Relations

Binary relation:

There are many relations in mathematics :"less than" , "is parallel to ","is a subset of", etc. These relations consider the existence or nonexistence of a certain connection between pairs of objects taken in a definite order. We define a relation simply in terms of ordered pairs of objects.

Product sets:

Consider two arbitrary sets A and B. The set of all ordered pairs (a,b) where $a \in A$ and $b \in B$ is called the product, or cartesian product, of A and B.

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$$

Example: Let A = {1,2} and B = {a ,b ,c} then

$$A \times B = \{(1,a), (1,b),(1,c),(2,a),(2,b),(2,c)\}$$

Also, $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

- The order in which the sets are considered is important, so $A \times B \neq B \times A$.

Let A and B be sets. A binary relation, R, from A to B is a subset of $A \times B$. If $(x,y) \in R$, we say that x is R-related to y and denote this by xRy

if $(x,y) \notin R$, we write $x \not R y$ and say that x is not R-related to y .

if R is a relation from A to A ,i.e. R is a subset of $A \times A$, then we say that R is a relation on A.

The **domain** of a relation R is the set of all first elements of the ordered pairs which belong to R, and the **range** of R is the set of second elements.

Example 1:

Let A = {1, 2, 3, 4}. Define a relation R on A by writing $(x, y) \in R$ if $x < y$. Then

$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$.

Example 2:

let A = {1,2,3} and R = {(1,2),(1,3),(3,2)}. Then R is a relation on A since it is a subset of $A \times A$ with respect to this relation:

1R2, 1R3, 3R2 but $(1,1) \notin R$ & $(2,1) \notin R$

The domain of R is {1,3} and

The range of R is {2,3}

Example 3:

Let A = {1, 2, 3}. Define a relation R on A by writing $(x, y) \in R$, such that $a \geq b$, list the element of R

$$aRb \leftrightarrow a \geq b \text{ , } a,b \in A$$

$\therefore R = \{(1,1),(2,1), (2,2), (3,1), (3,2), (3,3)\}$.

Example 4:

A relation on the set Z of integers is “m divides n.” A common notation for this relation is to write m|n when m divides n. Thus 6 | 30 but $7 \nmid 25$.

Representation of relations:

1) By language
2) By ordered pairs
3) By arrow form
4) By matrix form

5) By coordinates
6) By graph form

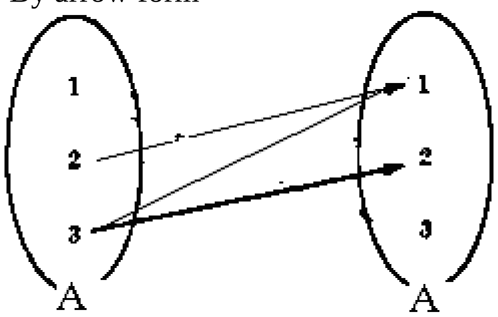
Example:

Let A = {1,2,3}, the relation R on A such that: aRb ↔ a>b; a,b∈A

1) By language:
   R={(a,b) : a,b ∈A and aRb ↔ a>b}

2) By ordered pairs
   R = {(2,1),(3,1),(3,2)}

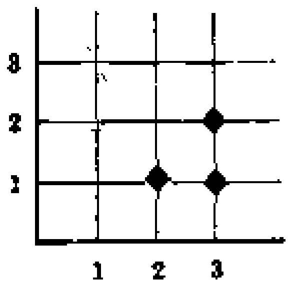
3) By arrow form



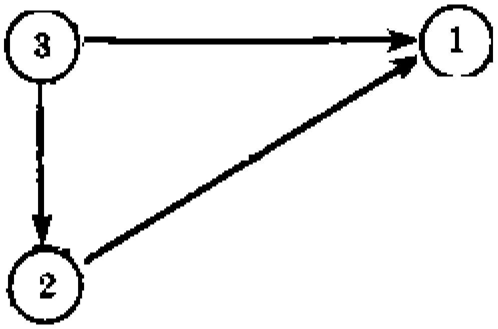
4) By matrix form

	1	2	3
1	0	0	0
2	1	0	0
3	1	1	0

5) By coordinates



6) By graph form



TYPES OF RELATIONS:
Properties of relations:

Let R be a relation on the set A

1) Reflexive : R is reflexive if : $\forall$ a $\in$A $\rightarrow$ aRa or (a,a) $\in$ R ; $\forall$ a, b $\in$A. . Thus R is not reflexive if there exists a $\in$ A such that (a, a) $\notin$ R.

2) Symmetric : aRb $\rightarrow$ bRa $\forall$ a,b $\in$A. if whenever *(a, b)* $\in$ *R* then *(b, a)* $\in$ *R*. Thus *R* is not symmetric if there exists *a, b* $\in$ *A* such that *(a, b)* $\in$ *R* but *(b, a)* $\notin$ *R*.

3) Transitive : aRb $\wedge$ bRc $\rightarrow$ aRc. that is, if whenever *(a, b), (b, c)* $\in$ *R* then *(a, c)* $\in$ *R*. Thus *R* is not transitive if there exist *a, b, c* $\in$ *R* such that *(a, b), (b, c)* $\in$ *R* but *(a, c)* $\notin$ *R*.

4) Equivalence relation : it is Reflexive & Symmetric & Transitive. That is, *R* is an equivalence relation on *S* if it has the following three properties:

a - For every *a* $\in$*S, aRa*.
b- If *aRb*, then *bRa*.
c- If *aRb* and *bRc*, then *aRc*.

5) Irreflexive : $\forall$ a $\in$A (a,a) $\notin$ R
6) AntiSymmetric : if aRb and bRa $\rightarrow$ a=b
the relations $\geq$,$\leq$ and $\subseteq$ are antisymmetric

Example 5: Consider the relation of C of set inclusion on any collection of sets:
1) A $\subset$ A for any set, so $\subset$ is reflexive
2) A $\subset$ B dose not imply B $\subset$ A, so $\subset$ is not symmetric
3) If A $\subset$ B and B $\subset$ C then A $\subset$ C, so $\subset$ is transitive
4) $\subset$ is reflexive, not symmetric & transitive, so $\subset$ is not equivalence relations
5) A $\subset$ A, so $\subset$ is not Irreflexive
6) If A $\subset$ B and B $\subset$ A then A = B, so $\subset$ is anti-symmetric

Example 6: If A ={1,2,3} and R={(1,1),(1,2),(2,1),(2,3)}
Is R equivalence relation ?
1) 2 is in A but (2,2) $\notin$ R, so R is not reflexive
2) (2,3) $\in$ R but (3,2) $\notin$ R, so R is not symmetric
3) (1,2) $\in$ R and (2,3) $\in$ R but (1,3) $\notin$ R, so R is not transitive

So R is not Equivalence relation