}$
$\bar{a} A \cdot \bar{a} A=1$
$\bar{A} \cdot \bar{a} x=A x, \bar{A} \cdot \bar{a} y=A y, \bar{A} \cdot \bar{a} z=A z$
The projection of any vector on other one can be given by using the dot product.

Projection of $\bar{B}$ on $\bar{a} y$ is
$|\bar{B}| \cos \theta$ but

$\bar{B} \cdot \bar{a} y=|\bar{B}||\bar{a} y| \cos \theta$
Then $|\bar{B}| \cos \theta=\frac{\bar{B} \cdot \bar{a} y}{|\bar{a} y|}=\bar{B} \cdot \bar{a} y=B y$ (scalar)
Vector projection is:-
$(\bar{B} \cdot \bar{a} y) \bar{a} y=B y \bar{a} y$
Projection of $\bar{A} o n \bar{B}$ is

$\bar{A} \cdot \bar{B}=|\bar{A}||\bar{B}| \cos \theta$
$\therefore|\bar{A}| \cos \theta=\frac{\bar{A} \cdot \bar{B}}{|\bar{B}|}=$ Scalar
Vector projection of $\bar{A}$ on $\bar{B}$ is
$\left(\frac{\bar{A} \cdot \bar{B}}{|B|}\right) \bar{a} B=$ Vector

Ex1:- given $\bar{F}=2 \bar{a} x-5 \bar{a} y-4 \bar{a} z$ and $\bar{G}=3 \bar{a} x+5 \bar{a} y+2 \bar{a} z$ find $a-$ $\bar{F} . \bar{G}$ b- the angle between $\bar{F}$ and $\bar{G}$ c- the length of the projection of $\bar{F}$ on $\bar{G}$ d- the vector projection of $\bar{F}$ on $\bar{G}$

## Solution:-

$$
\begin{aligned}
& \text { a- } \bar{F} \cdot \bar{G}=F x G x+F y G y+F z G z \\
& \bar{F} \cdot \bar{G}=6-25-8=-27
\end{aligned}
$$

b- $\quad \cos \theta=\frac{\bar{F} \cdot \bar{G}}{|\bar{F}||\bar{G}|}=\frac{-27}{\sqrt{4+25+16} \sqrt{9+25+4}}$
$\cos \theta=\frac{-27}{\sqrt{45} \sqrt{38}}=\frac{-27}{(6.708)(6.164)}=-0.6$

$$
\therefore \theta=130.8
$$

C-
proj of $\bar{F}$ on $\bar{G}$ is $\frac{\bar{F} \cdot \bar{G}}{|\bar{G}|}=\frac{27}{\sqrt{38}}=4.38$ (Length)
d-
$\bar{a} G=\frac{\bar{G}}{|\bar{G}|}=\frac{3}{6.164} \bar{a} x+\frac{5}{6.164} \bar{a} y+\frac{2}{6.164} \bar{a} z$
$\left(\frac{\bar{F} \cdot \bar{G}}{|\bar{G}|}\right) \bar{a} G=\left(\frac{-27}{\sqrt{38}}\right) \bar{a} G$
$=-2.13 \bar{a} x-3.55 \bar{a} y-1.42 \bar{a} z$

The cross product:-

It is also called the vector product, for the two vector $\bar{A}$ and $\bar{B}$ then :
$\bar{A} \times \bar{B}=$ vector $\bar{C}$ normal to $\bar{A}, \bar{B}$
And

$$
\bar{A} \times \bar{B}=|\bar{A}||\bar{B}| \sin \theta \bar{a} n
$$

$|\bar{A} \times \bar{B}|=A B \sin \theta=|\bar{G}|$
$\bar{B} \times \bar{A}=-(\bar{A} \times \bar{B})$
If the definition is applied to the unit vectors then
$\bar{a} x \times \bar{a} x=0, \quad \bar{a} y \times \bar{a} y=0, \quad \bar{a} z \times \bar{a} z=0$
and $\bar{a} x \times \bar{a} y=\bar{a} z, \bar{a} y \times \bar{a} z=\bar{a} x, \bar{a} z \times \bar{a} x=\bar{a} y$
and:
$\bar{a} x$

$\bar{a} y \times \bar{a} x=-\bar{a} z$
For $\bar{A}=A x \bar{a} x+A y \bar{a} y+A z \bar{a} z$
and $\bar{B}=B x \bar{a} x+B y \bar{a} y+B z \bar{a} z$
We have:-
$\bar{A} \times \bar{B}=(A x \bar{a} x+A y \bar{a} y+A z \bar{a} z) \times(B x \bar{a} x+B y \bar{a} y+B z \bar{a} z)$
$=(\mathrm{AyBz}-\mathrm{AzBy}) \bar{a} x+(A z B x-A x B z) \bar{a} y+(A x B y-A y B x) \bar{a} z$
or written as a determinant:-
$\bar{A} \times \bar{B}=\left|\begin{array}{lll}\bar{a} x & \bar{a} y & \bar{a} z \\ A x & A y & A z \\ B x & B y & B z\end{array}\right|$
Ex 1:- For $\bar{A}=2 \bar{a} x-3 \bar{a} y+\bar{a} z$ and $\bar{B}=-4 \bar{a} x-2 \bar{a} y+5 \bar{a} z$ find $\bar{A} \times$ $\bar{B},|\bar{A} \times \bar{B}|$, angle $\theta$ between $\bar{A}, \bar{B}, \bar{a} n$.

## Solution:-

$\bar{A} \times \bar{B}=\left|\begin{array}{ccc}\bar{a} x & \bar{a} y & \bar{a} z \\ 2 & -3 & 1 \\ -4 & -2 & 5\end{array}\right|=-13 \bar{a} x-14 \bar{a} y-16 \bar{a} z=\bar{C}$
$|\bar{A} \times \bar{B}|=|\bar{C}|=\sqrt{(13)^{2}+(14)^{2}+(16)^{2}}=24.92$
$|\bar{A} \times \bar{B}|=A B \sin \theta$ Then $\sin \theta=\frac{|\bar{A} \times \bar{B}|}{|\bar{A}||\bar{B}|}$
$\sin \theta=\frac{24.92}{(3.74)(6.7)} \rightarrow \sin \theta=0.99$
$\theta=82$
$\bar{a} n=\bar{a} c=\frac{\bar{C}}{|\bar{C}|}=\frac{\bar{A} \times \bar{B}}{|\bar{A} \times \bar{B}|}=\frac{-13 \bar{a} x-14 \bar{a} y-16 \bar{a} z}{24.92}$
Then $\bar{a} n=-0.52 \bar{a} x-0.56 \bar{a} y-0.642 \bar{a} z$

## Circular cylindrical coordinate:-

In this system ,the position can be defined by radius of cylinder $(\rho)$ angle $(\varnothing)$ with x -axis and the z - axis. $\mathrm{P}(\rho, \varnothing, z)$
$\rho$-axis
$\emptyset-a x i s$
z - axis
planes :
$\rho \emptyset-p l a n e(z=$ constant $)$
$\rho z-$ plane $\quad(\emptyset=$ constant $)$
$\mathrm{z} \emptyset-$ plane $\quad(\rho=\mathrm{constant})$
unit vector:
$\bar{a} \rho$ in $\rho$-direction.

All these axes and plane are perpendicular to each other. Also $\bar{a} \rho \cdot \bar{a} \rho=$ 1, $\bar{a} \rho \cdot \bar{a} \emptyset=0$

$$
\bar{a} \rho \times \bar{a} \emptyset=\bar{a} z, \quad \bar{a} \emptyset \times \bar{a} z=\bar{a} \rho, \quad \bar{a} z \times \bar{a} \emptyset=\bar{a} \rho
$$



A differential elements can be given as
$\rho \longrightarrow d \rho$
$\emptyset \longrightarrow \rho d \emptyset$
$\mathrm{z} \longrightarrow d z$
$\mathrm{d} \bar{l}=d \rho \bar{a} \rho+\rho d \emptyset \bar{a} \emptyset+d z \bar{a} z$
$\mathrm{d} \bar{s} 1=\rho d \rho d \emptyset \bar{a} Z$
$\mathrm{d} \bar{s} 2=\mathrm{d} \rho d z \bar{a} \emptyset$
$\mathrm{d} \bar{s} 3=\rho d \emptyset d z \bar{a} \rho$
Relation of coordinate axes with Cartesian:
$\mathrm{x}=\rho \cos \emptyset \quad \rho=\sqrt{x^{2}+y^{2}}$
$\mathrm{y}=\rho \sin \emptyset \quad \emptyset=\tan ^{-1} \frac{y}{x}$
$\mathrm{Z}=\mathrm{Z} \quad \mathrm{Z}=\mathrm{Z}$
Transformation of vectors can be given as

$$
\mathrm{A}=A x \bar{a} x+A y \bar{a} y+A z \bar{a} z
$$

$\bar{A}=A \rho \bar{a} \rho+A \emptyset \bar{a} \emptyset+A z \bar{a} z$
where $\mathrm{A} \rho=\overline{\bar{A}} \cdot \bar{a} \rho=(A x \bar{a} x+A y \bar{a} y+A z \bar{a} z$
$=\mathrm{Ax} \bar{a} x \cdot \bar{a} \rho+A y \bar{a} y \cdot \bar{a} \rho$
$\mathrm{A} \emptyset=\bar{A} \cdot \bar{a} \emptyset=(A x \bar{a} x+A y \bar{a} y+A z \bar{a} z) \cdot \bar{a} \emptyset$
$=\operatorname{Ax} \bar{a} x \cdot \bar{a} \emptyset+A y \bar{a} y \cdot \bar{a} \emptyset$
and $\mathrm{Az}=\bar{A} \cdot \bar{a} z=A z$
$\bar{a} x \cdot \bar{a} \rho=\cos \emptyset, \bar{a} y \cdot \bar{a} \rho=\sin \emptyset$
$\bar{a} x \cdot \bar{a} \emptyset=-\sin \emptyset, \bar{a} y \cdot \bar{a} \emptyset=\cos \emptyset$

|  | $\bar{a} \rho$ | $\bar{a} \emptyset$ | $\bar{a} z$ |
| :--- | :--- | :--- | :--- |
| $\bar{a} x$ | $\cos \emptyset$ | $-\sin \varnothing$ | 0 |
| $\bar{a} y$ | $\sin \varnothing$ | $\cos \emptyset$ | 0 |
| $\bar{a} z$ | 0 | 0 | 1 |

Ex1:- Transform the vector
$\bar{B}=y \bar{a} x-x \bar{a} y+z \bar{a} z$ into cylindrical coordinate

