FINITE ELEMENTS FOR HEAT TRANSFER PROBLEMS

HEAT CONDUCTION ANALYSIS

- Analogy between Stress and Heat Conduction Analysis

Structural problem	Heat transfer problem
Displacement	Temperature (scalar)
Stress/strain	Heat flux (vector)
Displacement B.C.	Fixed temperature B.C.
Surface traction force	Heat flux B.C.
Body force	Internal heat generation
Young’s modulus	Thermal conductivity

  – In finite element viewpoint, two problems are identical if a proper interpretation is given.

- More Complex Problems
  – Coupled structural-thermal problems (thermal strain).
  – Radiation problem

THERMAL PROBLEM

- Goals:

$$\boxed{[\underbrace{\mathbf{K}_T}_{\text{Conductivity matrix}}]\{\underbrace{\mathbf{T}}_{\text{Nodal temperature}}\} = \{\underbrace{\mathbf{Q}}_{\text{Thermal load}}\}}$$

  - Solve for temperature distribution for a given thermal load.
- Boundary Conditions
  - Essential BC: Specified temperature
  - Natural BC: Specified heat flux

STEADY-STATE HEAT TRANSFER PROBLEM

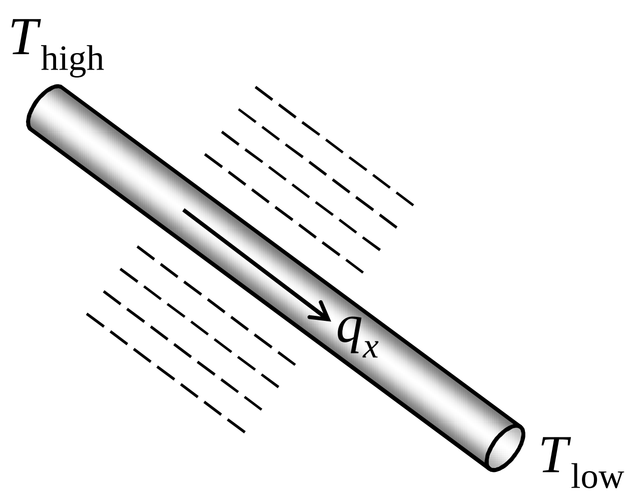
- Fourier Heat Conduction Equation:
  - Heat flow from high temperature to low temperature

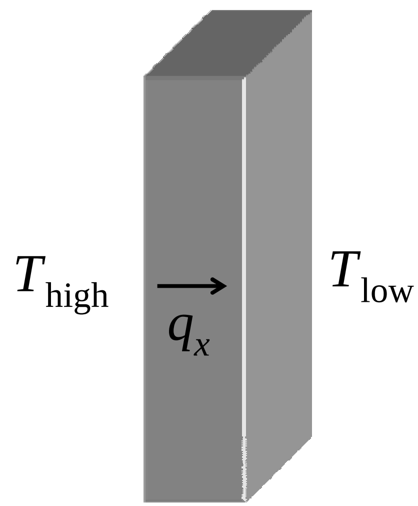
$$\boxed{q_x = -kA\frac{dT}{dx}}$$

Thermal conductivity (W/m/°C)

Heat flux (Watts)

- Examples of 1D heat conduction problems



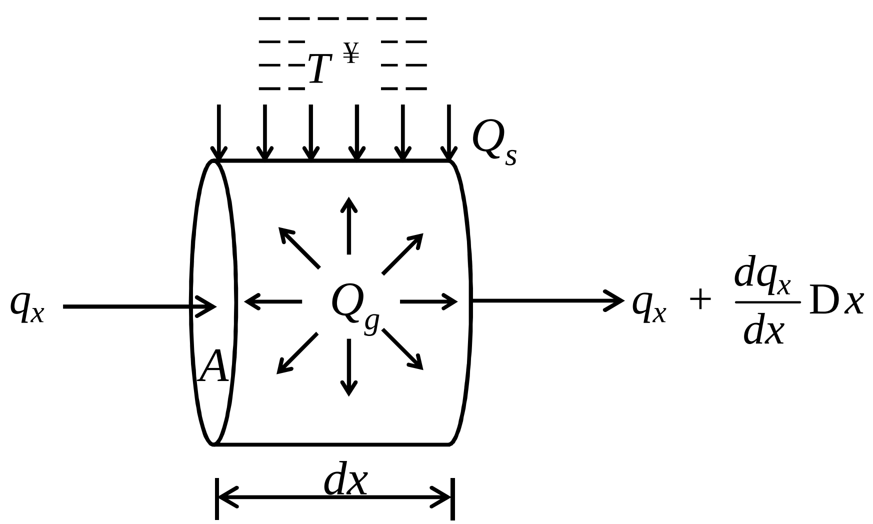


GOVERNING DIFFERENTIAL EQUATION

- Conservation of Energy
  - Energy In + Energy Generated = Energy Out + Energy Increase

$$\boxed{E_{in} + E_{generated} = E_{out} + \Delta U}$$

- Two modes of heat transfer through the boundary
  - Prescribed surface heat flow Qs per unit area
  - Convective heat transfer $Q_h = h\left(T^{\infty} - T\right)$
  - h: convection coefficient (W/m^2/°C)



GOVERNING DIFFERENTIAL EQUATION cont.

- Conservation of Energy at Steady State
  - No change in internal energy ($\Delta U = 0$)

$$\underbrace{q_x + Q_s P\Delta x + h\left(T^{\infty} - T\right)P\Delta x}_{E_{in}} + \underbrace{Q_g A\Delta x}_{E_{gen}} = \underbrace{\left(q_x + \frac{dq_x}{dx}\Delta x\right)}_{E_{out}}$$

  - P: perimeter of the cross-section

$$\frac{dq_x}{dx} = Q_g A + hP\left(T^{\infty} - T\right) + Q_s P, \quad 0 \le x \le L$$

- Apply Fourier Law

$$\boxed{\frac{d}{dx}\left(kA\frac{dT}{dx}\right) + Q_g A + hP\left(T^{\infty} - T\right) + Q_s P = 0,} \qquad 0 \le x \le L$$

  - Rate of change of heat flux is equal to the sum of heat generated and heat transferred

