6 DEFLECTIONS USING ENERGY METHODS

6.1 Method of Virtual Work: Beams and Frames

The method of virtual work can be applied to deflection problems involving beams and frames.

The principle of virtual work, or more exactly, the method of virtual force, may be formulated for beam and frame deflections by considering the beam shown in **Fig.6-1*b***. Here the displacement Δ of point *A* is to be determined.

To compute Δ a virtual unit load acting in the direction of Δ is placed on the beam at *A*, and the ***internal virtual moment*** **m** is determined by the method of sections at an arbitrary location ***x*** from the left support, **Fig. 6-1*a***.When the real loads act on the beam, **Fig. 6–1*b***, point *A* is displaced **Δ.**

Provided these loads cause ***linear elastic material response,*** then from the equation below , the element ***dx*** deforms or rotates

$$d\theta = \left(\frac{M}{EI}\right) dx$$

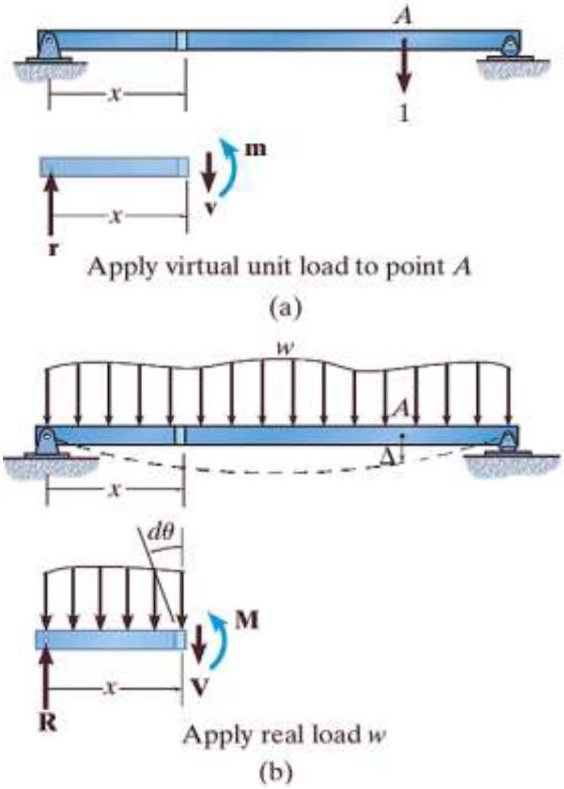


Fig. 6-1

Here ***M*** is the internal moment at ***x*** caused by the real loads. Consequently, the ***external virtual work*** done by the unit load is **1 .** Δ, and the ***internal virtual work*** done by the moment **m** is

$$m\,d\theta = m\left(\frac{M}{EI}\right) dx$$

Summing the effects on all the elements ***dx*** along the beam requires an integration,

$$1\ .\ \Delta = \int_0^L \frac{mM}{EI}\,dx \qquad ...(6\text{-}1)$$

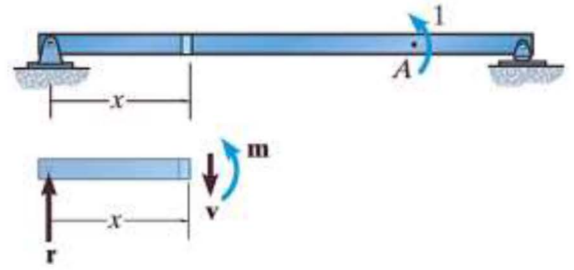
where

- **1** = external virtual unit load acting on the beam or frame in the direction of Δ
- ***m*** = internal virtual moment in the beam or frame, expressed as a function of ***x*** and caused by the external virtual unit load.
- Δ = external displacement of the point caused by the real loads acting on the beam or frame.
- ***M*** = internal moment in the beam or frame, expressed as a function of ***x*** and caused by the real loads.
- ***E*** = modulus of elasticity of the material.
- ***I*** = moment of inertia of cross-sectional area, computed about the neutral axis.

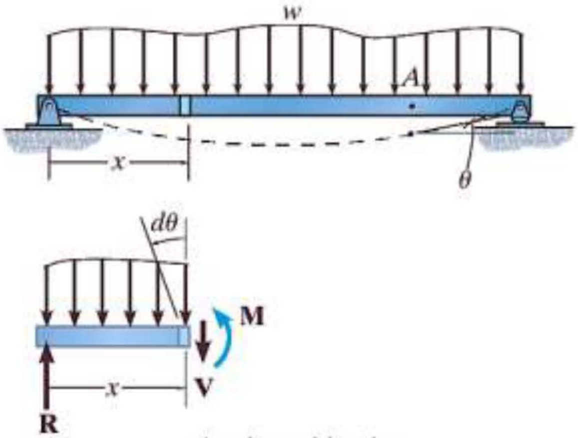
In a similar manner, if the tangent rotation or slope angle at a point *A* on the beam's elastic curve is to be determined, **Fig. 6-2,** a ***unit couple moment*** is first applied at the point, and the corresponding internal moments m_θ have to be determined. Since the work of the unit couple is **1 . *θ*,** then

$$1\,.\,\theta = \int_0^L \frac{m_\theta\, M}{EI}\, dx \qquad ...(6\text{-}2)$$

When applying **Eqs. 6-1** and **6-2**, it is important to realize that the definite integrals on the right side actually represent the amount of virtual strain energy that is ***stored*** in the beam. If concentrated forces or couple moments act on the beam or the distributed load is discontinuous, a single integration cannot be performed across the beam's entire length. Instead, separate *x* coordinates will have to be chosen within regions that have no discontinuity of loading. Also, it is not necessary that each ***x*** have the ***same origin***; however, the ***x*** selected for determining the real moment ***M*** in a particular region must be the ***same x*** as that selected for determining the virtual moment ***m* or** m_θ within the same region.



Apply virtual unit couple moment to point *A*
(a)

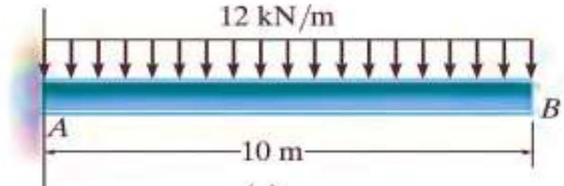


Apply real load *w*
(b)

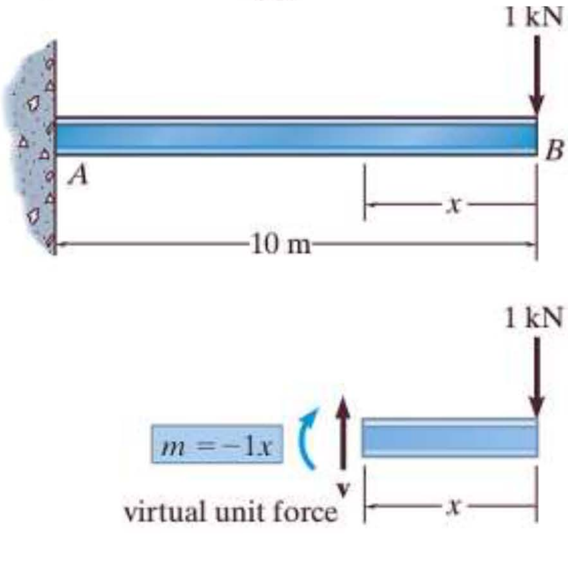
Fig. 6-2

EXAMPLE 6.1.1

Determine the displacement of point ***B*** of the steel beam shown in **Fig. *a*.** Take **E = 200 GPa, I = 500(10^6^) mm^4^.**



(a)



(b)

Solution

Virtual Moment m.

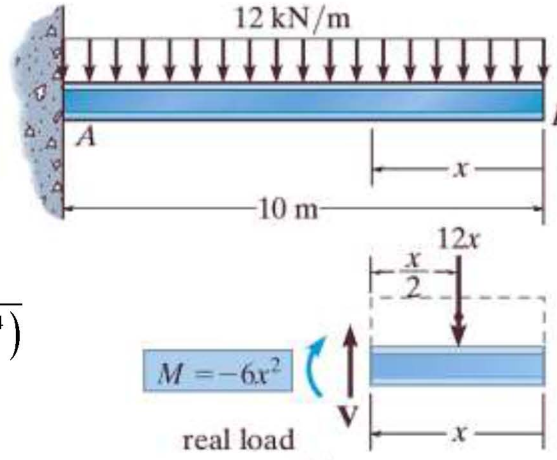
The vertical displacement of point ***B*** is obtained by placing a virtual unit load of **1 kN** at ***B*, Fig.*b***. By inspection there are no discontinuities of loading on the beam for *both* the real and virtual loads. Thus, a ***single x*** coordinate can be used to determine the virtual strain energy. This coordinate will be selected with its origin at ***B***, since then the reactions at *A* do not have to be determined in order to find the internal moments ***m*** and ***M*.** Using the method of sections, the internal moment ***m*** is formulated as shown in **Fig.*b***.

Real Moment M. Using the ***same x*** coordinate, the internal moment ***M*** is formulated as shown in **Fig.*c***.

Virtual-Work Equation. The vertical displacement of ***B*** is thus,

$$1\text{kN}.\Delta_B = \int_0^L \frac{mM}{EI}dx = \int_0^{10} \frac{(-1x)(-6x^2)}{EI}dx$$

$$1\text{kN}.\Delta_B = \frac{15(10^3)\text{kN}^2.m^3}{EI}$$



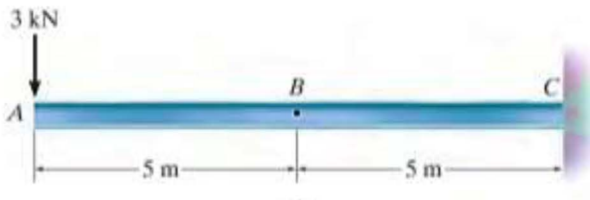
(c)

$$\Delta_B = \frac{15(10^3)\text{kN.m}^3}{200(10^6)\text{kN/ m}^2\left(500(10^6)\text{mm}^4\right)\left(10^{-12}\text{m}^4 / \text{mm}^4\right)}$$

$$= 0.150 \text{ m} = 150 \text{ mm}$$

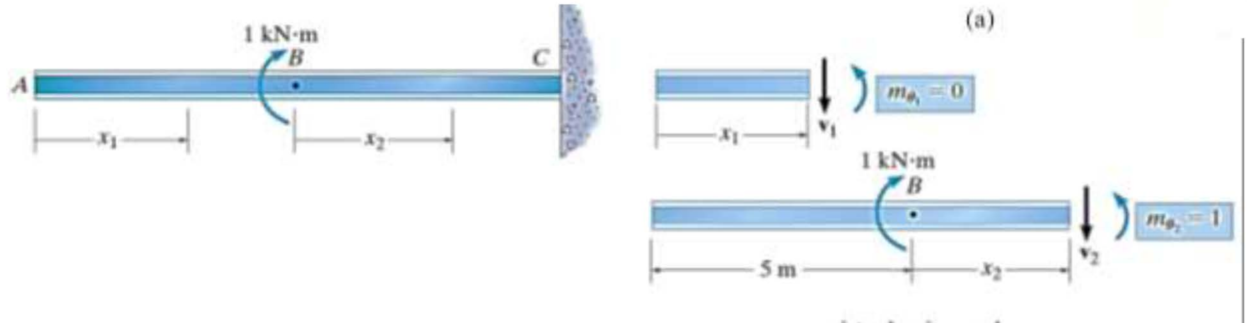
EXAMPLE 6.1.2

Determine the slope ***θ*** at point ***B*** of the steel beam shown in **Fig. *a***. Take **E = 200 GPa, I = 60(10^6) mm^4**

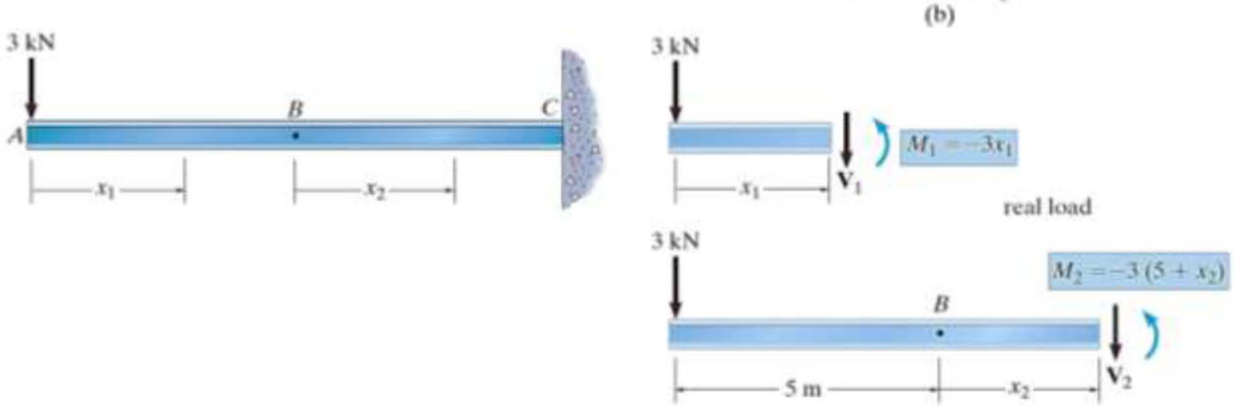


(a)

Solution



virtual unit couple
(b)



(c)

$$(1 \text{ kN.m}).\theta_B = \int_0^L \frac{m_\theta M}{EI}dx = \int_0^5 \frac{(0)(-3x_1)}{EI}dx_1 + \int_0^5 \frac{(1)[-3(5+x_2)]}{EI}dx_2$$

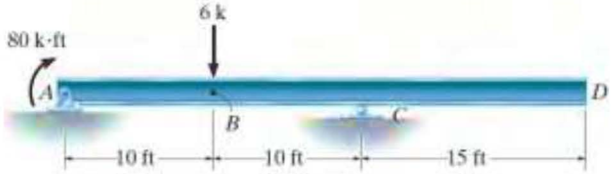
$$\theta_B = \frac{-112.5\text{kN.m}^2}{EI} = -0.00938 \text{ rad}$$

Note: The *negative sign* indicates is ***θ$_B$ opposite*** to the direction of the virtual couple moment shown in **Fig. *b***

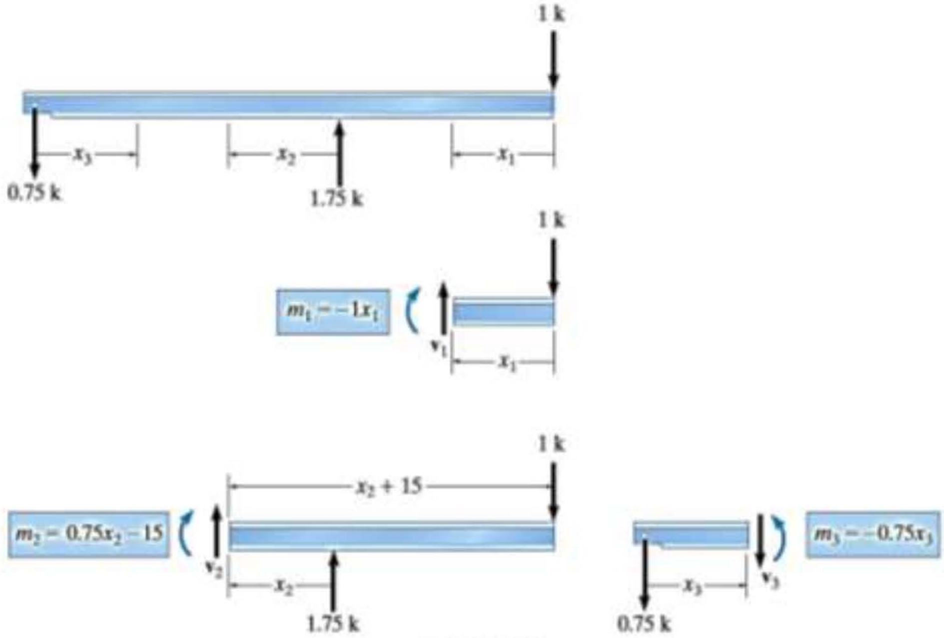
EXAMPLE 6.1.3

Determine the displacement at ***D*** of the steel beam in **Fig.*a***. Take **E = 29(10^3) ksi, I = 800 in^4**.

Solution

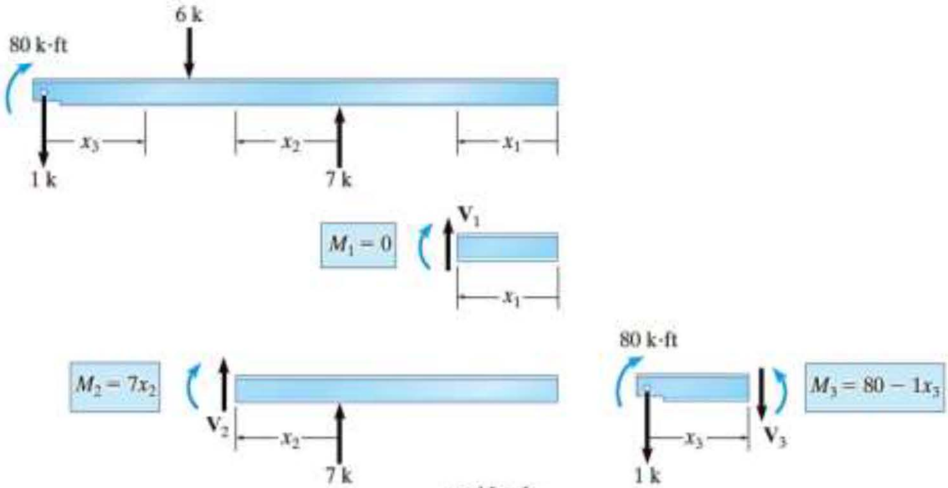


(a)



virtual loads

(b)



real loads

(c)

$$1\text{kN}.\Delta_D = \int_0^L \frac{mM}{EI}dx = \int_0^{15} \frac{(-1x_1)(0)}{EI}dx_1 + \int_0^{10} \frac{(0.75x_2 - 15)(7x_2)}{EI}dx_2 + \int_0^{10} \frac{(-0.75x_3)(80 - 1x_3)}{EI}dx_3$$

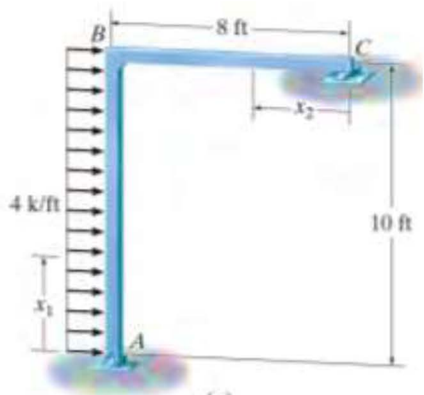
$$\Delta_D = \frac{0}{EI} - \frac{3500}{EI} - \frac{2750}{EI} = -\frac{6250\ \text{k.ft}^3}{EI}$$

$$\Delta_D = \frac{-6250\ \text{k.ft}^3 (12)^3\ \text{in}^3/\text{ft}^3}{29(10^3)\text{k/in}^2 (800\ \text{in}^4)} = -0.466\ \text{in}$$

Note: The negative sign indicates the displacement is upward, opposite to the downward unit load, **Fig. *b***. Also note that m_1 did not actually have to be calculated since $M_1 = 0$.

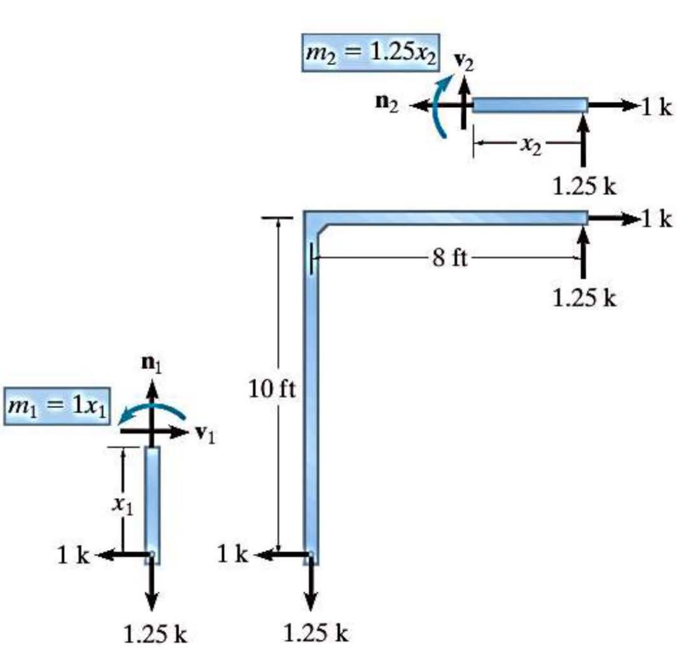
EXAMPLE 6.1.4

Determine the horizontal displacement of point *C* on the frame shown in **Fig. *a***. Take **E = 29(10³) ksi, I = 600 in⁴** for both members.