Solution:-
$\mathrm{B} \rho=\bar{B} \cdot \bar{a} \rho=y(\bar{a} x \cdot \bar{a} \rho)-x(\bar{a} y \cdot \bar{a} \rho)$
$=y \cos \varnothing-x \sin \varnothing=\rho \sin \varnothing \cos \varnothing-\rho \cos \emptyset \sin \varnothing=0$
$\mathrm{B} \emptyset=\bar{B} \cdot \bar{a} \emptyset=y(\bar{a} x \cdot \bar{a} \emptyset)-x(\bar{a} y \cdot \bar{a} \emptyset)$
$=-y \sin \varnothing-x \cos \emptyset=-\rho \sin ^{2} \varnothing-\rho \cos ^{2} \varnothing=-\rho$
Thus $\bar{B}=-\rho \bar{a} \emptyset+z \bar{a} z$

Ex2 :- given $P\left(6,125^{\circ},-3\right), Q(3,-1,4)$ find the distance from :a- P to the origin $\mathrm{b}-\mathrm{Q}$ perpendicular to the z -axis $\mathrm{c}-\mathrm{P}$ to the Q

Solution:-

$$
\begin{aligned}
& \text { a- } \mathrm{P}(\rho, \emptyset, z) \\
& \mathrm{P}\left(6,125^{\circ},-3\right)^{\text {* }} \\
& \mathrm{x}=\rho \cos \emptyset=6 \cos 125=-3.44 \\
& \mathrm{y}=\rho \sin \emptyset=6 \sin 125=4.915 \\
& \mathrm{z}=-3 \\
& \mathrm{r}=\sqrt{(-3.44)^{2}+(4.915)^{2}+(3)^{2}}=\sqrt{11.84+24.15+9}=6.71 \\
& \text { b- } \rho=\sqrt{x^{2}+y^{2}}=\sqrt{9+1}=3.16 \\
& \text { c- } \bar{r} P Q=(3+3.44) \bar{a} x+(-1-4.915) \bar{a} y+(4+3) \bar{a} z \\
& =6.44 \bar{a} x-5.915 \bar{a} y+7 \bar{a} z \\
& \therefore r P Q=\sqrt{41.47+34.98+49}=11.2
\end{aligned}
$$

The spherical coordinate system:-

Three coordinate axes ( $\mathrm{r}, \theta, \varnothing$ )

r : is the distance from the origin to any point ( $\mathrm{x}, \mathrm{y}, \mathrm{z}$ )
$\mathrm{r}=\sqrt{\mathrm{x}^{2}+\mathrm{y}^{2}+\mathrm{z}^{2}}$
Angle $\theta$ measured from the $z$ - axis
the third coordinate $\emptyset$ is also an angle is exactly the same angle $\emptyset$ of cylindircal coordinate

Three planes:
$\theta \emptyset-p l a n e(r=$ constant plane $)$ surface of sphere
$\theta r-p l a n e(\varnothing=$ constant plane $)$
$\mathrm{r} \emptyset-$ plane $(\theta=$ constant plane $)$
unit vectors: $\bar{a} r$ in direction of $\bar{r}$
$\bar{a} \theta$ in $\theta$ - direction
$\bar{a} \emptyset$ in $\emptyset$ - direction
These axes, plane and unit vectors are perpendicular to each other
Differential elements

$\mathrm{r} \longrightarrow d r$ and $d \bar{l}=d r \bar{a} r$
$\theta \rightarrow r d \theta=d l$
$\emptyset \rightarrow r \sin \theta d \emptyset=d l$
$\bar{a} r \cdot \bar{a} r=1, \bar{a} r \cdot \bar{a} \theta=0$
$\bar{a} r \times \bar{a} \theta=\bar{a} \emptyset \ldots$
$\bar{a} r \times \bar{a} r=0$

## Differential surfaces:-

$\mathrm{d} \bar{s} 1=r^{2} \sin \theta d \theta d \emptyset \bar{a} r \quad$ in $\theta \emptyset-p l a n e$
$\mathrm{d} \bar{s} 2=r d \theta d r \bar{a} \emptyset \quad$ in $r \theta-$ plane
$\mathrm{d} \bar{s} 3=r \sin \theta d \emptyset d r \bar{a} \theta \quad$ in $r \varnothing-$ plane
And $d v=r^{2} \sin \theta d \theta d \emptyset d r$
Transformations can be done:
$\mathrm{x}=\mathrm{r} \sin \theta \cos \emptyset$
$y=r \sin \theta \sin \varnothing$
$\mathrm{z}=\mathrm{r} \cos \theta$
and $\mathrm{r}=\sqrt{\mathrm{x}^{2}+\mathrm{y}^{2}+z^{2}} \quad r \geq 0(0 \rightarrow \infty)$
$\theta=\cos ^{-1} \frac{z}{\sqrt{x^{2}+y^{2}+z^{2}}} \quad 0 \leq \theta \leq 180^{\circ} \quad(0 \rightarrow \pi)$

|  | $\bar{a} r$ | $\bar{a} \theta$ | $\bar{a} \emptyset$ |
| :--- | :--- | :--- | :--- |
| $\bar{a} x$. | $\sin \theta \cos \emptyset$ | $\cos \theta \cos \emptyset$ | $-\sin \emptyset$ |
| $\bar{a} y$ | $\sin \theta \sin \varnothing$ | $\cos \theta \sin \varnothing$ | $\cos \emptyset$ |
| $\bar{a} z$ | $\cos \theta$ | $-\sin \theta$ | 0 |

$\varnothing=\tan ^{-1}\left(\frac{y}{x}\right) \quad 0 \leq \emptyset \leq 360 \quad(0 \rightarrow 2 \pi)$
$\bar{a} z \cdot \bar{a} r=\cos \theta$
$\bar{a} z \cdot \bar{a} \theta=-\sin \theta$
$\bar{a} z \cdot \bar{a} \emptyset=0$

Projecting on the x -axis
Ex:- Given the vector $\bar{G}=(x z / y) \bar{a} x$. express this vector in spherical coordinates.

## Solution:-

$\mathrm{Gr}=\bar{G} \cdot \bar{a} r=\frac{x z}{y} \bar{a} x \cdot \bar{a} r=\frac{x z}{y} \sin \theta \cos \emptyset=r \sin \theta \cos \theta \frac{\cos ^{2} \varnothing}{\sin \varnothing}$
$\mathrm{G} \theta=\bar{G} \cdot \bar{a} \theta=\frac{x z}{y} \bar{a} x \cdot \bar{a} \theta=\frac{x z}{y} \cos \theta \cos \emptyset=r \cos ^{2} \theta \frac{\cos ^{2} \phi}{\sin \phi}$
$\mathrm{G} \emptyset=\bar{G} \cdot \bar{a} \theta=\frac{x z}{y} \bar{a} x \cdot \bar{a} \emptyset=\frac{x z}{y}(-\sin \varnothing)=-r \cos \theta \cos \varnothing$
$\therefore \bar{G}=r \cos \theta \cos \emptyset(\sin \theta \cot \varnothing a r+\cos \theta \cot \varnothing a \theta-\bar{a} \emptyset$

## 2. Coulomb's law and Electric field intensity:-

Charges either positive or negative with different shapes (points, line, Surface, or volume) distributions. These charges give electric force on each other electric field electric flux.....


Charge of one electron is $-1.6 \times 10^{-19} c=$ charge of proton
A negative charge of one coulomb represents about $6 \times 10^{18}$ elect.

## Coulomb`s law:-

Stated that the force between two very small objects separated in a vacuum or free space by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the distance between them or.
$|\bar{F}|=K \frac{Q 1 Q 2}{R^{2}}=|\bar{F} 1|=|\bar{F} 2|$

F1 Q1

$$
\mathrm{K}=\frac{1}{4 \pi \epsilon_{0}}=9 \times 10^{9}
$$

Where $\epsilon_{0}=8.85 \times 10^{-12} \frac{F}{m}=$ permittivity offree space
Then $|\bar{F}|=\frac{Q 1 Q 2}{4 \pi \epsilon_{0} R^{2}}$
Along the line join the two charges so the force
On Q2 is:
$\bar{F} 2=\frac{Q 1 Q 2}{4 \pi \epsilon_{0} R_{12}^{2}} \bar{a}_{12}=-\bar{F} 1$
$\bar{a}_{12}=\frac{\bar{R}_{12}}{\left|R_{12}\right|}=$ unit vector alng $\bar{R}_{12}$
For many point charges then $\bar{F}$ on each one is the sum of the forces from other
$\bar{F}=\bar{F} 1+\bar{F} 2+\bar{F} 3+\cdots$
Ex:- Two point charges in free space $\mathrm{Q} 1=3 \times 10^{-4} c$ at $\mathrm{p}(1,2,3)$ and $\mathrm{Q} 2=-10^{-4} c$ at $(2,0,5)$ find $\bar{F} 1$ on $Q 1$ and $\bar{F} 2$ on $Q 2$.

## Solution:-


$\bar{R}_{12}$ from Q1 to Q2 is
$\bar{R}_{12}=(2-1) \bar{a} x+(0-2) \bar{a} y+(5-3) \bar{a} z$
$\therefore \bar{R}_{12}=\bar{a} x-2 \bar{a} y+2 \bar{a} z$
$\bar{a}_{12}=\frac{\bar{R}_{12}}{\left|\bar{R}_{12}\right|}=\frac{\bar{a} x-2 \bar{a} y+2 \bar{a} z}{3}$
$\therefore \bar{F} 2=\frac{Q 1 Q 2 \bar{a}_{12}}{4 \pi \epsilon_{0} \bar{R}_{12}}=9 \times 10^{9} \frac{3 \times 10^{-4}\left(-10^{-4}\right)}{9}\left(\frac{\bar{a} r-2 \bar{a} y+2 \bar{a} z}{3}\right)$
$\bar{F} 2=-30\left(\frac{\bar{a} x-2 \bar{a} y+2 \bar{a} z}{3}\right)=-10 \bar{a} x+20 \bar{a} y-20 \bar{a} z$
$\bar{F} 1=-\bar{F} 2=10 \bar{a} x-20 \bar{a} y+20 \bar{a} z N$

## Electric field intensity:-

It is defined as the force per unit charge

$$
\bar{E}=\frac{\bar{F}}{Q} \quad \text { Or } \quad \bar{F}=Q \bar{E}
$$

$\bar{E} 2=\frac{\bar{F} 2}{Q}$
$\bar{E}=\bar{E} 1+\bar{E} 2+\bar{E} 3+\cdots$

$$
\bar{a} R=\frac{\bar{R}}{|R|}
$$


$\bar{E} t$ at $P(x, y, z)$ is
$\bar{E} t=\bar{E} 1+\bar{E} 2+\bar{E} 3+\cdots$.
Where $\bar{E} 1=K \frac{Q 1}{R_{1}^{2}} \bar{a}_{1}, \bar{E} 2=K \frac{Q 2}{R_{2}^{2}} \bar{a}_{2}, \bar{E} 3=\frac{Q 3}{R_{3}^{2}} \bar{a}_{3}$
Ex:-A point charge $\mathrm{Q} 1=2 \mathrm{Mc}$ is located at $\mathrm{P} 1(-3,7,-4)$ in free space while $\mathrm{Q} 2=-5 \mathrm{Mc}$ is at $\mathrm{P} 2(2,4,-1)$ at the point $\mathrm{P} 3(12,15,18)$ find $\bar{E},|\bar{E}|, \bar{a} E$ the Force on 25 nc at this point (P3).

