Lectuer (1) :Differential Equations

A *differential equation* is an equation involving an unknown function and its derivatives.

Example : The following are differential equations involving the unknown function *y*.

$$\frac{dy}{dx} = 5x + 3 \tag{1.1}$$

$$e^y \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 1 \tag{1.2}$$

$$4\frac{d^3y}{dx^3} + (\sin x)\frac{d^2y}{dx^2} + 5xy = 0 \tag{1.3}$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + 3y\left(\frac{dy}{dx}\right)^7 + y^3\left(\frac{dy}{dx}\right)^2 = 5x \tag{1.4}$$

$$\frac{\partial^2 y}{\partial t^2} - 4\frac{\partial^2 y}{\partial x^2} = 0 \tag{1.5}$$

A differential equation is an *ordinary differential equation* if the unknown function depends on only one independent variable. If the unknown function depends on two or more independent variables, the differential equation is a *partial differential equation*. *In this book we will be concerned solely with ordinary differential equations.*

Example 1.2: Equations 1.1 through 1.4 are examples of ordinary differential equations, since the unknown function *y* depends solely on the variable *x*. Equation 1.5 is a partial differential equation, since *y* depends on both the independent variables *t* and *x*.

Note!

The *order* of a differential equation is the order of the highest derivative appearing in the equation.

Example 1.3: Equation 1.1 is a first-order differential equation; 1.2, 1.4, and 1.5 are second-order differential equations. (Note in 1.4 that the order of the highest derivative appearing in the equation is two.) Equation 1.3 is a third-order differential equation.

Notation

The expressions $y', y'', y''', y^{(4)}, \ldots, y^{(n)}$ are often used to represent, respectively, the first, second, third, fourth, . . ., nth derivatives of y with respect to the independent variable under consideration. Thus, y'' represents d^2y/dx^2 if the independent variable is x, but represents d^2y/dp^2 if the independent variable is p. Observe that parenthesis are used in $y^{(n)}$ to distinguish it from the nth power, y^n. If the independent variable is time, usually denoted by t, primes are often replaced by dots. Thus, $\dot{y}$, $\ddot{y}$, and $\dddot{y}$ represent, dy/dt, d^2y/dt^2, and d^3y/dt^3, respectively.

Solutions

A *solution* of a differential equation in the unknown function y and the independent variable x on the interval P is a function $y(x)$ that satisfies the differential equation identically for all x in P.

Example 1.4: Is $y(x) = c_1 \sin 2x + c_2 \cos 2x$, where c_1 and c_2 are arbitrary constants, a solution of $y'' + 4y = 0$?

Differentiating y, we find $y' = 2c_1 \cos 2x - 2c_2 \sin 2x$ and $y'' = -4c_1 \sin 2x - 4c_2 \cos 2x$. Hence,

$$
\begin{aligned}
y'' + 4y &= (-4c_1 \sin 2x - 4c_2 \cos 2x) + 4(c_1 \sin 2x + c_2 \cos 2x) \\
&= (-4c_1 + 4c_1)\sin 2x + (-4c_2 + 4c_2)\cos 2x \\
&= 0
\end{aligned}
$$

Thus, $y = c_1 \sin 2x + c_2 \cos 2x$ satisfies the differential equation for all values of x and is a solution on the interval $(-\infty, \infty)$.

Example 1.5: Determine whether $y = x^2 - 1$ is a solution of $(y')^4 + y^2 = -1$.

Note that the left side of the differential equation must be nonnegative for every real function $y(x)$ and any x, since it is the sum of terms raised to the second and fourth powers, while the right side of the equation is negative. Since no function $y(x)$ will satisfy this equation, the given differential equation has no solutions.

We see that some differential equations have infinitely many solutions (Example 1.4), whereas other differential equations have no solutions (Example 1.5). It is also possible that a differential equation has exactly one solution. Consider $(y')^4 + y^2 = 0$, which for reasons identical to those given in Example 1.5 has only one solution $y \equiv 0$.

You Need to Know

A *particular solution* of a differential equation is any one solution. The *general solution* of a differential equation is the set of all solutions.

Example 1.6: The general solution to the differential equation in Example 1.4 can be shown to be (see Chapters Four and Five) $y = c_1 \sin 2x + c_2 \cos 2x$. That is, every particular solution of the differential equation has this general form. A few particular solutions are: (a) $y = 5\sin 2x - 3\cos 2x$ (choose $c_1 = 5$ and $c_2 = -3$), (b) $y = \sin 2x$ (choose $c_1 = 1$ and $c_2 = 0$), and (c) $y \equiv 0$ (choose $c_1 = c_2 = 0$).

The general solution of a differential equation cannot always be expressed by a single formula. As an example consider the differential equation $y' + y^2 = 0$, which has two particular solutions $y = 1/\ x$ and $y \equiv 0$.