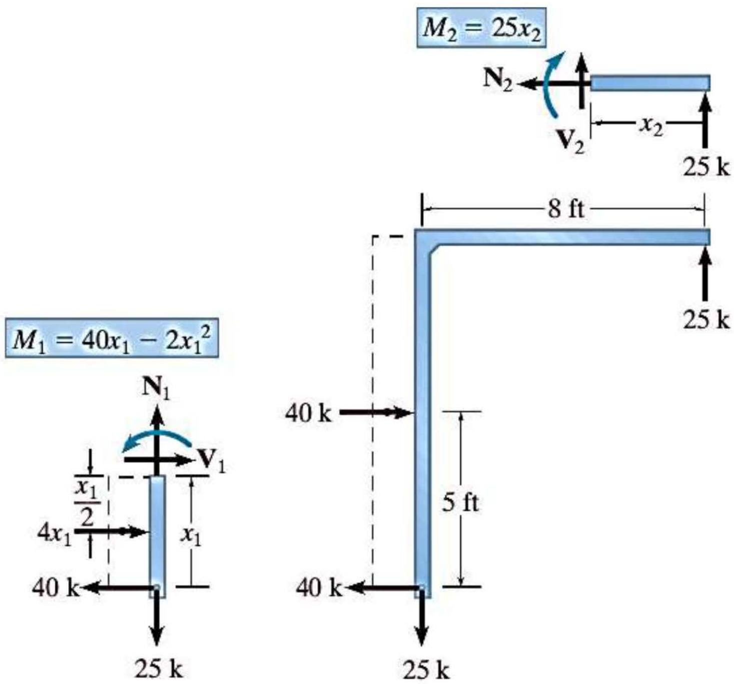
(a)

Solution



virtual loadings

(b)



real loadings

(c)

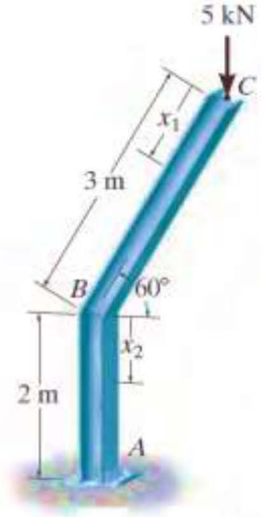
$$1\text{kN}.\Delta_{C_h} = \int_0^L \frac{mM}{EI}dx = \int_0^{10} \frac{(-1x_1)(40x_1 - 2x_1^2)}{EI}dx_1 + \int_0^8 \frac{(1.25x_2)(25x_2)}{EI}dx_2$$

$$\Delta_{C_h} = \frac{8333.3}{EI} + \frac{5333.3}{EI} = \frac{13666.7\ \text{k.ft}^3}{EI}$$

$$\Delta_{C_h} = \frac{13666.7\ \text{k.ft}^3(12)^3\ \text{in}^3/\text{ft}^3}{29(10^3)\text{k/in}^2(600\ \text{in}^4)} = 1.357\ \text{in}$$

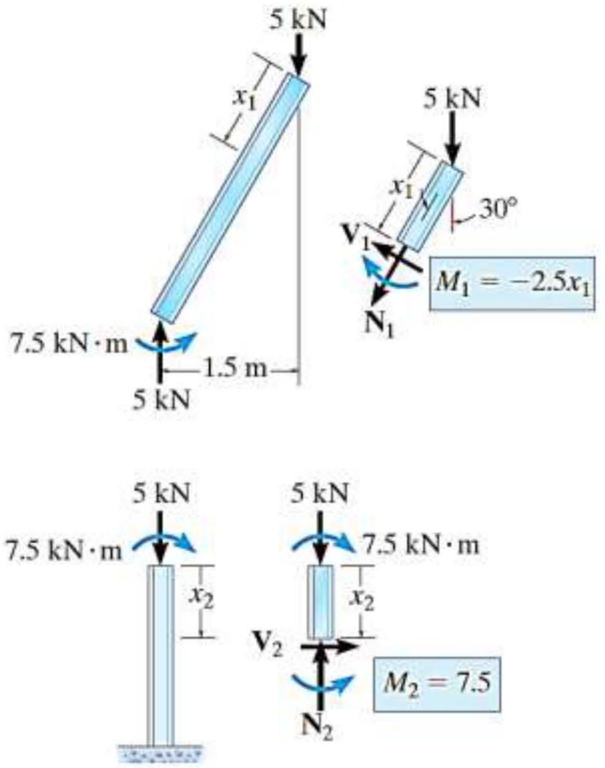
EXAMPLE 6.1.5

Determine the tangential rotation at point C of the frame shown in **Fig. *a***. Take E **= 200 GPa,** I **= (15)10⁶ mm⁴**.

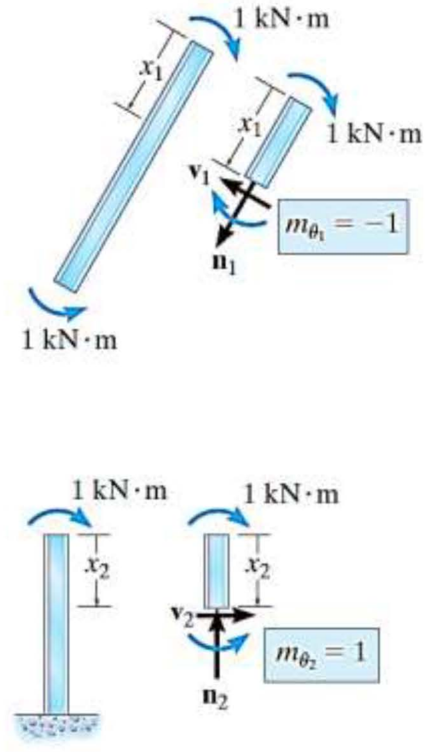


(a)

Solution



real loads
(b)



virtual loads
(c)

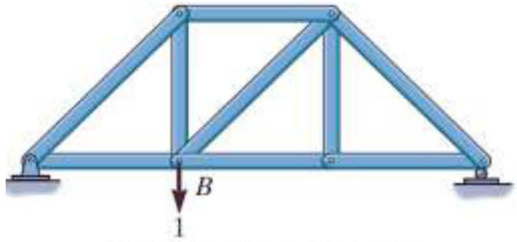
$$(1\ \text{kN.m}).\theta_C = \int_0^L \frac{m_\theta M}{EI}dx = \int_0^3 \frac{(-1)(-2.5x_1)}{EI}dx_1 + \int_0^2 \frac{(1)(7.5)}{EI}dx_2$$

$$\theta_C = \frac{11.25}{EI} + \frac{15}{EI} = \frac{26.25\,\text{kN.m}^2}{EI}$$

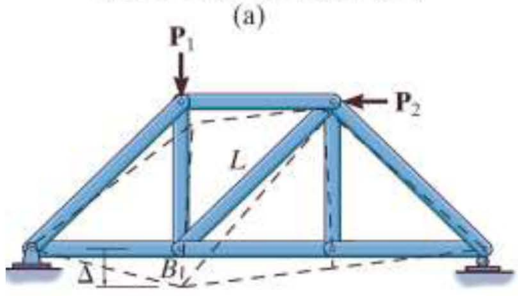
$$\theta_C = \frac{26.25\,\text{kN.m}^2}{200(10^6)\,\text{kN/m}^2\left[15\left(10^6\right)\text{mm}^4\right]\left(10^{-12}\ \text{m}^4/\text{mm}^4\right)} = 0.00875\ \text{rad}$$

6.2 Method of Virtual Work: Trusses

The method of virtual work can be used to determine the displacement of a truss joint when the truss is subjected to an external loading, temperature change, or fabrication errors. Each of these situations will now be discussed.



Apply virtual unit load to B
(a)



Apply real loads P_1, P_2
(b)

External Loading.

For the purpose of explanation let us consider the vertical displacement Δ of joint ***B*** of the truss in **Fig.*a***. Here a typical element of the truss would be one of its ***members*** having a length ***L***, **Fig.*b***. If the applied loadings $\mathbf{P_1}$ and $\mathbf{P_2}$ cause a ***linear elastic material response,*** then this element deforms an amount,

$$\Delta L = \frac{NL}{AE}$$

where ***N*** is the normal or axial force in the member, caused by the loads. The virtual-work equation for the truss is therefore

$$\mathbf{1.\,\Delta = \sum \frac{nNL}{AE}} \qquad \text{...(6-3)}$$

where

1	= external virtual unit load acting on the truss joint in the stated direction of Δ.
n	= internal virtual normal force in a truss member caused by the external virtual unit load.
Δ	= external joint displacement caused by the real loads on the truss.
N	= internal normal force in a truss member caused by the real loads.
E	= modulus of elasticity of a member.
A	= cross-sectional area of a member.
L	= length of a member.

Temperature.

In some cases, truss members may change their length due to temperature. If α is the coefficient of thermal expansion for a member and ΔT is the change in its temperature, the change in length of a member is $\Delta L = \alpha\ \Delta T\ L$

Hence, we can determine the displacement of a selected truss joint due to this temperature change from.

$$\mathbf{1.\,\Delta = \sum n\ \alpha\ \Delta T\ L} \qquad \text{...(6-4)}$$

where

1	= external virtual unit load acting on the truss joint in the stated direction of Δ.
n	= internal virtual normal force in a truss member caused by the external virtual unit load.
Δ	= external joint displacement caused by the temperature change.
α	= coefficient of thermal expansion of member.
ΔT	= change in temperature of member.
L	= length of a member.

Note: If any of the members undergoes an ***increase in temperature,*** ΔT will be ***positive,*** whereas a ***decrease in temperature*** results in a ***negative*** value for ΔT.

Fabrication Errors and Camber.

Occasionally, errors in fabricating the lengths of the members of a truss may occur. Also, in some cases truss members must be made slightly longer or shorter in order to give the truss a camber. If a truss member is shorter or longer than intended, the displacement of a truss joint from its expected position can be determined from ,

$$1 \cdot \Delta = \sum n \, \Delta L \qquad ...(6-5)$$

where

- **1** = external virtual unit load acting on the truss joint in the stated direction of Δ.
- n = internal virtual normal force in a truss member caused by the external virtual unit load.
- Δ = external joint displacement caused by the fabrication errors.
- ΔL = difference in length of the member from its intended size as caused by a fabrication error.

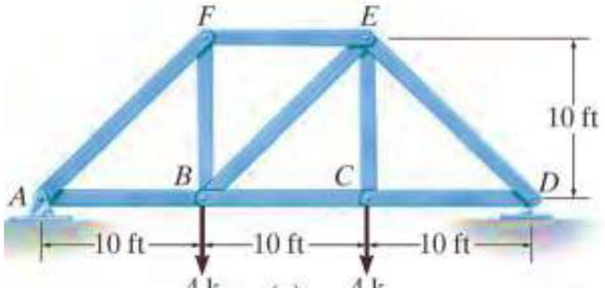
Note: When a fabrication error ***increases the length*** of a member, ΔL **is** ***positive,*** whereas a ***decrease in length*** **is** ***negative***.

A combination of the right sides of **Eqs. 6–3** through **6–5** will be necessary if both external loads act on the truss and some of the members undergo a thermal change or have been fabricated with the wrong dimensions.

$$1 \cdot \Delta = \sum \frac{nNL}{AE} + \sum n \, \alpha \, \Delta T \, L + \sum n \, \Delta L$$

EXAMPLE 6.2.1

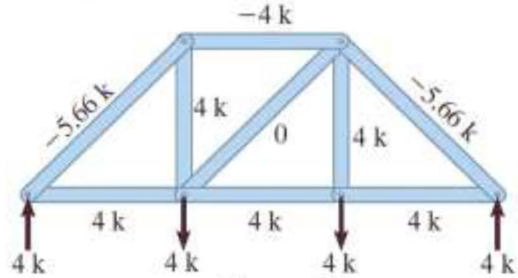
Determine the vertical displacement of joint *C* of the steel truss shown in **Fig. *a***. The cross-sectional area of each member is $A = 0.5$ **in²** and $E = 29(10^3)$ **ksi.**



(a)

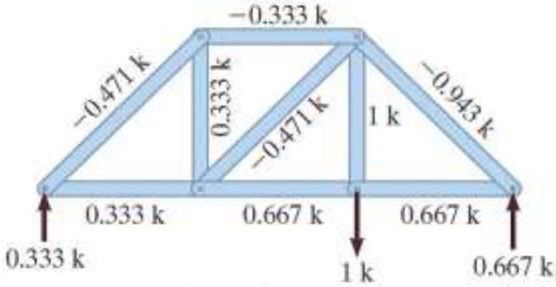
Solution

Real Forces N. The real forces in the members are calculated using the method of joints. The results are shown in **Fig. *b*.**



real forces **N**
(b)

Virtual Forces n. Only a vertical **1-k** load is placed at joint *C*, and the force in each member is calculated using the method of joints. The results are shown in **Fig. *c*.** Positive numbers indicate tensile forces and negative numbers indicate compressive forces.



virtual forces **n**
(c)

Virtual-Work Equation. Arranging the data in tabular form, we have

Member	*n* (k)	*N* (k)	*L* (ft)	*nNL* (k². ft)
AB	0.333	4	10	13.320
BC	0.667	4	10	26.680
CD	0.667	4	10	26.680
DE	-0.943	-5.66	14.14	75.471
FE	-0.333	-4	10	13.320
EB	-0.471	0	14.14	0.000
BF	0.333	4	10	13.320
AF	-0.471	-5.66	14.14	37.695
CE	1	4	10	40.000
				∑ **246.486**

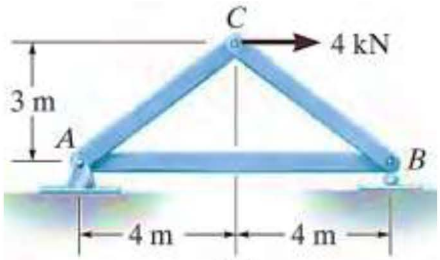
Thus

$$1\text{ k. }\Delta_{C_v} = \sum \frac{nNL}{AE} = \frac{246.486\text{ k}^2\text{.ft}}{AE}$$

$$1\text{ k. }\Delta_{C_v} = \frac{\left(246.486\text{ k}^2\text{.ft}\right)\left(12\text{ in/ft}\right)}{\left(0.5\text{ in}^2\right)\left(29\left(10^3\right)\text{ k/in}^2\right)} = 0.204\text{ in.}$$

EXAMPLE 6.2.2

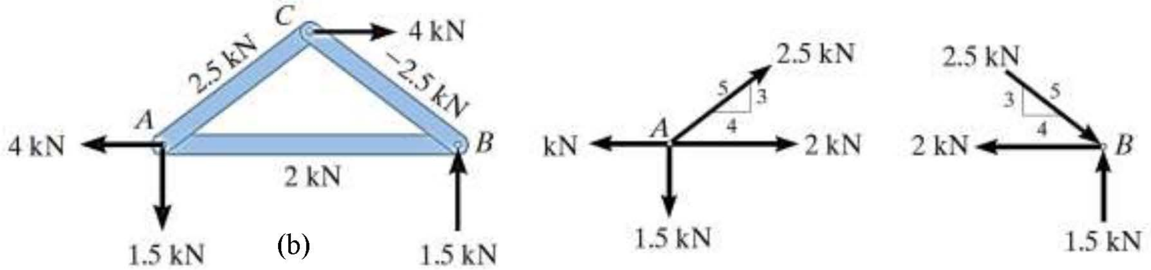
The cross-sectional area of each member of the truss shown in **Fig.*a*** is A = **400 mm²** and E = **200 GPa. (a)** Determine the vertical displacement of joint *C* if a **4-kN** force is applied to the truss at *C*. **(b)** If no loads act on the truss, what would be the vertical displacement of joint *C* if member *AB* were **5 mm** too short?



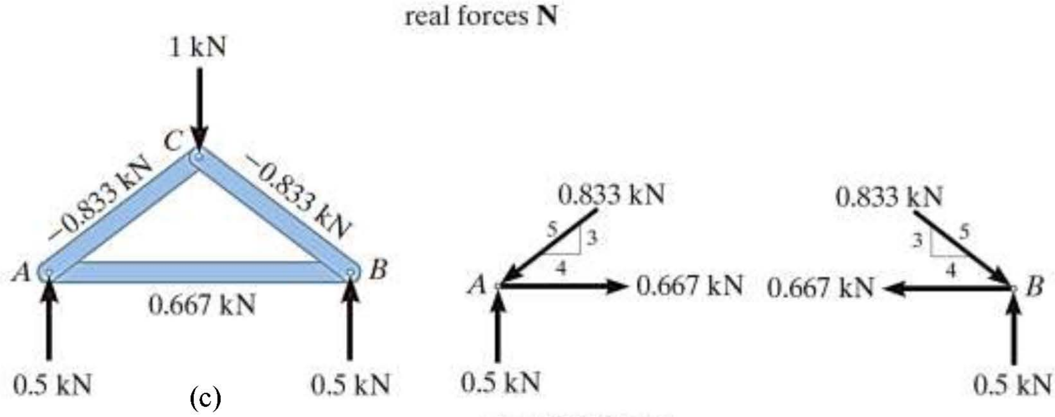
(a)

Solution

(a)



real forces **N**



virtual forces **n**

Member	*n* (kN)	*N* (kN)	*L* (m)	*nNL* (kN^2. m)
AB	0.667	2	8	10.672
AC	-0.833	2.5	5	-10.413
CB	-0.833	-2.5	5	10.413
				∑ **10.672**

$$1\text{ kN}.\ \Delta_{C_v} = \sum \frac{nNL}{AE} = \frac{10.672\text{ kN}^2.\text{m}}{AE}$$

$$1\text{ kN}.\ \Delta_{C_v} = \frac{10.672\text{ kN}^2.\text{m}}{400(10^{-6})\text{ m}^2(200(10^{-6})\text{ kN/m}^2)}$$

$$\Delta_{C_v} = 0.000133\text{ m} = 0.133\text{ mm}$$

(b)
Since the vertical displacement of ***C*** is to be determined, we can use the results of **Fig. *c***. Only member ***AB*** undergoes a change in length, namely, of **Δ*L* = 0.005 m**.
Thus,

$$1.\ \Delta = \sum n\ \Delta L$$

$$1\text{ kN}.\ \Delta_{C_v} = (0.667\text{ kN})(-0.005\text{ m})$$

$$\Delta_{C_v} = -0.00333\text{ m} = -3.33\text{ mm}$$

The negative sign indicates joint ***C*** is displaced ***upward***, opposite to the **1-kN** vertical load.

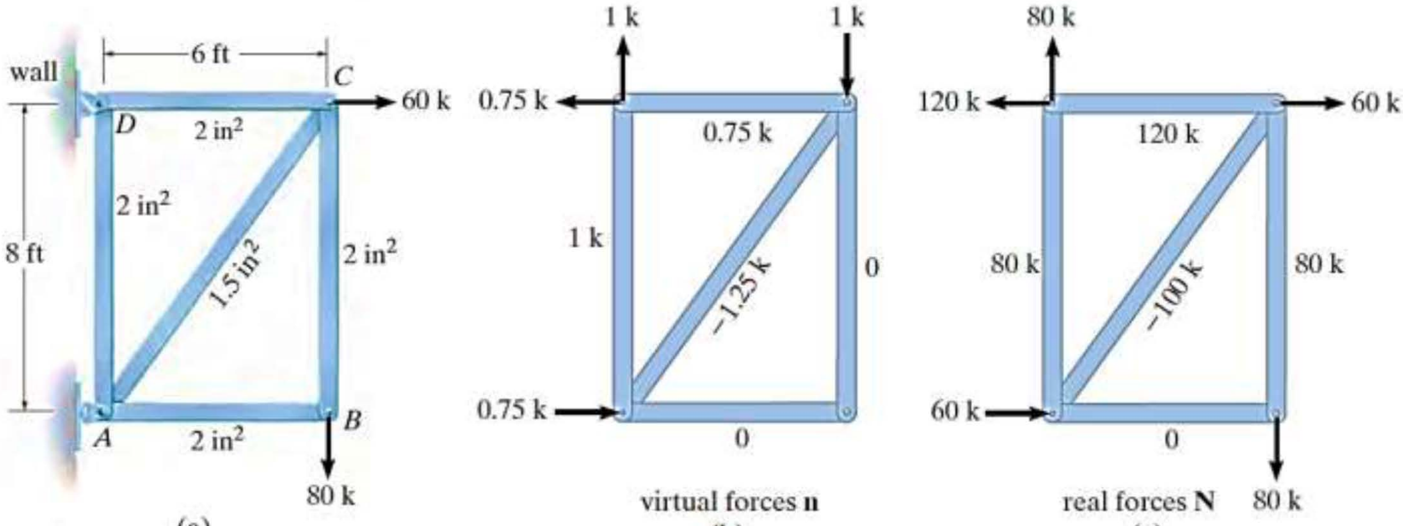
Note: If the **4-kN** load and fabrication error are both accounted for, the resultant displacement is then

$$\Delta_{C_v} = 0.133 - 3.33 = -3.20\text{ mm (upward)}.$$

EXAMPLE 6.2.3

Determine the vertical displacement of joint *C* of the steel truss shown in **Fig. *a***. Due to radiant heating from the wall, member ***AD*** is subjected to an ***increase*** in temperature of $\Delta T = +120°F$. Take $\alpha = 0.6(10^{-5})/°F$ and $E = 29(10^3)$ **kis.** The cross-sectional area of each member is indicated in the figure.