## Solution:-

$\bar{E}=K \frac{Q}{|\bar{R}|^{2}} \bar{a} R$

Then $\bar{R} 1=15 \bar{a} x+8 \bar{a} y+22 \bar{a} z$
$|\bar{R} 1|=\sqrt{15^{2}+8^{2}+22^{2}}=27.8$

$$
\bar{a}_{1}=\frac{\bar{R}}{|\bar{R}|}=\frac{15 \bar{a} x}{27.8}+\frac{8 \bar{a} y}{27.8}+\frac{22 \bar{a} z}{27.8}=0.539 \bar{a} x+0.287 \bar{a} y+0.79 \bar{a} z
$$

$$
\therefore \bar{E} 1=9 \times 10^{9} \frac{2 \times 10^{-6}}{773}(0.539 \bar{a} x+0.287 \bar{a} y+0.79 \bar{a} z)
$$

$$
\bar{E} 2=K \frac{Q 2}{R_{2}^{2}} \bar{a}_{2} \text { but } \bar{R}_{2}=10 \bar{a} x+11 \bar{a} y+19 \bar{a} z
$$

$$
\left|\bar{R}_{2}\right|=\sqrt{10^{2}+11^{2}+19^{2}}=24.1
$$

$\therefore \bar{E} 2=9 \times 10^{9}\left(\frac{-5 \times 10^{-6}}{582}\right)\left(\frac{10}{24.1} \bar{a} x+\frac{11}{24.1} \bar{a} y+\frac{19}{24.1} \bar{a} z\right)$
$\therefore \bar{E}=\bar{E} 1+\bar{E} 2=-19.5 \bar{a} x-28.5 \bar{a} y-42.4 \bar{a} z \frac{v}{m}$
$|\bar{E}|=54.7$
$\bar{a} E=\frac{\bar{E}}{|\bar{E}|}=-0.356 \bar{a} x-0.521 \bar{a} y-0.776 \bar{a} z$
$\bar{F}=Q \bar{E}$
So the field of n point charges is given by:
$\bar{E}(r)=\frac{Q 1}{4 \pi \epsilon_{0} R_{1}^{2}} \bar{a}_{1}+\frac{Q 2}{4 \pi \epsilon_{0} R_{2}^{2}} \bar{a}_{2}+\frac{Q 3}{4 \pi \epsilon_{0} R_{3}^{2}} \bar{a}_{3}+\cdots$.
Or $\bar{E}(r) \sum_{m=1}^{n} \quad \frac{Q m}{4 \pi \epsilon_{0} R_{m}^{2}} \bar{a}_{m} \quad \frac{v}{m}$

## Field due to a continuous volume charge distribution:

Charges may be given as volume charge distribution with density of $\boldsymbol{\rho}_{v}\left(\frac{c}{m^{3}}\right)$ or can be distributed on surface with density $\boldsymbol{\rho}_{\boldsymbol{s}}\left(\frac{c}{m^{2}}\right)$ or distributed on line with density $\rho_{l}\left(\frac{c}{m}\right)$
if we have small amount of charge $\Delta Q$ in a small volume $\Delta \mathrm{v}$ then
$\Delta Q=\rho_{v} \Delta \mathrm{v}$
Or $\mathrm{Q}=\iiint \rho_{V} d v=\int_{v} \rho_{v} d v$
For the surface
$\mathrm{Q}=\iint \rho_{s} d s$ and for the line
$\int \rho_{l} d l$
Then ${ }_{v} \int$ for volume, ${ }_{s} \int$ for surface , $\int$ for line
$\mathrm{dl}, \mathrm{ds}$ and dv are given for the three coordinate system.

Ex1: - calculate the total charge with in each distribution
a- $10 \frac{\mathrm{Mc}}{\mathrm{m}}$ on circular region $\rho=2,0 \leq \emptyset \leq 180^{\circ}$
b- $10 \frac{M c}{m^{2}}$ on $\mathrm{r}=40 \leq \theta \leq 90^{\circ}$
c- $10 \frac{M C}{m^{3}}$ through $1 \leq x \leq 2,2 \leq y \leq 3,1 \leq z \leq 4$
Solution:-
a- $\mathrm{Q}=\int \rho l d l=\int \rho l(\rho d \emptyset)=10 \times 10^{-6} \int_{0}^{\pi} \rho d \emptyset$

$$
\therefore Q=10 \times 10^{-6}(2)[\varnothing]_{0}^{\pi}=20 \pi M c
$$

$\mathrm{b}-\mathrm{Q}=\iint \rho s d s=\int_{\emptyset} \int_{\theta} 10 \times 10^{-6}\left(r^{2} \sin \theta d \theta d \varnothing\right)$

$$
\begin{aligned}
& =160 \times 10^{-6}\left(\int_{0}^{\frac{\pi}{2}} \sin \theta d \theta\right)\left(\int_{0}^{2 \pi} d \emptyset\right)=160(2 \pi) \times 10^{-6}[-\cos \theta]_{0}^{\frac{\pi}{2}} \\
& =320 \pi \times 10^{-6}{ }^{\circ} \mathrm{C} \\
& \text { c- } \mathrm{Q}=\iiint \rho v d v=\iiint \rho v(d x d y d z)
\end{aligned}
$$

$10 \times 10^{-6} \int_{z} \int_{y} \int_{z} d x d y d z$
$\therefore Q=10 \times 10^{-6}[x]_{1}^{2}[y]_{2}^{3}[z]_{1}^{4}=30 \mathrm{Mc}$
Ex2:-calculate the total charge in electron beam shown with $\mathrm{l}=2 \mathrm{~cm}$
length assuming that the charge density be:
$\rho v=-5 \times 10^{-6}\left(e^{-10^{5 \rho z}}\right) \mathrm{c} / m^{3}$, radius $=1 \mathrm{~cm}$

## Solution:-

$\mathrm{Q}=\iiint \rho v d v$
But dv $=\rho d \rho d \emptyset d z$
$\mathrm{Q}=\int_{0.02}^{0.04} \int_{0}^{2 \pi} \int_{0}^{0,01}-5 \times 10^{-6} e^{-10^{5 \rho z}} \rho d \rho d \emptyset d z$
$=\int_{0.02}^{0.04} \int_{0}^{0.01}-10^{-5} \pi e^{-10^{5 \rho z}} \rho d \rho d z$
$\mathrm{Q}=\int_{0}^{0.01}\left(\frac{-10^{-5} \pi}{-10^{5} \rho} e^{-10^{5} \rho z} \rho d \rho\right)^{z=0.04}$
$=\int_{0}^{0.01}-10^{-10} \pi\left(e^{-2000 \rho}-e^{-4000 \rho}\right) d \rho$
$=-10^{-10} \pi\left[\frac{e^{-2000 \rho}}{-2000}-\frac{e^{-4000 \rho}}{-4000}\right]_{0}^{0.01}$
$=-10^{-10} \pi\left[\frac{1}{2000}-\frac{1}{4000}\right]=\frac{-\pi}{40} p c$

## Field of a line charge:-

Let us assume a straight line charge extending a long the z-axis in a cylindrical coordinate system from -L1 to L2 (for finit line ) or from $\bowtie t o \bowtie$ (for infinit line), this line with uniform charge density $\rho l \frac{c}{m}$.

To find the electric field $(\bar{E})$ at any point near this as shown in figure below then use the differential element dl with total charge dq where
$\mathrm{dQ}=\rho l d l, d \bar{E}=\frac{d Q}{4 \pi \epsilon^{0} R_{2}^{2}} \quad \bar{a} R$
and
$\mathrm{d} \bar{E}=\frac{\rho l d l}{4 \pi \epsilon^{0} R_{2}^{2}} \bar{a} R$
$\bar{R}=\rho \bar{a} \rho \pm z \bar{a} z$ and $|\bar{R}|=\sqrt{\rho^{2}+z^{2}}$
$\bar{a} R=\frac{\rho \bar{a} \rho}{\sqrt{\rho^{2}+z^{2}}} \pm \frac{z}{\sqrt{\rho^{2}+z^{2}}} \bar{a} z$
$d \mathrm{~d}=\mathrm{d} / \mathrm{z}$
$\bar{E}=\bar{E}_{\rho}+\bar{E}_{Z}$
$\mathrm{dE} \rho=\frac{\rho l d z \rho}{4 \pi \epsilon^{0}\left(\rho^{2}+z^{2}\right)^{\frac{3}{2}}}=\frac{\rho l d z \sin \theta}{4 \pi \epsilon^{0}\left(\rho^{2}+z^{2}\right)}$
for all the element then

$$
\begin{aligned}
& E_{\rho}=\int_{-l 1}^{l 2} \frac{\rho l d z \rho}{4 \pi \epsilon^{0}\left(\rho^{2}+z^{2}\right)^{\frac{3}{2}}} \int \frac{d z}{\left(a z^{2}+b\right)^{m}}=\frac{z}{a(m-1)\left(a z^{2}+b\right)^{m}} \\
& \bar{E}=\frac{\rho l}{4 \pi \epsilon^{0} \rho}\left[\frac{z}{\sqrt{\rho^{2}+z^{2}}}\right]_{-\infty}^{\infty} \bar{a} \rho
\end{aligned}
$$

$$
\therefore \bar{E}=\frac{\rho l}{2 \pi \epsilon^{0} \rho} \bar{a} \rho \quad \mathrm{v} / \mathrm{m}
$$

$\mathrm{Ez}=\int_{-\infty}^{\infty} \frac{\rho l d z(z)}{4 \pi \epsilon_{0}\left(\rho^{2}+z^{2}\right)^{\frac{3}{2}}}=0 \quad$ (for summetry line)
There are many other ways of obtaining this field one of other, using the angle $\theta$ where $z=\rho \cot \theta$ and $\mathrm{dz}=-\rho \csc ^{2} \theta d \theta$ and $\mathrm{R}=\rho \csc \theta$ use these in the above relation to get the same answer .

Ex:- 20 Mc is distributed on infinite line $(\mathrm{x}=\mathrm{z}=0)$ find $\bar{E}$ at $(3,4,8)$ then estimate $\bar{F}$ on 5 nc located at this point.