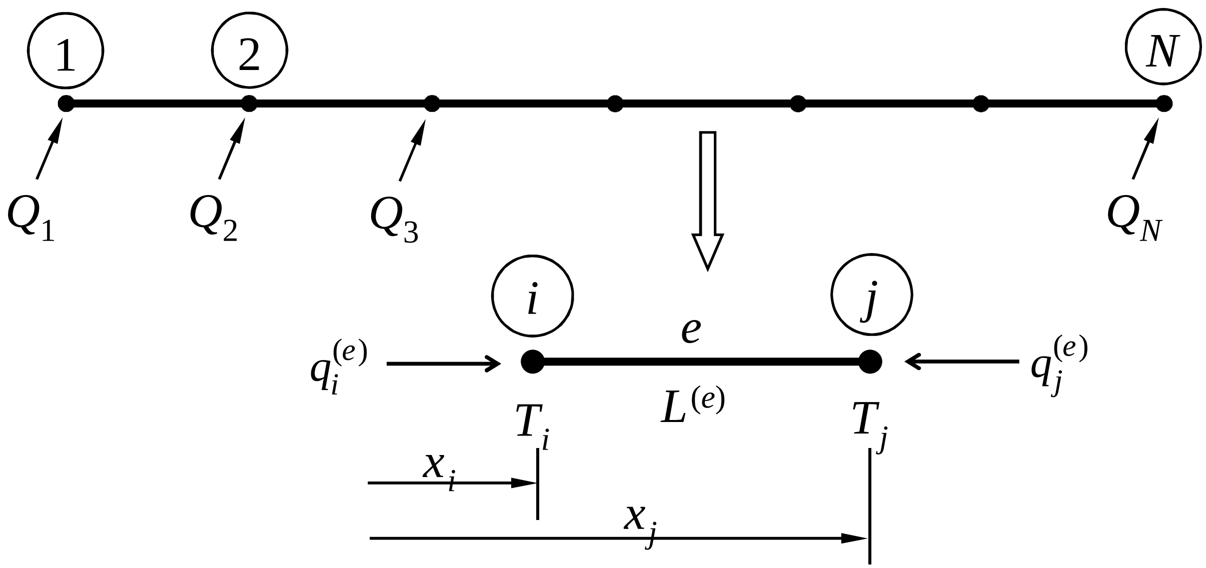
GOVERNING DIFFERENTIAL EQUATION cont.

- Boundary conditions
  - Temperature at the boundary is prescribed (essential BC)
  - Heat flux is prescribed (natural BC)
  - Example: essential BC at x = 0, and natural BC at x = L:

$$\begin{cases} T(0) = T_0 \\ kA\dfrac{dT}{dx}\bigg|_{x=L} = q_L \end{cases}$$

DIRECT METHOD

- Follow the same procedure with 1D bar element
  - No need to use differential equation
- Element conduction equation
  - Heat can enter the system only through the nodes
  - Q_i: heat enters at node i (Watts)
  - Divide the solid into a number of elements
  - Each element has two nodes and two DOFs (T_i and T_j)
  - For each element, heat entering the element is positive



ELEMENT EQUATION

- Fourier law of heat conduction

$$q_i^{(e)} = -kA\frac{dT}{dx} = -kA\frac{\left(T_j - T_i\right)}{L^{(e)}}$$

- From the conservation of energy for the element

$$q_i^{(e)} + q_j^{(e)} = 0 \quad \Longrightarrow \quad q_j^{(e)} = +kA\frac{(T_j - T_i)}{L^{(e)}}$$

- Combine the two equation

$$\boxed{\begin{Bmatrix} q_i^{(e)} \\ q_j^{(e)} \end{Bmatrix} = \frac{kA}{L^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \end{Bmatrix}}$$

Element conductance matrix

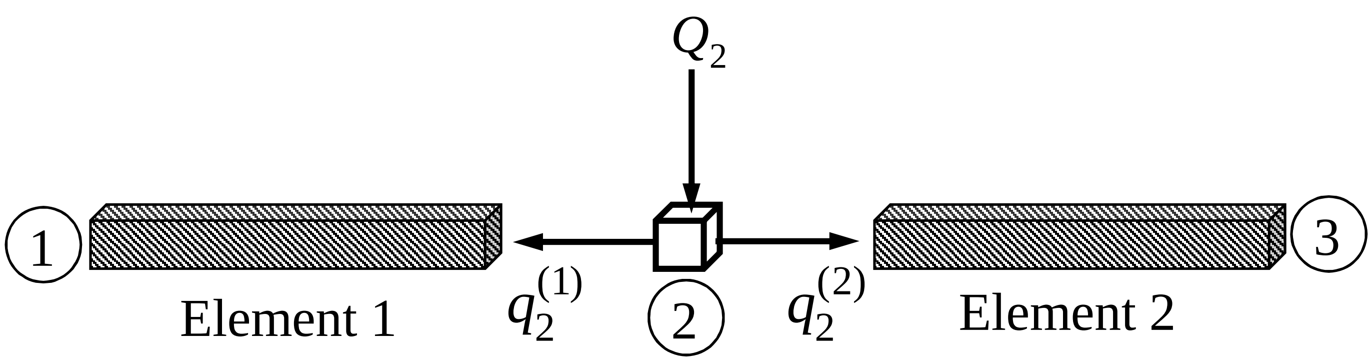
  – Similar to 1D bar element (k = E, T = u, q = f)

ASSEMBLY

- Assembly using heat conservation at nodes
  - Remember that heat flow into the element is positive
  - Equilibrium of heat flow:

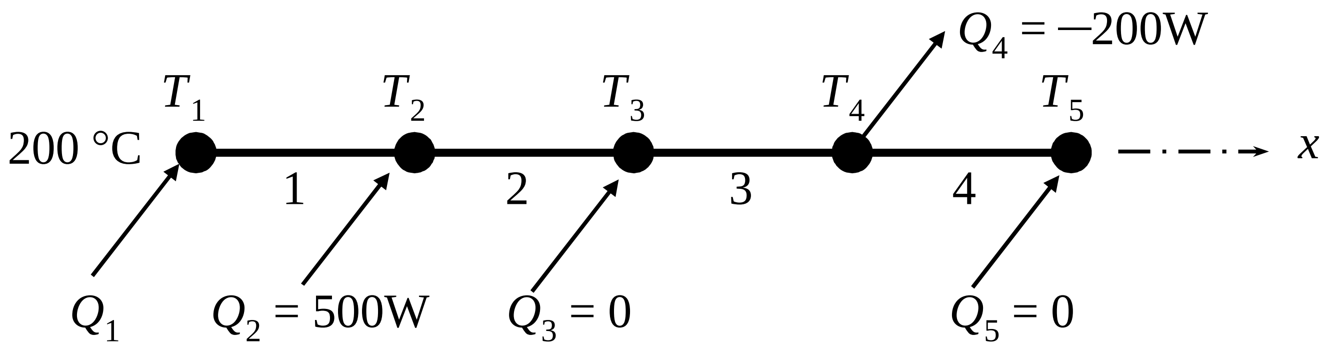
$$Q_i = \sum_{e=1}^{N_i} q_i^{(e)} \quad \Longrightarrow \quad \underset{(N\times N)}{[\mathbf{K}_T]} \begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_N \end{Bmatrix}$$