Solution



(a) virtual forces **n** (b) real forces **N** (c)

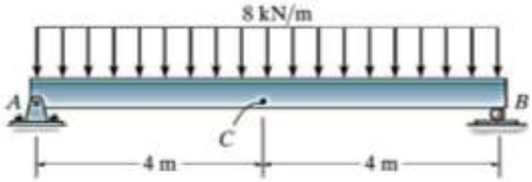
$$1.\ \Delta_{C_v} = \sum \frac{nNL}{AE} + \sum n\ \alpha\ \Delta T\ L$$

$$= \frac{(0.75)(120)(6)(12)}{2[29(10^3)]} + \frac{(1)(80)(8)(12)}{2[29(10^3)]}$$

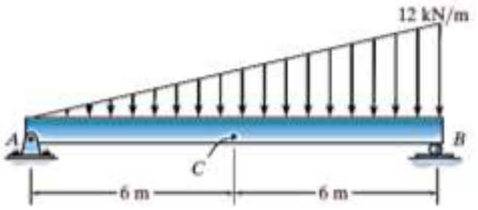
$$+ \frac{(-1.25)(-100)(10)(12)}{1.5[29(10^3)]} + (1)[0.6(10^{-5})](8)(12)$$

$$\Delta_{C_v} = 0.658\,\text{in}$$

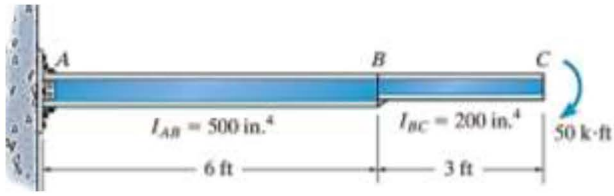
Hw.15
Determine the slope at *A* and displacement at point *C*. *EI* is constant. Use the principle of virtual work.



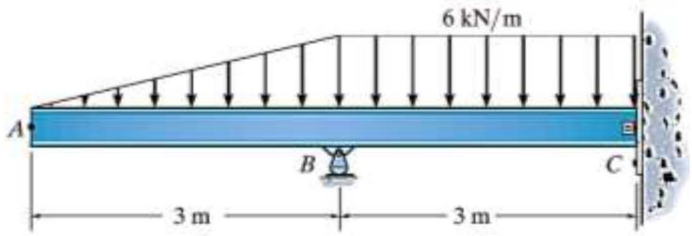
Hw.16
Determine the displacement at point *C*. *EI* is constant. Use the principle of virtual work.



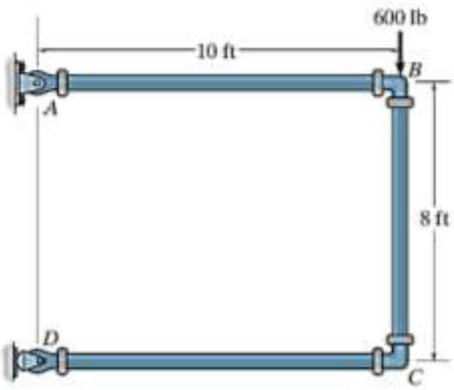
Hw.17
Determine the displacement and slope at point *C* of the cantilever beam.The moment of inertia of each segment is indicated in the figure. Take **E = 29(10^3) ksi**. Use the principle of virtual work.



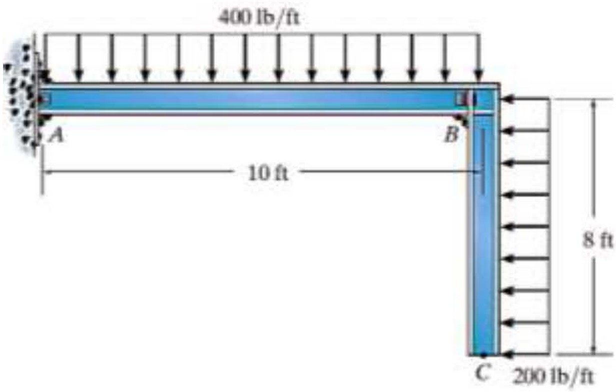
Hw.18
Determine the slope and displacement at point A. Assume *C* is pinned. Use the principle of virtual work. *EI* is constant.



Hw.19
Use the method of virtual work and determine the vertical deflection at the rocker support *D. EI* is constant

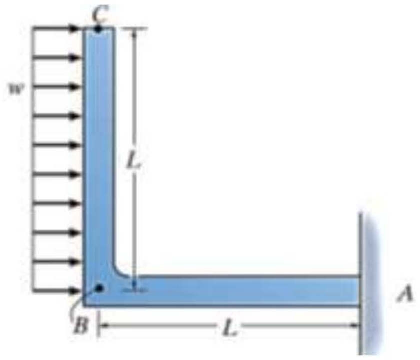


Hw.20
Determine the horizontal displacement of point *C. EI* is constant. Use the method of virtual work.



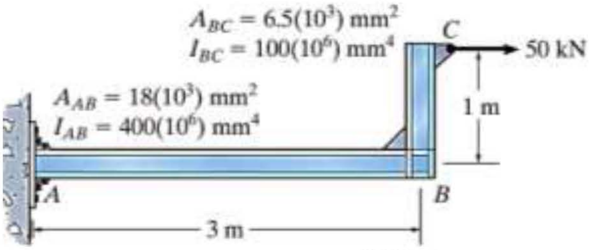
Hw.21

The **L-shaped** frame is made from two segments, each of length ***L*** and flexural stiffness ***EI***. If it is subjected to the uniform distributed load, determine the horizontal displacement of the end ***C***, and the vertical displacement of point ***B***. Use the method of virtual work.



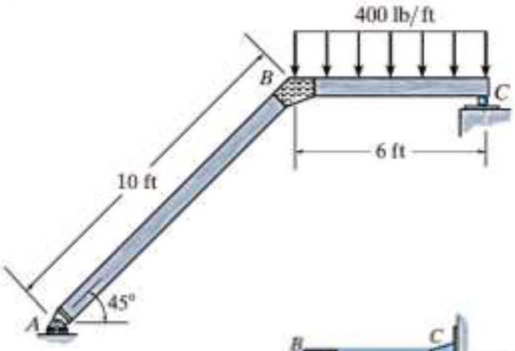
Hw.22

Determine the vertical deflection at ***C***. The cross-sectional area and moment of inertia of each segment is shown in the figure. Take ***E*** **= 200 GPa** Assume ***A*** is a fixed support. Use the method of virtual work.



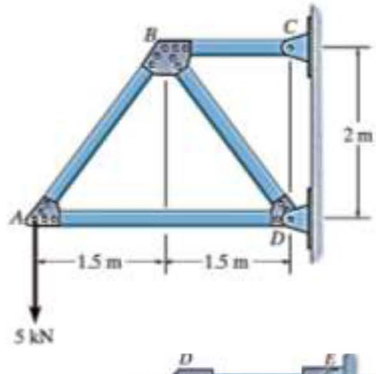
Hw.23

Use the method of virtual work and determine the horizontal deflection at ***C***. ***EI*** is constant.There is a pin at ***A***. Assume ***C*** is a roller and ***B*** is a fixed joint.



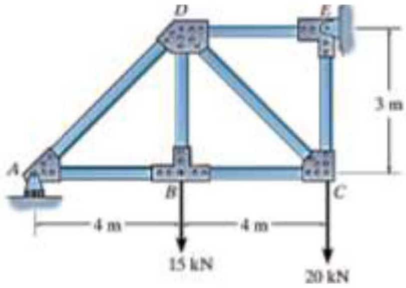
Hw.24

Determine the vertical displacement of joint ***A***. Each bar is made of steel and has a cross-sectional area of **600 mm2**.Take ***E*** **= 200 GPa**. Use the method of virtual work.



Hw.25

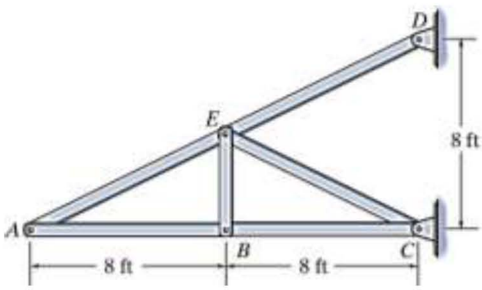
Determine the vertical displacement of joint ***D***. Use the method of virtual work. ***AE*** is constant. Assume the members are pin connected at their ends.



Hw.26

(A) Determine the vertical displacement of joint *A* if members *AB* and *BC* experience a temperature increase of ΔT = **200°F**. Take A **= 2 in^2** and E **= 29(10^3) ksi.** Also, **α= 6.60 (10-6)/°F.**

(B) Determine the vertical displacement of joint *A* if member *AE* is fabricated **0.5 in**. too short.



DEFORMATION OF STRUCTURES

Conjugate Beam Method

Properties of Conjugate Beam

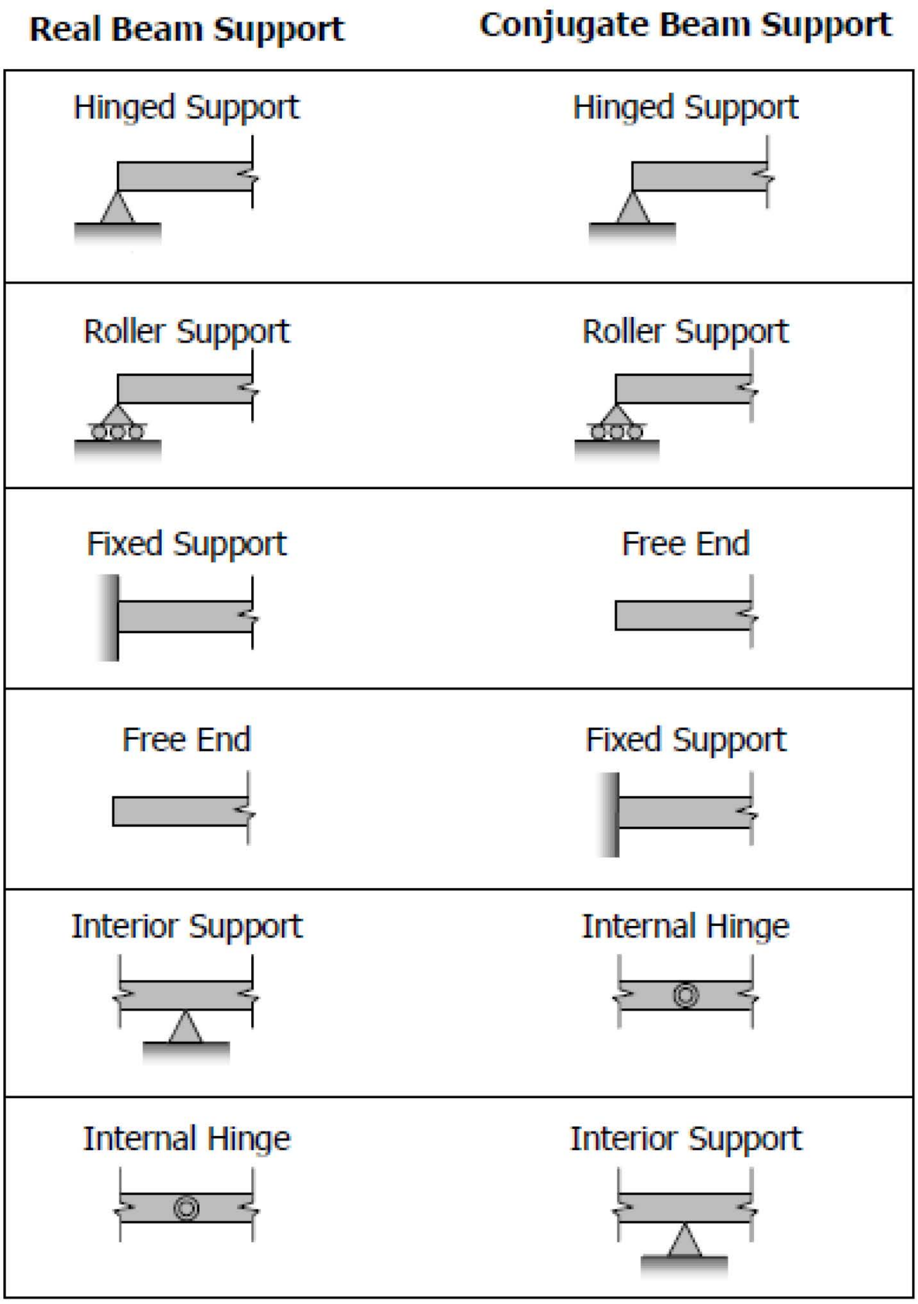
1. The length of a conjugate beam is always equal to the length of the actual beam.
2. The load on the conjugate beam is the M/EI diagram of the loads on the actual beam.
3. A simple support for the real beam remains simple support for the conjugate beam.
4. A fixed end for the real beam becomes free end for the conjugate beam.
5. The point of zero shear for the conjugate beam corresponds to a point of zero slope for the real beam.
6. The point of maximum moment for the conjugate beam corresponds to a point of maximum deflection for the real beam.

Slope on real beam = Shear on conjugate beam

Deflection on real beam = Moment on conjugate beam

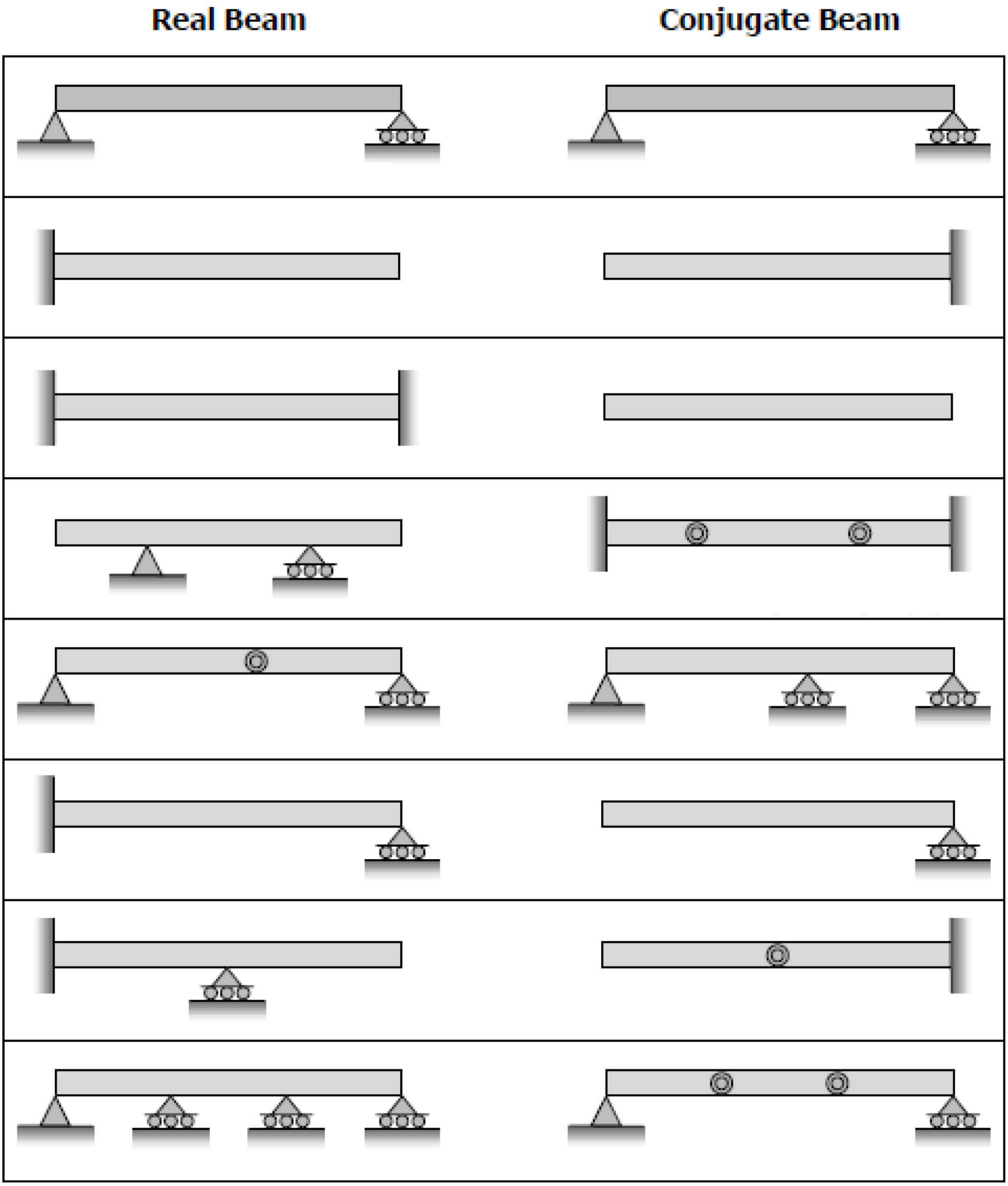
Supports of Conjugate Beam

Knowing that the slope on the real beam is equal to the shear on conjugate beam and the deflection on real beam is equal to the moment on conjugate beam, the shear and bending moment at any point on the conjugate beam must be consistent with the slope and deflection at that point of the real beam. Take for example a real beam with fixed support; at the point of fixed support there is neither slope nor deflection, thus, the shear and moment of the corresponding conjugate beam at that point must be zero. Therefore, the conjugate of fixed support is free end.



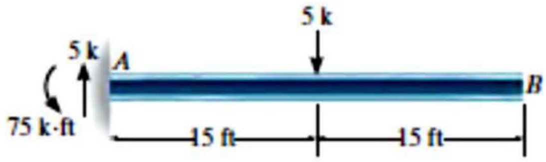
Examples of Beam and its Conjugate

The following are some examples of beams and its conjugate. Loadings are omitted.



Example 1

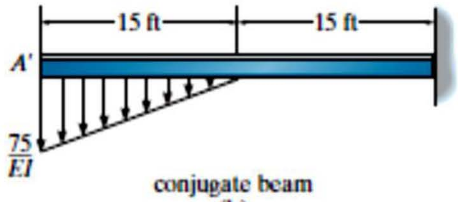
Determine the slope and deflection at point ***B*** of the steel beam shown in **Fig. a**. The reactions have been computed. $E = 29(10^3)$ **ksi**, $I = 800$ **in**^{4}.



real beam
(a)

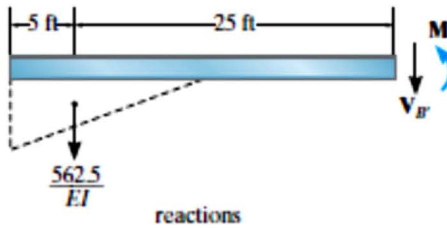
Solution

Conjugate Beam. The conjugate beam is shown in **Fig. b**. The supports at ***A'*** and ***B'*** correspond to supports ***A*** and ***B*** on the real beam. The ***M/EI*** diagram is *negative,* so the distributed load acts *downward,* i.e., away from the beam.



conjugate beam
(b)

Equilibrium. Since θ_B and Δ_B are to be determined, we must compute $V_{B'}$ and $M_{B'}$ in the conjugate beam, **Fig. c**.



reactions
(c)

$$+\uparrow\sum F_y=0;\quad -\frac{562.5\text{ k.ft}^2}{EI}-V_{B'}=0$$

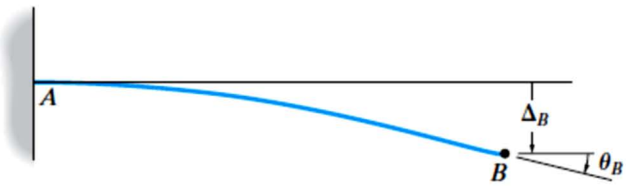
$$\theta_B=V_{B'}=-\frac{562.5\text{ k.ft}^2}{EI}$$

$$=\frac{-562.5\text{ k.ft}^2}{29(10^3)\text{k/in}^2(144\text{ in}^2/\text{ft}^2)\,800\text{ in}^4(1\text{ft}^4/(12)^4\text{ in}^4)}=-0.00349\text{ rad}$$

$$\circlearrowleft+\quad \sum M_{B'}=0;\qquad \frac{562.5\text{ k.ft}^2}{EI}(25\text{ ft})+M_{B'}=0$$

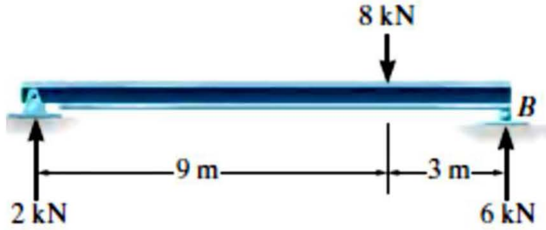
$$\Delta_B=M_{B'}=\frac{14062.5\text{ k.ft}^3}{EI}=-0.0873\text{ ft}=-1.05\text{ in}$$

The negative signs indicate the slope of the beam is measured clockwise and the displacement is downward, **Fig. d**.



(d)

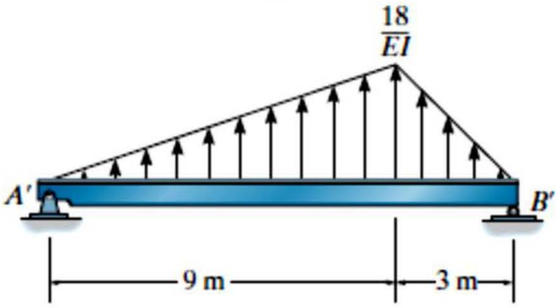
Example 2



Determine the maximum deflection of the steel beam shown in **Fig. a**. The reactions have been computed. $E = 200$ GPa, $I = 60(10^6)$ mm^4.