## Solution:-

$\bar{E}=\frac{\rho l}{2 \pi \epsilon_{0} \rho} \bar{a} \rho$
But $\bar{\rho}=3 \bar{a} x+8 \bar{a} z$
$\rho=|\bar{\rho}|=\sqrt{9+64}=\sqrt{73}=8.54$
$\bar{a} \rho=\frac{\bar{\rho}}{|\bar{\rho}|}=\frac{3}{8.54} \bar{a} x+\frac{8}{8.54} \bar{a} z=0.53 \bar{a} x+0.936 \bar{a} z$
$\therefore \bar{E}=\frac{20 \times 10^{-6}}{2 \pi \epsilon_{0}}(0.35 \bar{a} x+0.936 \bar{a} z)=() \bar{a} x+() \bar{a} z$

## Field of a sheet of charg :-

Consider infinite sheet of charge having auniform desity of $\rho s \frac{c}{m^{2}}$. if this sheet is placed in the yz-plane and symmetry in the $y, z$ then the resultant field will be in the x-direction only. Hence only Ex is present and it is a function of $x$.

By dividing the infinite sheet into differential-width (dy) strips then
$\rho l=\rho s d y$

And $\bar{R}=x \bar{a} x+y \bar{a} y$
$|\bar{R}|=\sqrt{x^{2}+y^{2}}$ and $\bar{a} R=\frac{x \bar{a} x}{\sqrt{x^{2}+y^{2}}} \pm \frac{y \bar{a} y}{\sqrt{x^{2}+y^{2}}}$
By symmetry there is no-y components.
$\therefore d \bar{E} x=\frac{\rho s d y}{2 \pi \epsilon_{0} \sqrt{x^{2}+y^{2}}}\left(\frac{x}{\sqrt{x^{2}+y^{2}}}\right) \quad$ (for on strip)
Then for all the strips
$\left.\bar{E}=\frac{\rho s}{2 \pi \epsilon_{0}} \int_{-\infty}^{\infty} \frac{x d y}{\left(x^{2}+y^{2}\right)} \bar{a} x=\frac{\rho s}{2 \pi \epsilon_{0}} \int_{-\infty}^{\infty} \frac{x d y}{\left(x^{2}+y^{2}\right)} \bar{a} x=\frac{\rho s}{2 \pi \epsilon_{0}} \tan ^{-1}\left(\frac{y}{x}\right)\right]_{-\infty}^{\infty} \bar{a} x$
$\therefore \bar{E}=\frac{\rho s}{2 \pi \epsilon_{0}} \bar{a} x \quad \frac{v}{m}($ for $x>0)$
And $\bar{E}=\frac{-\rho s}{2 \epsilon_{0}} \bar{a} x \quad \frac{v}{m} \quad($ for $x<0)$
Or in general

$$
\bar{E}=\frac{\rho s}{2 \epsilon_{0}} \bar{a} n \frac{v}{m}
$$

Ex:- Three uniform sheets of charge are located in free space as follows: $2 \mu \mathrm{c} / m^{2}$ at $\mathrm{x}=-3,-5 \mu \mathrm{c} / \mathrm{m}^{2}$ at $\mathrm{x}=1$, and $4 \mu \mathrm{c} / m^{2}$ at $\mathrm{x}=5$. Determine $\bar{E}$ at the point

$$
\begin{aligned}
& \text { a- At }(0,0,0) \\
& \overline{E 1}=\frac{\rho s 1}{2 \epsilon_{0}} \bar{a} x=\frac{2 \bar{a} x}{2 \epsilon_{0}} \times 10^{-6} \\
& =\frac{10^{6}}{8.85} \bar{a} x \\
& \bar{E} 2=\frac{\rho s 2}{2 \epsilon_{0}}(-a x)=\frac{-5 \times 10^{-6}}{2 \epsilon_{0}}(-a x) \\
& =\frac{5 \times 10^{6}}{2 \times 8.85} \bar{a} x
\end{aligned}
$$

$$
\begin{aligned}
& \bar{E} 3=\frac{\rho s 3}{2 \epsilon_{0}}(-a x)=\frac{4 \times 10^{-6}}{2 \epsilon_{0}}(-a x)=\frac{-2 \times 10^{6}}{8.85} \bar{a} x \\
& \therefore \bar{E}=\bar{E} 1+\bar{E} 2+\bar{E} 3=169.4 \bar{a} x \frac{k v}{m} \\
& \text { b- At }(2.5,-1.6,4.7) \\
& \therefore \bar{E}=\frac{1}{2 \epsilon_{0}}\left[2 \times 10^{-6} \bar{a} x-5 \times 10^{-6} \bar{a} x-4 \times 10^{-6} a x\right] \\
& =-395 \bar{a} x \frac{k v}{m} \\
& \text { c- } \bar{E}=\bar{E} 1+\bar{E} 2+\bar{E} 3=\frac{1}{2 \epsilon_{0}}\left[2 \times 10^{-6} \bar{a} x-5 \times 10^{-6} \bar{a} x+4 \times\right. \\
& \left.10^{-6} \bar{a} x\right] \\
& =56.5 \bar{a} x \frac{k v}{m} \\
& \mathrm{~d}-\bar{E}=\frac{1}{2 \epsilon_{0}}\left[-2 \times 10^{-6} \bar{a} x+5 \times 10^{-6} \bar{a} x-4 \times 10^{-6} \bar{a} x\right] \\
& =-56.5 \bar{a} x \frac{k v}{m} \\
& \bar{F}=Q \bar{E} \quad(a t \text { any point })
\end{aligned}
$$

## Streamlines and sketches of fields

Any charge of certain location and distribution has electric field lines can be drawn either by using a test charge or mathematically using line equation, for simple cases such as point charges charges these lines are:
$\mathrm{aR}=\frac{\bar{R}}{|R|}$
In two dimensional field (xy-plane) , the streamlines can be given using the relation :
$\frac{E y}{E x}=\frac{d y}{d x}$

Using the values of Ex, Ey then solve the resultant differential equation to get the equation of the streamlines

Ex:- uniform infinite line charge with $\rho l=2 \pi \epsilon_{0}$ draw the streamlines in the $x y$-plane.

## Solution:-

$\bar{E}=\frac{\rho l}{2 \pi \epsilon_{0} \rho} \bar{a} \rho=\frac{\bar{a} \rho}{\rho}=\frac{1}{\sqrt{x^{2}+y^{2}}}(\cos \emptyset \bar{a} x+\sin \emptyset \bar{a} y)$

$$
\therefore \bar{E}=\frac{x}{x^{2}+y^{2}} \bar{a} x+\frac{y}{x^{2}+y^{2}} \bar{a} y\left(\text { then using } \frac{E x}{E y}=\frac{d x}{d y}\right)
$$

But $\mathrm{Ex}=\frac{x}{x^{2}+y^{2}}$ and $E y=\frac{y}{x^{2}+y^{2}}$
$\therefore \frac{d y}{d x}=\frac{E y}{E x}=\frac{y}{x}$ or $\frac{d y}{y}=\frac{d x}{x}$
Therefore
$\operatorname{Ln} y=\ln x+\ln c$
$\therefore y=c x$ where $c:$ constant $0,1,-1 \ldots$
Ex:- a- find the general from of the equation for the stream lines of the field $\bar{E}=10 x y \bar{a} x+5 x^{2} \bar{a} y$
b- Sprcify the direction of $\bar{E}$ at $(2,3,-5)$

## Solution:-

a- $\frac{E y}{E x}=\frac{d y}{d x}$ then

$$
\begin{aligned}
& \frac{5 x^{2}}{10 x y}=\frac{d y}{d x} \rightarrow \frac{x}{2 y}=\frac{d y}{d x} \\
& \therefore \int 2 y d y=\int x d x
\end{aligned}
$$

$\therefore y^{2}=\frac{x^{2}}{2}+c \quad($ where $c:$ constant $)$
b- $\bar{E}=10(2)(3) \bar{a} x+5\left(2^{2}\right) \bar{a} y$
$=60 \bar{a} x+20 \bar{a} y$
$|\bar{E}|=\sqrt{60^{2}+20^{2}}=63.24$
$\therefore \bar{a} E=\frac{60}{63.24} \bar{a} x+\frac{20}{63.24} \bar{a} y$


## 3. Electric Flux Density, Gauss's law and Divergence:-

Electric flux $\psi$ is a streamline coincident with the field lines measured in coulomb. If we have appoint charge Q (coulomb). This charge gives total electric flux of

$$
\psi=Q
$$



Electric flux- density: $(\bar{D})$ measure in coulomb per $\mathrm{m}^{2}$ or lines per $\mathrm{m}^{2} . \bar{D}$ is a vector field its direction at apoint is the direction of flux lines at that point and the magnitude is given by the flux lines crossing a surface normal to the lines divided by the surface area
$|\bar{D}|=\frac{Q}{4 \pi a^{2}} \bar{a} \mathrm{r}$

$|\bar{D}|=\frac{Q}{4 \pi b^{2}} \bar{a} r \quad$ at $\mathbf{r}=\mathbf{b}$

Or at any redial distance $|\bar{D}|=\frac{Q}{4 \pi a^{2}} \bar{a} \mathrm{r} \mathrm{c} / \mathrm{m}^{2}$
But $\bar{E}$ at distance r from the point charge is
$\bar{E}=\frac{Q}{4 \pi a^{2}} \bar{a} \mathrm{r}$ therefore
$\bar{D}=\epsilon_{0} \bar{E} \quad \mathrm{C} / \mathrm{m}^{2}$

Also we can get the electric flux as

$$
\boldsymbol{\Psi}=\iint \overline{\boldsymbol{D}} \cdot \overline{\mathbf{d s}} \text { colomb }
$$


$\boldsymbol{\Psi}=\epsilon_{0} \iint \bar{E} \cdot \overline{\mathbf{d} \mathbf{s}}$
$\boldsymbol{\Psi}=\iint D d s \cos \theta$

Ex1: Given that $\bar{D}=2 x^{2} y^{2} \bar{a} x+4 x^{3} z^{2} y \bar{a} y$. Find $\boldsymbol{\psi}$ crosses the surface es:
a) $x=3,1 \leq y \leq 2,0 \leq z \leq 4$.
b) $y=2,1 \leq x \leq 3,1 \leq z \leq 4$.
c) $z=5,2 \leq x \leq 3,1 \leq y \leq 5$.