Example 1.7: The problem $y'' + 2y' = e^x;\ y(\pi) = 1,\ y'(\pi) = 2$ is an initial value problem, because the two subsidiary conditions are both given at $x = \pi$. The problem $y'' + 2y' = e^x;\ y(0) = 1,\ y(1) = 1$ is a boundary-value problem, because the two subsidiary conditions are given at $x = 0$ and $x = 1$.

A solution to an initial-value or boundary-value problem is a function $y(x)$ that both solves the differential equation and satisfies all given subsidiary conditions.

Standard and Differential Forms

Standard form for a first-order differential equation in the unknown function $y(x)$ is

$$y' = f(x, y) \tag{1.6}$$

where the derivative y' appears only on the left side of 1.6. Many, but not all, first-order differential equations can be written in standard form by algebraically solving for y' and then setting $f(x,y)$ equal to the right side of the resulting equation.

The right side of 1.6 can always be written as a quotient of two other functions $M(x,y)$ and $-N(x,y)$. Then 1.6 becomes $dy / dx = M(x, y) / -N(x, y)$, which is equivalent to the *differential form*

$$M(x, y)dx + N(x, y)dy = 0 \tag{1.7}$$

Linear Equations

Consider a differential equation in standard form 1.6. If $f(x,y)$ can be written as $f(x, y) = -p(x)y + q(x)$ (that is, as a function of x times y, plus another function of x), the differential equation is linear. First-order linear differential equations can always be expressed as

$$y' + p(x)y = q(x) \tag{1.8}$$

Linear equations are solved in Chapter Two.

Bernoulli Equations

A Bernoulli differential equation is an equation of the form

$$y' + p(x)y = q(x)y^n \tag{1.9}$$

where n denotes a real number. When $n = 1$ or $n = 0$, a Bernoulli equation reduces to a linear equation. Bernoulli equations are solved in Chapter Two.

Homogeneous Equations

A differential equation in standard form (1.6) is *homogeneous* if

$$f(tx, ty) = f(x, y) \tag{1.10}$$

for every real number t. Homogeneous equations are solved in Chapter Two.

Note!

In the general framework of differential equations, the word "homogeneous" has an entirely different meaning (see Chapter Four). Only in the context of first-order differential equations does "homogeneous" have the meaning defined above.

Separable Equations

Consider a differential equation in differential form (1.7). If $M(x,y) = A(x)$ (a function only of x) and $N(x,y) = B(y)$ (a function only of y), the differential equation is *separable*, or has its *variables separated*. Separable equations are solved in Chapter Two.

Exact Equations

A differential equation in differential form (1.7) is exact if

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} \tag{1.11}$$

Exact equations are solved in Chapter Two (where a more precise definition of exactness is given).

Chapter 2

Solutions of First-Order Differential Equations

In This Chapter:

- ✔ *Separable Equations*
- ✔ *Homogeneous Equations*
- ✔ *Exact Equations*
- ✔ *Linear Equations*
- ✔ *Bernoulli Equations*
- ✔ *Solved Problems*

Separable Equations

General Solution

The solution to the first-order separable differential equation (see Chapter One).

$$A(x)dx + B(y)dy = 0 \tag{2.1}$$

is

$$\int A(x)dx + \int B(y)dy = c \tag{2.2}$$

where c represents an arbitrary constant.

(See Problem 2.1)

The integrals obtained in Equation 2.2 may be, for all practical purposes, impossible to evaluate. In such case, numerical techniques (see Chapter 14) are used to obtain an approximate solution. Even if the indicated integrations in 2.2 can be performed, it may not be algebraically possible to solve for y explicitly in terms of x. In that case, the solution is left in implicit form.

The solution to the initial-value problem

$$A(x)dx + B(y)dy = 0; \qquad y(x_0) = y_0 \tag{2.3}$$

can be obtained, as usual, by first using Equation 2.2 to solve the differential equation and then applying the initial condition directly to evaluate c.

Alternatively, the solution to Equation 2.3 can be obtained from

$$\int^{x} A(s)ds + \int^{y} B(t)dt = 0 \tag{2.4}$$

where s and t are variables of integration.

Example:Find G.S.for D.E

$dy/dx = x^2 + 2 / y$

$=$

This equation may be rewritten in the differential form

$$(x^2 + 2)dx - ydy = 0$$

which is separable with $A(x) = x^2 + 2$ and $B(y) = -y$. Its solution is

$$\int (x^2 + 2)dx - \int ydy = c$$

or

$$\frac{1}{3}x^3 + 2x - \frac{1}{2}y^2 = c$$

Solving for y, we obtain the solution in implicit form as

$$y^2 = \frac{2}{3}x^3 + 4x + k$$

with $k = -2c$. Solving for y explicitly, we obtain the two solutions

$$y = \sqrt{\frac{2}{3}x^3 + 4x + k} \quad \text{and} \quad y = -\sqrt{\frac{2}{3}x^3 + 4x + k}$$