  - Same assembly procedure with 1D bar elements
- Applying BC
  - Striking-the-rows works, but not striking-the-columns because prescribed temperatures are not usually zero



EXAMPLE

- Calculate nodal temperatures of four elements
  - $A = 1\text{m}^2$, $L = 1\text{m}$, $k = 10\text{W/m/°C}$



- Element conduction equation

$$10\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} \end{Bmatrix} \qquad 10\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} q_2^{(2)} \\ q_3^{(2)} \end{Bmatrix}$$

$$10\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} q_3^{(3)} \\ q_4^{(3)} \end{Bmatrix} \qquad 10\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} q_4^{(4)} \\ q_5^{(4)} \end{Bmatrix}$$

EXAMPLE cont.

- Assembly

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix} = \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} + q_2^{(2)} \\ q_3^{(2)} + q_3^{(3)} \\ q_4^{(3)} + q_4^{(4)} \\ q_5^{(4)} \end{Bmatrix} = 10\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix}$$

- Boundary conditions (T_1 = 200 °C, Q1 is unknown)

$$10\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 200 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ 500 \\ 0 \\ -200 \\ 0 \end{Bmatrix}$$

EXAMPLE cont.

- Boundary conditions
  - Strike the first row

$$10\begin{bmatrix} \boxed{\begin{matrix}-1\\0\\0\\0\end{matrix}} & \begin{matrix}2\\-1\\0\\0\end{matrix} & \begin{matrix}-1\\2\\-1\\0\end{matrix} & \begin{matrix}0\\-1\\2\\-1\end{matrix} & \begin{matrix}0\\0\\-1\\1\end{matrix} \end{bmatrix}\begin{Bmatrix}200\\T_2\\T_3\\T_4\\T_5\end{Bmatrix} = \begin{Bmatrix}500\\0\\-200\\0\end{Bmatrix}$$

  - Instead of striking the first column, multiply the first column with T_1 = 200 °C and move to RHS

$$10\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix}T_2\\T_3\\T_4\\T_5\end{Bmatrix} = \begin{Bmatrix}500\\0\\-200\\0\end{Bmatrix} + \begin{Bmatrix}2000\\0\\0\\0\end{Bmatrix}$$

  - Now, the global matrix is positive-definite and can be solved for nodal temperatures

EXAMPLE cont.

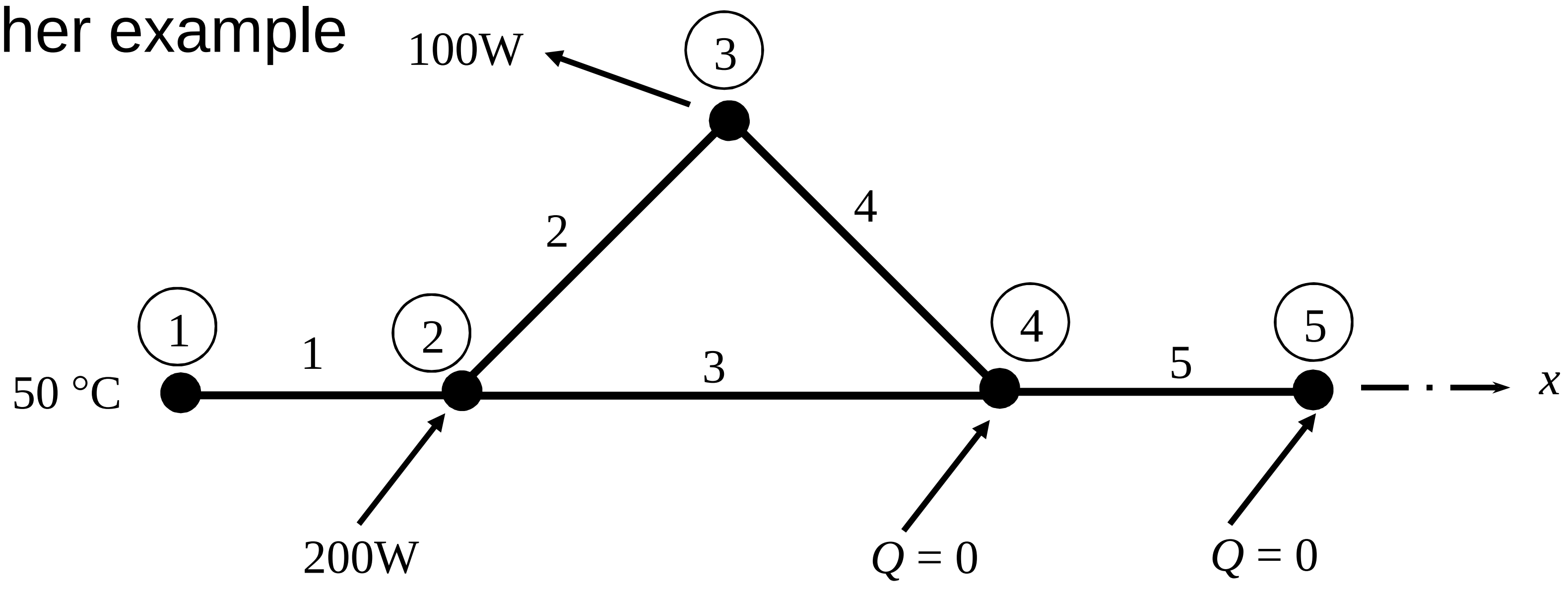
- Nodal temperatures

$$\{\mathbf{T}\}^T = \{200 \quad 230 \quad 210 \quad 190 \quad 190\}^\circ\mathrm{C}$$

- How much heat input is required to maintain $T_1 = 200^oC$?
  - Use the deleted first row with known nodal temperatures