Solution:-
$\boldsymbol{\Psi}=\iint \bar{D} \cdot \overline{\mathbf{d s}}$
a) $\bar{d} s=d y d z \bar{a} x$
$\boldsymbol{\Psi}=\iint 2 x^{2} y^{2} d y d z=2(9)[z]_{0}^{4}\left[\frac{y^{3}}{3}\right]_{1}^{2}=168 c$
b) $\bar{d} s=d x d z \bar{a} y$

$$
\boldsymbol{\Psi}=\iint 4 x^{3} z^{2} y d x d z=\text { (4) (2) }\left[\frac{x^{4}}{4}\right]_{1}^{3}\left[\frac{z^{3}}{3}\right]_{1}^{4}=3360 c
$$

c) $\bar{d} s=d x d y \bar{a} z$

$$
\boldsymbol{\psi}=0
$$

Ex2:- $10 \mu \mathrm{c}$ point charge at $(0,0,0)$. Find $\boldsymbol{\psi}$ at :-
a) $r=5,0 \leq \theta \leq 90^{\circ}$
b) $\theta=30^{\circ}, 1 \leq r \leq 4,0 \leq \phi \leq 90^{\circ}$

Solution:-
a)

$$
\boldsymbol{\Psi}=\int_{0}^{2 \pi} \int_{0}^{\frac{\pi}{2}} \frac{Q}{4 \pi r^{2}}\left(r^{2} \sin \theta d \theta d \phi\right)=\frac{Q}{2}=5 \mu c
$$

$\boldsymbol{\Psi}=0$ where $\bar{D}=\frac{Q}{4 \pi r^{2}} \bar{a} r \quad$ and $d \bar{s}=r \sin \theta d \phi d r \bar{a} \theta$

## Gauss's law:-

It states that the electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

$$
\psi=\int d \psi=\oint_{s} \bar{D} \cdot d \bar{s}=Q e n c
$$

Q enclosed may be point charges line, surface or volume then

$$
\begin{aligned}
& \psi=\int d \psi=\oint_{s} \bar{D} \cdot d \bar{s}=Q=\sum_{n=1}^{N} \text { Qn point charges } \\
& \int \rho l d l \text { line } \\
& \iint \rho s d s \quad \text { surface } \\
& \iiint \rho v d v \quad \text { volume }
\end{aligned}
$$

For volume charge distribution $(\rho v)$ then

$$
\psi_{\text {total }}=\oiint \bar{D} \cdot d \bar{s}=\iiint \rho v d v=Q e n c
$$

## Applications:

a- Point charge:- consider point charge Q (coulomb) at the origin then at distance (r) the electric field $\bar{E}$ is

$$
\begin{aligned}
& \bar{E}=\frac{Q}{4 \pi \epsilon_{0} r^{2}} \bar{a} r \\
& \bar{D}=\epsilon_{0} \bar{E}=\frac{Q}{4 \pi r^{2}} \bar{a} r
\end{aligned}
$$

Choose a closed spherical surface with radius (r) then apply Gauss's law:
$\psi=\oiint \bar{D} \cdot d \bar{s}=\oiint \bar{D} \cdot\left(r^{2} \sin \theta d \theta d \emptyset \bar{a} r\right)$
$\therefore \psi=\int_{0}^{2 \pi} \int_{0}^{\pi}\left(\frac{Q}{4 \pi r^{2}} \bar{a} r\right) \cdot\left(r^{2} \sin \theta d \theta d \emptyset \bar{a} r\right)$
$=\frac{Q}{4 \pi} \int_{0}^{2 \pi} \int_{0}^{\pi} \sin \theta d \theta d \emptyset$
$=\frac{\mathrm{Q}}{4 \pi}[2 \pi][-\cos \theta]_{0}^{\pi}$
$=\frac{\mathrm{Q}}{4 \pi}[4 \pi]=Q$
So $=\psi_{\text {total }}=Q$ enclosed
b- Line charge :- for infinite line charge with $=\rho l \quad c / m$ then choose Gauss's surface as cylinder with radius $\rho$ then on this closed surface
$\mathrm{Q}=\oiint \bar{D} \cdot d \bar{s}=\iint_{\text {sides }} D s d s+0 \int_{\text {top }} d s+0 \int_{\text {bottom }} d s$
$\therefore Q=D_{s} \int_{0}^{l} \int_{0}^{2 \pi} \rho d \emptyset d z=D_{s}\left(2 \pi \rho_{l}\right)$
$\therefore D_{\rho}=\frac{Q}{2 \pi \rho_{l}} \quad$ but $\quad Q=\int \rho_{l} d l=\rho_{l} l$
$\therefore D_{\rho}=\frac{\rho_{l}}{2 \pi \rho} \quad$ or $\quad \bar{E}=\frac{\rho_{l}}{2 \pi \epsilon_{0} \rho} \bar{a} \rho$
c- Surface charge :- if we use cylindrical surface with $\rho_{s}$ on the cylinder $\rho=a$ then apply Gauss's law at $\rho<a$ and at $\rho>a$ at $\rho \leq a$ then $\oiint \bar{D} . d \bar{s}=0$ then
$\mathrm{D}=0, \mathrm{E}=0$

At $\rho \geq a$ then $Q=\iint \rho_{s} d s=\rho_{s} \int_{0}^{l} \int_{0}^{2 \pi} a d \emptyset d z$
$\mathrm{Q}=\rho_{s}[2 \pi a l]$
Then $\oiint \bar{D} \cdot d \bar{s}=2 \pi a l \rho_{s}$
$\mathrm{D}[2 \pi \rho l]=2 \pi a l \rho_{s}$

$$
\bar{D}=\frac{a \rho_{s}}{\rho} \bar{a} \rho
$$

This result can be given in terms

$$
\text { af } \rho_{l} \text { as } \rho_{l}=2 \pi a \rho_{s} \text { then } \rho_{s}=\frac{\rho_{l}}{2 \pi a}
$$

so $\bar{D}=\frac{\rho_{l}}{2 \pi \rho} \bar{a} \rho$

## d- Differential volume element:-

We are now going to apply the methods of Gauss's law to un symmetry problem, the differential volume with charge $\rho_{v} \mathrm{c} / \mathrm{m}^{3}$ for the volume $\Delta v=\Delta x \Delta y \Delta z$. and point $p(x, y, z)$ is the center of this volume where $\bar{D}$ at this point is $D_{0}=D_{x 0} \bar{a} x+D_{y 0} \bar{a} y+$ $D_{z 0} \bar{a} z c / m^{2}$ apply Gauss's law to this surface then :
$\oint_{S} \bar{D} . d \bar{s}=Q$

the closed integral must be broken up into six intgrals on over
Each face

$$
\begin{aligned}
\oint \bar{D} \cdot d \bar{s}= & \iint \bar{D} \cdot d \bar{s}_{1}+\iint \bar{D} \cdot d s_{2}+\iint \bar{D} \cdot d \bar{s}_{3}+\iint \bar{D} \cdot d \bar{s}_{4}+\iint \bar{D} \cdot d \bar{s}_{5} \\
& +\iint \bar{D} \cdot d \bar{s}_{6}
\end{aligned}
$$

Where $\mathrm{d} \bar{s}_{1}=d y d z \bar{a} x=-d \bar{s}_{4}$

$$
\begin{gathered}
\mathrm{d} \bar{s}_{2}=d x d z \bar{a} y=-d \bar{s}_{5} \\
\mathrm{~d} \bar{s}_{3}=d x d y \bar{a} z=-d \bar{s}_{6}
\end{gathered}
$$

then $\iint \bar{D} . d \bar{s}_{1}+\iint \bar{D} . d \bar{s}_{4}=\frac{\partial D x}{\partial x} \Delta x \Delta y \Delta z$
$\iint \bar{D} . d \bar{s}_{2}+\iint \bar{D} . d \bar{s}_{5}=\frac{\partial D y}{\partial y} \Delta x \Delta y \Delta z$
$\iint \bar{D} \cdot d \bar{s}_{3}+\iint \bar{D} \cdot d \bar{s}_{6}=\frac{\partial D z}{\partial z} \Delta x \Delta y \Delta z$
$\therefore \oiint \bar{D} \cdot d \bar{s}=Q=\left(\frac{\partial D x}{\partial x}+\frac{\partial D y}{\partial y}+\frac{\partial D z}{\partial z}\right) \Delta v$

## Divergence:-

from the above relation we have :
$\frac{\partial D x}{\partial x}+\frac{\partial D y}{\partial y}+\frac{\partial D z}{\partial z}=\frac{\oiint \bar{D} \cdot d \bar{s}}{\Delta v}=\frac{Q}{\Delta v}$
Or , as a limit
$\frac{\partial D x}{\partial x}+\frac{\partial D y}{\partial y}+\frac{\partial D z}{\partial z}=\lim _{\Delta v \rightarrow 0} \frac{\oiint \bar{D} \cdot d \bar{s}}{\Delta v}=\lim _{\Delta v \rightarrow 0} \frac{Q}{\Delta v}$
And

$$
\frac{\partial D x}{\partial x}+\frac{\partial D y}{\partial y}+\frac{\partial D z}{\partial z}=\rho_{v} \quad=\text { divergence } \bar{D}
$$

or
$\nabla . \bar{D}=\operatorname{div} \bar{D}=\frac{\partial D x}{\partial x}+\frac{\partial D y}{\partial y}+\frac{\partial D z}{\partial z}=\rho_{v}$
$\nabla . \bar{D}=\rho_{v}($ maxwell'sfirst equation)
$\operatorname{div} \bar{D}=\frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho D \rho)+\frac{1}{\rho} \frac{\partial D \emptyset}{\partial \emptyset}+\frac{\partial D z}{\partial z}$ (cylindrical)
$\operatorname{div} \bar{D}=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} D r\right)+\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}(\sin \theta D \theta)+\frac{1}{r \sin \theta} \frac{\partial D \emptyset}{\partial \emptyset} \quad$ (spherical)

## The divergence theorem:-

For $\oiint \bar{D} \cdot d \bar{s}=Q$ and letting
$\mathrm{Q}=\iiint \rho_{v} d v$ and $\nabla \cdot \bar{D}=\rho_{v}$ then

$$
\iint \bar{D} \cdot d \bar{s}=Q=\iint \rho_{v} d v=\iiint(\nabla \cdot \bar{D}) d v
$$

Or

$$
\oiint \bar{D} \cdot d \bar{s}=\iiint(\nabla \cdot \bar{D}) d v=\text { total } f u x \psi_{t o t}=Q_{e n c}
$$

This is the diver gene theorem in electrostatic.
Ex1:- determine on expression for the volume charge density that gives rise to the field:

$$
\begin{aligned}
& \text { a- } \bar{D}=e^{4 x} e^{-5 y} e^{-2 z}(2 \bar{a} x-2.5 \bar{a} y-\bar{a} z) \\
& \text { b- } \bar{D}=e^{-2 z}\left(2 \rho \emptyset \bar{a} \rho+\rho \bar{a} \emptyset-2 \rho^{2} \emptyset \bar{a} z\right)
\end{aligned}
$$

## Solution:-

a- $\nabla \cdot \bar{D}=\rho v=\frac{\partial D x}{\partial x}+\frac{\partial D y}{\partial y}+\frac{\partial D z}{\partial z}$
But $D_{x}=2 e^{4 x} e^{-5 y} e^{-2 z}$ then
$\frac{\partial D x}{\partial x}=8 e^{4 x} e^{-5 y} e^{-2 z}$
$\frac{\partial D y}{\partial y}=12.5 e^{4 x} e^{-5 y} e^{-2 z}$
$\frac{\partial D z}{\partial z}=2 e^{4 x} e^{-5 y} e^{-2 z}$

$$
\therefore \rho v=(8+12.5+2) e^{4 x} e^{-5 y} e^{-2 z}=22.5 e^{4 x} e^{-5 y} e^{-2 z}
$$

$$
\mathrm{b}-\nabla \cdot \bar{D}=\rho v=\frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho D \rho)+\frac{1}{\rho} \frac{\partial D \emptyset}{\partial \emptyset}+\frac{\partial D z}{\partial z}
$$

