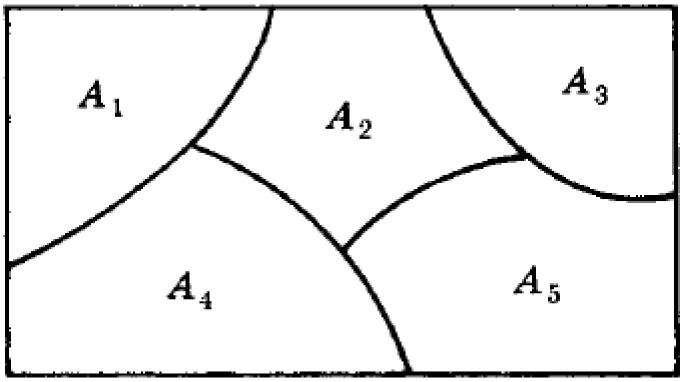
Partitions of set:

Let S be any nonempty set. A partition ($\prod$) of S is a subdivision of S into nonoverlapping, nonempty subsets. A partition of S is a collection {Ai} of non-empty subsets of S such that:

1) $A_i \neq \emptyset$, where i=1,2,3,……
2) The sets of {Ai } are mutually disjoint
   or $A_i \cap A_j = \emptyset$ where $i \neq j$.
3) $\cup A_i = S$, where $A_1 \cup A_2 \cup \ldots\ldots\ldots \cup A_i = S$

The partition of a set into five cells, A1, A2,A3,A4,A5, can be represented by Venn diagram



Example 1: let A = {1,2,3,n}

A1 = {1}, A2 = {3,n}, A3 = {2}

$\prod$ = {A1, A2, A3} is a partition on A because it satisfy the three above conditions.

Example 2 : Consider the following collections of subsets of S = {1,2,3,4,5,6,7,8,9}

(i) [{1,3,5},{2,6},{4,8,9}]
(ii) [{1,3,5},{2,4,6,8},{5,7,9}]
(iii) [{1,3,5},{2,4,6,8},{7,9}]

Then

(i) is not a partition of S since 7 in S does not belong to any of the subsets.
(ii) is not a partition of S since {1,3,5} and {5,7,9} are not disjoint.
(iii) is a partition of S.

FINITE SETS, COUNTING PRINCPLE:

A set is said to be finite if it contains exactly m distinct elements where m denotes some nonnegative integer. Otherwise, a set is said to be infinite. For example, the empty set $\emptyset$ and the set of letters of English alphabet are finite sets, whereas the set of even positive integers, {2,4,6,…..}, is infinite.

If a set A is finite, we let $n(A)$ or #(A) denote the number of elements of A.

Example: If A ={1,2,a,w} then

$n(A) = \#(A) = |A| = 4$

Lemma: If A and B are finite sets and disjoint Then $A \cup B$ is finite set and:

$$\boldsymbol{n(A \cup B) = n(A) + n(B)}$$

Theorem (Inclusion–Exclusion Principle): Suppose A and B are finite sets. Then $A \cup B$ and $A \cap B$ are finite and

$$|A \cup B| = |A| + |B| - |A \cap B|$$

That is, we find the number of elements in A or B (or both) by first adding n(A) and n(B) (inclusion) and then subtracting $n(A \cap B)$ (exclusion) since its elements were counted twice.
We can apply this result to obtain a similar formula for three sets:

Corollary:
If A,B,C are finite sets then
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Example (1) :
$A= \{1,2,3\}$
$B= \{3,4\}$
$C= \{5,6\}$

$A \cup B \cup C = \{1,2,3,4,5,6\}$
$|A \cup B \cup C| = 6$

$|A| =3$, $|B| = 2$, $|C| = 2$
$A \cap B = \{3\}$, $|A \cap B| = 1$
$A \cap C = \{ \}$, $|A \cap C| = 0$
$B \cap C = \{ \}$, $|B \cap C| = 0$
$A \cap B \cap C = \{ \}$, $|A \cap B \cap C| = 0$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
$|A \cup B \cup C| = 3 + 2 +2 -1 – 0 – 0 + 0 = 6$

