Mathematic induction:

It is useful for proving propositions that must be true for all integers or for a range of integer.
Proposition: is any statement P(n) which can be either true or false for each n in N.
Suppose P has the following two properties.

(i) P(1) is true
(ii) P(k+1) is true whenever P(k) is true

Then P is true for every positive integer $\forall$ n $\geq$ k .

Example 1: Let P be the proposition that the sum of the first n odd numbers is n^2; that is,

$$P(n): \quad 1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

Prove P (for n $\geq$ 1)

Solution:
(The nth odd number is 2n – 1, and the next odd number is 2n +1.) Observe that P(n) is true for n = 1,

(i) n=1; P(1): $2*n-1 = 1^2$
(ii) n=k; Assuming P(k) is true,
We add (2k-1)+2 = 2K + 1 to both sides of P(k), obtaining:

$$\begin{aligned} 1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) &= k^2 + (2k + 1) \\ &= (k + 1)^2 \end{aligned}$$

Which is P(k + 1). That is, P(k + 1) is true whenever P(k) is true. By the principle of mathematical induction, P is true for all n $\geq$.k.

Example 2:.

$$P(n): \quad 1 + 2 + 3 + 4 + \ldots\ldots + n = 1/2\, n(n+1)$$

$$\text{or} \quad \sum_{i=1}^{n} i = 1/2\, n\, (n + 1)$$

Prove P (for n $\geq$ 1)

solution :
n =1 (i) P(1) : left side =1
Right side =1/2 * 1 * (2) =1
(ii) let P(k) is true ; n=k

$$1 + 2 + 3 + 4 + \ldots\ldots + k = 1/2 * k * (k+1)$$

to prove that P(k+1) is true

$$\begin{aligned} 1 + 2 + 3 + 4 + \ldots\ldots + k + (k+1) &= 1/2 * k * (k+1) + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \end{aligned}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= 1/2\,(k + 1)(k + 2)$$

So P is true for all $n \geq k$

Example 3:

Prove the following proposition (for $n \geq 0$):

$$P(n) : 1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$$

solution :

(i) P(0) : left side =1

Right side $= 2^1 - 1 = 1$

(ii) Assuming P(k) is true ; n=k

$$P(k) : 1 + 2 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 1$$

We add 2k+1 to both sides of P(k), obtaining

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$
$$= 2(2^{k+1}) - 1 = 2^{k+2} - 1$$

which is P(k +1). That is, P(k +1) is true whenever P(k) is true. By the principle of induction, P(n) is true for all n.

Homework:

Prove by induction:

1) $2 + 4 + 6 + \ldots\ldots + 2n = n\,(n + 1)$
2) $1 + 4 + 7 + \ldots\ldots + (3n - 2) = 1/2\, n\,(3n - 1)$