Matrices and Digital Images

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The images you see on internet pages and the photos you take with your mobile phone are examples of digital images. It is possible to represent this kind of image using matrices. For example, the small image of Felix the Cat (on the left) can be represented by a 35×35 matrix whose elements are the numbers 0 and 1. These numbers specify the color of each pixel (a pixel is the smallest graphical element of a matricial image, which can take only one color at a time): the number 0 indicates black, and the number 1 indicates white. Digital images using only two colors are called *binary images* or *boolean images.*

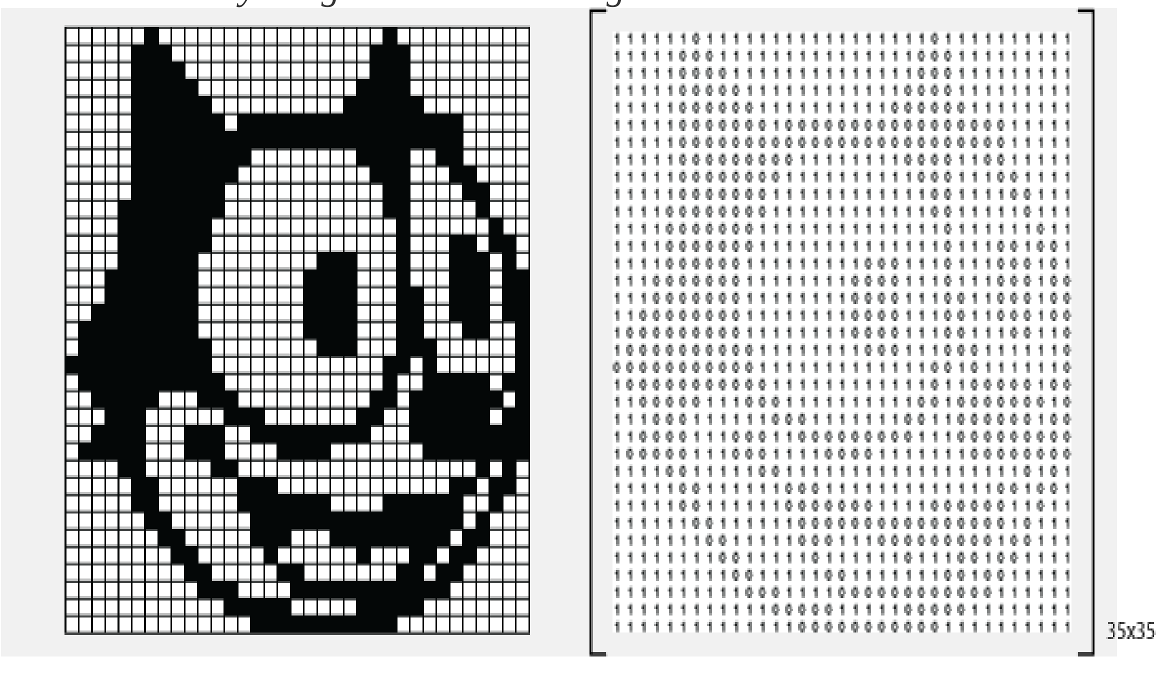

Figure 2: The matrix corresponding to Felix The Cat

Grayscale images can also be represented by matrices. Each element of the matrix determines the intensity of the corresponding pixel. For convenience, most of the current digital files use integer numbers between 0 (to indicate black, the color of minimal intensity) and 255 (to indicate white, maximum intensity), giving a total of $256 = 2^8$ different levels of gray (This quantity of levels of gray is reasonable to work with images in WEB pages. However, there are certain specific applications

that need more levels of gray in order to reproduce the image with more details and avoid rounding errors in numerical calculations, as is the case of medical images).

Color images, in turn, can be represented by three matrices. Each matrix specifies the amount of **red**, **green** and **blue** that makes up the image. This color system is known as RGB (There are many other color systems that are used depending on the application: CMYK (for printing), Y'IQ (for TV analog transmission in NTSC), etc). The elements of these matrices are integer numbers between 0 and 255, and they determine the intensity of the pixel with respect to the color of the matrix. Thus, in the RGB system, it is possible to represent $256^3 = 2^{24} = 16777216$ different colors.