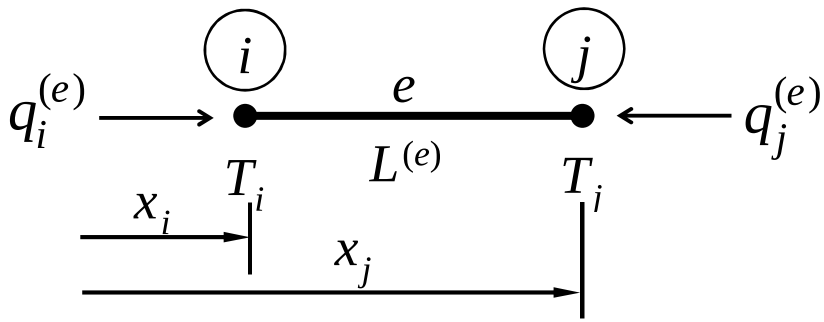
$$Q_1 = 10T_1 - 10T_2 + 0T_3 + 0T_4 + 0T_5 = -300\,\mathrm{W}$$

- Other example



GALERKIN METHOD FOR HEAT CONDUCTION

- Direct method is limited for nodal heat input
- Need more advanced method for heat generation and convection heat transfer
- Galerkin method in Chapter 3 can be used for this purpose
- Consider element (e)
- Interpolation



$$\tilde{T}(x) = T_i N_i(x) + T_j N_j(x)$$

$$N_i(x) = \left(1 - \frac{x - x_i}{L^{(e)}}\right), \quad N_j(x) = \frac{x - x_i}{L^{(e)}}$$

$$\tilde{T}(x) = \lfloor \mathbf{N} \rfloor \{\mathbf{T}\} = \lfloor N_i(x) \quad N_j(x) \rfloor \begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$$

Temperature varies linearly

- Heat flux

$$\frac{d\tilde{T}}{dx} = \left\lfloor -\frac{1}{L^{(e)}} \quad \frac{1}{L^{(e)}} \right\rfloor \{\mathbf{T}\} = \lfloor \mathbf{B} \rfloor \{\mathbf{T}\}$$

Heat flow is constant

GALERKIN METHOD cont.

- Differential equation with heat generation

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right)+Q_gA=0, \quad 0\le x\le L$$

- Substitute approximate solution

$$\frac{d}{dx}\left(kA\frac{d\tilde{T}}{dx}\right)+AQ_g=R(x) \longrightarrow \text{Residual}$$

- Integrate the residual with $N_i(x)$ as a weight

$$\int_{x_i}^{x_j}\left(\frac{d}{dx}\left(kA\frac{d\tilde{T}}{dx}\right)+AQ_g\right)N_i(x)dx=0$$

- Integrate by parts

$$kA\frac{d\tilde{T}}{dx}N_i(x)\bigg|_{x_i}^{x_j}-\int_{x_i}^{x_j}kA\frac{d\tilde{T}}{dx}\frac{dN_i}{dx}dx=-\int_{x_i}^{x_j}AQ_gN_i(x)dx$$

GALERKIN METHOD cont.

- Substitute interpolation relation

$$\int_{x_i}^{x_j} kA\left(T_i \frac{dN_i}{dx} + T_j \frac{dN_j}{dx}\right)\frac{dN_i}{dx}\,dx = \int_{x_i}^{x_j} AQ_g N_i(x)dx - q(x_j)N_i(x_j) + q(x_i)N_i(x_i)$$

- Perform integration

$$\frac{kA}{L^{(e)}}\left(T_i - T_j\right) = Q_i^{(e)} + q_i^{(e)} \qquad Q_i^{(e)} = \int_{x_i}^{x_j} AQ_g N_i(x)dx$$

- Repeat with $N_j(x)$ as a weight

$$\frac{kA}{L^{(e)}}\left(T_j - T_i\right) = Q_j^{(e)} + q_j^{(e)} \qquad Q_j^{(e)} = \int_{x_i}^{x_j} AQ_g N_j(x)dx$$

GALERKIN METHOD cont.

- Combine the two equations

$$\frac{kA}{L^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \end{Bmatrix} = \begin{Bmatrix} Q_i^{(e)} + q_i^{(e)} \\ Q_j^{(e)} + q_j^{(e)} \end{Bmatrix}$$

$$\boxed{[\mathbf{k}_T^{(e)}]\{\mathbf{T}\} = \{\mathbf{Q}^{(e)}\} + \{\mathbf{q}^{(e)}\}}$$

Similar to 1D bar element

  - $\{\mathbf{Q}^{(e)}\}$: thermal load corresponding to the heat source
  - $\{\mathbf{q}^{(e)}\}$: vector of nodal heat flows across the cross-section

- Uniform heat source

$$\{\mathbf{Q}^{(e)}\} = \int_{x_i}^{x_j} AQ_g \begin{bmatrix} N_i(x) \\ N_j(x) \end{bmatrix} dx = \frac{AQ_g L^{(e)}}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

  - Equally divided to the two nodes

- **Temperature varies linearly in the element, and the heat flux is constant**

EXAMPLE

- Heat chamber

Wall temperature = 200 °C

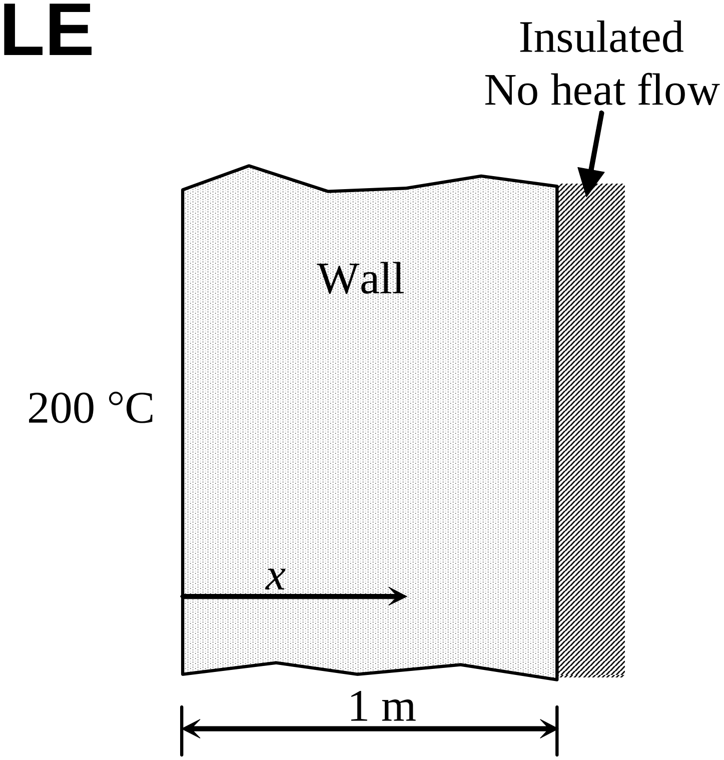
Uniform heat source inside the wall Q = 400 W/m³.