$D_{\rho}=2 \rho \emptyset e^{-2 z}$ then $(\rho D \rho)=2 \rho^{2} \emptyset e^{-2 z}$ and
$\frac{\partial}{\partial \rho}(\rho D \rho)=4 \rho \emptyset e^{-2 z}$
$\therefore \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho D \rho)=4 \emptyset e^{-2 z}$
$\frac{\partial D \emptyset}{\partial \emptyset}=0 \quad \tan \frac{1}{\rho} \frac{\partial D \emptyset}{\partial \emptyset}=0$
$\frac{\partial D z}{\partial z}=4 \rho^{2} \emptyset e^{-2 z}$
$\therefore \rho_{v}=4 \emptyset e^{-2 z}+4 \rho^{2} \emptyset e^{-2 z}=4 \emptyset e^{-2 z}\left(\rho^{2}+1\right) c / m^{3}$
Ex2:- The electric flux density in free space is given as
$\bar{D}=2 x y \bar{a} x+x^{2} \bar{a} y \mathrm{c} / \mathrm{m}^{2}$. Evaluate both sides of the divergence theorem for the rectangular parallelepiped formed by the planes $\mathrm{x}=0$ and $1 \mathrm{y}=0$ and 2 and $\mathrm{z}=0$ and 3 .

Solution:-
$\oiint \bar{D} \cdot d \bar{s}=\iiint(\nabla \cdot \bar{D}) d v$
L.H side R.H.side

$$
\begin{aligned}
& \iint \bar{D} \cdot d \bar{s}=\iint \bar{D} \cdot d \bar{s}_{1}+\iint \bar{D} \cdot d \bar{s}_{2}+\iint \bar{D} \cdot d \bar{s}_{3}+\iint \bar{D} \cdot d \bar{s}_{4}+\iint \bar{D} \cdot d \bar{s}_{5} \\
& \quad+\iint \bar{D} \cdot d \bar{s}_{6}
\end{aligned}
$$

But d $\bar{s}_{1}=d y d z \bar{a} x=-d \bar{s}_{4}$

$$
\begin{aligned}
& \mathrm{d} \bar{s}_{2}=d x d z \bar{a} y=-\mathrm{d} \bar{s}_{5} \\
& \mathrm{~d} \bar{s}_{3}=d x d y \bar{a} z=-\mathrm{d} \bar{s}_{6}
\end{aligned}
$$

$\therefore \oiint \bar{D} \cdot d \bar{s}=\int_{0}^{3} \int_{0}^{2}(\bar{D})_{x=1} \cdot(d y d z \bar{a} x)+\int_{0}^{3} \int_{0}^{2}(\bar{D})_{x=0}(-d y d z \bar{a} x)$
$+\int_{0}^{3} \int_{0}^{1}(\bar{D})_{y=2} \cdot(d x d z \bar{a} y)+\int_{0}^{3} \int_{0}^{1}(\bar{D})_{y=0} \cdot(-d x d z \bar{a} y)$
$\iint(\bar{D})_{z=3} \cdot(d x d y \bar{a} z)+\iint(\bar{D})_{z=0}(-d x d y \bar{a} z)$
$\therefore \oiint \bar{D} \cdot d \bar{s}=\int_{0}^{3} \int_{0}^{2}(D x)_{x=1} d y d z-\int_{0}^{3} \int_{0}^{2}(D x)_{x=0} d y d z$
$+\int_{0}^{3} \int_{0}^{1}(D y)_{y=2} d x d z-\int_{0}^{3} \int_{0}^{1}(D y)_{y=0} d x d z+0+0$
however, $(D x)_{x=0}$ and $(D y)_{y=0}=(D y)_{y=2}$, wich leaves

$$
\oiint \bar{D} \cdot d \bar{s}=\int_{0}^{3} \int_{0}^{2} 2 y d y d z=\int_{0}^{3} 4 d z=12 C
$$

R.H side

$$
\begin{aligned}
& \nabla \cdot \bar{D}=\frac{\partial D x}{\partial x}+\frac{\partial D y}{\partial y}+\frac{\partial D z}{\partial z} \\
&=\frac{\partial}{\partial x}(2 x y)+\frac{\partial}{\partial y}\left(x^{2}\right)=2 y \\
& \therefore \iiint_{0}(\nabla \cdot \bar{D}) d v=\int_{0}^{3} \int_{0}^{2} \int_{0}^{1} 2 y d y d x d z=\int_{0}^{3} \int_{0}^{2} 2 y d y d z \\
&= \int_{0}^{3} 4 d z=12 C
\end{aligned}
$$

$$
\therefore \text { L.H.S }=\text { R.H.S }=12 C
$$

The it is a prove of Gauss'slaw.

## 4- Energy and potential, Gradient, Dipole:-

if we attempt to move the test charge or any charge $(Q)$ against the electric field, we have to exert a force equal and opposite to that exerted by the field and this requires us to expend energy or do work if we wish to move the charge in the direction of the field, our energy expenditure turns out to be negative we do not do the work, the field does.

Suppose we wish to move a charge $Q$ a distance $\mathrm{d} \bar{l}$ in an electric field $\bar{E}$.
The force on Q due to the electric field is:-

$$
\begin{gathered}
\bar{F}_{E}=Q \bar{E} \text { the component of this force in the direction } d \bar{l} \text { is: }- \\
F_{E L}=\bar{F} \cdot \bar{a}_{l}=Q \bar{E} \cdot \bar{a}_{l} \quad\left(\bar{a}_{l}: \text { unit vector in } d \bar{l}\right)
\end{gathered}
$$

The force which we must apply is equal and opposite to the force due to field.

$$
F_{a p p l}=-Q \bar{E} \cdot \bar{a}_{l}
$$

And our expenditure of energy is the product of the force and distance:
$\mathrm{d} \mathrm{w}=-\mathrm{Q} \bar{E} \cdot \bar{a}_{l} d l=-Q \bar{E} \cdot d \bar{l}$
energy $=$ work done $=\mathrm{d} \mathrm{w}=-\mathrm{Q} \bar{E} \cdot d \bar{l}$
or $\mathrm{w}=-\mathrm{Q} \int_{\text {initial }}^{\text {final }} \bar{E} \cdot d \bar{l}$ Joul


So the work required to move the charge a finite distance in the electric field is

$$
w_{21}=-Q \int_{(1)}^{(2)} \bar{E} \cdot d \bar{l}=-Q \int_{(1)}^{(2)} E d l \cos \theta
$$

If $\theta=0$
$w_{21}=$ negative (maximum)
If $\theta=90^{\circ}, w_{21}=0$ (no work done)


If $\theta=180^{\circ}, w_{21}=\max (+v e)$
$\mathrm{d} \bar{l}=d x \bar{a} x+d y \bar{a} y+d z \bar{a} z($ cartesian $)$
$\mathrm{d} \bar{l}=d \rho \bar{a} \rho+\rho d \emptyset \bar{a} \emptyset+d z \bar{a} z \quad($ cylindrical)
$\mathrm{d} \bar{l}=d r \bar{a} r+r d \theta \bar{a} \theta+r \sin \theta d \emptyset \bar{a} \emptyset(s p h e r i c a l)$
Ex1:-A nonuniform electric field is given by:
$\bar{E}=y \bar{a} x+x \bar{a} y+2 \bar{a} z \frac{v}{m}$. Determine the work expended (energy)
In carrying 2 C point charge from $\mathrm{B}(1,0,1)$ to $\mathrm{A}(0.8,0.6,1)$ along the shorter arc of the circle $x^{2}+y^{2}=1, z=1$

## Solution:-

in Cartesian, $\mathrm{d} \bar{l}=\mathrm{dx} \bar{a} x+d y \bar{a} y+d z \bar{a} z$
$w_{A B}=-Q \int_{B}^{A} \bar{E} \cdot d \bar{l}=-2 \int_{B}^{A}(y \bar{a} x+x \bar{a} y+2 \bar{a} z) \cdot(\mathrm{dx} \bar{a} x+d y \bar{a} y+$ $d z \bar{a} z)$
$\therefore w_{A B}=-2 \int_{1}^{0.8} y d x-2 \int_{0}^{0.6} x d y-4 \int_{1}^{1} d \bar{z}$
Using the equation of the circle then
$\mathrm{y}=\sqrt{1-\mathrm{x}^{2}}, x=\sqrt{1-y^{2}}$
we have

$$
\mathrm{w}=-2 \int_{1}^{0.8} \sqrt{1-x^{2}} d x-2 \int_{0}^{0.6} \sqrt{1-y^{2}} d y-0
$$

using table of integration $\int \sqrt{1-x^{2}} \mathrm{dx}=\frac{1}{2}\left[x \sqrt{1-x^{2}}+\sin ^{-1} x\right]$

$$
\therefore w=-\left[x \sqrt{1-x^{2}}+\sin ^{-1} x\right]_{1}^{0.8}-\left[y \sqrt{1-y^{2}}+\sin ^{-1} y\right]_{0}^{0.6}
$$

$=-(0.48+0.927-0-1.571)-(0.48+0.644-0-0)$
$\therefore w=-0.96 \mathrm{~J}$
Ex2:- Use Ex1, then find $w_{A B}$ by slecting the straight line path

## Solution:-

We must determine the equation of the straight-line.
$X-X_{B}=\frac{X_{A}-X_{B}}{Y_{A}-Y_{B}}\left(Y-Y_{B}\right)$
$(\mathrm{X}-1)=\frac{(0.8-1)}{(0.6-0)}(y-0)$
$(\mathrm{X}-1)=\frac{-0.2}{0.6} y$
$\therefore y=-3(X-1), z=1$ (from $y z$ equ.)
Thus $\mathrm{w}=-\mathrm{Q} \int \bar{E} . d \bar{l}$ which gives
$\mathrm{W}=-2 \int_{1}^{0.8} y d x-2 \int_{0}^{0.6} x d y-4 \int_{1}^{1} d z$
$\mathrm{W}=6 \int_{1}^{0.8}(x-1) d x-2 \int_{0}^{0.6}\left(1-\frac{y}{3}\right) d y$
$\therefore w=-0.96 J$ (the same result as in ex1)

Ex3:- infinite line charge of $\rho_{l}=20 \mathrm{Mc} / \mathrm{m}$ along the z -axis. Calculate the work done to move 50 nc point charge from A to B where:

$$
\begin{aligned}
& \text { a- } \mathrm{A}\left(3,60^{\circ},-8\right), B\left(3,120^{\circ},-8\right) \\
& \text { b- } \mathrm{A}\left(3,60^{\circ},-8\right), B\left(7,60^{\circ},-8\right)
\end{aligned}
$$