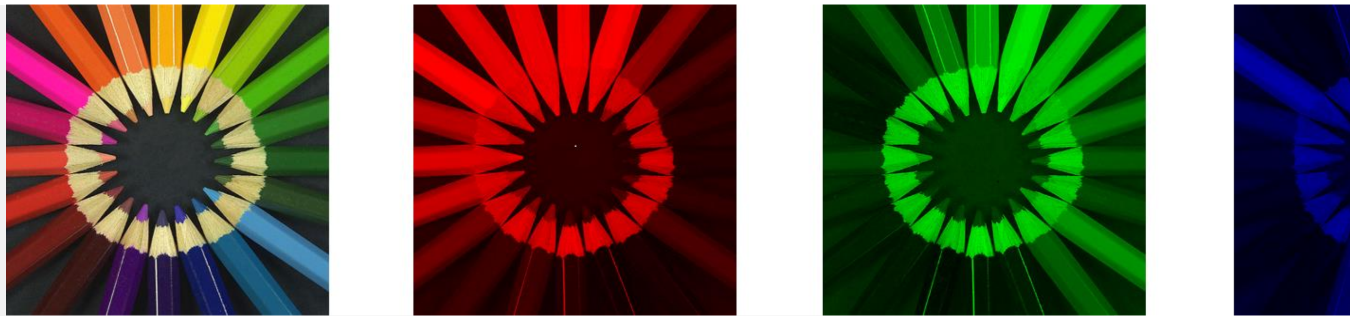

Figure 3: Original picture, Red, Green and Blue components

Digital image processing and operations with matrices

Once a digital image can be represented by matrices, we may ask how operations on their elements affect the corresponding image. For example, if we consider the binary image Abelow as a matrix, say $A = (a_{i,j})$, then the image B corresponds to the *transposed matrix*of A, that is, $B = (b_{i,j}) = (a_{j,i}) = A^T$. The image H, by its turn, corresponds to the matrix $(a_{j,35-i+1})$. Try to discover the matricial relationships between the image A and the other images!

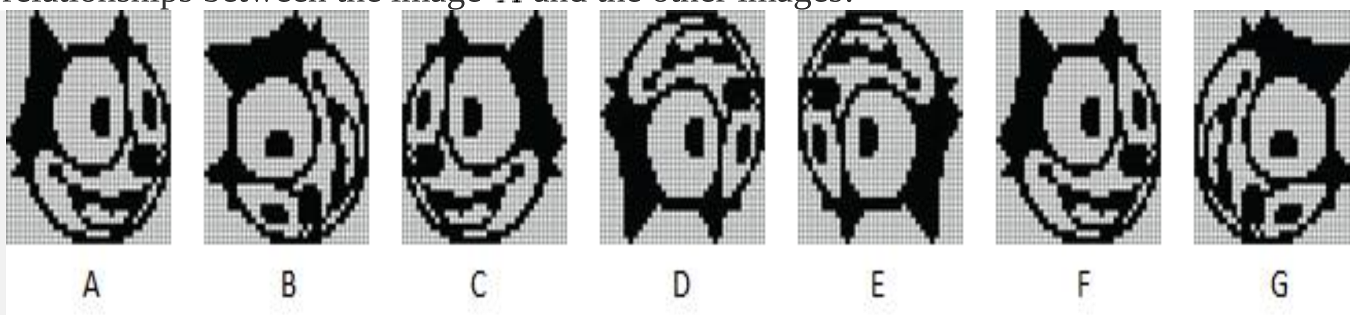

Figure 4: Matrix transformations

Another example: if we take the standard arithmetic mean of the component matrices R, G and B from a color image A, we will get a

grayscale version of the image (non-integer values are rounded to the nearest integer):

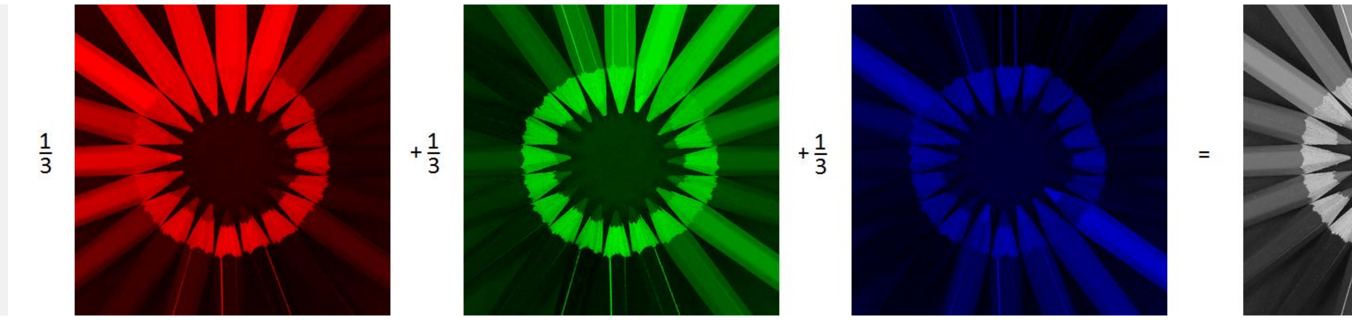

Figure 5: arithmetic mean of the component matrices

One more example: using the operations of multiplication by a scalar and sum of matrices, it is possible to create an image transition effect commonly used, for instance, in PowerPoint© presentations and *slide shows*. More precisely, consider two grayscale images of the same size, represented by the matrices A and Z. For each scalar (real number) t in the interval $[0, 1]$, define the matrix $M(t) = (1-t)A + tZ$.