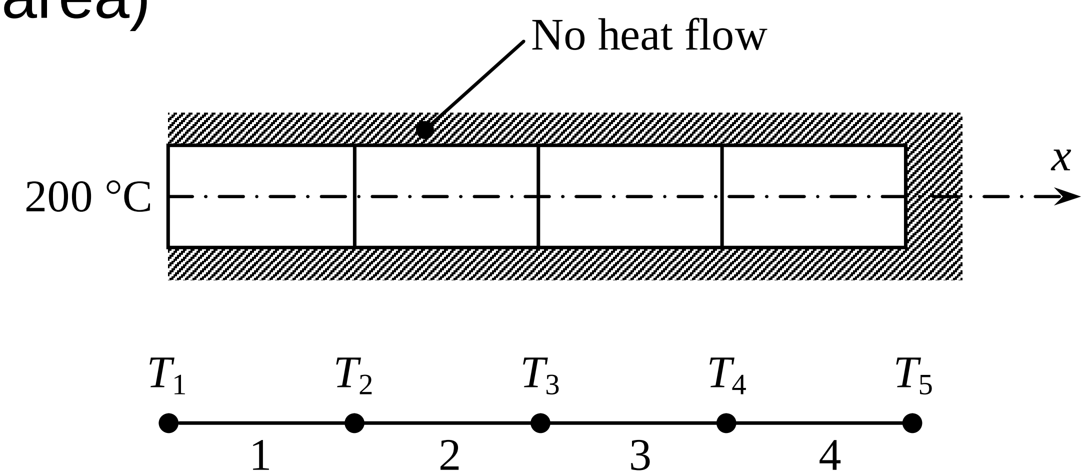
Thermal conductivity of the wall is k = 25 W/m·°C.

Use four elements through the thickness (unit area)

Boundary Condition:

$T_1 = 200$, $q_{x=1} = 0$.





EXAMPLE *cont.*

- Element Matrix Equation
  - All elements are identical

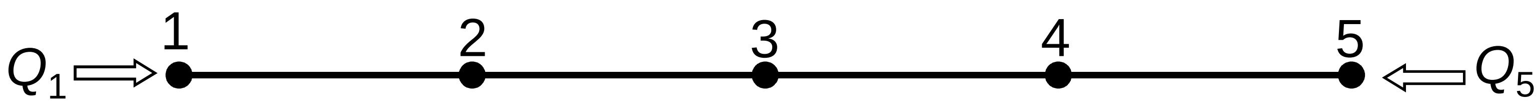
$$100\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 50 \\ 50 \end{Bmatrix} + \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} \end{Bmatrix}$$

  - Assembly

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix} = \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} + q_2^{(2)} \\ q_3^{(2)} + q_3^{(3)} \\ q_4^{(3)} + q_4^{(4)} \\ q_5^{(4)} \end{Bmatrix} = 100\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} - \begin{Bmatrix} 50 \\ 100 \\ 100 \\ 100 \\ 50 \end{Bmatrix}$$

EXAMPLE *cont*.

- Boundary Conditions
  - At node 1, the temperature is given (T_1 = 200). Thus, the heat flux at node 1 (Q_1) should be unknown.
  - At node 5, the insulation condition required that the heat flux (Q_5) should be zero. Thus, the temperature at node 5 should be unknown.
  - At nodes 2 – 4, the temperature is unknown (T_2, T_3, T_4). Thus the heat flux should be known.



$$100\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 200 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 50 + Q_1 \\ 100 \\ 100 \\ 100 \\ 50 \end{Bmatrix}$$

EXAMPLE *cont.*

- Imposing Boundary Conditions
  - Remove first row because it contains unknown Q_1.
  - Cannot remove first column because T_1 is not zero.

$$100\begin{bmatrix} -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 200 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 100 \\ 100 \\ 50 \end{Bmatrix}$$

  - Instead, move the first column to the right.

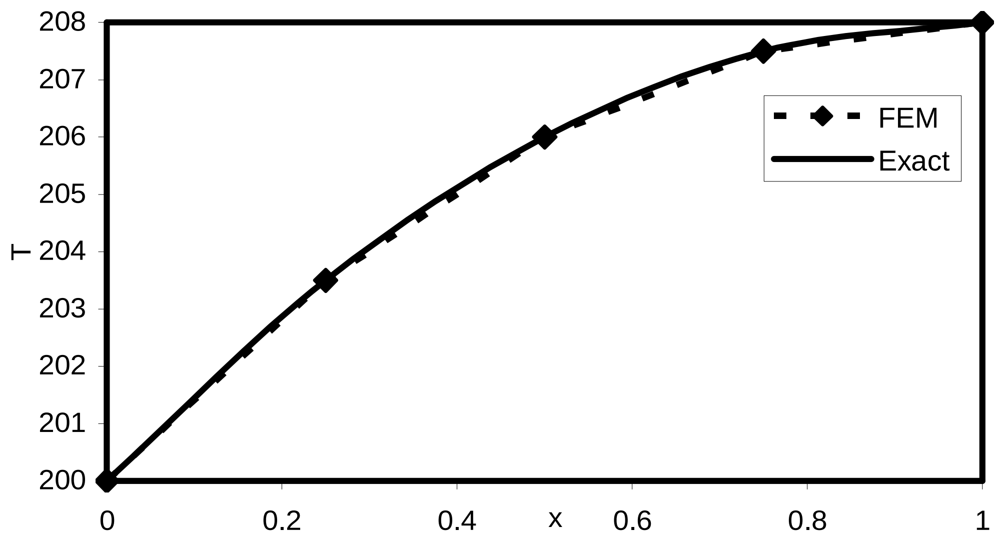
$$100(-1\times 200 + 2\times T_2 - 1\times T_3) = 100$$
$$100(2\times T_2 - 1\times T_3) = 100 + 20000$$

$$100\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 100 \\ 100 \\ 50 \end{Bmatrix} + \begin{Bmatrix} 20000 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 20100 \\ 100 \\ 100 \\ 50 \end{Bmatrix}$$