Solution

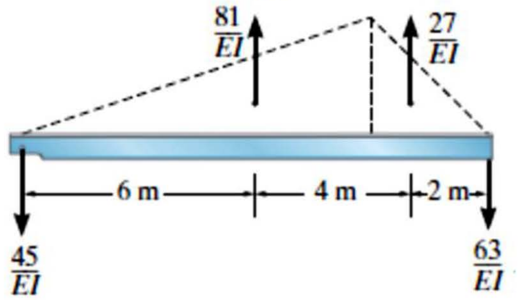
Conjugate Beam. The conjugate beam loaded with the ***M/EI*** diagram is shown in **Fig. b**. Since the ***M/EI*** diagram is positive, the distributed load acts upward (away from the beam).



conjugate beam

(b)

Equilibrium. The external reactions on the conjugate beam are determined first and are indicated on the free-body diagram in Fig. c. ***Maximum deflection*** of the real beam occurs at the point where the ***slope*** of the beam is ***zero***. This corresponds to the same point in the conjugate beam where the ***shear*** is ***zero***.



external reactions

(c)

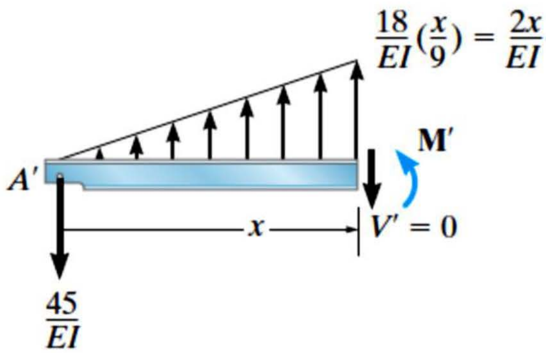
Assuming this point acts within the region **0 ≤ *x* ≥ 9 m** from ***A′***. we can isolate the section shown in Fig. d. Note that the peak of the distributed loading was determined from proportional triangles, that is,

w/x = (18/EI)/9

We require ***V′* = 0** so that,

$$+\uparrow \sum F_y = 0; \quad -\frac{45}{EI}+\frac{1}{2}\left(\frac{2x}{EI}\right)x=0$$

$$x = 6.71 \text{ m} \qquad (0 \le x \ge 9 \text{ m}) \text{ OK}$$



internal reactions

(d)

Using this value for ***x***, the maximum deflection in the real beam corresponds
to the moment ***M′***. Hence,

$$+\circlearrowleft$$

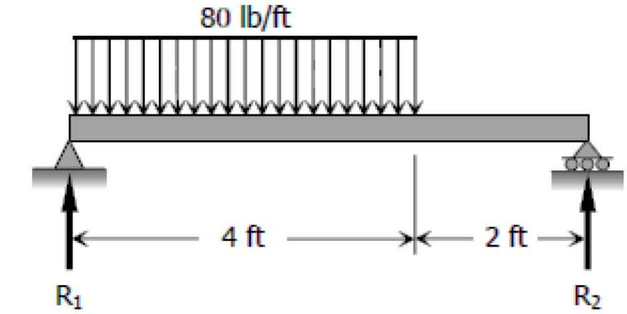
$$\sum M = 0; \quad \frac{45}{EI}(6.71)-\left[\frac{1}{2}\left(\frac{2(6.71)}{EI}\right)6.71\right]\frac{1}{3}(6.71)+M'$$

$$\Delta_{max} = M' = -\frac{201.2 \text{ kN.m}^3}{EI} = -0.0168 \text{ m} = -16.8 \text{ mm}$$

The negative sign indicates the deflection is downward.

Example 3

For the beam in the figure find the value of ***EIδ*** at **2 *ft*** from ***R₂***



Solution

Solving for reactions

$\Sigma M_{R2} = 0$

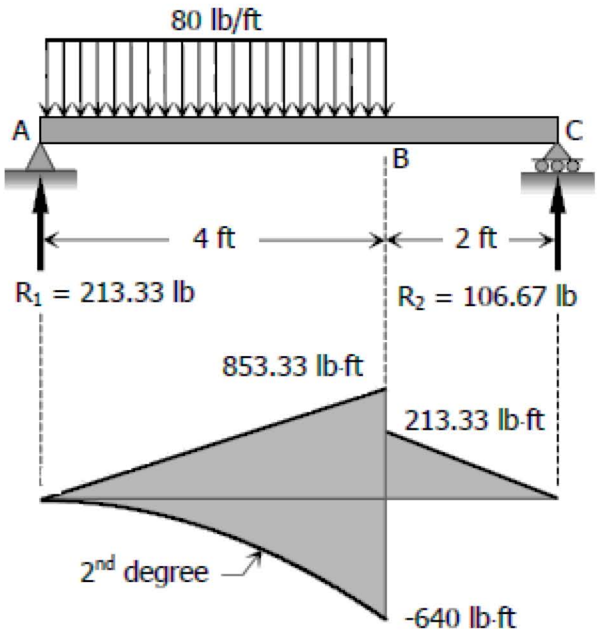
$6R_1 = 80(4)(4)$

$R1 = 213.33\,\text{lb}$

$\Sigma M_{R1} = 0$

$6R_2 = 80(4)(2)$

$R_2 = 106.67\,\text{lb}$



Moment Diagram by Parts

From the conjugate beam

$\Sigma M_A = 0$

$$6F_2 + \frac{1}{3}(4)(640)\left[\frac{3}{4}(4)\right] =$$

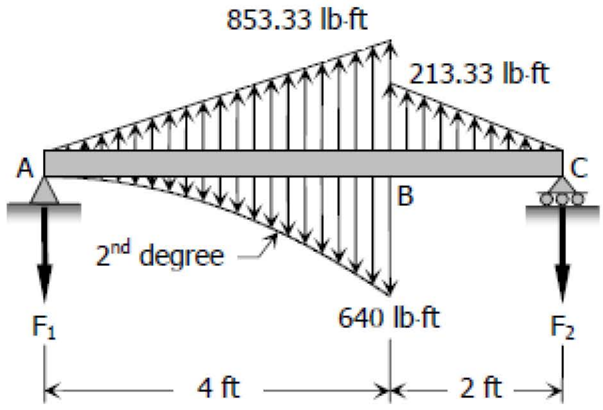
$$\frac{1}{2}(4)(853.33)\left[23(4)\right] + \frac{1}{2}(2)(213.33)\left[4 + \frac{1}{3}(2)\right]$$

$F_2 = 497.77\,\text{lb.ft}^2$

$$M_B = \frac{1}{2}(2)(213.33\left[\frac{1}{3}(2)\right] - 2F_2$$

$$M_B = \frac{1}{2}(2)(213.33\left[\frac{1}{3}(2)\right] - 2(497.77)$$

$M_B = -853.32\ \text{lb.ft}^3$



Conjugate Beam Loaded with M/EI Diagram

Thus, the deflection at B is

$EI\,\delta_B = M_B$

$EI\,\delta_B = -853.32\ \text{lb.ft}^3$

$EI\,\delta_B = 853.32\ \text{lb.ft}^3$ downward

8 ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES

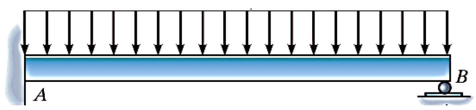
8.1 Statically Indeterminate Structures

A structure of any type is classified as *statically indeterminate* when the number of unknown reactions or internal forces exceeds the number of equilibrium equations available for its analysis. The most of the structures designed today are statically indeterminate. This indeterminacy may arise as a result of added supports or members, or by the general form of the structure. For example, reinforced concrete buildings are almost always statically indeterminate since the columns and beams are poured as continuous members through the joints and over supports.

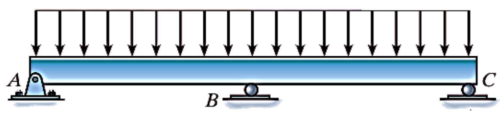
8.2 Slope-Deflection Method (Displacement Method)

All structures must satisfy equilibrium, load-displacement, and compatibility of displacements requirements in order to ensure their safety.

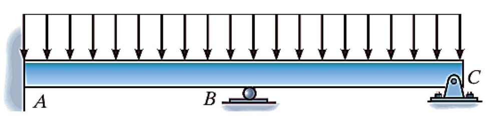
- ✓ This method considers the deflection as the primary unknowns.
- ✓ In this method, if the slopes at the ends and the relative displacement of the ends are known, the end moment can be found in terms of slopes, deflection, stiffness and length of the members.
- ✓ In the slope-deflection method, the relationship is established between moments at the ends of the members and the corresponding rotations and displacements.
- ✓ The *basic assumption* used in the slope-deflection method is that a typical member can flex but the shear and axial deformation are negligible. It is no different from that used with the force method.
- ✓ *Kinematically indeterminate* structures versus statically indeterminate structures:



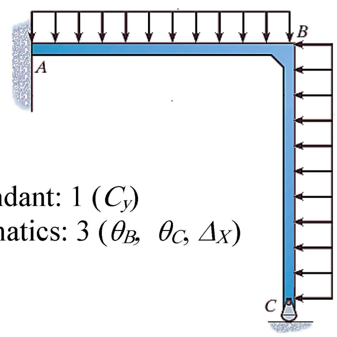
Redundant: 1 (B_y)
Kinematics: 1 (θ_B)



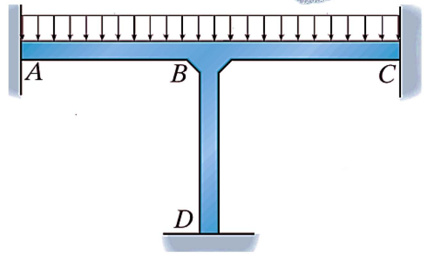
Redundant: 1 (B_y)
Kinematics: 3 (θ_A, θ_B, θ_C)



Redundant: 2 (B_y , C_y)
Kinematics: 2 (θ_B, θ_C)



Redundant: 1 (C_y)
Kinematics: 3 (θ_B, θ_C, Δ_X)



Redundant: 6 (A_x, A_y, M_A, B_x, B_y, M_B)
Kinematics: 1 (θ_B)

Sign convention

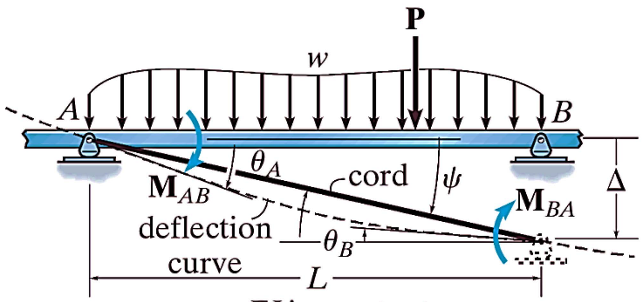
All **clockwise internal moments** and **end rotation** are ***positive***.

Basic Idea of Slope Deflection Method

The basic idea of the slope deflection method is to write the equilibrium equations for ***each node*** in terms of the ***deflections and rotations***. Solve for the generalized displacements. Using moment-displacement relations, moments are then known. The structure is thus reduced to a determinate structure.

8.2.1 Fundamental Slope-Deflection Equations:

The slope-deflection method is so named since it relates the unknown slopes and deflections to the applied load on a structure. In order to develop the general form of the slope-deflection equations, we will consider the typical span ***AB*** of a continuous beam as shown in the figure, which is subjected to the arbitrary loading and has a constant ***EI***. We wish to relate the beam's internal end moments M_{AB} and M_{BA} in terms of its three degrees of freedom, namely, its angular displacements θ_A and θ_B and linear displacement Δ which could be caused by a relative settlement between the supports. Since we will be developing a formula, ***moments*** and ***angular displacements*** will be considered ***positive*** when they act ***clockwise on the span***, as shown in the Figure Furthermore, the ***linear displacement*** Δ is considered ***positive*** as shown, since this displacement causes the cord of the span and the span's cord angle ψ to rotate ***clockwise***.



EI is constant
positive sign convention

The slope-deflection equations can be obtained by using the principle of superposition by considering *separately* the moments developed at each support due to each of the displacements θ_A, θ_B, and Δ and then the loads

Case A*: Rotation at A*

θ_A, Unknown

$\theta_B = 0, \quad \Delta_{AB} = 0, \quad P = 0$

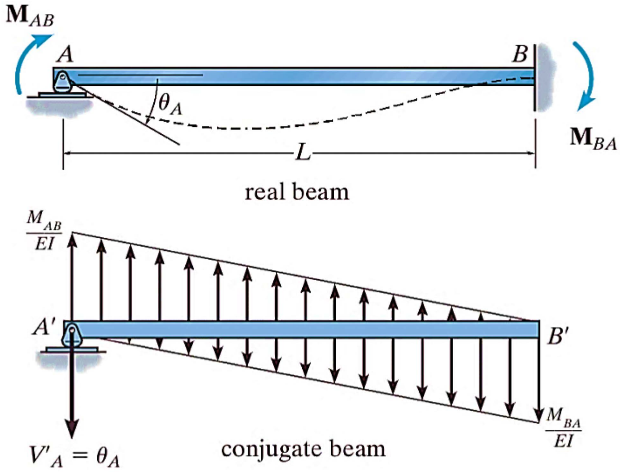
$$\theta_A = \frac{M_{AB}}{2EI}L - \frac{M_{BA}}{2EI}L$$

$$\Delta = 0 = \frac{M_{AB}L}{2EI}\cdot\frac{L}{3} - \frac{M_{BA}L}{2EI}\cdot\frac{2L}{3}$$

$$M_{AB} = 2M_{BA}$$

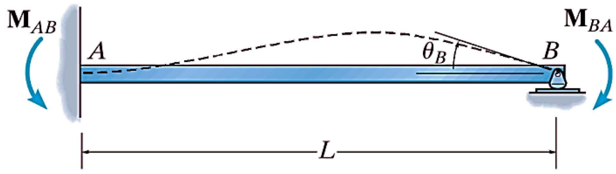
$$\theta_A = \frac{M_{AB}L}{4EI}$$

$$M_{AB} = \frac{4EI\cdot\theta_A}{L}, \qquad M_{BA} = \frac{2EI\cdot\theta_A}{L}$$



θ_B, Unknown

$\theta_A = 0, \quad \Delta_{BA} = 0, \quad P = 0$



$$M_{AB} = \frac{2EI \cdot \theta_B}{L}, \qquad M_{BA} = \frac{4EI \cdot \theta_B}{L}$$

Case C*: Displacement of End B Related to End A* (Relative Linear Displacement, Δ)

Δ_{AB}, Unknown

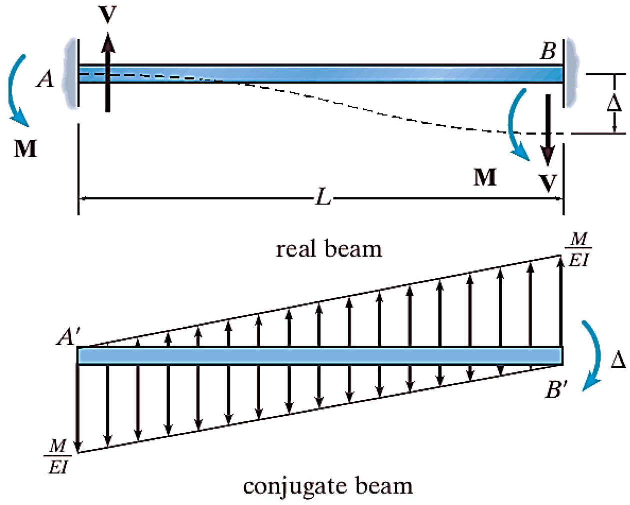
$[\theta_A = \theta_B = 0, \quad P = 0]$

$$\theta = 0 = \frac{-M_{AB}}{EI}\frac{L}{2} + \frac{M_{BA}}{EI}\frac{L}{2}$$

$$\Delta_{AB} = \frac{-M_{AB}L}{2EI} \cdot \frac{2L}{3} + \frac{M_{BA}L}{2EI} \cdot \frac{L}{3}$$

$$\Delta_{AB} = \frac{-M_{AB}L^2}{6EI}$$

$$M_{AB} = M_{BA} = \frac{-6EI\,\Delta_{AB}}{L^2}$$



Case D: *Fixed-End Moments* (FEM)

In order to develop the slope-deflection equations, we must transform these ***span loadings*** into equivalent moments acting at the nodes and then use the load-displacement relationships just derived. This is done simply by finding the reaction moment that each load develops at the nodes.

This moment is called a ***fixed-end moment* (FEM)**. Note that according to our sign convention, it is ***negative*** at ***node A (counterclockwise)*** and ***positive*** at ***node B (clockwise).***

For convenience in solving problems, fixed-end moments have been calculated for many loadings and are tabulated.

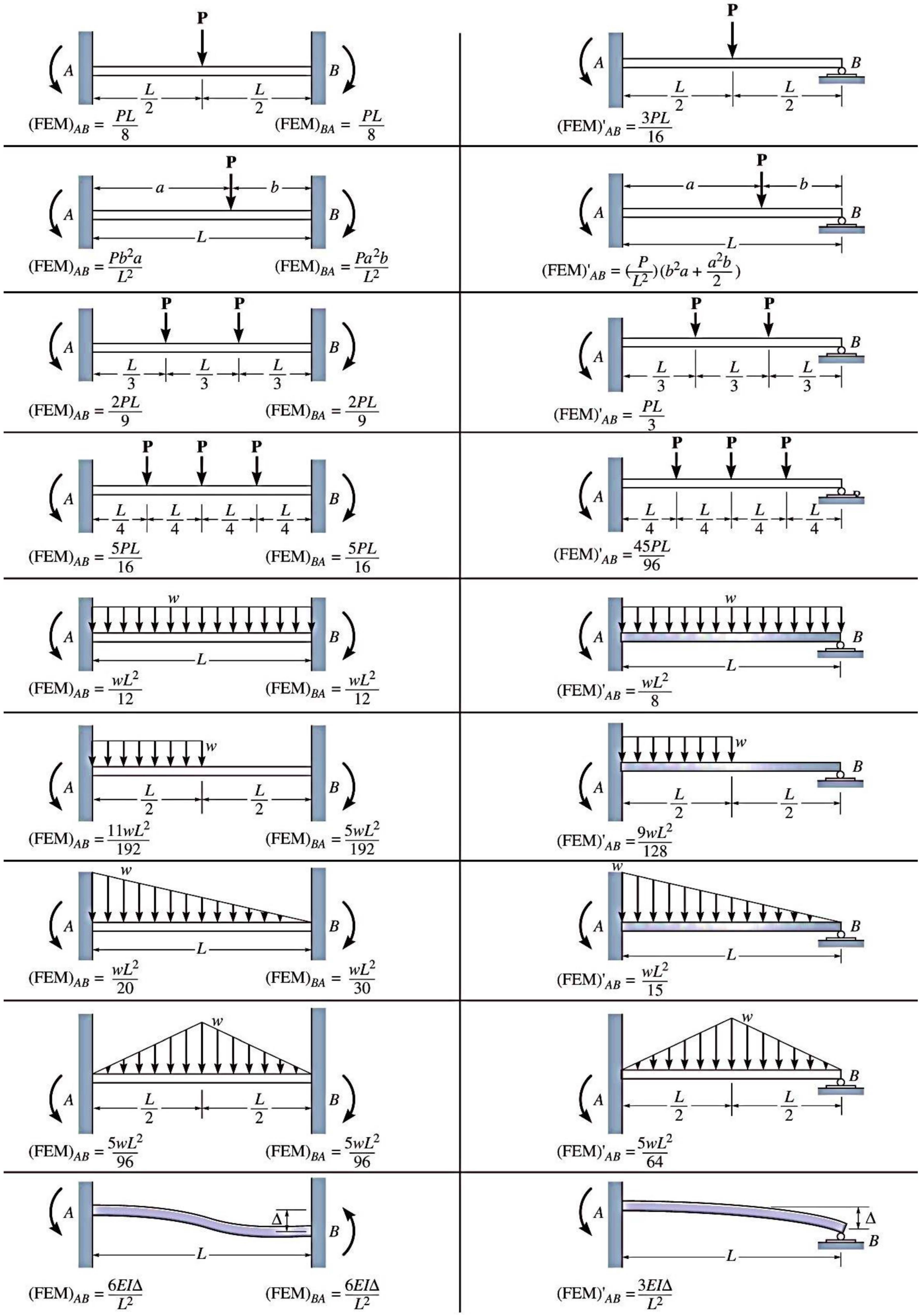


Table (8-1): Fixed-End Moments (FEM)

8.2.2 Slope-Deflection Equation.

$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta_{AB}}{L}\right)\right] + (FEM)_{AB}$$

$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta_{AB}}{L}\right)\right] + (FEM)_{BA}$$

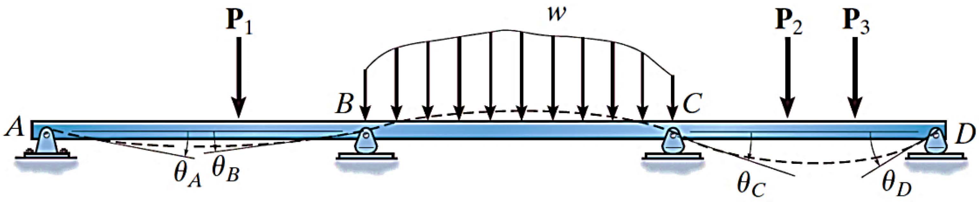
or

$$M_N = 2Ek\left[2\theta_N + \theta_F - 3\psi\right] + (FEM)_N$$

$$M_F = 2Ek\left[2\theta_F + \theta_N - 3\psi\right] + (FEM)_F$$

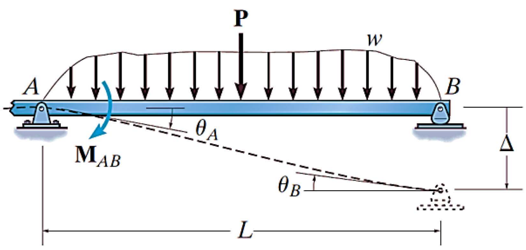
where

M_N	= internal moment in the near end of the span; this moment is *positive clockwise* when acting on the span.
M_F	= internal moment in the far end of the span; this moment is *positive clockwise* when acting on the span.
E , k	= modulus of elasticity of material and span stiffness $k = I/L$
θ_N , θ_F	= near- and far-end slopes or angular displacements of the span at the supports; the angles are measured in *radians* and are *positive clockwise.*
ψ	= span rotation of its cord due to a linear displacement, that is, $\psi = \Delta/L$; this angle is measured in *radians* and is *positive clockwise.* (ψ **'psi'**)
$(FEM)_N$	= fixed-end moment at the near-end support; the moment is *positive clockwise* when acting on the span; refer to the table for various loading conditions.