Notice that $M(0) = A$, $M(1) = Z$ and, for each t between 0 and 1, the elements of the matrix $M(t)$ are between the elements of the matrices A and Z. Therefore, when t varies from 0 to 1, the matrix $M(t)$ varies from A to Z. For the case of color images, the transformation above must be applied to the matrices R, G and B that compose each image.

Figure 6: M(0) = A, M(0.13), M(0.25), M(0.38), M(0.50), M(0.63), M(0.75), M(0.88), M(1)=Z

Multiplication of matrices also has applications in digital image processing. Although our next example will be more elaborate (with a rationale based on more advanced mathematical techniques usually only studied in Linear Algebra university courses), we believe, still, that it will be of interest to the reader, since this will have the opportunity to enjoy an amazing application derived from the ability to decompose a matrix as

a product of matrices with special structures. The omitted details may be found in the references [Lay, 2011] and [Poole, 2005]. Consider, therefore, a *singular value decomposition* (SVD), that consists of writing a matrix $A_{m\times n}$ as the product of three matrices:

$$A_{m\times n} = U_{m\times m} S_{m\times n} V_{n\times n}^T,$$

where U and V are *orthogonal matrices* (that is, $U^T U$ and $V^T V$ are $m\times m$ and $n\times n$ identity matrices, respectively) and S is a matrix whose elements $s_{i,j}$ are equal to zero for $i \neq j$ and $s_{1,1} \geq s_{2,2} \geq \cdots \geq s_{k,k} \geq 0$, with $k = \min\{m, n\}$. Here is an example of an SVD decomposition:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = USV^T$$

$$= \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \end{bmatrix}^T.$$

It can be shown that every matrix has an SVD decomposition ([Lay, 2011], [Poole, 2005]). Moreover, algorithms exist that allow us to calculate such decompositions using a computer. The key point of our example is to observe that if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are the columns of the matrix U and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are the columns of the matrix V, then

$$A = USV^T = s_{1,1}\mathbf{u}_1\mathbf{v}_1^T + s_{2,2}\mathbf{u}_2\mathbf{v}_2^T + \cdots + s_{k,k}\mathbf{u}_k\mathbf{v}_k^T.$$

Why is that? Suppose that A, a grayscale image of size 1000×1000, must be transmitted from a satellite to a laboratory on Earth. In principle, the satellite would have to send 1million numbers (one for each pixel). As typically only the first elements $s_{i,i}$ of the matrix S of the SVD decomposition for A are significant (the others are “small”), it is enough, then, that the satellite sends, say, the 20 first columns of U and V, and the 20 first numbers $s_{i,i}$ (totaling only $20 \cdot 1000 + 20 \cdot 1000 + 20 = 40020$ numbers that must be sent). Upon receiving these data, the laboratory on Earth calculates the matrix $s_{1,1}\mathbf{u}_1\mathbf{v}_1^T + s_{2,2}\mathbf{u}_2\mathbf{v}_2^T + \cdots + s_{20,20}\mathbf{u}_{20}\mathbf{v}_{20}^T$ that will give an approximation of the original image.

Let’s see an example: the picture below of the mathematician Christian Felix Klein (1849-1925) has $720 \times 524 = 377280$ pixels.

Figure 7: Felix Klein

From the SVD decomposition of the corresponding matrix of this image, we can calculate the matrices $s_{1,1}\mathbf{u}_1\mathbf{v}_1^T + s_{2,2}\mathbf{u}_2\mathbf{v}_2^T + \cdots + s_{r,r}\mathbf{u}_r\mathbf{v}_r^T$ for $r = 1, 5, 10$ and 20. These matrices generate approximations to the original image, as illustrated in the following figures. Notice that the original image corresponds to the case $r = 524$. It is quite impressive, is it not?

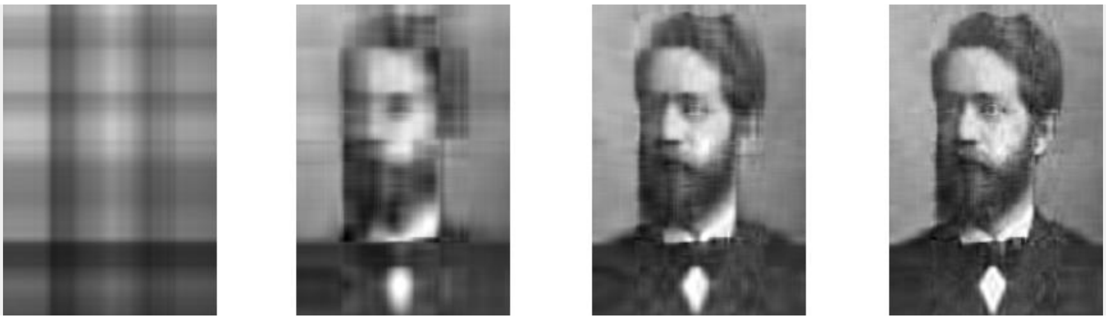

Figure 8: cases r=1, 5, 1