EXAMPLE *cont.*

- Solution

$$T_1 = 200\ °C,\ T_2 = 203.5\ °C,\ T_3 = 206\ °C,\ T_4 = 207.5\ °C,\ T_5 = 208\ °C$$



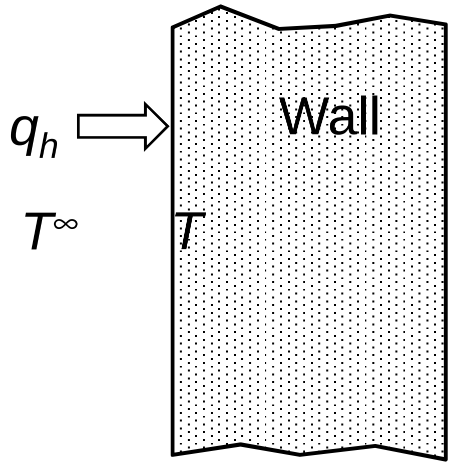
- Discussion
  - In order to maintain 200 degree at node 1, we need to remove heat

$$Q_1 + 50 = 100T_1 - 100T_2 = -350$$

$$\Rightarrow Q_1 = -400\ W$$

CONVECTION BC

- Convection Boundary Condition
  - Happens when a structure is surrounded by fluid
  - Does not exist in structural problems !!!
  - BC includes unknown temperature (mixed BC)



$$\boxed{q_h = hS(T^{\infty} - T)}$$

Fluid Temperature

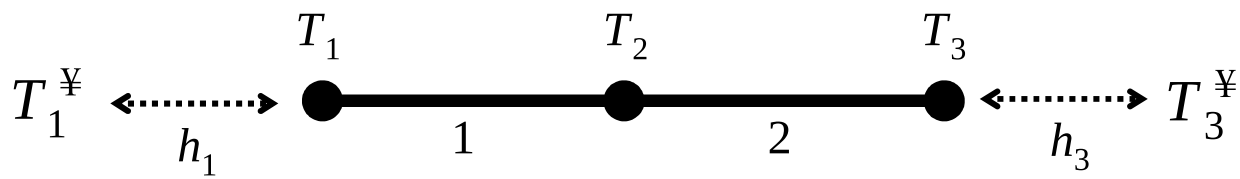
Convection Coefficient

  - Heat flow is not prescribed. Rather, it is a function of temperature on the boundary, which is unknown
- 1D Finite Element
  - When both Nodes 1 and 2 are convection boundary

$$\begin{cases} q_1 = hAT_1^{\infty} - hAT_1 \\ q_2 = hAT_2^{\infty} - hAT_2 \end{cases}$$

EXAMPLE (CONVECTION ON THE BOUNDARY)

- Element equation



$$\frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} \end{Bmatrix} \qquad \frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} q_2^{(2)} \\ q_3^{(2)} \end{Bmatrix}$$

- Balance of heat flow
  - Node 1: $q_1^{(1)} = h_1 A(T_1^{\infty} - T_1)$
  - Node 2: $q_2^{(1)} + q_2^{(2)} = 0$
  - Node 3: $q_3^{(2)} = h_3 A(T_3^{\infty} - T_3)$
- Global matrix equation

$$\frac{kA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} h_1 A(T_1^{\infty} - T_1) \\ 0 \\ h_3 A(T_3^{\infty} - T_3) \end{Bmatrix}$$

EXAMPLE cont.

- **Move unknown nodal temperatures to LHS**

$$\begin{bmatrix} \frac{kA}{L}+h_1A & -\frac{kA}{L} & 0 \\ -\frac{kA}{L} & \frac{2kA}{L} & -\frac{kA}{L} \\ 0 & -\frac{kA}{L} & \frac{kA}{L}+h_3A \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} h_1AT_1^\infty \\ 0 \\ h_3AT_3^\infty \end{Bmatrix}$$

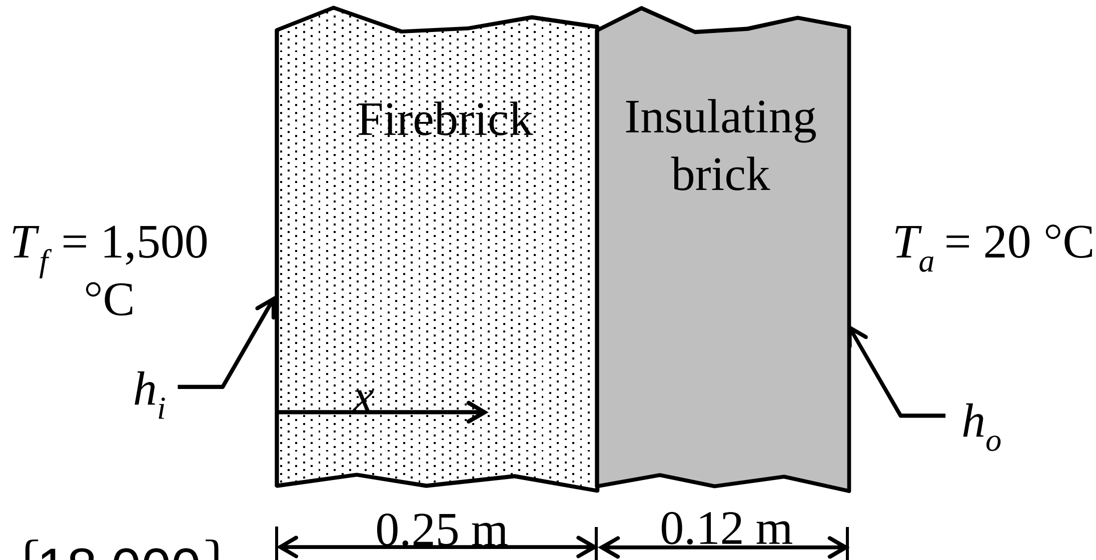
- The above matrix is P.D. because of additional positive terms in diagonal
- How much heat flow through convection boundary?
  - After solving for nodal temperature, use

$$q_1^{(1)} = h_1A(T_1^\infty - T_1)$$

- This is convection at the end of an element

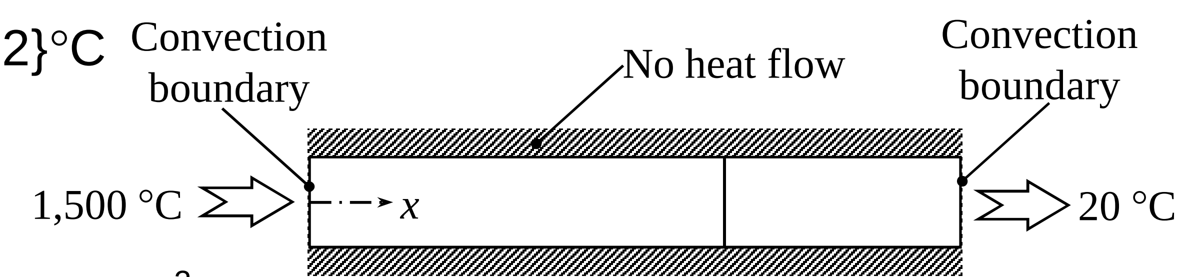
EXAMPLE: FURNACE WALL

- Firebrick
  k_1=1.2W/m/°C
  h_i=12W/m²/°C
- Insulating brick
  k_2=0.2W/m/°C
  h_o=2.0W/m²/°C