Pin-Supported End Span.

Occasionally an end span of a beam or frame is supported by a pin or roller at its ***far end,*** **Fig.*a***. When this occurs, the moment at the roller or pin must be **zero**; and **provided** the angular displacement θ_B at this support does not have to be determined, we can modify the general slope-deflection equation so that it has to be applied ***only once*** to the span rather than twice.

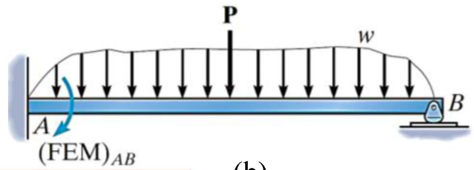


(a)

$$M_N = 2Ek\left[2\theta_N + \theta_F - 3\psi\right] + (FEM)_N$$

$$0 = 2Ek\left[2\theta_F + \theta_N - 3\psi\right] + (FEM)_F$$

Multiplying the first equation by **2** and subtracting the second equation from it yields



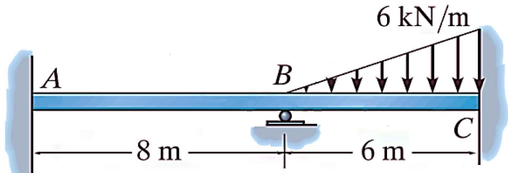
(b)

$$M_N = 3Ek\left[\theta_N - \psi\right] + (FEM)'_N$$

Only for End Span with Far End Pinned or Roller Supported

EXAMPLE 8.2.1

Draw the shear and moment diagrams for the beam shown in **Fig. *a*. *EI*** is constant.



(a)

Solution

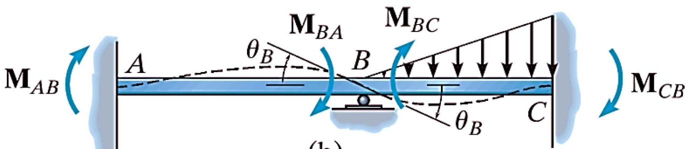
$$M_N = 2Ek\left[2\theta_N + \theta_F - 3\psi\right] + \left(FEM\right)_N$$

$$M_F = 2Ek\left[2\theta_F + \theta_N - 3\psi\right] + \left(FEM\right)_F$$

For member AB

$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta_{AB}}{L}\right)\right] + \left(FEM\right)_{AB}$$

$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta_{AB}}{L}\right)\right] + \left(FEM\right)_{BA}$$



(b)

(FEM)**$_{AB}$* = ***(FEM)**$_{BA}$* **= 0** (There is no load on span ***AB.)

$$M_{AB} = \frac{2EI}{8}\left[2(0) + \theta_B - 3(0)\right] + 0 = \frac{EI}{4}\cdot\theta_B \qquad ...(1)$$

$$M_{BA} = \frac{2EI}{8}\left[0 + 2\theta_B - 3(0)\right] + 0 = \frac{EI}{2}\cdot\theta_B \qquad ...(2)$$

$$\Rightarrow 2M_{AB} = M_{BA}$$

For member BC

$$\left(FEM\right)_{BC} = -\frac{wL^2}{30} = -\frac{6(6)^2}{30} = -7.2 \text{ kN.m}$$

$$\left(FEM\right)_{CB} = \frac{wL^2}{20} = \frac{6(6)^2}{30} = 10.8 \text{ kN.m}$$

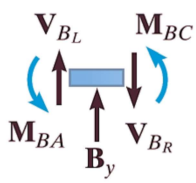
> **Note: (*FEM*)$_{BC}$** is negative since it acts counterclockwise *on the beam* at ***B***.

$$M_{BC} = \frac{2EI}{6}\left[2\theta_B + 0 - 3(0)\right] - 7.2 = \frac{2EI}{3}\cdot\theta_B - 7.2 \qquad ...(3)$$

$$M_{CB} = \frac{2EI}{6}\left[\theta_B + 2(0) - 3(0)\right] + 10.8 = \frac{EI}{3}\cdot\theta_B + 10.8 \qquad ...(4)$$

Equilibrium Equations. The above **four** equations contain **five** unknowns. The necessary fifth equation comes from the condition of ***moment equilibrium*** at support ***B***.

$$+\circlearrowleft \sum M_B = 0 \qquad M_{BA} + M_{BC} = 0 \qquad ...(5)$$



(c)

To solve, substitute **Eqs. (2)** and **(3)** into **Eq. (5)**, which yields

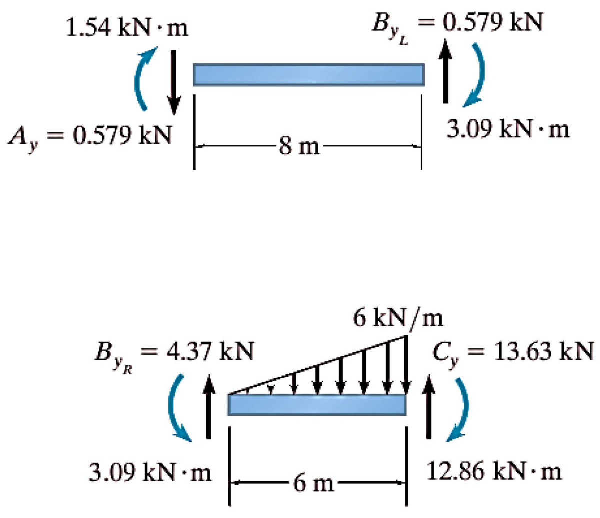
$$\frac{EI}{2}\cdot\theta_B + \frac{2EI}{3}\cdot\theta_B - 7.2 = 0 \qquad \Rightarrow \qquad \boldsymbol{\theta_B = \frac{6.17}{EI}}$$

Substituting this value into **Eqs. (1)–(4)** yields

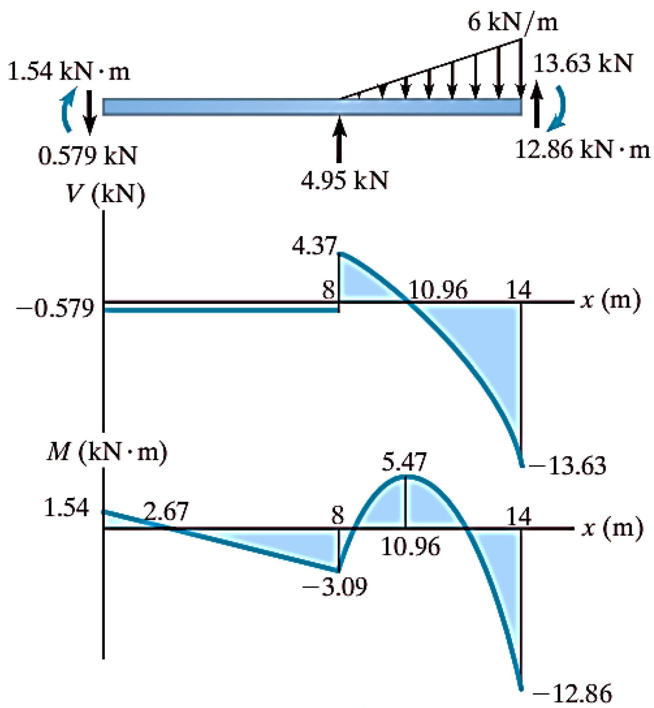
$$M_{AB} = 1.54 \text{ kN}.m \qquad M_{BA} = 3.09 \text{ kN}.m$$

$$M_{BC} = -3.09 \text{ kN}.m \qquad M_{CB} = 12.86 \text{ kN}.m$$

Using these results, the shears at the end spans are determined from the equilibrium equations, **Fig. *d***. The free-body diagram of the entire beam and the shear and moment diagrams are shown in **Fig. *e***.

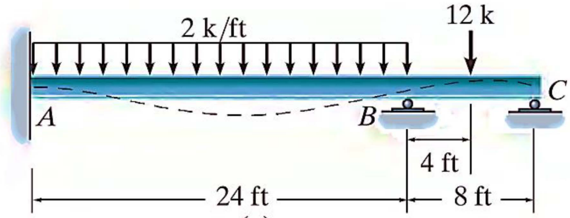


(d)



(e)

EXAMPLE 8.2.2
Draw the shear and moment diagrams for the beam shown in **Fig. *a***. ***EI*** is constant.



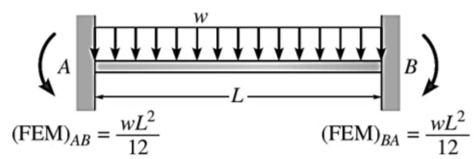
(a)

Solution

$$M_N = 2Ek\left[2\theta_N + \theta_F - 3\psi\right] + \left(FEM\right)_N$$

$$M_F = 2Ek\left[2\theta_F + \theta_N - 3\psi\right] + \left(FEM\right)_F$$

For member AB



$$M_{AB} = 2E\left(\frac{I}{L}\right)\left[2\theta_A + \theta_B - 3\left(\frac{\Delta_{AB}}{L}\right)\right] + \left(FEM\right)_{AB}$$

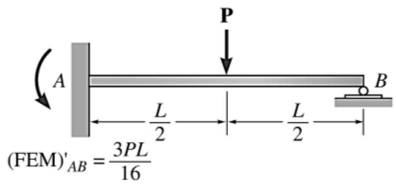
$$M_{BA} = 2E\left(\frac{I}{L}\right)\left[2\theta_B + \theta_A - 3\left(\frac{\Delta_{AB}}{L}\right)\right] + \left(FEM\right)_{BA}$$

$$M_{AB} = \frac{2EI}{24}\left[0 + \theta_B - 0\right] - \frac{2(24)^2}{12} = \frac{EI}{12}\cdot\theta_B - 96 \qquad ...(1)$$

$$M_{BA} = \frac{2EI}{24}\left[0 + 2\theta_B - 0\right] + \frac{2(24)^2}{12} = \frac{EI}{6}\cdot\theta_B + 96 \qquad ...(2)$$

For member BC

$$M_N = 3Ek\left[\theta_N - \psi\right] + \left(FEM\right)'_N$$

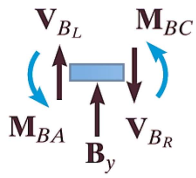


$$\therefore \quad M_{BC} = \frac{3EI}{L}\left[\theta_B - \frac{\Delta}{L}\right] + \left(FEM\right)'_{BC}$$

$$M_{BC} = \frac{3EI}{8}\left[\theta_B - 0\right] - \frac{3(12)(8)}{16} = \frac{3EI}{8}\cdot(\theta_B) - 18 \qquad ...(3)$$

$$M_{CB} = 0$$

Equilibrium Equations. The above **three** equations contain **four** unknowns. The necessary **fourth** equation comes from the condition of ***moment equilibrium*** at support ***B***.



$$+\circlearrowleft \sum M_B = 0 \qquad M_{BA} + M_{BC} = 0 \qquad ...(4)$$

To solve, substitute **Eqs. (2)** and **(3)** into **Eq. (4)**, which yields

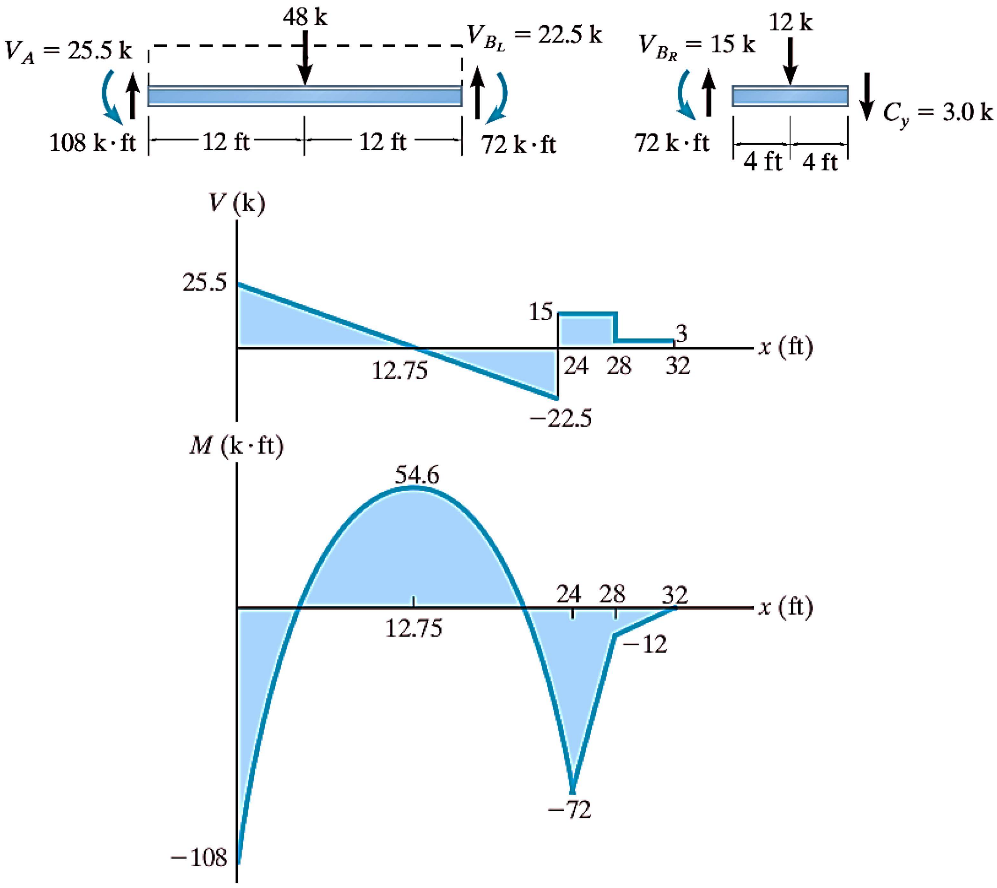
$$\left[\frac{EI}{6}\cdot\theta_B + 96\right] + \left[\frac{3EI}{8}\cdot(\theta_B) - 18\right] = 0$$

$$\frac{26EI}{48}\cdot(\theta_B) + 78 = 0 \quad \Rightarrow \quad \boldsymbol{\theta_B = \frac{-144}{EI}}$$

Note: Since θ_B is negative (counterclockwise) the elastic curve for the beam has been correctly drawn in **Fig. *a***. Substituting θ_B into **Eqs. (1)–(3),** we get

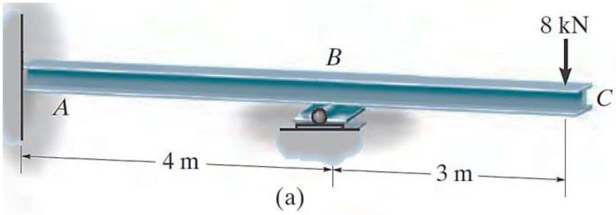
$M_{AB} = -108.0 \text{ k.ft}$ $\qquad M_{BA} = -72.0 \text{ k.ft}$

$M_{BC} = 72.0 \text{ kN.ft}$ $\qquad M_{CB} = 0$



EXAMPLE 8.2.3

Determine the moment at *A* and *B* for the beam shown in **Fig. *a***. The support at ***B*** is displaced (settles) **80 mm**. Take **E = 200 GPa , I = 5(10⁶) mm⁴.**

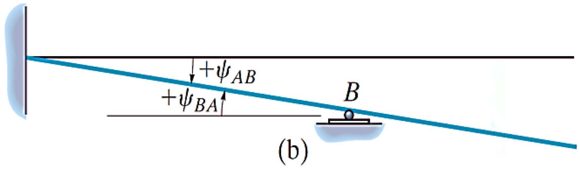


(a)

Solution

Only one span (***AB***) must be considered in this problem since the moment M_{BC} due to the overhang can be calculated from statics. Since there is no loading on span ***AB***, the **FEMs** are ***zero***.

As shown in **Fig.*b***, the downward displacement (settlement) of ***B*** causes the cord for span ***AB*** to rotate ***clockwise***. Thus,



(b)

$$\psi_{AB} = \psi_{BA} = \frac{0.08 \text{ m}}{4 \text{ m}} = 0.02 \text{ rad}$$

The stiffness for AB is

$$k = \frac{I}{L} = \frac{5(10^6)\ \text{mm}^4(10^{-12})\ \text{m}^4/\text{mm}^4}{4 \text{ m}} = 1.25(10^{-6})\ \text{m}^3$$

Applying the slope-deflection equation, to span ***AB***, with $\theta_A = 0$ we have,

$$M_N = 2Ek\left[2\theta_N + \theta_F - 3\psi\right] + \left(FEM\right)_N$$

$$M_F = 2Ek\left[2\theta_F + \theta_N - 3\psi\right] + \left(FEM\right)_F$$

$$M_{AB} = 2(200(10^9)\ \text{N/m}^2)[1.25(10^{-6})\ \text{m}^3][2(0) + \theta_B - 3(0.02)] + 0 \qquad ...(1)$$

$$M_{BA} = 2(200(10^9)\ \text{N/m}^2)[1.25(10^{-6})\ \text{m}^3][2\theta_B + (0) - 3(0.02)] + 0 \qquad ...(2)$$

The free-body diagram of the beam at support ***B*** is shown in **Fig.*c***. Moment equilibrium requires

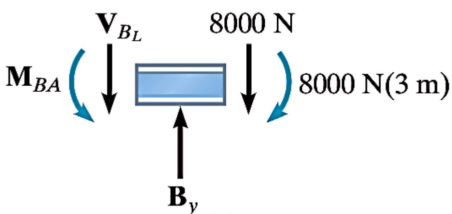
$$+\circlearrowleft \sum M_B = 0 \qquad M_{BA} - 8000 \text{ N}(3 \text{ m}) = 0 \qquad ...(3)$$

$$\Rightarrow M_{BA} = 24000 \text{ N. m}$$

Substituting in **Eq. (2)** yields

$$1(10^6)\theta_B - 30(10^3) = 24(10^3)$$

$$\Rightarrow \theta_B = 0.054 \text{ rad}$$



(c)

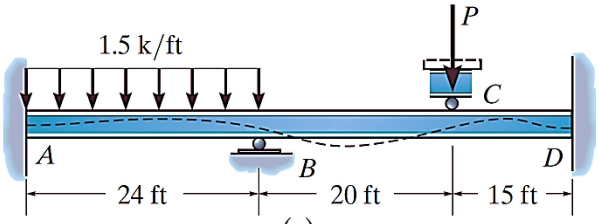
Thus, from **Eqs. (1)** and **(2),**

$$M_{AB} = -3.00 \text{ kN.m}$$

$$M_{BA} = 24.0 \text{ kN.m}$$

EXAMPLE 8.2.4

Determine the internal moments at the supports of the beam shown in **Fig.*a***. The roller support at ***C*** is pushed downward **0.1 ft** by the force **P**. Take ***E* = 29(10³) ksi, *I* = 1500 in⁴.**



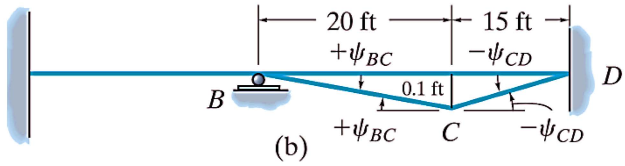
(a)

Solution

For member BC

$$(FEM)_{AB} = -\frac{wL^2}{12} = -\frac{1(1.5)(24)^2}{12} = -72.0 \text{ k.ft}$$

$$(FEM)_{BA} = -\frac{wL^2}{12} = \frac{1(1.5)(24)^2}{12} = 72.0 \text{ k.ft}$$



(b)

As shown in **Fig. *b***, the displacement (or settlement) of the support ***C*** causes ψ_{BC} to be ***positive***, since the cord for span ***BC*** rotates ***clockwise***, and ψ_{CD} to be ***negative***, since the cord for span ***CD*** rotates ***counterclockwise***. Hence,

$$\psi_{BC} = \frac{0.1 \text{ ft}}{20 \text{ ft}} = 0.005 \text{ rad} \quad , \quad \psi_{CD} = -\frac{0.1 \text{ ft}}{15 \text{ ft}} = -0.00667 \text{ rad}$$

Also, expressing the units for the stiffness in feet, we have

$$k_{AB} = \frac{1500}{24(12)^4} = 0.003014 \text{ ft}^3, \quad k_{BC} = \frac{1500}{20(12)^4} = 0.003617 \text{ ft}^3, \quad k_{CD} = \frac{1500}{15(12)^4} = 0.004823 \text{ ft}^3$$

Noting that $\theta_A = \theta_D = 0$ since ***A*** and ***D*** are fixed supports, and applying the slope-deflection twice to each span, we have