Solution:-

$$
w_{B A}=-Q \int_{A}^{B} \bar{E} \cdot d \bar{l}
$$

$\operatorname{But} \bar{E}=\frac{\rho_{l}}{2 \pi \epsilon_{0} \rho} \bar{a} \rho$

And $\mathrm{d} \bar{l}=d \rho \bar{a} \rho+\rho d \emptyset \bar{a} \emptyset+d z \bar{a} z$

$$
\begin{aligned}
& \text { a- } w_{B A}= \\
& -50 \times 10^{-9}\left[\int_{\rho 1}^{\rho 2} \frac{\rho_{l}}{2 \pi \epsilon_{0} \rho} \bar{a} \rho . d \rho \bar{a} \rho+\int_{\emptyset 1}^{\varnothing 2}{\frac{\rho_{l}}{2 \pi \epsilon_{0} \rho}=0}^{=} \rho . \rho d \emptyset \bar{a} \emptyset\right. \\
& \left.+\int_{z 1}^{z 2} \frac{\rho_{l}}{2 \pi \epsilon_{0} \rho} \bar{a} \rho . d z \bar{a} z=0\right] \\
& \therefore w_{B A}=-50 \times 10^{-9} \int_{3}^{3} \frac{\rho_{l} d \rho}{2 \pi \epsilon_{0} \rho}=0 \\
& \text { b- } w_{B A}=-50 \times 10^{-9} \int_{3}^{7} \frac{\rho_{l} d \rho}{2 \pi \epsilon_{0} \rho} \\
& =-50 \times 10^{-9}\left(\frac{\rho_{l}}{2 \pi \epsilon_{0}}\right)[\ln \rho]_{3}^{7} \\
& =50 \times 10^{-9}\left(18 \times 10^{9}\right)\left(20 \times 10^{-6}\right)[\ln 3-\ln 7] \\
& =18 \times 10^{-3}[\ln 3-\ln 7] \text { Joul }
\end{aligned}
$$

## Potential difference and potential:-

The potential difference $V_{a b}$ is defined as the work done (by an external source ) in moving a unit positive charge from one point (b) to another (a)in an electric field.

$$
V_{a b}=\frac{w_{a b}}{Q}=-\int_{b(\text { init })}^{a(\text { final })} \bar{E} \cdot d \bar{l} \quad(\text { volts })
$$

$V_{a b}=V_{a}-V_{b}=V_{\text {final }}-V_{\text {init }}$


Where $V_{b}$ : potential at (b) and $V_{a}$ is the potential of (a)
Ex1:- For the point charge $\mathrm{Q}=30 \mu \mathrm{c}$ at $(0,0,0)$ find the potential difference between $b(8,6,7)$ to $a\left(4,30^{\circ}, 60^{\circ}\right)$ then estimate the potential in each point .

Solution:-
$V_{a b}=-\int_{b}^{a} \bar{E} \cdot \overline{d l}$


But $\bar{E}=\frac{Q}{4 \pi \varepsilon_{0} r^{2}} \bar{a} r(($ in spherical coordinate $))$
Then

$$
\begin{gathered}
\mathrm{d} \bar{l}=d r \bar{a} r+r d \theta \bar{a} \theta+r \sin \theta d \emptyset \bar{a} \emptyset \\
\therefore V_{a b}=-\left[\int_{r b}^{r a} \frac{Q d r}{4 \pi \varepsilon_{0} r^{2}}\right]=\frac{Q}{4 \pi \varepsilon_{0}}\left[\frac{1}{r}\right]_{r a}^{r b} \\
\therefore \frac{Q}{4 \pi \varepsilon_{0} r_{a}}-\frac{Q}{4 \pi \varepsilon_{0} r_{b}}=V_{a}-V_{b} \\
r_{b}=\sqrt{8^{2}+6^{2}+7^{2}}=12.2 \text { and } r_{a}=4 \\
V_{a b}=\frac{Q}{4 \pi \varepsilon_{0}}\left[\frac{1}{4}-\frac{1}{12.2}\right] \\
=9 \times 10^{9}\left(30 \times 10^{-6}\right)(0.168) \\
=45.36 \mathrm{KV}
\end{gathered}
$$

$$
\begin{gathered}
V_{a}=\frac{Q}{4 \pi \varepsilon_{0} r_{a}}=9 \times 10^{9}\left(30 \times 10^{-6}\right)(0.25)=67.5 \mathrm{KV} \\
V_{b}=\frac{Q}{4 \pi \varepsilon_{0} r_{b}}=9 \times 10^{9}\left(30 \times 10^{-6}\right)(0.08196)=22.14 \mathrm{KV}
\end{gathered}
$$

$$
\therefore V_{a b}=-V_{b a}=-\int_{b}^{a} \bar{E} \cdot \overline{d l}=V_{a}-V_{b}
$$

Ex2:- Two point charge $\mathrm{Q} 1=10 \mu \mathrm{c}$ at $(2,4,5)$ and $\mathrm{Q} 2=20 \mu \mathrm{c}$ at $(-3,6$, $2)$. Find the potential at $(0,0,0)$.

Solution:-

$$
\begin{aligned}
& V_{p}=\frac{Q}{4 \pi \varepsilon_{0} r_{p}} \\
& V_{1}=\frac{Q}{4 \pi \varepsilon_{0} r_{1}}=\frac{9 \times 10^{9}\left(10 \times 10^{-6}\right)}{\sqrt{2^{2}+4^{2}+5^{2}}}=\frac{9 \times 10^{4}}{6.7}=1.34 \times 10^{4} \mathrm{~V} \\
& V_{2}=\frac{Q}{4 \pi \varepsilon_{0} r_{2}}=\frac{9 \times 10^{9}\left(20 \times 10^{-6}\right)}{\sqrt{9+36+4}}=\frac{18}{7} \times 10^{4}=2.57 \times 10^{4} \mathrm{~V} \\
& \mathrm{~V}=V_{1}+V_{2}
\end{aligned}
$$

If there is a reference point then

$$
V_{a b}=-\int_{b}^{a} \bar{E} \cdot \overline{d l}=\frac{Q}{4 \pi \varepsilon_{0} r}+c
$$

Where c is constant of the integration to be calculate.

## Potential field of a system of charge:-

The potential field of a single point charge $Q_{1}$ located at $r_{1}$ is

$$
\mathrm{V}=\frac{Q_{1}}{4 \pi \epsilon^{0} r_{1}}=k \frac{Q_{1}}{r_{1}}
$$

For a group of point charges:
$V_{\rho}=\frac{Q_{1}}{4 \pi \epsilon^{0} r_{1}}+\frac{Q_{2}}{4 \pi \epsilon_{0} r_{2}}+\ldots \ldots .+\frac{Q_{N}}{4 \pi \epsilon_{0} Y_{N}}$
$\operatorname{Or} V_{\rho}=\sum_{n=1}^{n=N} \frac{Q_{n}}{4 \pi \epsilon_{0} r_{n}}$
If we have line charge with $\rho_{l} \mathrm{c} / \mathrm{m}$ then
$V_{\rho}=\int_{l} \frac{\rho_{l} d_{l}}{4 \pi \epsilon_{0} R} \quad$ volt
For a surface charge distribution then the potential volt is:

$$
V_{\rho}=\iint \frac{\rho_{s} d_{s}}{4 \pi \epsilon_{0} R}
$$

For volume $V_{\rho}$ is:-

$$
V_{\rho}=\iiint \frac{\rho_{v} d_{v}}{4 \pi \epsilon_{0} R}
$$



These are the potential at one point if we need the potential difference then $V_{\rho}$ should be calculate

Ex:- find the potential at $(0,0,10)$ caused by each of the following charge distribution in free space
a- Ring: $\rho_{l}=5^{n c} / m, \rho=4, z=0$
b- Disc: $\rho_{s}=2^{n c} / m^{2}, 0 \leq \rho \leq 4, z=0$
Solution:-
a- $V_{\rho}=\int \frac{\rho_{l} d_{l}}{4 \pi \epsilon_{0} R}$
$\mathrm{dl}=\rho d \emptyset \quad$ and $\bar{R}=-\rho \bar{a} \rho+z \bar{a} z$
$|\bar{R}|=R=\sqrt{\rho^{2}+z^{2}}=\sqrt{4^{2}+10^{2}}=\sqrt{116}$
$\therefore V_{\rho}=5 \times 10^{-9}\left(9 \times 10^{9}\right) \int_{0}^{2 \pi} \frac{4 d \emptyset}{\sqrt{116}}=\frac{45 \times 4}{\sqrt{116}}[\emptyset]_{0}^{2 \pi}=33.43 \pi$ volt
$V_{\rho}=104.9$ volt
b- $V_{\rho}=\iint \frac{\rho_{s} d_{s}}{4 \pi \epsilon_{0} R}$
$\mathrm{ds}=\rho d \rho d \emptyset$
$\mathrm{R}=\sqrt{\rho^{2}+z^{2}}=\sqrt{\rho^{2}+100}$
$\therefore V_{\rho}=2 \times 10^{-9} \times 9 \times 10^{9} \int_{\rho_{0}}^{4} \int_{\phi_{0}}^{2 \pi} \frac{\rho d \rho d \emptyset}{\sqrt{\rho^{2}+100}}=18(2 \pi) \int_{0}^{4} \frac{\rho d \rho}{\left(\rho^{2}+100\right)^{\frac{1}{2}}}$
$V_{\rho}=36 \pi \int_{0}^{4}\left(\rho^{2}+100\right)^{-\frac{1}{2}} \rho d \rho=36 \pi\left[\rho^{2}+100\right]_{0}^{4}$
$\therefore V_{\rho}=36 \pi[\sqrt{116}-\sqrt{100}]=87$ volt
H.W:- calculate the potential difference from $(0,0,10)$ to $(0,0,6)$ for both cases.

Hint: calculate $V_{\rho}$ at $(0,0,6)$ then find the difference for the tw solution.