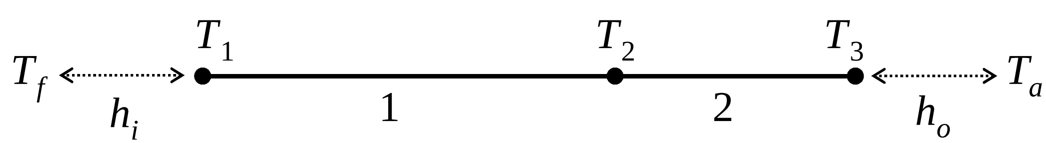


$$\begin{bmatrix} 16.8 & -4.8 & 0 \\ -4.8 & 6.47 & -1.67 \\ 0 & -1.67 & 3.67 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 18{,}000 \\ 0 \\ 40 \end{Bmatrix}$$

$$\{\mathbf{T}\}^T = \{1{,}411 \quad 1{,}190 \quad 552\}°C$$



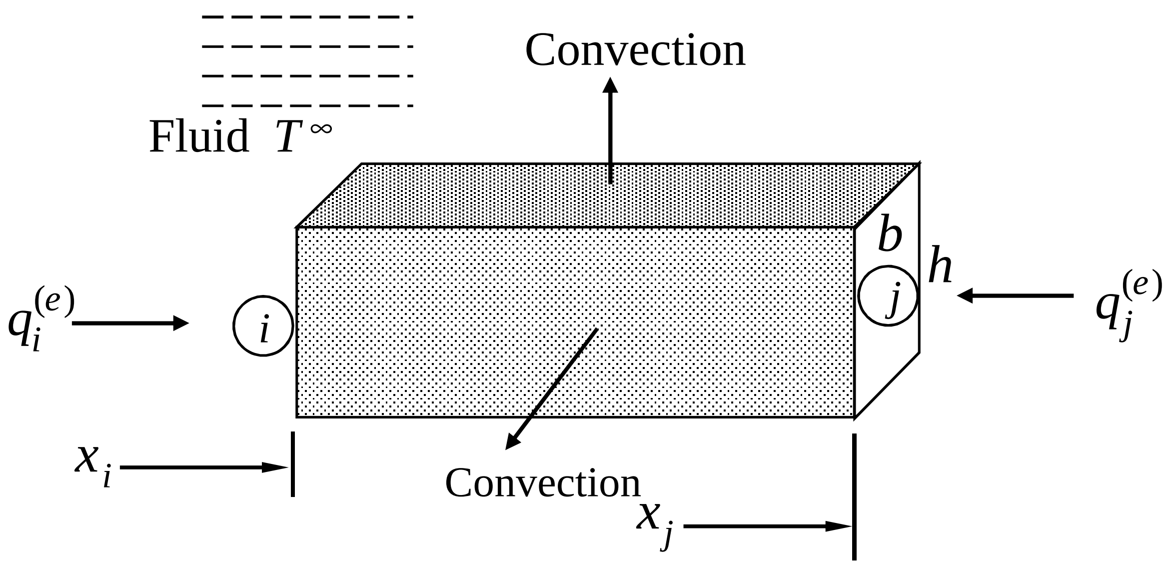
$$q_3^{(2)} = h_0(T_a - T_3) = -1054 \text{ W/m}^2$$



CONVECTION ALONG A ROD

- Long rod is submerged into a fluid
- Convection occurs across the entire surface
- Governing differential equation

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) + AQ_g + hP\left(T^{\infty} - T\right) = 0, \quad 0 \le x \le L \qquad P = 2(b+h)$$



CONVECTION ALONG A ROD cont.

- DE with approximate temperature

$$\frac{d}{dx}\left(kA\frac{d\tilde{T}}{dx}\right)+AQ_g+hP\left(T^{\infty}-\tilde{T}\right)=R(x)$$

- Minimize the residual with interpolation function $N_i(x)$

$$\int_{x_i}^{x_j}\left(\frac{d}{dx}\left(kA\frac{d\tilde{T}}{dx}\right)+AQ_g+hP(T^{\infty}-\tilde{T})\right)N_i(x)dx=0$$

- Integration by parts

$$kA\frac{d\tilde{T}}{dx}N_i(x)\Big|_{x_i}^{x_j}-\int_{x_i}^{x_j}kA\frac{d\tilde{T}}{dx}\frac{dN_i}{dx}dx-\int_{x_i}^{x_j}hP\tilde{T}N_i dx=-\int_{x_i}^{x_j}AQ_gN_i(x)dx-\int_{x_i}^{x_j}hPT^{\infty}N_i dx$$

CONVECTION ALONG A ROD cont.

- Substitute interpolation scheme and rearrange

$$\int_{x_i}^{x_j} kA\left(T_i \frac{dN_i}{dx} + T_j \frac{dN_j}{dx}\right)\frac{dN_i}{dx}dx + \int_{x_i}^{x_j} hP(T_iN_i + T_jN_j)N_i\,dx$$

$$= \int_{x_i}^{x_j} (AQ_g + hPT^{\infty})N_i\,dx - q(x_j)N_i(x_j) + q(x_i)N_i(x_i)$$

- Perform integration and simplify

$$\frac{kA}{L^{(e)}}\left(T_i - T_j\right) + hpL^{(e)}\left(\frac{T_i}{3} + \frac{T_j}{6}\right) = Q_i^{(e)} + q_i^{(e)}$$

$$Q_i^{(e)} = \int_{x_i}^{x_j} (AQ_g + hPT^{\infty})N_i(x)dx$$

- Repeat the same procedure with interpolation function $N_j(x)$

CONVECTION ALONG A ROD cont.