For span *AB*:

$$M_{AB} = 2\left[29(10^3)(12)^2\right](0.003014)\left[2(0) + \theta_B - 3(0)\right] - 72 = 25173.6\theta_B - 72 \qquad ...(1)$$

$$M_{BA} = 2\left[29(10^3)(12)^2\right](0.003014)\left[2\theta_B + 0 - 3(0)\right] + 72 = 50347.2\theta_B + 72 \qquad ...(2)$$

For span *BC*:

$$M_{BC} = 2\left[29(10^3)(12)^2\right](0.003617)\left[2(\theta_B) + \theta_C - 3(0.005)\right] + 0$$

$$M_{BC} = 60416.7\theta_B + 30208.3\theta_C - 453.1 \qquad ...(3)$$

$$M_{CB} = 2\left[29(10^3)(12)^2\right](0.003617)\left[2(\theta_C) + \theta_B - 3(0.005)\right] + 0$$

$$M_{CB} = 60416.7\theta_C + 30208.3\theta_B - 453.1 \qquad ...(4)$$

For span *CD*:

$$M_{CD} = 2\left[29(10^3)(12)^2\right](0.004823)\left[2(\theta_C) + 0 - 3(-0.00667)\right] + 0 = 80555.6\theta_C + 805.6 \qquad ...(5)$$

$$M_{DC} = 2\left[29(10^3)(12)^2\right](0.004823)\left[2(0) + \theta_C - 3(-0.00667)\right] + 0 = 40277.8\theta_C + 805.6 \qquad ...(6)$$

Equilibrium Equations. These six equations contain eight unknowns. Writing the moment equilibrium equations for the supports at ***B*** and ***C***, **Fig. *c***, we have

$+\circlearrowleft \sum M_B = 0; \qquad M_{BA} + M_{BC} = 0 \qquad ...(7)$

$+\circlearrowleft \sum M_C = 0; \qquad M_{CB} + M_{CD} = 0 \qquad ...(8)$

In order to solve, substitute **Eqs. (2)** and **(3)** into **Eq. (7)**, and **Eqs. (4)** and **(5)** into **Eq. (8)**.This yields

$\theta_C + 3.667\theta_B = 0.01262 \qquad ...(9)$

$-\theta_C - 0.214\theta_B = 0.00250 \qquad ...(10)$

solve, **Eqs. (9)** and **(10)** yields,

$$\boldsymbol{\theta_B = 0.00438\,\text{rad} \quad , \quad \theta_C = -\,0.00344\,\text{rad}}$$

The negative value for θ_C indicates ***counterclockwise*** rotation of the tangent at ***C***, **Fig. *a***. Substituting these values into **Eqs. (1)–(6)** yields

$\mathbf{M_{AB} = 38.2\ k.ft}$

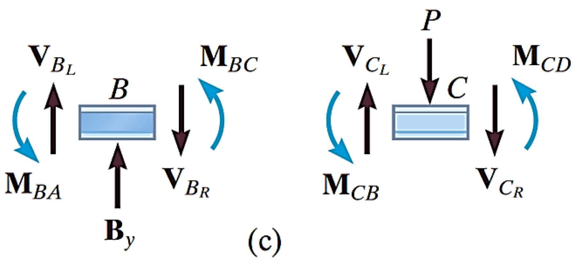
$\mathbf{M_{BA} = 292\ k.ft}$

$\mathbf{M_{BC} = -292\ k.ft}$

$\mathbf{M_{CB} = -529\ k.ft}$

$\mathbf{M_{CD} = 529\ k.ft}$

$\mathbf{M_{DC} = 667\ k.ft}$



(c)

Apply these end moments to spans ***BC*** and ***CD*** and show that

$\mathbf{V_{CL} = 41.05\ k}$

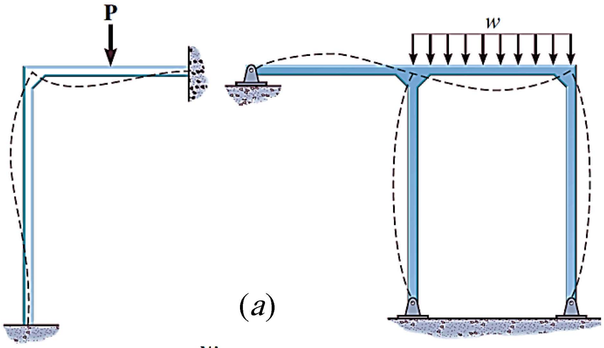
$\mathbf{V_{CR} = -79.73\ k}$

and the force on the roller is

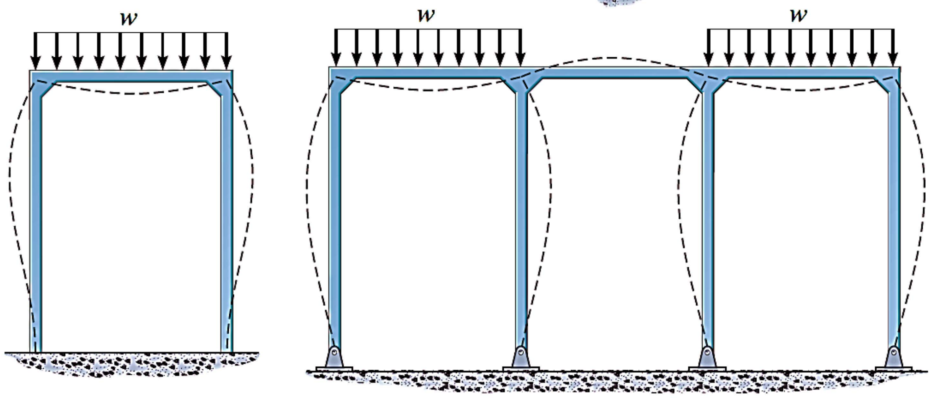
$\mathbf{P = 121\ k}$ **.**

8.3 Analysis of Frames: No Sidesway

A frame will not sidesway, or be displaced to the left or right, provided it is properly restrained. Examples are shown in **Fig. *a***. Also, no sidesway will occur in an unrestrained frame provided it is symmetric with respect to both loading and geometry, as shown in **Fig. *b***. For both cases the term ψ in the slope-deflection equations is equal to ***zero***, since bending does not cause the joints to have a linear displacement.



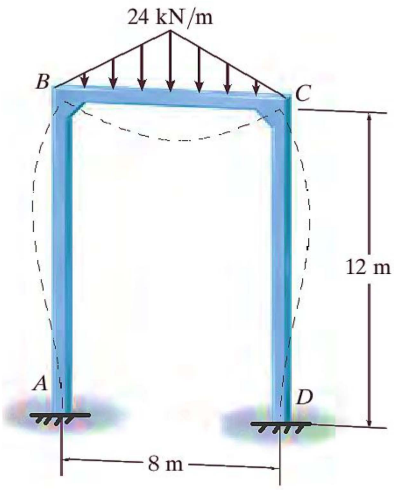
(*a*)



(*b*)

EXAMPLE 8.3.1

Determine the moments at each joint of the frame shown in **Fig. *a***. ***EI*** is constant .



(*a*)

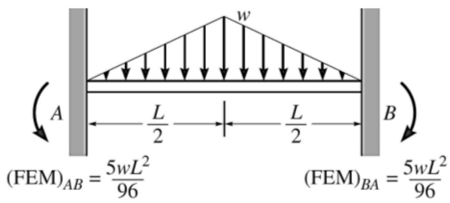
Solution

$$M_N = 2Ek\left[2\theta_N + \theta_F - 3\psi\right] + \left(FEM\right)_N$$

$$M_F = 2Ek\left[2\theta_F + \theta_N - 3\psi\right] + \left(FEM\right)_F$$

$\boldsymbol{\theta_A = \theta_D = 0}$

$\boldsymbol{\psi_{AB} = \psi_{BC} = \psi_{CD} = 0}$**,** since no sidesway will occur.



For member AB

$$M_{AB} = \frac{2EI}{12}\left[0 + \theta_B - 0\right] - 0 = \frac{EI}{6}\cdot(\theta_B) \qquad ...(1)$$

$$M_{BA} = \frac{2EI}{12}\left[0 + 2\theta_B - 0\right] + 0 = \frac{EI}{3}\cdot(\theta_B) \qquad ...(2)$$

For member BC

$$M_{BC} = \frac{2EI}{8}\left[2\theta_B + \theta_C - 0\right] - \frac{5wL^2}{96} = \frac{EI}{4}\cdot(2\theta_B + \theta_C) - 80 \qquad ...(3)$$

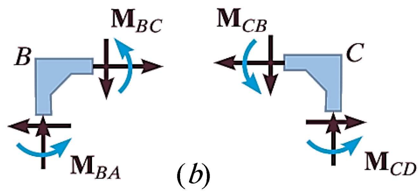
$$M_{CB} = \frac{2EI}{8}\left[\theta_B + 2\theta_C - 0\right] + \frac{5wL^2}{96} = \frac{EI}{4}\cdot(\theta_B + 2\theta_C) + 80 \qquad ...(4)$$

For member CD

$$M_{CD} = \frac{2EI}{12}\left[2\theta_C + 0 - 0\right] - 0 = \frac{EI}{3}\cdot(\theta_C) \qquad ...(5)$$

$$M_{DC} = \frac{2EI}{12}\left[\theta_C + 0 - 0\right] + 0 = \frac{EI}{6}\cdot(\theta_C) \qquad ...(6)$$

Equilibrium Equations. The preceding **six** equations contain **eight** unknowns. The remaining **two** equilibrium equations come from moment equilibrium at joints ***B*** and ***C***, **Fig. *b***. We have



(*b*)

$$M_{BA} + M_{BC} = 0$$

$$\left(\frac{EI}{3}\cdot(\theta_B)\right) + \left(\frac{EI}{4}\cdot(2\theta_B + \theta_C) - 80\right) = 0$$

$$\frac{5EI}{6}\cdot\theta_B + \frac{EI}{4}\cdot\theta_C - 80 = 0 \qquad ...(7)$$

$$M_{CB} + M_{CD} = 0$$

$$\left(\frac{EI}{4}\cdot(\theta_B + 2\theta_C) + 80\right) + \left(\frac{EI}{3}\cdot(\theta_C)\right) = 0$$

$$\frac{EI}{4}\cdot\theta_B + \frac{5EI}{6}\cdot\theta_C + 80 = 0 \qquad ...(8)$$

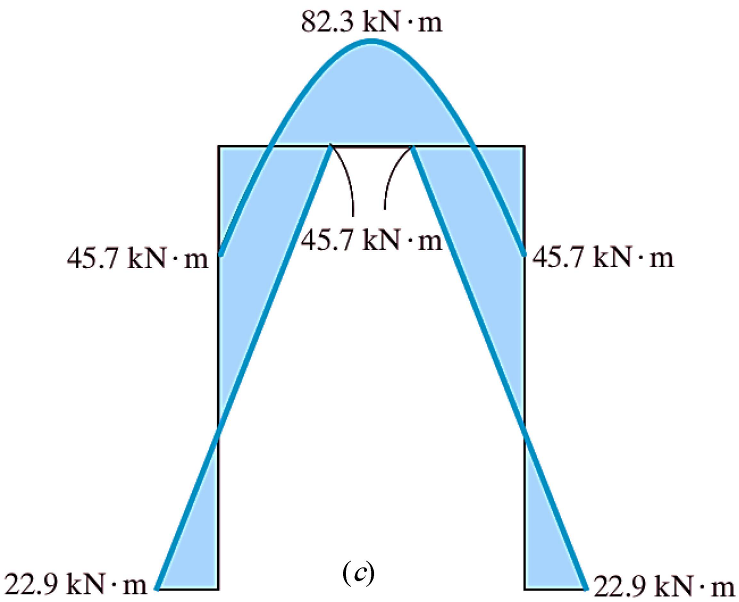
Solving **Eqs. (7)** and **(8)**, simultaneously yields,

$$\boldsymbol{\theta_B = -\theta_C = \frac{137.1}{EI}}$$

Substituting into **Eqs. (1)–(6)**, we get

$$\mathbf{M_{AB} = 22.9\ kN.m}$$
$$\mathbf{M_{BA} = 45.7\ kN.m}$$
$$\mathbf{M_{BC} = -45.7\ kN.m}$$
$$\mathbf{M_{CB} = 45.7\ kN.m}$$
$$\mathbf{M_{CD} = -45.7\ kN.m}$$
$$\mathbf{M_{DC} = -22.9\ kN.m}$$

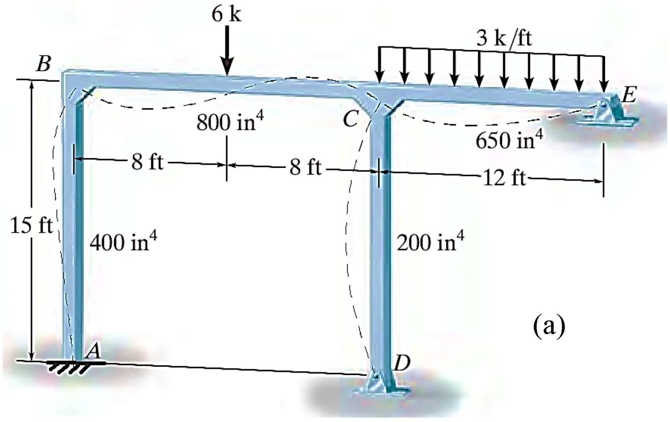
Using these results, the reactions at the ends of each member can be determined from the equations of equilibrium, and the moment diagram for the frame can be drawn, **Fig. *c***.



(*c*)

EXAMPLE 8.3.2

Determine the internal moments at each joint of the frame shown in **Fig. *a***. The moment of inertia ***I*** for each member is given in the figure. Take ***E* = 29(10³) ksi**.



(a)

Solution

$\boldsymbol{\theta_A = 0}$

$\boldsymbol{\psi_{AB} = \psi_{BC} = \psi_{CD} = \psi_{CE} = 0}$**,** since no sidesway will occur.

The member stiffness's,

$$k_{AB} = \frac{400}{15(12)} = 0.001286 \text{ ft}^3 \quad , \quad k_{BC} = \frac{800}{16(12)} = 0.002411 \text{ ft}^3$$

$$k_{CD} = \frac{200}{15(12)} = 0.000643 \text{ ft}^3 \quad , \quad k_{CE} = \frac{650}{12(12)} = 0.002612 \text{ ft}^3$$

The *FEMs* due to the loadings are

$$(FEM)_{BC} = -\frac{PL}{8} = -\frac{6(16)}{8} = -12 \text{ k.ft} \qquad (FEM)_{CB} = \frac{PL}{8} = \frac{6(16)}{8} = 12 \text{ k.ft}$$

$$(FEM)_{CE} = -\frac{wL^2}{12} = -\frac{3(12)^2}{12} = -36 \text{ k.ft} \qquad (FEM)_{EC} = \frac{wL^2}{12} = \frac{3(12)^2}{12} = 36 \text{ k.ft}$$

For member AB

$$M_{AB} = 2[29(10^3)(12)^2](0.001286)[2(0) + \theta_B - 3(0)] + 0 = 10740.70\ \theta_B \qquad ...(1)$$

$$M_{BA} = 2[29(10^3)(12)^2](0.001286)[2(\theta_B) + 0 - 3(0)] + 0 = 12481.50\ \theta_B \qquad ...(2)$$

For member BC

$$M_{BC} = 2[29(10^3)(12)^2](0.002411)[2(\theta_B) + \theta_C - 3(0)] - 12 = 40277.8\ \theta_B + 20138.9\ \theta_C - 12 \qquad ...(3)$$

$$M_{CB} = 2[29(10^3)(12)^2](0.002411)[2(\theta_C) + \theta_B - 3(0)] + 12 = 20138.9\ \theta_B + 40277.8\ \theta_C + 12 \qquad ...(4)$$

For member CD

$$M_{CD} = 2[29(10^3)(12)^2](0.000643)[2(\theta_C) + \theta_D - 3(0)] + 0 = 10740.74\ \theta_C + 5370.37\ \theta_D \qquad ...(5)$$

$$M_{DC} = 2[29(10^3)(12)^2](0.000643)[2(\theta_D) + \theta_C - 3(0)] + 0 = 5370.37\ \theta_C + 10740.74\ \theta_D \qquad ...(6)$$

For member CE

$$M_{CE} = 2[29(10^3)(12)^2](0.02612)[2(\theta_C) + \theta_E - 3(0)] - 36 = 43634.26\ \theta_C + 21817.13\ \theta_E - 36 \qquad ...(7)$$

$$M_{EC} = 2[29(10^3)(12)^2](0.02612)[2(\theta_E) + \theta_C - 3(0)] + 36 = 21817.13\theta_C + 43634.26\theta_E + 36 \qquad ...(8)$$

$\because M_{DC} = 0,$

From **Eqs (5)**, and **(6)** ***eliminates*** the unknown and $\boldsymbol{\theta_D}$

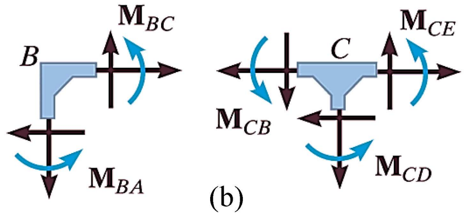
$$M_{CD} = 8055.6\ \theta_C \qquad ...(9)$$

$\because M_{EC} = 0,$

From **Eqs (7)**, and **(8)** ***eliminates*** the unknown and $\boldsymbol{\theta_E}$

$$M_{CE} = 32725.7\ \theta_C - 54 \qquad ...(10)$$

Equations of Equilibrium. These ***six*** equations contain ***eight*** unknowns. ***Two moment equilibrium*** equations can be written for joints ***B*** and ***C*, Fig. *b***. We have



(b)

$$+\circlearrowleft \sum M_B = 0 \qquad M_{BA} + M_{BC} = 0 \qquad ...(11)$$

$$+\circlearrowleft \sum M_C = 0 \qquad M_{CB} + M_{CD} + M_{CE} = 0 \qquad ...(12)$$

In order to solve, substitute **Eqs. (2)** and **(3)** into **Eq. (11)**, and **Eqs. (4)**,**(9)** and **(10)** into **Eq. (12)**.This gives

$$61\,759.3\theta_B + 20\,138.9\theta_C = 12$$

$$20\,138.9\theta_B + 81\,059.0\theta_C = 42$$

Solving these equations simultaneously yields

$$\boldsymbol{\theta_B = 2.758(10^{-5})\ \text{rad} \qquad \theta_C = 5.113(10^{-4})\ \text{rad}}$$

These values, being ***clockwise***, tend to distort the frame as shown in **Fig. *a***. Substituting these values into **Eqs. (1), (2), (3), (4), (9),** and **(10)** and solving,

$$\boldsymbol{M_{AB} = 0.296\ \text{k.ft}}$$

$$\boldsymbol{M_{BA} = 0.592\ \text{k.ft}}$$

$$\boldsymbol{M_{BC} = -0.592\ \text{k.ft}}$$

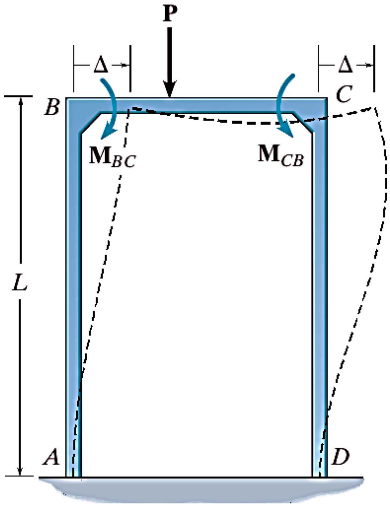
$$\boldsymbol{M_{CB} = 33.1\ \text{k.ft}}$$

$$\boldsymbol{M_{CD} = 4.12\ \text{k.ft}}$$

$$\boldsymbol{M_{CE} = -37.3\ \text{k.ft}}$$

8.4 Analysis of Frames: Sidesway

A frame will sidesway, or be displaced to the side, when it or the loading acting on it is nonsymmetric. When applying the slope-deflection equation to each column of this frame, we must therefore consider the column rotation ψ (since $\psi = \Delta/L$) as unknown in the equation. As a result an extra equilibrium equation must be included for the solution. In the previous sections it was shown that unknown *angular displacements* θ were related by joint *moment equilibrium equations*. In a similar manner, when unknown joint *linear displacements* Δ (or span rotations ψ) occur, we must write *force equilibrium equations* in order to obtain the complete solution. The unknowns in these equations, however, must only involve the internal *moments* acting at the ends of the columns, since the slope-deflection equations involve these moments.



EXAMPLE 8.4.1

Determine the moments at each joint of the frame shown in **Fig. *a***. ***EI*** is constant.