Example (2):
Suppose a list A contains the 30 students in a mathematics class, and a list B contains the 35 students in an English class, and suppose there are 20 names on both lists. Find the number of students:
(a) only on list A
(b) only on list B
(c) on list $A \cup B$

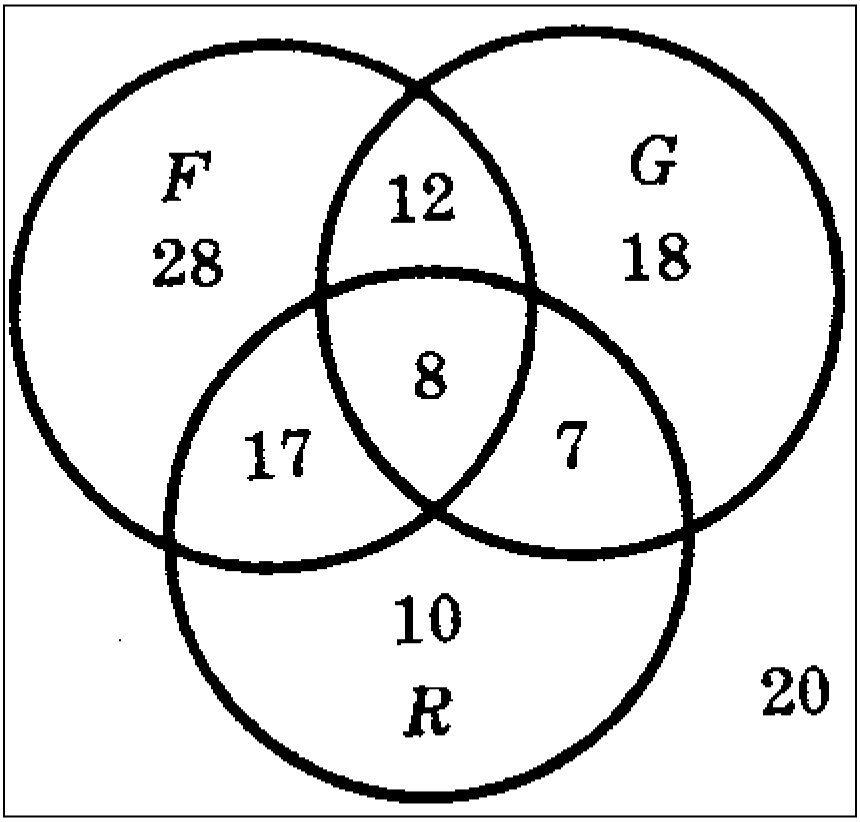
Solution:
(a) List A has 30 names and 20 are on list B; hence $30 - 20 = 10$ names are only on list A.
(b) Similarly, $35 - 20 = 15$ are only on list B.
(c) We seek $n(A \cup B)$. By inclusion–exclusion,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45.$$

Example (3):
Suppose that 100 of 120 computer science students at a college take at least one of languages: French, German, and Russian and:
65 study French (F).
45 study German (G).
42 study Russian (R).

20 study French & German $F \cap G$.
25 study French & Russian $F \cap R$.
15 study German & Russian $G \cap R$.
Find the number of students who study:

1) All three languages ($F \cap G \cap R$)
2) The number of students in each of the eight regions of the Venn diagram



Solution:
$|F \cup G \cup R| = |F| + |G| + |R| - |F \cap G| - |F \cap R| - |G \cap R| + |F \cap G \cap R|$
$100 = 65 + 45 + 42 - 20 - 25 - 15 + |F \cap G \cap R|$
$100 = 92 + |F \cap G \cap R|$
$\therefore |F \cap G \cap R| = 8$ students study the 3 languages

20 – 8 = 12 $(F \cap G) - R$
25 – 8 = 17 $(F \cap R) - G$
15 – 8 = 7 $(G \cap R) - F$

65 – 12 – 8 – 17 = 28 students study French only
45 – 12 – 8 7 = 18 students study German only
42 – 17 – 8 7 = 10 students study Russian only
120 – 100 = 20 students do not study any language