- Finite element equation with convection along the rod

$$\left[\frac{kA}{L^{(e)}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{hPL^{(e)}}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right]\begin{Bmatrix}T_i\\ T_j\end{Bmatrix}=\begin{Bmatrix}Q_i^{(e)}+q_i^{(e)}\\ Q_j^{(e)}+q_j^{(e)}\end{Bmatrix}$$

$$\left[[\mathbf{k}_T^{(e)}]+[\mathbf{k}_h^{(e)}]\right]\{\mathbf{T}\}=\{\mathbf{Q}^{(e)}\}+\{\mathbf{q}^{(e)}\}$$

- Equivalent conductance matrix due to convection

$$\left[\mathbf{k}_h^{(e)}\right]=\frac{hPL^{(e)}}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

- Thermal load vector

$$\{\mathbf{Q}^{(e)}\}=\begin{Bmatrix}Q_i\\ Q_j\end{Bmatrix}=\frac{AQ_gL^{(e)}+hPL^{(e)}T^{\infty}}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

EXAMPLE: HEAT FLOW IN A COOLING FIN

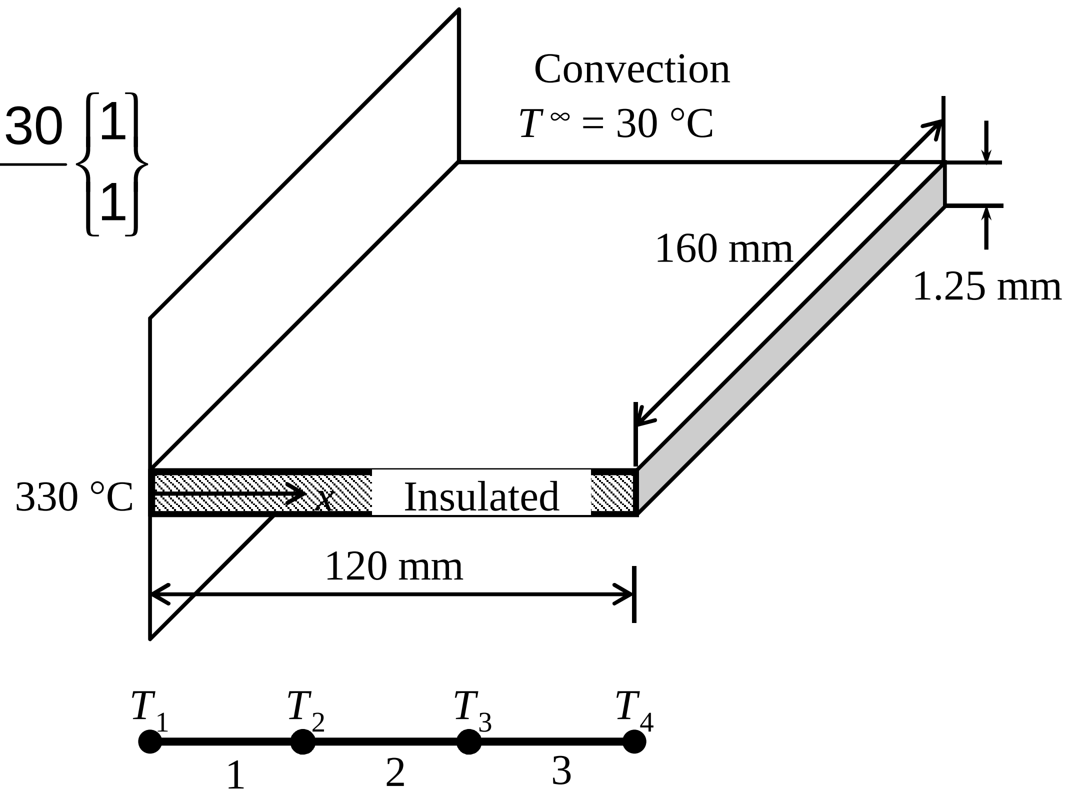
- k = 0.2 W/mm/°C, h = 2×10^{-4} W/mm^2/°C
- Element conductance matrix

$$[\mathbf{k}_T^{(e)}]+[\mathbf{k}_h^{(e)}]=\frac{0.2\times 200}{40}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{2\times 10^{-4}\times 320\times 40}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

- Thermal load vector

$$\{\mathbf{Q}^{(e)}\}=\frac{2\times 10^{-4}\times 320\times 40\times 30}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

- Element 1



EXAMPLE: HEAT FLOW IN A COOLING FIN cont.

- Element conduction equation
  - Element 1 $$\begin{bmatrix} 1.8533 & -0.5733 \\ -0.5733 & 1.8533 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 38.4 \\ 38.4 \end{Bmatrix} + \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} \end{Bmatrix}$$
  - Element 2 $$\begin{bmatrix} 1.8533 & -0.5733 \\ -0.5733 & 1.8533 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 38.4 \\ 38.4 \end{Bmatrix} + \begin{Bmatrix} q_2^{(2)} \\ q_3^{(2)} \end{Bmatrix}$$
  - Element 3 $$\begin{bmatrix} 1.8533 & -0.5733 \\ -0.5733 & 1.8533 \end{bmatrix} \begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 38.4 \\ 38.4 \end{Bmatrix} + \begin{Bmatrix} q_3^{(3)} \\ q_4^{(3)} \end{Bmatrix}$$
- Balance of heat flow
  - Node 1 $q_1^{(1)} = Q_1$
  - Node 2 $q_2^{(1)} + q_2^{(2)} = 0$
  - Node 3 $q_3^{(2)} + q_3^{(2)} = 0$
  - Node 4 $q_4^{(3)} = hA(T^{\infty} - T_4)$

EXAMPLE: HEAT FLOW IN A COOLING FIN cont.

- Assembly

$$\begin{bmatrix} 1.853 & -.573 & 0 & 0 \\ -.573 & 3.706 & -.573 & 0 \\ 0 & -.573 & 3.706 & -.573 \\ 0 & 0 & -.573 & 1.853 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 38.4 + Q_1 \\ 76.8 \\ 76.8 \\ 38.4 + hA(T^{\infty} - T_4) \end{Bmatrix}$$

- Move T_4 to LHS and apply known $T_1 = 330$

$$\begin{bmatrix} 1.853 & -.573 & 0 & 0 \\ -.573 & 3.706 & -.573 & 0 \\ 0 & -.573 & 3.706 & -.573 \\ 0 & 0 & -.573 & 1.893 \end{bmatrix} \begin{Bmatrix} 330 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 38.4 + Q_1 \\ 76.8 \\ 76.8 \\ 39.6 \end{Bmatrix}$$

- Move the first column to RHS after multiplying with T_1=330

$$\begin{bmatrix} 3.706 & -.573 & 0 \\ -.573 & 3.706 & -.573 \\ 0 & -.573 & 1.893 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 265.89 \\ 76.8 \\ 39.6 \end{Bmatrix}$$

EXAMPLE: HEAT FLOW IN A COOLING FIN cont.

- Solve for temperature

$$T_1 = 330°C,\ \ T_2 = 77.57°C,\ \ T_3 = 37.72°C,\ \ T_4 = 32.34°C$$