Solution

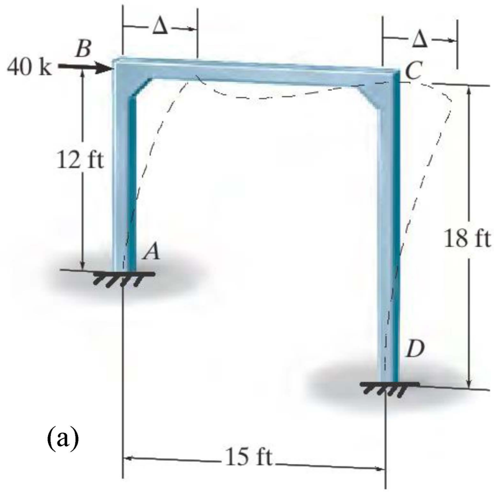
Both joints ***B*** and ***C*** are assumed to be displaced an ***equal amount*** Δ. Consequently,

$\psi_{AB} = \Delta/12$ and $\psi_{DC} = \Delta/18$

Both terms are ***positive*** since the cords of members ***AB*** and ***CD*** "***rotate*** " ***clockwise.***

Relating ψ_{AB} to ψ_{DC}

$$\psi_{AB} = (18/12)\,\psi_{DC}$$



(a)

$$M_{AB} = 2E\left(\frac{I}{12}\right)\left[2(0)+\theta_B-3\left(\frac{18}{12}\psi_{DC}\right)\right]+0 = EI\left(0.1667\theta_B-0.75\psi_{DC}\right) \quad ...(1)$$

$$M_{BA} = 2E\left(\frac{I}{12}\right)\left[2(\theta_B)+0-3\left(\frac{18}{12}\psi_{DC}\right)\right]+0 = EI\left(0.333\theta_B-0.75\psi_{DC}\right) \quad ...(2)$$

$$M_{BC} = 2E\left(\frac{I}{15}\right)\left[2(\theta_B)+\theta_C-3(0)\right]+0 = EI\left(0.267\theta_B+0.133\theta_C\right) \quad ...(3)$$

$$M_{CB} = 2E\left(\frac{I}{15}\right)\left[2(\theta_C)+\theta_B-3(0)\right]+0 = EI\left(0.267\theta_C+0.133\theta_B\right) \quad ...(4)$$

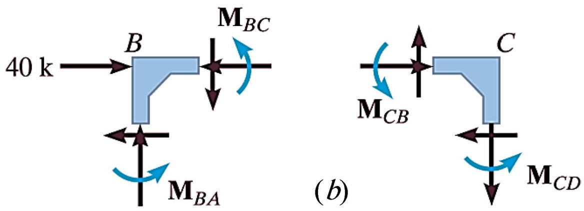
$$M_{CD} = 2E\left(\frac{I}{18}\right)\left[2(\theta_C)+0-3\psi_{DC}\right]+0 = EI\left(0.222\theta_C-0.333\psi_{DC}\right) \quad ...(5)$$

$$M_{DC} = 2E\left(\frac{I}{18}\right)\left[2(0)+\theta_C-3\psi_{DC}\right]+0 = EI\left(0.111\theta_C-0.333\psi_{DC}\right) \quad ...(6)$$

Equations of Equilibrium. The ***six*** equations contain ***nine*** unknowns. ***Two*** moment equilibrium equations for joints ***B*** and ***C***, **Fig.*b***, can be written, namely,

$$M_{BA} + M_{BC} = 0 \qquad ...(7)$$

$$M_{CB} + M_{CD} = 0 \qquad ...(8)$$



(*b*)

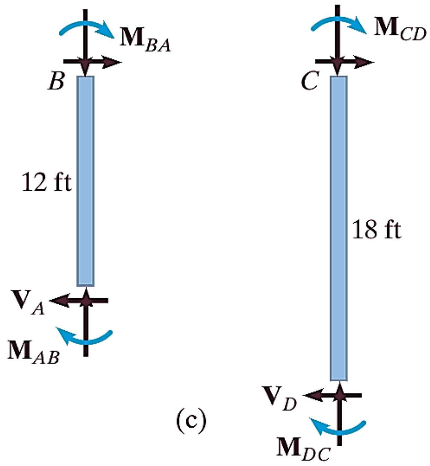
Since a horizontal displacement △ occurs, we will consider summing forces on the ***entire frame*** in the ***x*** direction. This yields

$$+\rightarrow \sum F_x = 0; \qquad 40 - V_A - V_D = 0$$

The horizontal reactions or column shears V_A and V_D can be related to the internal moments by considering the free-body diagram of each column separately, **Fig. *c***. We have

$$\sum M_B = 0; \qquad V_A = -\frac{M_{AB} + M_{BA}}{12}$$

$$\sum M_C = 0; \qquad V_D = -\frac{M_{DC} + M_{CD}}{18}$$



(c)

Thus,

$$40 + \frac{M_{AB} + M_{BA}}{12} + \frac{M_{DC} + M_{CD}}{18} = 0 \qquad ...(9)$$

In order to solve, substitute **Eqs. (2)** and **(3)** into **Eq. (7), Eqs. (4)** and **(5)** into **Eq. (8)**, and **Eqs. (1), (2), (5), (6)** into **Eq. (9)**. This yields

$$0.6\theta_B + 0.133\theta_C - 0.75\psi_{DC} = 0$$

$$0.133\theta_B + 0.489\theta_C - 0.333\psi_{DC} = 0$$

$$0.5\theta_B + 0.222\theta_C - 1.944\psi_{DC} = -\frac{480}{EI}$$

Solving simultaneously, we have

$$\boldsymbol{EI\,\theta_B = 438.81 \qquad EI\,\theta_C = 136.18 \qquad EI\,\psi_{DC} = 375.26}$$

Finally, using these results and solving **Eqs. (1)–(6)** yields

$$M_{AB} = -208 \text{ k.ft}$$

$$M_{BA} = -135 \text{ k.ft}$$

$$M_{BC} = 135 \text{ k.ft}$$

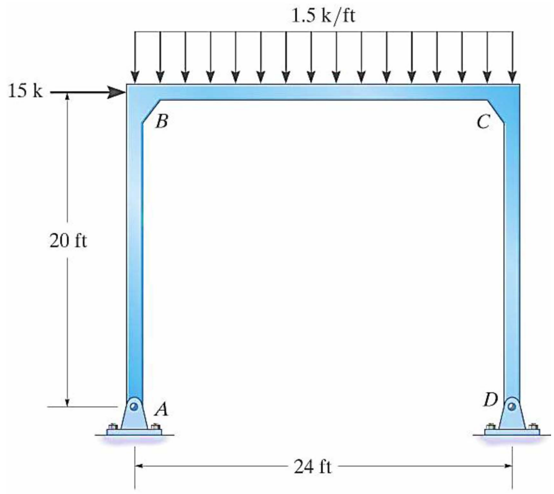
$$M_{CB} = 94.8 \text{ k.ft}$$

$$M_{CD} = -94.8 \text{ k.ft}$$

$$M_{DC} = -110 \text{ k.ft}$$

EXAMPLE 8.4.2

Determine the rotation and the horizontal displacement at joints ***B*** and ***C*** of the frame shown in **Fig. *a***. ***EI*** is constant.



Solution

For member AB

$M_{AB} = 0$

$$M_{BA} = \frac{3EI}{20}\left[\theta_B - \frac{3\Delta}{20}\right] + 0 \qquad ...(1)$$

For member BC

$$M_{BC} = \frac{2EI}{24}[2\theta_B + \theta_C - 0] - \frac{wL^2}{12} = \frac{EI}{12}\cdot(2\theta_B + \theta_C) - 72 \quad ...(2)$$

$$M_{CB} = \frac{2EI}{24}[\theta_B + 2\theta_C - 0] + \frac{wL^2}{12} = \frac{EI}{12}\cdot(\theta_B + 2\theta_C) + 72 \quad ...(3)$$

For member CD

$$M_{CD} = \frac{3EI}{20}\left[\theta_C - \left(\frac{3\Delta}{20}\right)\right] + 0 \qquad ...(4)$$

$M_{DC} = 0$

Equations of Equilibrium. These ***four*** equations contain ***seven*** unknowns. ***Two moment equilibrium*** equations can be written for joints ***B*** and ***C***,

$\because M_{BA} + M_{BC} = 0$

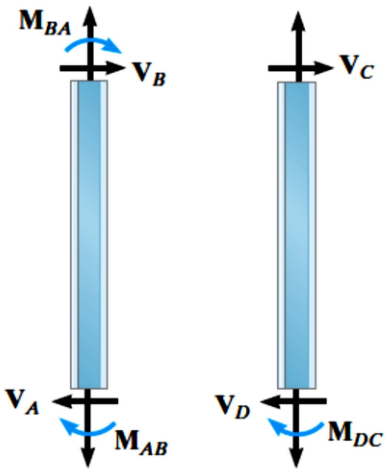
$$\frac{3EI}{20}\left[\theta_B - \frac{3\Delta}{20}\right] + \frac{EI}{12}\cdot(2\theta_B + \theta_C) - 72 = 0$$

$$0.3166\theta_B + 0.0833\theta_C - 0.0225\Delta = \frac{72}{EI} \qquad ...(5)$$

$\because M_{CB} + M_{CD} = 0$

$$\frac{EI}{12}\cdot(\theta_B + 2\theta_C) + 72 + \frac{3EI}{20}\left[\theta_C - \left(\frac{3\Delta}{20}\right)\right] = 0$$

$$0.0833\theta_B + 0.3166\theta_C - 0.0225\Delta = -\frac{72}{EI} \qquad ...(6)$$



$\because 15 - V_A - V_D = 0$

$$\sum M_B = 0; \qquad V_A = -\frac{M_{AB} + M_{BA}}{20}$$

$$\sum M_C = 0; \qquad V_D = -\frac{M_{DC} + M_{CD}}{20}$$

Thus,

$$15 + \frac{M_{AB} + M_{BA}}{20} + \frac{M_{DC} + M_{CD}}{20} = 0$$

$$0.15\theta_B + 0.15\theta_C - 0.045\Delta = -\frac{18}{EI} \qquad ...(7)$$

Solve **Eq. (5), (6),** and **(7)**

$$EI\,\theta_B = 338.63 \qquad EI\,\theta_C = -278.60 \qquad EI\,\Delta = 533.41$$

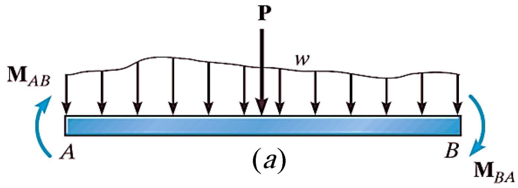
9 DISPLACEMENT METHOD OF ANALYSIS: MOMENT DISTRIBUTION

9.1 Moment Distribution

Moment distribution is a method of successive approximations that may be carried out to any desired degree of accuracy. Essentially, the method begins by assuming each joint of a structure is fixed. Then, by unlocking and locking each joint in succession, the internal moments at the joints are "distributed" and balanced until the joints have rotated to their final or nearly final positions. It will be found that this process of calculation is both repetitive and easy to apply.

Sign Convention.

We will establish the same sign convention as that established for the slope-deflection equations: ***Clockwise moments*** that act ***on the member*** are considered ***positive,*** whereas ***counterclockwise moments*** are ***negative,*** **Fig. *a*.**

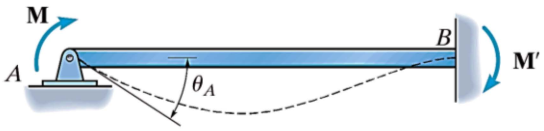


Fixed-End Moments (*FEMs*).

The moments at the "walls" or fixed joints of a loaded member are called ***fixed-end moments***. These moments can be determined from the **table (8-1)**, depending upon the type of loading on the member.

Member Stiffness Factor.

The amount of moment required to rotate the end *A* of the beam by $\theta = 1$ **rad**

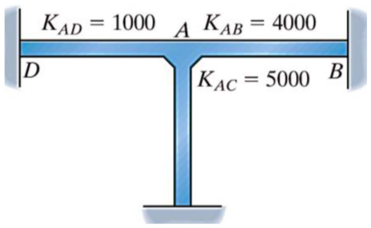


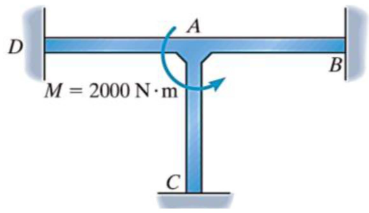
$$M = \frac{4EI\theta}{L}$$

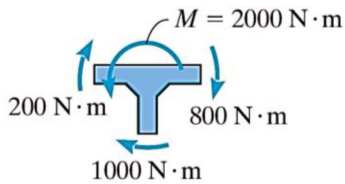
$$K = \frac{4EI}{L} \quad \text{Far End Fixed}$$

Joint Stiffness Factor.

If several members are fixed connected to a joint and each of their far ends is fixed, then by the principle of superposition, the ***total stiffness factor*** at the joint is the sum of the member stiffness factors at the joint, that is, $K_T = \sum K$.







$$M = M_{AD} + M_{AB} + M_{AC}$$

$$K_t = \sum K_i$$

$$K_t = 4000 + 5000 + 1000 = 10\ 000.$$

Distribution Factor (*DF*).

If a moment ***M*** is applied to a fixed connected joint, the connecting members will each supply a portion of the resisting moment necessary to satisfy moment equilibrium at the joint.That fraction of the total resisting moment supplied by the member is called the ***distribution factor* (DF)**.

$$M = M_1 + M_2 + M_3 + \dots$$

$$M = K_1\theta + K_2\theta + K_3\theta + .. = \theta\sum K$$

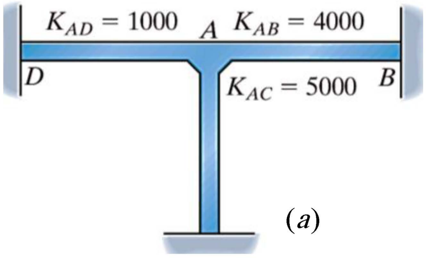
$$DF = \frac{M_i}{M} = \frac{\theta K_i}{\theta\sum K} = \frac{K_i}{\sum K} \quad \Rightarrow \quad \boxed{DF = \frac{K_i}{\sum K}}$$

For example, the distribution factors for members ***AB***, ***AC***, and ***AD*** at joint ***A*** in **Fig. *a*** are,

$$DF_{AB} = \frac{K_i}{\sum K} = \frac{4000}{10000} = 0.4$$

$$DF_{AC} = \frac{K_i}{\sum K} = \frac{5000}{10000} = 0.5$$

$$DF_{AD} = \frac{K_i}{\sum K} = \frac{1000}{10000} = 0.1$$



(*a*)

Member Relative-Stiffness Factor.

For the same material, the term **4*E*** of the stiffness factor will cancel then it is easier to determine the relative stiffness factor as:

$$\boxed{K_R = \frac{I}{L} \quad \text{Far End Fixed}}$$

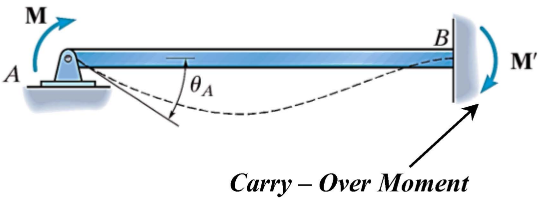
and use this for the computations of the ***DF.***

Carry-Over Factor.

The fraction of the moment that is carried out from the joint to the end:

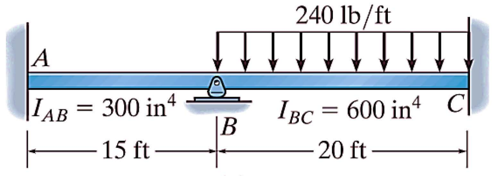
$$\boxed{M' = \frac{1}{2}M}$$

in the case of a beam with ***the far end fixed,*** the carry-over factor is +½. The plus sign indicates both moments act in the same direction.



EXAMPLE 9.1.1

Use moment distribution method to determine the moment at joint ***A***, ***B***, and ***C***, for the beam shown in **Fig. *a***. ***EI*** is constant.



(a)

Solution

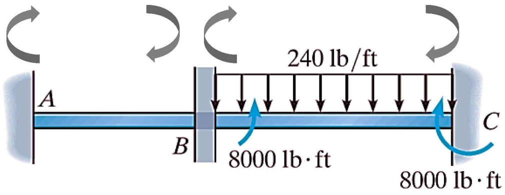
Stiffness Factor:

$$K_R = \frac{I}{L}$$

$$K_{AB} = \frac{300}{15} = 20 \qquad , K_{BC} = \frac{600}{20} = 30$$

$$K_{AB} : K_{BC}$$

$$2:3$$



joint *B* held fixed
(b)

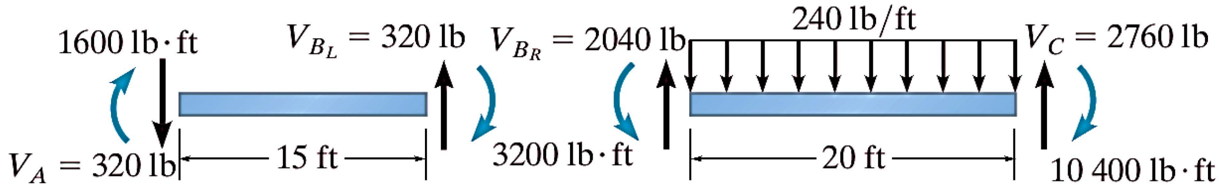
Distribution Factor:

$$DF_{AB} = \frac{2}{5} = 0.4 \,, \qquad DF_{BC} = \frac{3}{5} = 0.6$$

Fixed-End Moments (FEMs):

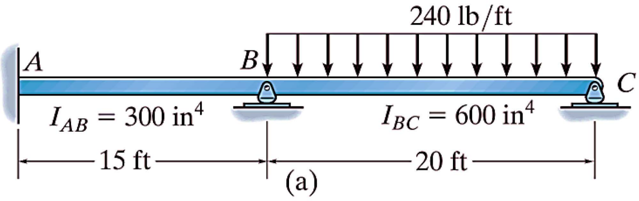
$$(FEM)_{BC} = \frac{wL^2}{12} = -\frac{240(20)^2}{12} = -8000 \text{ lb.ft} \,, \qquad (FEM)_{CB} = \frac{wL^2}{12} = \frac{240(20)^2}{12} = 8000 \text{ lb.ft}$$

Joint	*A*	*B*		*C*
Member	***AB***	***BA***	***BC***	***CB***
DF	**0**	**0.4**	**0.6**	**0**
FEM			**-8000**	**8000**
Dist. CO.		**3200**	**4800**	
Dist. CO.	**1600**			**2400**
$\sum M$	**1600**	**3200**	**-3200**	**10400**



EXAMPLE 9.1.2

Use moment distribution method to determine the moment at joint ***A***, ***B***, and ***C***, for the beam shown in **Fig. *a***. ***E*** is constant.



Solution

Stiffness Factor:

$$K_R = \frac{I}{L}$$

$$K_{AB} = \frac{300}{15} = 20 \qquad , K_{BC} = \frac{600}{20} = 30 \quad \Rightarrow K_{AB} : K_{BC} = 2:3$$

Distribution Factor:

$$DF_{AB} = \frac{20}{50} = 0.4 \,, \qquad DF_{BC} = \frac{30}{50} = 0.6$$

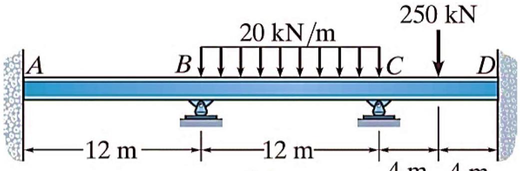
Fixed-End Moments (FEMs):

$$(FEM)_{BC} = \frac{wL^2}{12} = -\frac{240(20)^2}{12} = -8000 \text{ lb.ft} \,, \qquad (FEM)_{CB} = \frac{wL^2}{12} = \frac{240(20)^2}{12} = 8000 \text{ lb.ft}$$

Joint	*A*	*B*		*C*
Member	*AB*	*BA*	*BC*	*CB*
DF	0	0.4	0.6	1
FEM			-8000	8000
Dist. CO.		3200	4800	-8000
FEM	1600		-4000	2400
Dist. CO.		1600	2400	-2400
FEM	800		-1200	1200
Dist. CO.		480	720	-1200
FEM	240		-600	360
Dist. CO.		240	360	-360
FEM	120		-180	180
Dist. CO.		72	108	-180
FEM	36		-90	54
Dist. CO.		36	54	-54
FEM	18		-27	27
Dist. CO.		10.8	16.2	-27
FEM	5.4		-13.5	8.1
Dist. CO.		5.4	8.1	-8.1
FEM	2.7		-4.05	4.05
Dist. CO.		1.62	2.43	-4.05
FEM	0.81		-2.025	1.22
Dist. CO.		0.81	1.22	-1.22
FEM	0.40		-0.61	0.61
Dist. CO.		0.244	0.37	-0.61
$\sum M$	2823.31	5646.87	-5646.87	0

EXAMPLE 9.1.3

Determine the internal moments at each support of the beam shown in **Fig. *a***. ***EI*** is constant.