## Potential Gradient:-

The electric field intensity $\bar{E}$ can be found from the potential field V using the reverse direction: for the general relation
$\mathrm{V}=-\int \bar{E} . d \bar{l}$ this may be applied to a very short element of length $\Delta \bar{l}$ along $\bar{E}$, the potential difference $\Delta \mathrm{V}$ is :-
$\Delta V=-\bar{E} . \Delta \bar{l}$ where $\Delta \bar{l}=\Delta \mathrm{l} \bar{a} l$
$\Delta V=-\bar{E} \cdot \Delta \bar{l} \cos \theta$
$\frac{\Delta V}{\Delta l}=-E \cos \theta$
Or $\frac{d v}{d l}=-E \cos \theta$ this can be maximum when $\theta=\pi$
Then $\left.\frac{d v}{d l}\right|_{\max }=E \quad(\bar{E}, \Delta \bar{l}$ in opposite direction $)$
$\bar{E}$ opposite to increasing in potential
$\left.\frac{d v}{d l}\right|_{\max }=\frac{d v}{d N}$
Also $\Delta V=\frac{d v}{d x} \Delta x+\frac{d v}{d y} \Delta y+\frac{d v}{d z} \Delta z$
$=\left(\frac{d v}{d x} \bar{a} x+\frac{d v}{d y} \bar{a} y+\frac{d v}{d z} \bar{a} z\right) .(\Delta x \bar{a} x+\Delta y=\Delta \bar{l} \bar{a} y+\Delta z \bar{a} z)$
$=-\bar{E} .(\Delta \bar{l})$

$$
\therefore \bar{E}=-\operatorname{grad} V=-\nabla V
$$

$\nabla$ is the deal operator:
$\nabla V=\frac{\partial v}{\partial x} \bar{a} x+\frac{\partial v}{\partial y} \bar{a} y+\frac{\partial v}{\partial z} \bar{a} z$
(Cartesian)
$\nabla V=\frac{\partial V}{\partial \rho} \bar{a} \rho+\frac{1}{\rho} \frac{\partial V}{\partial \emptyset} \bar{a} \emptyset+\frac{\partial V}{\partial z} \bar{a} z \quad$ (cylindrical)
$\nabla V=\frac{\partial V}{\partial r} \bar{a} r+\frac{1}{r} \frac{\partial V}{\partial \theta} \bar{a} \theta+\frac{1}{r \sin \theta} \frac{\partial V}{\partial \emptyset} \bar{a} \emptyset \quad$ (spherical)
Ex1:- Given that the potential $\mathrm{V}=2 x^{2} y^{3}+4 x y z^{2} m V$ find $\bar{E}$ at $(4,5,8)$.
Give expression for $\rho_{v} \mathrm{c} / \mathrm{m}^{3}$.
Solution:-
$\bar{E}=-\nabla V=-\left[\frac{\partial V}{\partial x} \bar{a} x+\frac{\partial V}{\partial y} \bar{a} y+\frac{\partial V}{\partial z} \bar{a} z\right]$
But $\frac{\partial V}{\partial x}=4 x y^{3}+4 y z^{2}$
$\frac{\partial V}{\partial y}=6 x^{2} y^{2}+4 x z^{2}$
$\frac{\partial V}{\partial z}=8 x y z$
$\therefore \bar{E}=-\left[\left(4 x y^{3}+4 y z^{2}\right) \bar{a} x+\left(6 x^{2} y^{2}+4 x z^{2}\right) \bar{a} y+(8 x y z) \bar{a} z\right]$
$\bar{D}=-E_{0}\left[\left(4 x y^{3}+4 y z^{2}\right) \bar{a} x+\left(6 x^{2} y^{2}+4 x z^{2}\right) \bar{a} y+(8 x y z) \bar{a} z\right]$
At $(4,5,8)$ then
$\bar{E}=-\left[16\left(5^{3}\right)+20(64)\right] \bar{a} x-[6(16)(25)+16(64)] \bar{a} y$ $-[(32(5) 8] \bar{a} z$
$=-3280 \bar{a} x=15424 \bar{a} y-1280 \bar{a} z$
$\rho_{v}=\nabla \cdot \bar{D}=\frac{\partial D x}{\partial x}+\frac{\partial D y}{\partial y}+\frac{\partial D z}{\partial z}$

$$
\begin{aligned}
& \frac{\partial D x}{\partial x}=4 y^{3}, \frac{\partial D y}{\partial y}=12 x^{2} y, \frac{\partial D z}{\partial z}=8 x y \\
& \therefore \rho_{v}=\left(4 y^{3}+12 x^{2} y+8 x y\right) \times 10^{-3} \mathrm{c} / \mathrm{m}^{3}
\end{aligned}
$$

## The dipole:-

An electric dipole (or a dipole) is the name given to two point charges of equal (magnitude and opposite sign, separated by a distance which is small compared to the distance to the point $p$ at which we want know the electric and potential field.

At point $p$, the potential from both charges $+Q,-Q$ is
$\mathrm{V}=\frac{Q}{4 \pi \epsilon_{0}}\left[\frac{1}{R_{1}}-\frac{1}{R_{2}}\right]$
$=\frac{Q}{4 \pi \epsilon_{0}}\left[\frac{R_{2}-R_{1}}{R_{1} R_{2}}\right]$
For $r \ggg d$ then

$$
R_{1}=r-\frac{d}{2} \cos \theta
$$

$R_{2}=r+\frac{d}{2} \cos \theta$
$\therefore R_{2}-R_{1}=d \cos \theta$
$R_{1} R_{2}=r^{2}-\left(\frac{d}{2} \cos \theta\right)^{2} \cong r^{2}$
$\therefore \quad V=\frac{Q d \cos \theta}{4 \pi \epsilon_{0} r^{2}}$ volt or $\mathrm{V}=\frac{\overline{Q d} \cdot \bar{a} r}{4 \pi \epsilon_{0} r^{2}}=\frac{\bar{P} \cdot \bar{a} r}{4 \pi \epsilon_{0} r^{2}}$
$\mathrm{E}=-\nabla \mathrm{V}=-\left[\frac{\partial V}{\partial r} \bar{a} r+\frac{1}{r} \frac{\partial V}{\partial \theta} \bar{a} \theta+\frac{1}{r \sin \theta} \frac{\partial V}{\partial \emptyset} \bar{a} \emptyset\right]$
$=-\left[-\frac{Q d \cos \theta}{2 \pi \epsilon_{0} r^{3}} \bar{a} r-\frac{Q d \sin \theta}{4 \pi \epsilon_{0} r^{3}} \bar{a} \theta\right]$
Or
$\bar{E}=\frac{Q d}{4 \pi \epsilon_{0} r^{3}}[2 \cos \theta \bar{a} r+\sin \theta \bar{a} \theta] v / m$
Ex1: A dipole at the origin in free space has a moment of $400 \pi$ ( $0.6 \bar{a} x-$ $0.75 \bar{a} y+0.8 \bar{a} z)$ c.m. Find :
a) The potential at $(0,0,5)$
b) The potential at $(2,3,4)$

## Solution

a)

$$
\bar{P}=400 \pi(0.6 \bar{a} x-0.75 \bar{a} y+0.8 \bar{a} z)
$$

$\bar{r}=5 \bar{a} z \quad$ then $\quad \bar{a} r=\bar{a} z$
$V=\frac{\bar{P} \cdot \bar{a} r}{4 \pi \varepsilon_{0} r^{2}}=\frac{0.8\left(400 \pi \varepsilon_{0}\right)}{4 \pi \varepsilon_{0} r^{2}}=\frac{80}{r^{2}}=\frac{80}{25}=2.3$ volt
b)
$\bar{r}=2 \bar{a} x+3 \bar{a} y+4 \bar{a} z$
$|\bar{r}|=\sqrt{4+9+16}=\sqrt{29}=5.385$
$\bar{a} r=\frac{2}{\sqrt{29}} \bar{a} x+\frac{3}{\sqrt{29}} \bar{a} y+\frac{4}{\sqrt{29}} \bar{a} z$

$$
\begin{aligned}
V= & \frac{\bar{P} \cdot \bar{a} r}{4 \pi \varepsilon_{0} r^{2}}=\frac{100\left[\frac{1.2}{\sqrt{29}}-\frac{2.25}{\sqrt{29}}+\frac{3.2}{\sqrt{29}}\right.}{29} \\
& =1.59 \text { volt }
\end{aligned}
$$

Ex2:- A dipole in free space is formed by a charge of 1 nC at at $(0,0,0.01)$ and -1 nC at $(0,0,-0.01)$. At point $\mathrm{p}\left(0.2,45^{0}, 0^{\circ}\right)$ find $\bar{E},|\bar{E}|$.

## Solution:-

$\bar{E}=\frac{Q d}{4 \pi \varepsilon_{0} r^{3}}[2 \cos \theta \bar{a} r+\sin \theta \bar{a} \theta]$
$\frac{1 \times 10^{-9}(0.02)}{(0.2)^{3}}\left(9 \times 10^{9}\right)[2 \cos 45 \bar{a} r+\sin 45 \bar{a} \theta]$
$=31.8 \bar{a} r+15.89 \bar{a} \theta \mathrm{~V} / \mathrm{m}$

$$
\begin{aligned}
|\bar{E}| & =\sqrt{(31.8)^{2}+(15.890)^{2}} \\
& =35.5 \mathrm{~V} / \mathrm{m}
\end{aligned}
$$

## Energy density in the Electrostatic field:-

The electric charges produces energy stored in a region these energy stored can be expressed as: for a group of point charges ( $Q_{1}, Q_{2}, \ldots \ldots Q_{m}$ )

$$
W_{e}=\frac{1}{2}\left[Q_{1} V_{1}+Q_{2} V_{2}+\cdots Q_{m} V_{m}\right]=\frac{1}{2} \sum_{N=1}^{N=m} Q_{N} V_{N}
$$

Where $V_{1}$ :- potential at position of $Q_{1}$ from other charges if we have continuous charge distribution then
$d v$ : differential volume
V:- potential
$W_{e}=\frac{1}{2} \iiint \rho_{v} V d v=\frac{1}{2} \iiint(\nabla \cdot \bar{D}) V d v$
Or
$W_{e}=\frac{1}{2} \iiint(\bar{D} \cdot \bar{E}) d v=\frac{1}{2} \iiint\left(\varepsilon_{0} E^{2}\right) d v$ Joul
The energy density is

$$
\frac{d W_{e}}{d v}=\frac{1}{2} \bar{D} \cdot \bar{E}=\frac{1}{2} \varepsilon_{0} E^{2} \mathrm{Joul} / \mathrm{m}^{3}
$$

Ex: The electric field of coaxial cable is $\bar{E}=\frac{\rho_{S a}}{\varepsilon_{0} \rho} \bar{a} \rho$ at any radius $\rho$ where $\mathrm{a} \leq \rho \leq b . \rho_{S}$ is surface charge density on inner conductor find the energy stored in L length of cable.

Solution:-

$$
W_{e}=\frac{1}{2} \iiint\left(\varepsilon_{0} E^{2}\right) d v
$$

$$
=\frac{1}{2} \int_{0}^{L} \int_{0}^{2 \pi} \int_{a}^{b} \varepsilon_{0} \frac{a^{2} \rho_{s}^{2}}{\varepsilon_{0}^{2} \rho^{2}} \quad \rho d \rho d \emptyset d z
$$

$$
=\frac{\pi L a^{2} \rho_{S}^{2}}{\varepsilon_{0}} \ln \frac{b}{a} \quad \text { Joul }
$$


[^0]:    * Multiplication of a vector by a vector are either dot or cross.