(a)

Solution

Members *AB* & *BC*:

$$K_{BA} = \frac{I}{12}, \quad K_{BC} = \frac{I}{12}$$

$$K_{BA} : K_{BC} = 1:1$$

$$DF_{BA} = \frac{1}{2} = 0.5$$

$$DF_{BC} = \frac{1}{2} = 0.5$$

Member *BC* & *CD*:

$$K_{CB} = \frac{I}{12}, \quad K_{CD} = \frac{I}{8}$$

$$K_{CB} : K_{CD} = 8:12 = 2:3$$

$$DF_{CB} = \frac{2}{5} = 0.4$$

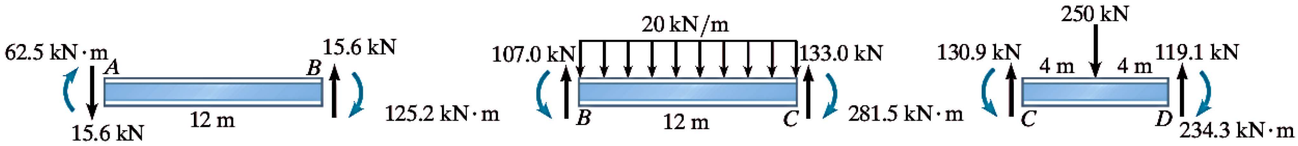
$$DF_{CD} = \frac{3}{5} = 0.6$$

Fixed-End Moments (FEMs):

$$(FEM)_{BC} = \frac{wL^2}{12} = -\frac{20(12)^2}{12} = -240 \text{ kN.m}, \qquad (FEM)_{CB} = \frac{wL^2}{12} = \frac{20(12)^2}{12} = 240 \text{ kN.m}$$

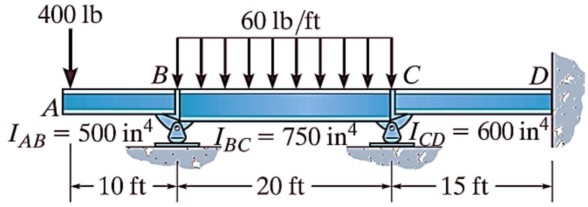
$$(FEM)_{CD} = \frac{PL}{8} = -\frac{250(8)}{8} = -250 \text{ kN.m}, \qquad (FEM)_{DC} = \frac{PL}{8} = \frac{250(8)}{8} = 250 \text{ kN.m}$$

Joint	***A***	***B***		***C***		***D***
Member	***AB***	***BA***	***BC***	***CB***	***CD***	***DC***
DF	**0**	**0.5**	**0.5**	**0.4**	**0.6**	**0**
FEM			**-240**	**240**	**-250**	**250.00**
Dist. CO.		**120**	**120**	**4**	**6**	
FEM	**60**		**2**	**60**		**3.00**
Dist. CO.		**-1**	**-1**	**-24**	**-36**	
FEM	**-0.5**		**-12**	**-0.5**		**-18.00**
Dist. CO.		**6**	**6**	**0.2**	**0.3**	
FEM	**3**		**0.1**	**3**		**0.15**
Dist. CO.		**-0.05**	**-0.05**	**-1.2**	**-1.8**	
FEM	**-0.025**		**-0.6**	**-0.025**		**-0.90**
Dist. CO.		**0.3**	**0.3**	**0.01**	**0.015**	
$\sum M$	**62.475**	**125.25**	**-125.25**	**281.485**	**-281.485**	**234.25**



EXAMPLE 9.1.3

Determine the internal moments at each support of the beam shown in **Fig. *a***. ***EI*** is constant.



(a)

Solution

Member *BC* & *CD*:

$$K_{CB}=\frac{750}{20}=37.5\ ,\qquad K_{CD}=\frac{600}{15}=40\ ,\qquad K_{CB}:K_{CD}=37.5:40$$

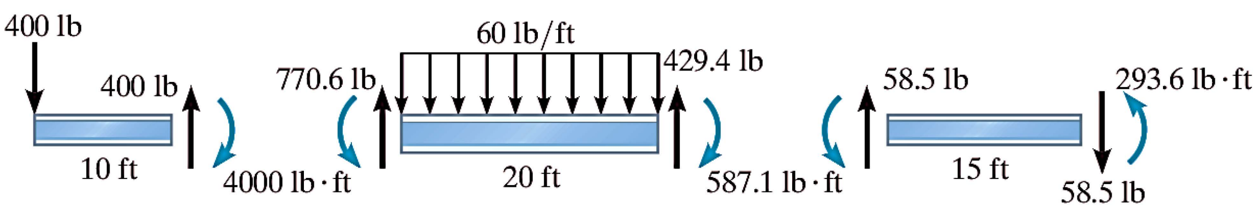
$$DF_{CB}=\frac{37.5}{77.5}=0.484\ ,\quad DF_{CD}=\frac{40}{77.5}=0.516$$

Fixed-End Moments (FEMs):

Due to the overhang, $(FEM)_{BA}$ = 400 lb (10 ft) = 4000 lb.ft

$$(FEM)_{BC}=\frac{wL^2}{12}=-\frac{60(20)^2}{12}=-2000\text{ lb.ft}\ ,\qquad (FEM)_{CB}=\frac{wL^2}{12}=\frac{60(20)^2}{12}=2000\text{ lb.ft}$$

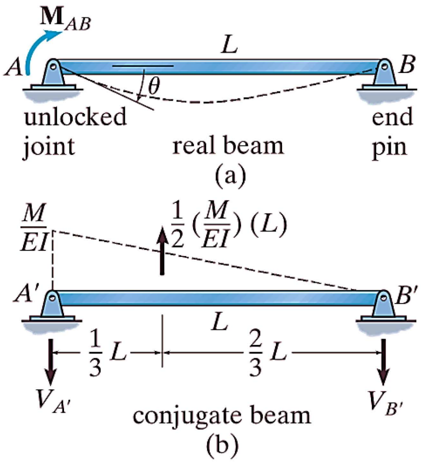
Joint	*B*		*C*		*D*
Member	***BA***	***BC***	***CB***	***CD***	***DC***
DF	**0**	**1**	**0.484**	**0.516**	**0**
FEM	**4000**	**-2000**	**2000**		
Dist. CO.		**-2000**	**-968**	**-1032**	
FEM		**-484**	**-1000**		**-516**
Dist. CO.		**484**	**484**	**516**	
FEM		**242**	**242**		**258**
Dist. CO.		**-242**	**-117.13**	**-124.87**	
FEM		**-58.56**	**-121**		**-62.44**
Dist. CO.		**58.56**	**58.56**	**62.44**	
FEM		**29.28**	**29.28**		**31.22**
Dist. CO.		**-29.28**	**-14.17**	**-15.11**	
FEM		**-7.09**	**-14.64**		**-7.55**
Dist. CO.		**7.09**	**7.09**	**7.55**	
FEM		**3.54**	**3.54**		**3.78**
Dist. CO.		**-3.54**	**-1.71**	**-1.83**	
FEM		**-0.86**	**-1.77**		**-0.91**
Dist. CO.		**0.86**	**0.86**	**0.91**	
FEM		**0.43**	**0.43**		**0.46**
Dist. CO.		**-0.43**	**-0.21**	**-0.22**	
FEM		**-0.10**	**-0.21**		**-0.11**
Dist. CO.		**0.10**	**0.10**	**0.11**	
$\sum M$	**4000**	**-4000**	**587.02**	**-587.02**	**-43.56**



9.2 Stiffness Factor Modification:

In the previous examples of moment distribution we have considered each beam span to be constrained by fixed support at its far end when distributing and carrying over the moments. It's possible to modify the stiffness factor as the following:

9.2.1 Member Pin supported at far end



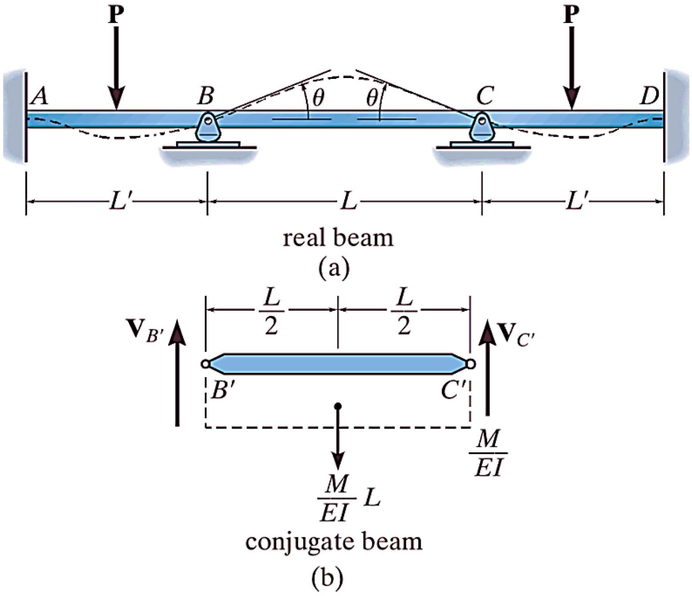
$$\sum M_{B'} = 0 \Rightarrow V'_A(L) - \frac{M}{2EI} \cdot L \cdot \frac{2}{3}L$$

$$V'_A = \theta = \frac{ML}{3EI}, \qquad M = \frac{3EI\,\theta}{L}$$

$$K = \frac{3EI}{L}$$ Far End Pinned or Roller Supported

9.2.2 Symmetric Beam and Loading.

If a beam is symmetric with respect to both its loading and geometry, the bending-moment diagram for the beam will also be symmetric. As a result, a modification of the stiffness factor for the center span can be made, so that moments in the beam only have to be distributed through joints lying on either half of the beam.



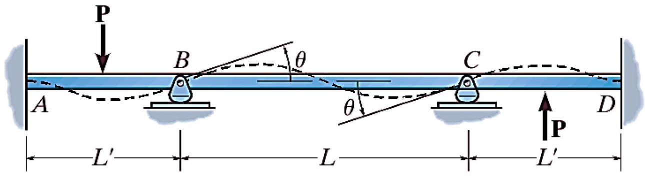
$$\sum M_{C'} = 0 \Rightarrow -V'_B(L) + \frac{M}{EI} \cdot L \cdot \frac{L}{2}$$

$$V'_A = \theta = \frac{ML}{2EI}, \quad \Rightarrow \; M = \frac{2EI\,\theta}{L}$$

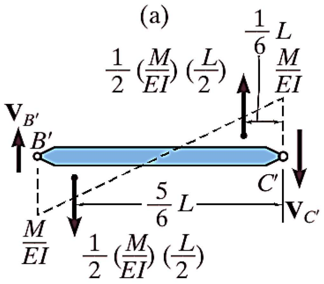
$$K = \frac{2EI}{L}$$, Symmetric Beam and Loading

9.2.3 Symmetric Beam with Antisymmetric Loading.

If a symmetric beam is subjected to antisymmetric loading, the resulting moment diagram will be antisymmetric.



real beam
(a)



conjugate beam
(b)

$$\sum M_{C'} = 0;$$

$$-V_{B'}(L) + \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{5L}{6}\right) - \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{6}\right) = 0$$

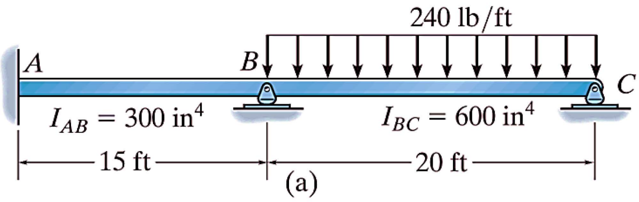
$$V_{B'} = \theta = \frac{ML}{6EI}$$

$$M = \frac{6EI}{L}\theta$$

$$K = \frac{6EI}{L}$$ Symmetric Beam with Antisymmetric Loading

EXAMPLE 9.2.1

Use moment distribution method to determine the moment at joint ***A***, ***B***, and ***C***, for the beam shown in **Fig. *a***. ***EI*** is constant.



Solution

Stiffness Factor:

$$K_{AB} = \frac{4EI}{L} = \frac{4(E)300}{15} = 80E \ , \quad K_{BC} = \frac{3EI}{L} = \frac{3(E)600}{20} = 90E$$

$$K_{AB} : K_{BC} = 80:90$$

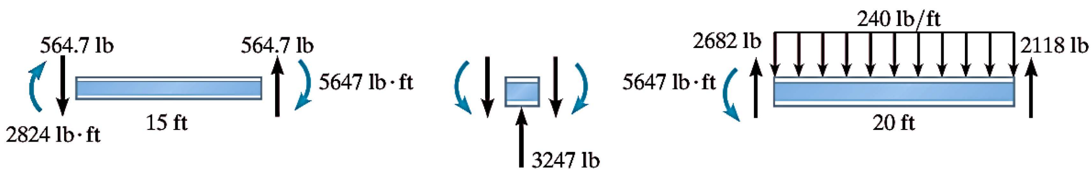
Distribution Factor:

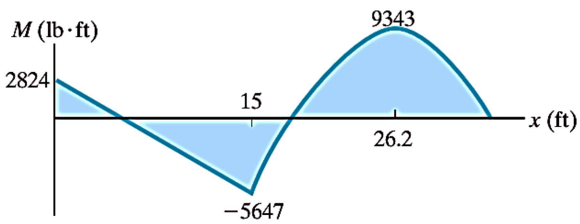
$$DF_{AB} = \frac{80}{170} = 0.4706 \ , \quad DF_{BC} = \frac{90}{170} = 0.5294$$

Fixed-End Moments (FEMs):

$$(FEM)_{BC} = -\frac{wL^2}{8} = -\frac{240(20)^2}{8} = -12000 \text{ lb.ft}$$

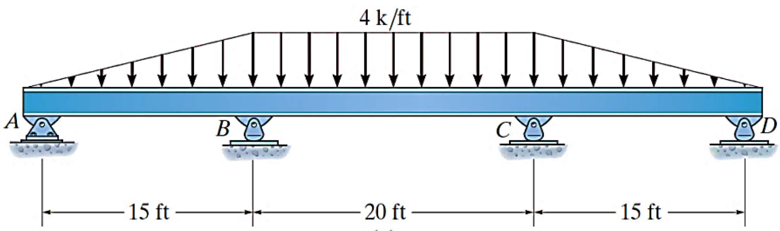
Joint	*A*	*B*		*C*
Member	***AB***	***BA***	***BC***	***CB***
DF	**0**	**0.4706**	**0.5294**	**1**
FEM			**-12000**	
Dist. CO.		**5647.2**	**6352.8**	
Dist. CO.	**2823.6**			**0**
$\sum M$	**2823.6**	**5647.2**	**-5647.2**	**0**





EXAMPLE 9.2.2

Determine the internal moments at the supports for the beam shown in **Fig. *a***. ***EI*** is constant**.**



(a)

Solution

The beam and loading and loading are symmetrical, we will apply ***K* = 2*EI*/*L*** to compute the stiffness factor of the center span ***BC*** and therefore use only the ***left half*** of the beam for the analysis. Furthermore, the distribution of moment at ***A*** can be skipped by using the ***FEM*** for a triangular loading on a span with one end fixed and the other pinned.

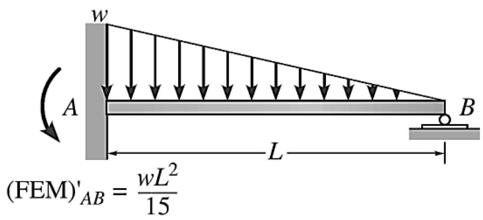
Member *AB* &*BC*
Stiffness Factor:

$$K_{BA} = \frac{3EI}{15} = \frac{EI}{5}\ , \qquad K_{BC} = \frac{2EI}{20} = \frac{EI}{10}$$

$$K_{BA} : K_{BC} = 10:5 = 2:1$$

Distribution Factor:

$$DF_{BA} = \frac{2}{3} = 0.667\ , \qquad DF_{BC} = \frac{1}{3} = 0.334$$



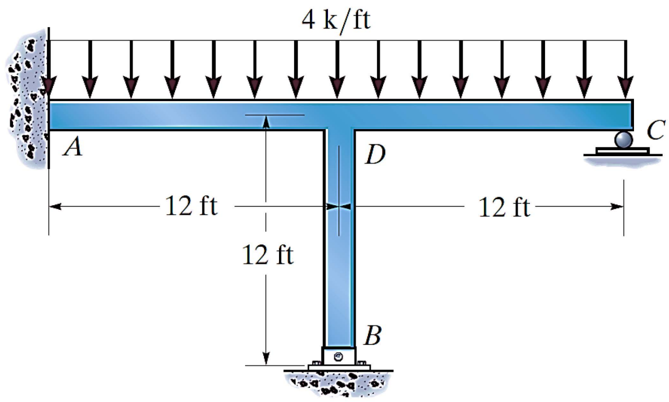
Fixed-End Moments (FEMs):

$$(FEM)'_{BA} = \frac{wL^2}{15} = \frac{4(15)^2}{15} = 60 \text{ k.ft}\ , \quad (FEM)_{BC} = -\frac{wL^2}{12} = -\frac{4(20)^2}{12} = -133.3 \text{ k.ft}$$

Joint	*A*	*B*	
Member	***AB***	***BA***	***BC***
DF	**1**	**0.667**	**0.333**
FEM		**60**	**-133.3**
Dist. CO.		**48.9**	**24.4**
$\sum M$	**0**	**108.9**	**-108.9**

EXAMPLE 9.2.3

Determine the moments at the fixed support ***A*** and joint ***D***. Assume ***B*** is pinned.



Solution

Member Stiffness Factor

$$K_{AD} = \frac{4EI}{12} = \frac{EI}{3} \;,\; K_{DC} = K_{DB} = \frac{3EI}{12} = \frac{EI}{4}$$

Distribution Factor:

$$DF_{AD} = 0$$

$$DF_{DA} = \frac{EI/3}{EI/3 + EI/4 + EI/4} = 0.4 \;,\; DF_{DC} = DF_{DB} = \frac{EI/4}{EI/3 + EI/4 + EI/4} = 0.3$$

$$DF_{CD} = DF_{BD} = 1$$

Fixed-End Moments (FEMs):

$$(FEM)_{AD} = -\frac{wL^2}{12} = -\frac{4(12)^2}{12} = -48 \text{ k.ft} \qquad (FEM)_{DA} = \frac{wL^2}{12} = \frac{4(12)^2}{12} = 48 \text{ k.ft}$$

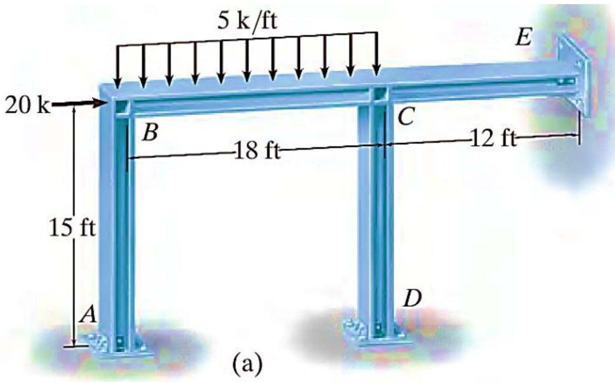
$$(FEM)_{DC} = -\frac{wL^2}{8} = -\frac{4(12)^2}{8} = -72 \text{ k.ft}$$

$$(FEM)_{CD} = (FEM)_{BD} = (FEM)_{DB} = 0$$

Joint	*A*	*D*			*C*	*B*
Member	***AD***	***DA***	***DB***	***DC***	***CD***	***BD***
DF	**0**	**0.4**	**0.3**	**0.3**	**1**	**1**
FEM **Dist. CO.**	**-48**	**48** **9.60**	**0** **7.20**	**-72** **7.20**	**0**	**0**
Dist. CO.	**4.80**					
$\sum M$	**-43.2**	**57.6**	**7.20**	**-64.8**	**0**	**0**

EXAMPLE 9.2.4

Determine the internal moments at the joints of the frame shown in **Fig. *a***. There is a pin at ***E*** and ***D*** and a fixed support at ***A***. ***EI*** is constant.



(a)

Solution

Member *AB* &*BC*

Stiffness Factor:

$$K_{BA} = \frac{4EI}{15}\ ,\ K_{BC} = \frac{4EI}{18}$$

$$K_{BA} : K_{BC} = 18:15 = 6:5$$

Distribution Factor:

$$DF_{BA} = \frac{6}{11} = 0.545\ ,\quad DF_{BC} = \frac{5}{11} = 0.455\ ,\quad DF_{AB} = 0$$

Member *CB* , *CD* & *CE*

Stiffness Factor:

$$K_{CB} = \frac{4EI}{18} = 0.222EI\ ,\ K_{CD} = \frac{3EI}{15} = 0.2EI\ ,\ K_{CE} = \frac{3EI}{12} = 0.25EI$$

$$K_{CB} : K_{CD} : K_{CE} = 0.222 : 0.2 : 0.25$$

Distribution Factor:

$$DF_{CB} = \frac{0.222}{0.222+0.2+0.25} = 0.33$$

$$DF_{CD} = \frac{0.2}{0.222+0.2+0.25} = 0.298 \qquad DF_{DC} = 1\ ,\ DF_{EC} = 1$$

$$DF_{CD} = \frac{0.25}{0.222+0.2+0.25} = 0.372$$

Fixed-End Moments (FEMs):

$$(FEM)_{BC} = -\frac{wL^2}{12} = -\frac{5(18)^2}{12} = -135 \text{ k.ft}\ ,\qquad (FEM)_{CB} = \frac{wL^2}{12} = \frac{5(18)^2}{12} = 135 \text{ k.ft}$$

Joint	*A*	*B*		*C*			*D*	*E*
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*CE*	*DC*	*EC*
DF	0	0.545	0.455	0.330	0.298	0.372	1	1
FEM Dist. CO.		73.6	-135 61.4	135 - 44.6	-40.2	-50.2		
FEM Dist. CO.	36.8	12.2	-22.3 10.1	30.7 -10.1	-9.1	-11.5		
FEM Dist. CO.	6.1	2.8	-5.1 2.3	5.1 -1.7	-1.5	-1.9		
FEM Dist. CO.	1.4	0.4	-0.8 0.4	1.2 -0.4	-0.4	-0.4		
FEM Dist. CO.	0.2	0.1	-0.2 0.1	0.2 -0.1	0.0	-0.1		
$\sum M$	44.5	89.1	-89.1	115	-51.2	-64.1		